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THE NORMALIZED RICCI FLOW ON FOUR-MANIFOLDS AND EXOTIC SMOOTH STRUCTURES*

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Abstract. We shall prove that, for every natural number ℓ there exists a closed topological 4-manifold X_ℓ which admits smooth structures for which non-singular solutions of the normalized Ricci flow exist, but also admits smooth structures for which no non-singular solution of the normalized Ricci flow exists. Hence, in dimension 4, smooth structures become definite obstructions to the existence of non-singular solutions to the normalized Ricci flow.

Key words. Ricci flow, non-singular solution, exotic smooth structure.

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1. Introduction.

Let X be a closed oriented Riemannian manifold X of dimension $n \geq 2$. The normalized Ricci flow [14] on X is the following evolution equation:

$$\frac{\partial}{\partial t} g = -2Ric_g + \frac{2}{n}\left(\frac{\int_X s_g d\mu_g}{\int_X d\mu_g}\right) g, \tag{1.1}$$

where Ric_g and s_g denote respectively the Ricci curvature and the scalar curvature of the evolving Riemannian metric g, $vol_g := \int_X d\mu_g$ and $d\mu_g$ is the volume measure with respect to g. Recall that a solution $\{g(t)\}$ to the normalized Ricci flow on a time interval $[0, T)$ is said to be maximal if it cannot be extended past time T. A maximal solution$\{g(t)\}, t \in [0, T)$, to the normalized Ricci flow on X is called non-singular [15] if $T = \infty$ and the Riemannian curvature tensor $Rm_{g(t)}$ of $g(t)$ satisfies

$$\sup_{X \times [0,T)} |Rm_{g(t)}| < \infty.$$

As a pioneer work, Hamilton [14] proved that there exists a unique non-singular solution to the normalized Ricci flow on 3-manifolds if the initial metric is positive Ricci curvature. Moreover, Hamilton [15] classified non-singular solutions to the normalized Ricci flow on 3-manifolds. These fundamental works were very important for understanding long-time behavior of solutions of the Ricci flow on 3-manifolds. On the other hand, though many authors study the properties of non-singular solutions in higher dimensions, the existence and non-existence of non-singular solutions to the normalized Ricci flow are still mysterious in general. The main purpose of the current article is to study, from the gauge theoretical point of view, this problem in case of dimension four. We shall point out that the difference between existence and non-existence of non-singular solutions to the normalized Ricci flow strictly depend on one's choice of smooth structure. The main result is Theorem 1.3 stated below.

In an important work [10], Fang, Zhang and Zhang studied the properties of non-singular solutions in higher dimensions. Let X be a closed oriented smooth 4-manifold and suppose that there is a non-singular solution to the normalized Ricci flow on X. Then, one of fundamental discoveries due to [10] is that if the solution satisfies $\hat{s}_{g(t)} \leq -c < 0$, where the constant c is independent of t and define as

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$\hat{s}_g := \min_{x \in X} s_g(x)$ for a given Riemannian metric g, the 4-manifold X must satisfy the following topological constraint on the Euler characteritic $\chi(X)$ and signature $\tau(X)$ of X:

$$2\chi(X) \geq 3|\tau(X)|.$$

In this article, we shall call this Fang-Zhang-Zhang inequality (or, for brevity, FZZ inequality) we shall also call $2\chi(X) > 3|\tau(X)|$ the strict FZZ inequality. The FZZ inequality gives us, under the above condition on $\hat{s}_{g(t)}$, the only known topological obstruction to the existence of non-singular solutions to the normalized Ricci flow. It is also known that any Einstein 4-manifold X must satisfy the same bound $2\chi(X) \geq 3|\tau(X)|$ which is so called Hitchin-Thorpe inequality [40, 16]. Since any Einstein metric is a non-singular solution, FZZ inequality can be seen as a generalization of Hitchin-Thorpe inequality to Ricci flow case.
On the other hand, there is a natural diffeomorphism invariant arising from a variational problem for the total scalar curvature of Riemannain metrics. As was conjectured by Yamabe [43], and later proved by Trudinger, Aubin, and Schoen [2, 30, 37, 41], every conformal class on any smooth compact manifold contains a Riemannian metric of constant scalar curvature. For each conformal class $[g] = \{vg \mid v : X \to \mathbb{R}^+\}$, we are able to consider an associated number $Y_{[g]}$ which is so called Yamabe constant of the conformal class $[g]$. The Trudinger-Aubin-Schoen theorem tells us that this number is actually realized as the constant scalar curvature of some unit volume metric in the conformal class $[g]$. Then, Kobayashi [21] and Schoen [38] independently introduced the following invariant which is so called Yamabe invariant of X:

$$\mathcal{Y}(X) = \sup_{\mathcal{C}} Y_{[g]},$$

where $\mathcal{C}$ is the set of all conformal classes on X. It is known that $\mathcal{Y}(X) \leq 0$ if and only if X does not admit a metric of positive scalar curvature. It is also known that the Yamabe invariant is sensitive to the choice of smooth structure of a 4-manifold. After the celebrated works of Donaldson [9] and Freedman [11], it now turns out that many exotic smooth structures exist in dimension 4. We are able to construct many examples of topological 4-manifolds admitting distinct smooth structures for which values of the Yamabe invariants are different by using the result [18] of LeBrun with the author of the current article. We shall observe that the above condition $\hat{s}_{g(t)} \leq -c < 0$ is closely related to the negativity of the Yamabe invariant. More precisely, in Proposition 2.2 proved in Section 2, we shall see that the condition $\hat{s}_{g(t)} \leq -c < 0$ is always satisfied for any solution to the normalized Ricci flow if a given smooth Riemannian manifold X of dimension $n \geq 3$ has $\mathcal{Y}(X) < 0$. By results proved by [1] and [10], we are also able to see that, if a compact topological 4-manifold M admits a smooth structure Z with $\mathcal{Y} < 0$ and for which there exists a non-singular solution to the normalized Ricci flow, then the strict FZZ inequality $2\chi(Z) > 3|\tau(Z)|$ must hold, where we denote the compact topological 4-manifold M admits the smooth structure Z by Z. Let us here emphasize that $2\chi(Z) > 3|\tau(Z)|$ is just a topological constraint, is not a differential topological one. This observation and the special feature of smooth structures in dimension 4 naturally lead us to ask the following:

QUESTION 1.2. Let X be any compact topological 4-manifold which admits at least two distinct smooth structures Z^i with negative Yamabe invariant $\mathcal{Y} < 0$.

Suppose that, for at least one of these smooth structures Z^i, there exist non-singular solutions to the the normalized Ricci flow. Then, for every other smooth structure Z^i with $\mathcal{Y} < 0$, are there always non-singular solutions to the normalized Ricci flow?

Since X admits, for at least one of these smooth structures Z^i, non-singular solutions to the the normalized Ricci flow, we are able to conclude that $2\chi(Z_i) > 3|\tau(Z_i)|$ holds for every i. Notice that this is equivalent to $2\chi(X) > 3|\tau(X)|$. Hence, even if there are always non-singular solutions to the normalized Ricci flow for every other smooth structure Z^i, it dose not contradict the strict FZZ inequality. Interestingly, the main result of the current article tells us that the answer to Question 1.2 is negative as follows:

THEOREM 1.3. *For every natural number ℓ, there exist a simply connected compact topological non-spin 4-manifold X_ℓ satisfying the following properties:*

(1) X_ℓ admits at least ℓ different smooth structures M^i_ℓ with $\mathcal{Y} < 0$ and for which there exist non-singular solutions to the the normalized Ricci flow. Moreover the existence of the solutions forces the strict FZZ inequality $2\chi > 3|\tau|$ as a topological constraint,

(2) X_ℓ also admits infinitely many different smooth structures N^j_ℓ with $\mathcal{Y} < 0$ and for which there exists no non-singular solution to the normalized Ricci flow.

Notice that Freedman's classification [11] implies that X_ℓ above must be homeomorphic to a connected sum $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$, where $\mathbb{C}P^2$ is the complex projective plane and $\overline{\mathbb{C}P^2}$ is the complex projective plane with the reversed orientation, and p and q are some appropriate positive integers which depend on the natural number ℓ. Notice also that, for the standard smooth structure on $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$, we have $\mathcal{Y} > 0$ because, by a result of Schoen and Yau [36] or Gromov and Lawson [13], there exists a Riemannian metric of positive scalar curvature for such a smooth structure. Hence, smooth structures which appear in Theorem 1.3 are far from the standard smooth structure. To the best of our knowledge, Theorem 1.3 is the first result which shows that, in dimension four, smooth structures become definite obstructions to the existence of non-singular solutions to the normalized Ricci flow. Namely, Theorem 1.3 teaches us that the existence or non-existence of non-singular solutions depends strictly on the diffeotype of a 4-manifold and it is not determined by homeotype alone. This gives a new insight into the property of solutions to the Ricci flow on 4-manifolds.

To prove the non-existence result in Theorem 1.3, we need to prove new obstructions to the existence of non-singular solutions to the normalized Ricci flow. Indeed, it is the main non-trivial step in the proof of Theorem 1.3. For instance, we shall prove that, for any closed symplectic 4-manifold with $b^+(X) \geq 2$ and $2\chi(X) + 3\tau(X) > 0$, where $b^+(X)$ stands for the dimension of a maximal positive definite subspace of $H^2(X, \mathbb{R})$ with respect to the intersection form, there is no non-singular solution of the normalized Ricci flow on a connected sum $M := X\#k\overline{\mathbb{C}P^2}$ if $3k > 2\chi(X) + 3\tau(X)$ holds. See Corollary 2.15 stated below. See also Theorems 2.12 and 2.16 below for more general obstructions. We shall use the Seiberg-Witten monopole equations [42] to prove the obstructions. We should notice that, under the same condition with the above, LeBrun [27] firstly proved that the above 4-manifold M cannot admit any Einstein metric by using Seiberg-Witten monopole equations [42]. However, notice that non-singular solutions do not necessarily converge to smooth Einstein metrics. See also [10]. Hence, non-existence result on non-singular solutions proved in the current article never follow from the obstruction of LeBrun in general. In this sense, the obstructions proved in Section 2 are new.

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2. Ricci flow solutions with bounded scalar curvature. Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$, and $[g] = \{ug \mid u : X \to \mathbb{R}^+\}$ is the conformal class of an arbitrary metric g. Trudinger, Aubin, and Schoen [2, 30, 37, 41] proved every conformal class on any smooth compact manifold contains a Riemannian metric of constant scalar curvature. Such a metric $\hat{g}$ can be constructed by minimizing the Einstein-Hilbert functional

$$\hat{g} \mapsto \frac{\int_X s_{\hat{g}}\ d\mu_{\hat{g}}}{\left(\int_X d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}}$$

among all metrics conformal to g. Notice that, by setting $\hat{g} = u^{4/(n-2)}g$, we have

$$\frac{\int_X s_{\hat{g}}\ d\mu_{\hat{g}}}{\left(\int_X d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}} = \frac{\int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2\right] d\mu_g}{\left(\int_X u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}}.$$

As was already mentioned in Introduction, associated to each conformal class $[g]$, we are able to define the following number which is called Yamabe constant of $[g]$:

$$Y_{[g]} = \inf_{\hat{g} \in [g]} \frac{\int_X s_{\hat{g}}\ d\mu_{\hat{g}}}{\left(\int_X d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}}.$$

Equivalently,

$$Y_{[g]} = \inf_{u \in C^\infty_+(X)} \frac{\int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2\right] d\mu_g}{\left(\int_X u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}},$$

where $C^\infty_+(X)$ is the set of all positive functions $u : X \to \mathbb{R}^+$. Then the Yamabe invariant [21, 38] of X is defined by $\mathcal{Y}(X) = \sup_{[g] \in \mathcal{C}} Y_{[g]}$, where $\mathcal{C}$ is the set of all conformal classes on X. It is also known [26] that the following holds if $\mathcal{Y}(X) \leq 0$:

$$\mathcal{Y}(X) = -\left(\inf_g \int_X |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}}, \tag{2.1}$$

where supremum is taken over all smooth metrics g on X.

PROPOSITION 2.2. *Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$ and with $\mathcal{Y}(X) < 0$. If there is a solution $\{g(t)\}$, $t \in [0, T)$, to the normalized Ricci flow, then the solution satisfies*

$$\hat{s}_{g(t)} := \min_{x \in X} s_{g(t)}(x) \leq \frac{\mathcal{Y}(X)}{(vol_{g(0)})^{2/n}} < 0.$$

Proof. Suppose that there is a solution $\{g(t)\}$, $t \in [0,T)$ to the normalized Ricci flow. Let us consider the Yamabe constant $Y_{[g(t)]}$ of a conformal class $[g(t)]$ of a metric $g(t)$ for any $t \in [0,T)$. By definition, we have

$$\mathcal{Y}(X) \geq Y_{[g(t)]} = \inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X \left[s_{g(t)} u^2 + 4\frac{n-1}{n-2} |\nabla u|^2 \right] d\mu_{g(t)}}{\left(\int_X u^{2n/(n-2)} d\mu_{g(t)} \right)^{(n-2)/n}}.$$

We therefore obtain

$$\begin{aligned} \mathcal{Y}(X) &\geq \inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X \left(\min_{x \in X} s_{g(t)} u^2 + 4\dfrac{n-1}{n-2} |\nabla u|^2 \right) d\mu_{g(t)}}{\left(\int_X u^{2n/(n-2)} d\mu_{g(t)} \right)^{n-2/n}} \\ &\geq \hat{s}_{g(t)} \left(\inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X u^2 d\mu_{g(t)}}{\left(\int_X u^{2n/(n-2)} d\mu_{g(t)} \right)^{n-2/n}} \right), \end{aligned}$$

where notice that $\hat{s}_g := \min_{x \in X} s_g(x)$. If $\hat{s}_{g(t)} \geq 0$ holds, then the above estimate tells us that $\mathcal{Y}(X) \geq 0$. Since we assume that $\mathcal{Y}(X) < 0$, we are able to conclude that $\hat{s}_{g(t)} < 0$ must hold.

On the other hand, the Hölder inequality tells us that the following inequality holds:

$$\begin{aligned} \int_X u^2 d\mu_{g(t)} &\leq \left(\int_X u^{2n/n-2} d\mu_{g(t)} \right)^{n-2/n} \left(\int_X d\mu_{g(t)} \right)^{2/n} \\ &= \left(\int_X u^{2n/n-2} d\mu_{g(t)} \right)^{n-2/n} (vol_{g(t)})^{2/n}. \end{aligned}$$

This implies that

$$\inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X u^2 d\mu_{g(t)}}{\left(\int_X u^{2n/n-2} d\mu_{g(t)} \right)^{n-2/n}} \leq (vol_{g(t)})^{2/n}.$$

Since we have $\hat{s}_{g(t)} < 0$, this also implies

$$\hat{s}_{g(t)} \left(\inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X u^2 d\mu_{g(t)}}{\left(\int_X u^{2n/n-2} d\mu_{g(t)} \right)^{n-2/n}} \right) \geq \hat{s}_{g(t)} (vol_{g(t)})^{2/n}.$$

We therefore obtain

$$\begin{aligned} \mathcal{Y}(X) &\geq \hat{s}_{g(t)} \left(\inf_{u \in C^{\infty}_{+}(X)} \frac{\int_X u^2 d\mu_{g(t)}}{\left(\int_X u^{2n/n-2} d\mu_{g(t)} \right)^{n-2/n}} \right) \\ &\geq \hat{s}_{g(t)} (vol_{g(t)})^{2/n}. \end{aligned}$$

On the other hand, notice that the normalized Ricci flow preserves the volume of the solution. Therefore, we have $vol_{g(t)} = vol_{g(0)}$ for ant $t \in [0,T)$. Hence, we get the desired bound for any $t \in [0,T)$:

$$\hat{s}_{g(t)} \leq \frac{\mathcal{Y}(X)}{(vol_{g(t)})^{2/n}} = \frac{\mathcal{Y}(X)}{(vol_{g(0)})^{2/n}} < 0. \qquad \square$$

To prove the main result of the current article, we shall use the Seiberg-Witten theory [42, 33]. For the convenience of the reader who is unfamiliar with Seiberg-Witten theory, we shall recall briefly the definition of the Seiberg-Witten monopole equations. Let X be a closed oriented Riemannian 4-manifold with $b^+(X) \geq 2$. Recall that a spin^c-structure Γ_X on a smooth Riemannian 4-manifold X induces a pair of spinor bundles $S^{\pm}_{\Gamma_X}$ which are Hermitian vector bundles of rank 2 over X. A Riemannian metric on X and a unitary connection A on the determinant line bundle $\mathcal{L}_{\Gamma_X} := det(S^+_{\Gamma_X})$ induce the twisted Dirac operator $\mathcal{D}_A : \Gamma(S^+_{\Gamma_X}) \longrightarrow \Gamma(S^-_{\Gamma_X})$. The Seiberg-Witten monopole equations over X are the following system for a unitary connection $A \in \mathcal{A}_{\mathcal{L}_{\Gamma_X}}$ and a spinor $\phi \in \Gamma(S^+_{\Gamma_X})$:

$$\mathcal{D}_A \phi = 0, \; F^+_A = iq(\phi), \tag{2.3}$$

where F^+_A is the self-dual part of the curvature of A and $q : S^+_{\Gamma_X} \to \wedge^+$ is a certain natural real-quadratic map satisfying $|q(\phi)| = (1/2\sqrt{2})|\phi|^2$, where $\wedge^+$ is the bundle of self-dual 2-forms. We recall the definition of monopole class [22, 27, 17, 18, 29].

Definition 2.4. Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 2$. An element $\mathfrak{a} \in H^2(X, \mathbb{Z})/\text{torsion} \subset H^2(X, \mathbb{R})$ is called monopole class of X if there exists a spin^c-structure Γ_X with $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X}) = \mathfrak{a}$ which has the property that the corresponding Seiberg-Witten monopole equations (2.3) have a solution for every Riemannian metric on X. Here $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X})$ is the image of the first Chern class $c_1(\mathcal{L}_{\Gamma_X})$ of the complex line bundle $\mathcal{L}_{\Gamma_X}$ in $H^2(X, \mathbb{R})$.

There are several ways to detect the existence of monopole classes. For example, if X is a closed symplectic 4-manifold X with $b^+(X) \geq 2$, then $c_1(X)$ is monopole class by the celebrated result of Taubes [39], where $c_1(X)$ is the first Chern class of the canonical bundle of X. This is proved by thinking the moduli space of solutions of (2.3) as a cycle which represents an element of the homology of a certain configuration space. For any closed oriented smooth 4-manifold X with $b^+(X) \geq 2$, one can define the integer valued Seiberg-Witten invariant $SW_X(\Gamma_X) \in \mathbb{Z}$ for any spin^c-structure Γ_X by integrating a cohomology class on the moduli space of solutions of (2.3) associated with Γ_X:

$$SW_X : Spin(X) \longrightarrow \mathbb{Z},$$

where $Spin(X)$ is the set of all spin^c-structures on X. Taubes [39] proved that $SW_X(\hat{\Gamma}_X) \equiv 1 \pmod 2$ holds for any closed symplectic 4-manifold X with $b^+(X) \geq 2$, where $\hat{\Gamma}_X$ is the canonical spin^c-structure induced from the symplectic structure. This implies that $c_1(X)$ is a monopole class of X. There is a sophisticated refinement of the idea of this construction. It detects the presence of a monopole class by an element of a stable cohomotopy group. This is due to Bauer and Furuta [3, 4]. They interpreted (2.3) as a map between two Hilbert bundles over the Picard tours of a 4-manifold X. The map is called the Seiberg-Witten map. Roughly speaking, the cohomotopy refinement of the Seiberg-Witten invariant is defined by taking an stable cohomotopy class of the finite dimensional approximation of the Seiberg-Witten map. The invariant takes its value in a certain complicated stable cohomotopy group. We shall call the invariant stable cohomotopy Seiberg-Witten invariant. The non-triviality of this invariant also implies the existence of monopole classes.
For a Riemannian metric g on X, the cohomology group $H^2(X; \mathbb{R})$ is identified

with the space of g-harmonic 2-forms, and is decomposed into the direct sum of the g-self-dual part $\mathcal{H}_g^+$ and the g-anti-self-dual part $\mathcal{H}_g^-$. For a cohomology class $\mathfrak{a} \in H^2(X;\mathbb{R})/\operatorname{Tor} \subset H^2(X;\mathbb{R})$, let $\mathfrak{a}^+ \in \mathcal{H}_g^+$ be the g-self-dual part of $\mathfrak{a}$. We shall use the following curvature bounds proved by LeBrun [27, 29]:

Proposition 2.5. *Let (X,g) be a closed oriented Riemannian 4-manifold. If $\mathfrak{a}$ is a monopole class of X, then*

$$\int_X s_g^2 d\mu_g \geq 32\pi^2(\mathfrak{a}^+)^2, \tag{2.6}$$

$$\int_X (s_g - \sqrt{6}|W_+|_g)^2 d\mu_g \geq 72\pi^2(\mathfrak{a}^+)^2 \tag{2.7}$$

hold for every Riemannian metric g on X, where s_g is the scalar curvature and W_+ is the self-dual part of the Weyl curvature.

By using (2.6), we obtain a result on upper bounds on Yamabe invariants:

Proposition 2.8. *Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. Let X be a closed almost-complex 4-manifold with $b^+(X) \geq 2$ and $c_1^2(X) = 2\chi(X) + 3\tau(X) \geq 0$. Assume that $SW_X(\Gamma_X) \neq 0$ holds, where Γ_X is the spin^c-structure compatible with the almost-complex structure. Then the Yamabe invariant of $M = X\#N$ satisfies*

$$\mathcal{Y}(M) \leq -4\pi\sqrt{2c_1^2(X)} \leq 0. \tag{2.9}$$

Proof. It is known [3, 5] that there is a comparison map between the stable cohomotopy refinement of Seiberg-Witten invariant and the integer valued Seiberg-Witten invariant. In particular, Proposition 5.4 in [5] tells us that the comparison map becomes isomorphism when the given 4-manifold is almost-complex and $b^+ > 1$. Hence, the value of the stable cohomotopy Seiberg-Witten invariant [3, 4] of X for the spin^c-structure Γ_X compatible with the almost complex structure is non-trivial if $SW_X(\Gamma_X) \neq 0$ holds. Then, the proofs of Proposition 6 and Corollary 8 in [18] (see also Theorem 8.8 in [5]) imply that M has non-trivial stable cohomotopy Seiberg-Witten invariants and

$$\mathfrak{a} := \pm c_1(X) + \sum_{i=1}^{k} \pm E_i$$

is a monopole class of $X\#N$, where $E_1, E_2, \cdots, E_k$ be a set of generators for $H^2(N,\mathbb{Z})$/torsion relative to which the intersection form is diagonal, $c_1(X)$ is the first Chern class of the canonical bundle of the almost-complex 4-manifold X, the $\pm$ signs are arbitrary and are independent of one another. Then, by the standard argument (for instance, see Corollary 11 of [18]), we are able to prove $(\mathfrak{a}^+)^2 \geq (c_1^+(X))^2$. Since $(c_1^+(X))^2 \geq c_1^2(X)$ holds, we obtain

$$(\mathfrak{a}^+)^2 \geq c_1^2(X). \tag{2.10}$$

By this and (2.6), we get

$$\inf_g \int_M s_g^2 d\mu_g \geq 32\pi^2 c_1^2(X). \tag{2.11}$$

On the other hand, notice that the non-triviality of stable cohomotopy Seiberg-Witten invariants of M forces that M cannot admit any metric of positive scalar curvature. This implies that $\mathcal{Y}(M) \leq 0$. Therefore, we are able to obtain the formula (2.1) in the case of $n = 4$. Hence the desired result now follows from (2.11). □

The main result of this section is the following:

THEOREM 2.12. *Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. Let X be a closed almost-complex 4-manifold with $b^+(X) \geq 2$ and $c_1^2(X) = 2\chi(X) + 3\tau(X) > 0$. Assume that $SW_X(\Gamma_X) \neq 0$ holds, where Γ_X is the spin^c-structure compatible with the almost-complex structure. Suppose that there is a long-time solution $\{g(t)\}$ of (1.1) on $M := X \# N$ with scalar curvature $|s_g| < C$ for a constant C independent of $t \in [0, +\infty)$. Then the following holds:*

$$4b_1(N) + b_2(N) \leq \frac{1}{3}c_1^2(X).$$

Proof. Suppose that there is a long-time solution $\{g(t)\}$ of (1.1) on M with scalar curvature $|s_g| < C$ for a constant C independent of t. Since we have $\mathcal{Y}(M) \leq -4\pi\sqrt{2c_1^2(X)} < 0$ by (2.9), where we also used $c_1^2(X) = 2\chi(X) + 3\tau(X) > 0$, Proposition 2.2 implies

$$\hat{s}_{g(t)} \leq -\frac{4\pi}{(vol_{g(0)})^{1/2}}\sqrt{2c_1^2(X)} < 0.$$

Hence $\{g(t)\}$ with $|s_g| < C$ satisfies $\hat{s}_{g(t)} \leq -D < 0$ by setting

$$D = -\frac{4\pi}{(vol_{g(0)})^{1/2}}\sqrt{2c_1^2(X)}.$$

Notice that D is independent of t. Therefore, the following holds by Lemma 3.1 of [10]:

$$\int_0^\infty \int_M | \overset{\circ}{r}_{g(t)} |^2 d\mu_{g(t)} dt < \infty.$$

This implies that

$$\int_m^{m+1} \int_M | \overset{\circ}{r}_{g(t)} |^2 d\mu_{g(t)} dt \longrightarrow 0 \tag{2.13}$$

holds when $m \to +\infty$. On the other hand, we also get the following by the Chern-Gauss-Bonnet formula and the Hirzebruch signature formula,

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s_{g(t)}^2}{24} + 2|W_+|_{g(t)}^2 - \frac{| \overset{\circ}{r}_{g(t)} |^2}{2} \right) d\mu_g(t).$$

Therefore, we have

$$\begin{aligned}(2\chi + 3\tau)(M) &= \int_m^{m+1} \Big((2\chi + 3\tau)(M) \Big) dt \\ &= \frac{1}{4\pi^2}\int_m^{m+1} \int_M \left(\frac{s_{g(t)}^2}{24} + 2|W_+|_{g(t)}^2 - \frac{| \overset{\circ}{r}_{g(t)} |^2}{2} \right) d\mu_g(t) dt.\end{aligned}$$

By taking $m \to +\infty$ and using (2.13), we get

$$(2\chi + 3\tau)(M) = \lim_{m\to\infty} \frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2_{g(t)}\right) d\mu_g(t)dt. \quad (2.14)$$

On the other hand, the Cauchy-Schwarz inequality and the triangle inequality imply that the following (see [27]):

$$\int_M \left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2\right) d\mu_{g(t)} \geq \frac{1}{27}\int_M \left(s_{g(t)} - \sqrt{6}|W^+|\right)^2 d\mu_{g(t)}.$$

This inequality, (2.7) and (2.10) imply

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2\right) d\mu_{g(t)} \geq \frac{2}{3}(2\chi + 3\tau)(X) = \frac{2}{3}c_1^2(X).$$

Therefore, we have

$$\frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2_{g(t)}\right) d\mu_{g(t)}dt \geq \int_m^{m+1} \frac{2}{3}c_1^2(X)dt = \frac{2}{3}c_1^2(X).$$

Hence we get

$$\lim_{m\to\infty} \frac{1}{4\pi^2}\int_m^{m+1}\int_M \left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2_{g(t)}\right) d\mu_{g(t)}dt \geq \frac{2}{3}c_1^2(X).$$

This inequality and (2.14) imply

$$(2\chi + 3\tau)(M) \geq \frac{2}{3}c_1^2(X).$$

We also have $2\chi(M)+3\tau(M) = c_1^2(X)-(4b_1(N)+b_2(N))$, where we used $b^+(N) = 0$. Therefore, the following holds:

$$c_1^2(X) - (4b_1(N) + b_2(N)) \geq \frac{2}{3}c_1^2(X).$$

This implies the desired result. □

Theorem 2.12 and the celebrated result of Taubes [39] imply

Corollary 2.15. *Let X be a closed symplectic 4-manifold with $b^+(X) \geq 2$ and $c_1^2(X) = 2\chi(X) + 3\tau(X) > 0$. Suppose that there is a long-time solution $\{g(t)\}$ of (1.1) on $M := X\#k\overline{\mathbb{C}P^2}$ with scalar curvature $|s_g| < C$ for a constant C independent of $t \in [0, +\infty)$. Then*

$$k \leq \frac{1}{3}c_1^2(X).$$

We are also able to prove the following result:

Theorem 2.16. *For $i = 1, 2, 3, 4$, let X_i be a closed almost-complex 4-manifold whose integer valued Seiberg-Witten invariant satisfies $SW_{X_i}(\Gamma_{X_i}) \equiv 1 \pmod 2$,* *where Γ_{X_i} is the spin^c-structure compatible with the almost-complex structure. Assume that the following conditions are satisfied:*

$$b_1(X_i) = 0,\ b^+(X_i) \equiv 3 \pmod 4,\ \sum_{i=1}^{4} b^+(X_i) \equiv 4 \pmod 8, \tag{2.17}$$

$$\sum_{i=1}^{j} c_1^2(X_i) = \sum_{i=1}^{j} (2\chi(X_i) + 3\tau(X_i)) > 0, \tag{2.18}$$

where $j = 2, 3, 4$. Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. Suppose that there is a long-time solution $\{g(t)\}$ of (1.1) on $M_j := \left(\#_{i=1}^{j} X_i\right) \# N$ with scalar curvature $|s_g| < C$ for a constant C independent of $t \in [0, +\infty)$, where $j = 2, 3, 4$. Then

$$4(j-1) + 4b_1(N) + b_2(N) \leq \frac{1}{3} \sum_{i=1}^{j} c_1^2(X_i).$$

Proof. By Corollary 11 of [18], there is a monopole class $\mathfrak{a}$ of M_j such that

$$(\mathfrak{a}^+)^2 \geq \sum_{i=1}^{j} c_1^2(X_i).$$

This and (2.6) tell us that Yamabe invariant of M_j satisfies

$$\mathcal{Y}(M_j) \leq -4\pi \sqrt{2 \sum_{i=1}^{j} c_1^2(X_i)} < 0,$$

where we used (2.18). For more details, see [18]. Suppose that there is a long-time solution $\{g(t)\}$ of (1.1) on M_j with $|s_g| < C$ for a constant C independent of $t \in [0, +\infty)$. By Proposition 2.2, we are able to get

$$\hat{s}_{g(t)} \leq -E < 0,$$

where

$$E = -\frac{4\pi}{(vol_{g(0)})^{1/2}} \sqrt{2 \sum_{i=1}^{j} c_1^2(X_i)}.$$

Then Lemma 3.1 of [10] tells us that

$$\int_0^{\infty} \int_{M_j} |\overset{\circ}{r}_{g(t)}|^2 d\mu_{g(t)} dt < \infty.$$

This implies that

$$\int_m^{m+1} \int_{M_j} |\overset{\circ}{r}_{g(t)}|^2 d\mu_{g(t)} dt \longrightarrow 0 \tag{2.19}$$

holds when $m \to +\infty$. Therefore, we have

$$(2\chi + 3\tau)(M_j) = \lim_{m\to\infty} \frac{1}{4\pi^2}\int_m^{m+1}\int_{M_j}\left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2_{g(t)}\right) d\mu_g(t)dt. \quad (2.20)$$

As in the proof of Theorem 2.12, we are also able to get

$$\frac{1}{4\pi^2}\int_M\left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2\right) d\mu_{g(t)} \geq \frac{2}{3}\sum_{i=1}^{j} c_1^2(X_i).$$

Therefore, we have

$$\lim_{m\to\infty} \frac{1}{4\pi^2}\int_m^{m+1}\int_{M}\left(\frac{s^2_{g(t)}}{24} + 2|W_+|^2_{g(t)}\right) d\mu_{g(t)}dt \geq \frac{2}{3}\sum_{i=1}^{j} c_1^2(X_i).$$

This and (2.20) imply

$$(2\chi + 3\tau)(M_j) \geq \frac{2}{3}\sum_{i=1}^{j} c_1^2(X_i).$$

On the other hand, a direct computation implies

$$\begin{aligned}(2\chi + 3\tau)(M_j) &= \sum_{i=1}^{j}(2\chi(X_i) + 3\tau(X_i)) + (2\chi(N) + 3\tau(N)) - 4j \\ &= -(4b_1(N) + b_2(N)) - 4(j-1) + \sum_{i=1}^{j} c_1^2(X_i),\end{aligned}$$

where we used $b^+(N) = 0$. Therefore, we get

$$-(4b_1(N) + b_2(N)) - 4(j-1) + \sum_{i=1}^{j} c_1^2(X_i) \geq \frac{2}{3}\sum_{i=1}^{j} c_1^2(X_i).$$

This implies the desired result as promised. □

3. Proof of Theorem 1.3 . In what follows, we shall use the following notations for any 4-manifold X:

$$\chi_h(X) := \frac{1}{4}\Big(\chi(X) + \tau(X)\Big),\ c_1^2(X) := 2\chi(X) + 3\tau(X).$$

We shall prove the following result:

PROPOSITION 3.1. *For every $\delta > 0$, there exists a constant $d_\delta > 0$ satisfying the following property: every lattice point (α, β) satisfying*

$$0 < \beta < (6-\delta)\alpha - d_\delta \quad (3.2)$$

is realized by (χ_h, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected symplectic 4-manifolds with the following properties:

(1) each symplectic 4-manifold N is non-spin,

(2) N has $\mathcal{Y}(N) < 0$,

(3) there exists no non-singular solution of the normalized Ricci flow on N.

Proof. Building upon symplectic sum construction due to Gompf [12] and gluing formula of Seinerg-Witten invariants due to Morgan-Mrowka-Szabó [31] and Morgan-Szabó-Taubes [32], a nice result on infinitely many pairwise non-diffeomorphic simply connected symplectic 4-manifolds is proved in [6]. In particular, infinitely many smooth structures are given by performing the logarithmic transformation in the sense of Kodaira. Theorem 4 of [6] tells us that, for every $\delta > 0$, there exists a constant $d_\delta > 0$ satisfying the following property: every lattice point (α, β) satisfying

$$0 < \beta \leq (9 - \delta)\alpha - d_\delta$$

is realized by (χ_h, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected symplectic 4-manifolds. In particular, each symplectic 4-manifold X satisfies $c_1^2(X) = \beta > 0$ and we are able to see that $b^+(X) \geq 2$ by the construction. By Corollary 2.15, we conclude that, if a positive integer k satisfies

$$k > \frac{1}{3}c_1^2(X) = \frac{\beta}{3},$$

then there exists no non-singular solution to the normalized Ricci flow on the symplectic 4-manifold $N := X \# k\overline{\mathbb{C}P^2}$. Moreover, the symplectic 4-manifold N is non-spin. These non-spin symplectic 4-manifolds N actually cover the area (3.2), where notice that

$$\chi_h(N) = \chi_h(X), \ c_1^2(N) = \beta - k < \frac{2}{3}\beta.$$

Moreover, under the connected sum with $\overline{\mathbb{C}P^2}$, the infinitely many different smooth structures remain distinct as was already noticed in [6]. Finally, since X has non-trivial valued Seiberg-Witten invariants by a result of Taubes [39], the bound (2.9) tells us that

$$\mathcal{Y}(N) \leq -4\pi\sqrt{2c_1^2(X)} = -4\pi\sqrt{2\beta} < 0.$$

Therefore, we get the desired result. □

We also have

PROPOSITION 3.3. *For every positive integer $\ell > 0$, there are ℓ-tuples of simply connected spin and non-spin algebraic surfaces with the following properties:*

(1) these are homeomorhic, but are pairwise non-diffeomorphic,

(2) for every fixed $\ell > 0$, the ratios c_1^2/χ_h of the ℓ-tuples are dense in the interval $J := [4, 8]$,

(3) each algebraic surface M has $\mathcal{Y}(M) < 0$,

(4) there exists a non-singular solution to the the normalized Ricci flow on M. Moreover the existence of the solution forces the strict FZZ inequality $2\chi(M) > 3|\tau(M)|$.

Proof. Salvetti [35] proved that, for any $k > 0$, there exists a pair (χ_h, c_1^2) such that for this pair one has at least k homeomorphic algebraic surfaces with different divisibilities for their canonical classes by taking iterated branched covers of the projective plane. This construction is fairly generalized in [6]. By Corollary 1 of [6],

we know that, for every ℓ, there are ℓ-tuples of simply connected spin and non-spin algebraic surfaces with ample canonical bundles which are homeomorphic, but are pairwise non-diffeomorphic. Moreover, it was also shown that, for every fixed ℓ, the ratios c_1^2/χ_h of the ℓ-tuples are dense in the interval J. Therefore, to prove this proposition, it is enough to prove (3) and (4). We notice that one can see that each such an algebraic surface M has $b^+(M) \geq 3$ by the construction. Now, the negativity of the Yamabe invariant of the algebraic surface M is a direct consequence of Proposition 2.8. In fact, the canonical bundle of each algebraic surface M is ample and hence $c_1(M) < 0$. In particular, since M is a Kähler surface with $b_2^+(M) \geq 3$ and $c_1^2(M) > 0$, Proposition 2.8 tells us that $\mathcal{Y}(M) = -4\pi\sqrt{2c_1^2(M)} < 0$. Hence we proved (3).
Moreover, (4) follows from the celebrated result of Cao [7, 8] because each algebraic surface M has ample canonical bundle and hence $c_1(M) < 0$. We therefore conclude that, for the initial metric g_0 which is chosen to represent the first Chern class, there always exists a non-singular solution to the normalized Ricci flow and it converges to an Einstein metric of negative scalar curvature as $t \to \infty$.
On the other hand, it was proved in [10] that the existence of non-singular solutions on 4-manifold X with $\overline{\lambda}(X) < 0$ implies

$$2\chi(X) - 3|\tau(X)| \geq \frac{1}{96\pi^2}\overline{\lambda}^2(X),$$

where $\overline{\lambda}(X)$ is the Perelman $\overline{\lambda}$-invariant [34, 20] of X. It is also known [1] that $\mathcal{Y}(X) = \overline{\lambda}(X)$ holds if $\mathcal{Y}(X) \leq 0$. Therefore, we are able to conclude that M must satisfy the strict FZZ inequality $2\chi(M) > 3|\tau(M)|$ because we have $\mathcal{Y}(M) < 0$. □

Propositions 3.1 and 3.3 enable us to prove the main result as follows:

Proof of Theorem 1.3. Proposition 3.3 tells us that, for every positive integer $\ell > 0$, we are always able to find ℓ-tuples M_ℓ^i of simply connected non-spin algebraic surfaces of general type and these are homeomorhic, but are pairwise non-diffeomorphic. And the ratios c_1^2/χ_h of M_ℓ^i are dense in the interval $J := [4, 8]$ for every fixed $\ell > 0$. Moreover, Proposition 3.3 tells us that $\mathcal{Y}(M_\ell^i) < 0$ holds and, on each of M_ℓ^i, there exists a non-singular solution of the the normalized Ricci flow and the existence of the solution forces the strict FZZ inequality $2\chi > 3|\tau|$.

On the other hand, Proposition 3.1 tells us that any pair (α, β) in the area (3.2) can be realized by (χ_h, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected non-spin symplectic 4-manifolds with $\mathcal{Y} < 0$ and on each of which there exists no non-singular solution of the normalized Ricci flow. Notice that the ratios c_1^2/χ_h of these non-spin symplectic 4-manifolds are not more than 6, here see again the area (3.2). By this fact and the density of the ratios c_1^2/χ_h of M_ℓ^i in the interval $J := [4, 8]$, we are able to find infinitely many pairwise non-diffeomorphic simply connected non-spin symplectic 4-manifolds N_ℓ^i such that $\mathcal{Y} < 0$ and, on each of N_ℓ^i, there exists no non-singular solution of the normalized Ricci flow, and moreover, M_ℓ^i and N_ℓ^i are both non-spin and have the same (χ_h, c_1^2). Freedman's classification [11] implies that they must be homeomorphic. However, each of M_ℓ^i is not diffeomorphic to any N_ℓ^i because, on each of M_ℓ^i, a non-singular solution exists and, on the other hand, no non-singular solution exists on each of N_ℓ^i. Therefore, we are able to conclude that, for every natural number ℓ, there exists a simply connected topological non-spin 4-manifold X_ℓ satisfying the desired properties. □

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