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1/4-PINCHED CONTACT SPHERE THEOREM*

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Abstract. Given a closed contact 3-manifold with a compatible Riemannian metric, we show that if the sectional curvature is 1/4-pinched, then the contact structure is universally tight. This result improves the Contact Sphere Theorem in [EKM12], where a 4/9-pinching constant was imposed. Some tightness results on positively curved contact open 3-manifold are also discussed.

Key words. Contact sphere theorem, curvature, tight contact structure.

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0. Introduction. A *contact metric manifold* (M, ξ, g) is a contact manifold equipped with a compatible Riemannian metric g, where ξ is the contact structure. See Definition 2.1 for the definition of compatibility. In [EKM12], the authors studied how the curvature bounds on g implies the (universal) tightness of ξ. In particular, the authors showed that if the sectional curvature of g is 4/9-pinched, then ξ is universally tight. Since any 1/4-pinched closed 3-manifold has the universal cover diffeomorphic to S^3 and there is a unique, up to contactomorphism, tight contact structure ξ_{std} on S^3 [Eli92], the authors therefore concluded that the universal cover of (M, ξ) must be contactomorphic to (S^3, ξ_{std}). The main goal of this note is to improve the pinching constant to 1/4. More precisely, we have

Theorem 0.1 (Contact Sphere Theorem). *Suppose (M, ξ, g) is a closed contact metric 3-manifold. If the sectional curvature* $\sec(g)$ *satisfies*

$$\frac{1}{4} < \sec(g) \leq 1,$$

then the universal cover of M, with the lifted contact structure, is contactomorphic to (S^3, ξ_{std}).

Remark 0.2. According to [Ham82], the universal cover of a closed Riemannian 3-manifold with sectional curvature pinched by any positive number is diffeomorphic to S^3. (In fact, Hamilton shows that the Ricci flow deforms any metric with positive Ricci curvature to the one with constant sectional curvature.) At this moment, we do not see why 1/4 should be the optimal pinching constant for Theorem 0.1 to hold, but we do believe that substantially new ideas (such as geometric flows) will be needed to further improve the pinching constant.

Our strategy of proving Theorem 0.1 essentially follows the arguments in [EKM12]. However, instead of trying to bound the tight radius by the convex radius from below as in [EKM12], we construct a shrinking family of (not necessarily smooth) strictly convex spheres and carefully estimate the convexity to ensure the tightness.

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Using this new tool, we are able to prove the following two theorems for open manifolds, which can be viewed as a counterpart of Corollary 1.4 in [EKM12]. Note also that there is no convexity radius estimate on such manifold since no upper curvature bound is assumed.

THEOREM 0.3. *Let* (M, ξ, g) *be an open contact metric manifold such that* g *is complete and* $\sec(g) > 0$. *Then* ξ *is tight.*

We note immediately that M being open and being positively curved imply that $M \simeq \mathbb{R}^3$. In fact using an argument of Wu [Wu79], we can weaken the curvature condition and get

THEOREM 0.4. *Let* (M, ξ, g) *be an open contact metric manifold such that* g *having nonnegative sectional curvature on* M *and positive sectional curvature in* $M \setminus K$, *where* K *is a compact subset of* M. *Then* ξ *is tight.*

This paper is organized as follows. In section 1 we review the notion of ϵ-convexity in Riemannian geometry and, in particular, we establish a convexity estimate which is the key ingredient in our proof Theorem 0.1. In section 2 we compare Riemannian convexity with pseudo-convexity in almost complex manifolds tamed by a symplectic form. The later will be crucial in proving the (universal) tightness of contact structures using the techniques of pseudo-holomorphic curves. Proofs of Theorem 0.1, Theorem 0.3 and Theorem 0.4 are given in section 3 and section 4.

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1. Riemannian Convexity. In this section, we will study some convexity properties of a 3-dimensional Riemannian manifold (M, g). We restrict ourselves to dimension three because we are interested in the geometry of contact 3-manifolds, but all the results in this section still hold in any dimension. We say a closed domain $D \subset M$ is *convex*, if any two points $x, y \in D$ can be joined by a minimal geodesic γ contained in D. If, in addition, the interior of γ lies in the interior of D, then we say D is *strictly convex*. In this note, we denote the distance function induced by the Riemannian metric by $\mathrm{d}(\cdot, \cdot)$ and the open ball of radius r centered at $p \in M$ by $B(p, r)$. By writing $\partial B(p, r)$ we mean the distance sphere of radius r centered at $p \in M$.

LEMMA 1.1. *Let* (M, g) *be a closed Riemannian manifold with*

$$\sec(g) \geq 1.$$

Then for any $p \in M$, *the closed set* $N := M \setminus B(p, \pi/2)$ *is convex.*

Proof. Without loss of generality, we may assume N contains at least two distinct points, say, $x, y \in N$. Let $\gamma : [0, a] \to M$ be a geodesic parameterized by arc-length with $\gamma(0) = x, \gamma(a) = y$ such that $a = \mathrm{d}(x, y)$. Since $\sec(g) \geq 1$, the classical Bonnet Theorem ([Pet06, Lemma 21]) and Cheng's Theorem([Pet06, Theorem 62]) imply that the diameter of M is less than or equal to π and equality holds if and only if M is isometric to the round sphere. Hence we can further assume $a < \pi$; otherwise if $a = \pi$ then N is a round hemisphere and the statement is clear. Choose the comparison triangle $\Delta\tilde{p}\tilde{x}\tilde{y}$ in $\mathbf{S}^2(1)$, the unit 2-sphere, i.e., the points in $\mathbf{S}^2(1)$

such that $\mathrm{d}_{S^2}(\tilde{p},\tilde{x}) = \mathrm{d}(p,x)$, $\mathrm{d}_{S^2}(\tilde{p},\tilde{y}) = \mathrm{d}(p,y)$ and $\mathrm{d}_{S^2}(\tilde{x},\tilde{y}) = \mathrm{d}(x,y)$, where d_{S^2} is induced by the round metric on $\mathbf{S}^2(1)$. Let $\tilde{\gamma}$ be the minimal geodesic in $\mathbf{S}^2(1)$ connecting $\tilde{x}$ and $\tilde{y}$. Then Toponogov's comparison theorem implies that

$$\mathrm{d}(p,\gamma(t)) \geq \mathrm{d}(\tilde{p},\tilde{\gamma}(t))$$

for all t. It is clear that $\mathrm{d}(\tilde{p},\tilde{\gamma}(t)) \geq \pi/2$ in $\mathbf{S}^2(1)$. Hence $\gamma(t) \in N$ for any $t \in [0,a]$, i.e., N is convex. □

For example, if we take M to be the round 3-sphere $\mathbf{S}^3(1)$, then the N defined above is a hemisphere with a totally geodesic boundary, which is a great 2-sphere. However if we shrink N a little bit, then we get a smaller hemisphere with strictly convex boundary. It is helpful to keep this example in mind because a similar argument will be used later to construct a convex ball in a contact metric 3-manifold.

For our later purposes, we need to study continuous convex functions. One quick way to define the convexity of a continuous function $f : M \to \mathbb{R}$ is to require that its restriction on any geodesic segment is convex as a function from $\mathbb{R}$ to $\mathbb{R}$. More precisely, we need the following quantitative definition from [Esc86].

Definition 1.2. *Let (M,g) be a Riemannian manifold and $\epsilon > 0$ be a constant. A continuous function $f : M \to \mathbb{R}$ is called* ϵ-convex *if for any point $p \in M$ and any $0 < \eta < \epsilon$, there exists a smooth function h, defined on an open neighborhood U of p, such that*

- *$h \leq f$ in U,*
- *$h(p) = f(p)$,*
- *$D^2h(v,v) \geq \eta\|v\|^2$ for any $v \in T_p(M)$.*

Such an h is called a η-supporting function *of f at p, or just* supporting function *when η is implicit.*

The following estimation of the convexity of distance function to the boundary is crucial for our proof. Some similar statements for the Busemann function have been proved by Cheeger-Gromoll [CG72] and Wu [Wu79]. For the distance function to boundary of an Alexandrov spaces with lower curvature bound, the convexity was first proved by Perelman in [Per93] and then made rigorous by Alexander-Bishop [AB03, Theorem 0.1 CBB 5B]. However the proof given in [AB03] used several important tools and the fundamental structural theory for Alexandrov spaces and hence require more background knowledge. In order to keep this note more self-contained we present a purely Riemannian geometric proof - This also has the advantage of explicitly constructing the supporting function, which will be used later.

Proposition 1.3 (Convexity Estimate). *Let (M,g) be a Riemannian manifold with $\sec(g) \geq 1$ and $D \subset M$ be a closed convex domain with nonempty boundary. Define $f : D \to \mathbb{R}$ by*

$$f(x) = e^{\pi/2 - \mathrm{d}(x,\partial D)}.$$

Then $f(x)$ is strictly convex. Moreover, if $\mathrm{d}(x,\partial D) = \ell$, then f is $\min(1,\ell)$-convex.

Proof. We first introduce some notation which will be used in the proof. Let $p \in D \setminus \partial D$, $q \in \partial D$ such that

$$\ell = \mathrm{d}(p,q) = \mathrm{d}(p,\partial D).$$

Let $\gamma : (-\epsilon, \epsilon) \to D$ be a unit speed geodesic with $\gamma(0) = p$, and $\sigma : [0, \ell] \to D$ be the geodesic from p to q. Hence at p, we have the following orthogonal decomposition

$$\gamma'(0) = a\sigma'(0) + bW,$$

where $W \in (\sigma'(0))^{\perp} \subset T_p(M)$ and $a^2 + b^2 = 1$. Using a method similar to the construction of a variational field in [Ge14], we construct a vector field V along σ by

$$V(t) = a\left(1 - \frac{t}{\ell}\right)\sigma'(t) + bW(t), \tag{1.1}$$

where $W(t)$ is the parallel translate of W along σ. Let $\alpha : [0, \ell] \times (-\delta, \delta) \to M$ be a variation of σ which induces $V(t)$ for $\delta < a$; in fact we can define $\alpha(t, s) = \exp_{\sigma(t)}(sV(t))$ for $t \in [0, \ell], s \in (-\delta, \delta)$. Since $V(\ell) = bW(\ell) \perp \sigma'(\ell)$, the convexity of D implies that

$$\alpha(\ell, s) \notin D \setminus \partial D \text{ for all } s. \tag{1.2}$$

Now the proof proceeds in two steps.

Step 1: Constructing a supporting function.

Denote the curve $t \mapsto \alpha(t, s)$ by σ_s. We define $L : (-\delta, \delta) \to \mathbb{R}$ by

$$L(s) = \frac{\pi}{2} - \int_0^{\ell} \sqrt{\langle \sigma_s'(t), \sigma_s'(t) \rangle}dt.$$

i.e., L is $\pi/2$ minus the length of the length of σ_s. Clearly $L(0) = \pi/2 - \ell$. Let

$$h(s) = \frac{\pi}{2} - \mathrm{d}(\gamma(s), \partial D),$$

Then by (1.2)

$$L(s) \leq \frac{\pi}{2} - \mathrm{d}(\gamma(s), \partial D) = h(s),$$

i.e., L is a supporting function of h at $\gamma(0)$.

Step 2: Convexity estimates of L and e^L.

By the second variational formula for arc-length, we have

$$L''(0) = \int_0^{\ell} \Big(R(\sigma', V', \sigma', V') - \langle V', V' \rangle + (\langle V', \sigma' \rangle)^2 \Big) dt.$$

Here $R(X, Y, Z, W) = \langle -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, W \rangle$ is the Riemannian curvature tensor. Note that our construction (1.1) implies $\langle V', V' \rangle = a^2/\ell^2$ and $(\langle V', \sigma' \rangle)^2 = a^2/\ell^2$. Moreover, by the assumption that $\sec(g) \geq 1$, we have

$$L''(0) \geq \int_0^{\ell} b^2 dt = b^2 \ell.$$

Consider the composition e^L. We calculate:

$$\begin{aligned}(e^L)''(0) &= e^L(L')^2(0) + e^L L''(0)\\ &\geq e^{\frac{\pi}{2}-\ell}(a^2 + b^2\ell)\\ &\geq \min(\ell, 1),\end{aligned} \tag{1.3}$$

where we used the first variational formula to get $L'(0) = a$. The last inequality follows from the fact that $\ell \leq \pi/2$ under given curvature condition, and $a^2 + b^2 = 1$. By Step 1, e^L supports e^h. Hence it follows that $f = e^h$ is $\min(\mathrm{d}(p, \partial D), 1)$-convex. □

Remark 1.4. Our choice of using the exponential function in the construction of f is not essential, and, in fact, any function $\kappa : \mathbb{R} \to \mathbb{R}$ with $\kappa > 0$, $\kappa' > 0$ and $\kappa'' > 0$ will also work. As a consequence the number $\min(1, \ell)$ is also not important since it clearly depends on the choice of κ. In fact, the most commonly used functions in metric geometry are the *generalized trigonometric functions*, which interpolate analytically between the usual trigonometric and hyperbolic functions. See for example [AB03] or [Ge14].

2. Pseudoconvexity in symplectizations. Let (M, ξ) be a contact 3-manifold. Following [EKM12] we have

Definition 2.1. *A Riemannian metric g is* compatible *with ξ if there is a contact form α defining ξ such that*

$$||R_\alpha|| = 1 \text{ and } *d\alpha = \theta'\alpha$$

for some positive constant θ', where R_α is the Reeb vector field and $*$ is the Hodge star operator associated with g. A compatible triple (M, α, g) is called a contact metric manifold.

Remark 2.2. A compatibility condition between contact structure and Riemannian metric first appeared in [CH85], where $\theta' = 2$.

Remark 2.3. In [EKM12], a notion of *weakly compatible metric* is also discussed, where θ' is not necessarily a constant and the length of R_α is allowed to vary. But we will not use the weak compatibility in this note.

Throughout this section we will assume that (M, α, g) is a contact metric manifold with $\sec(g) \geq 1$.

Recall the symplectization $W = \mathbb{R}_+ \times M$ of M is a symplectic manifold with symplectic form $\omega = d(t\alpha)$ where $t \in \mathbb{R}_+$. Also fix a metric-preserving compatible almost complex structure J on W such that

$$J\partial_t = R_\alpha \text{ and } J\xi = \xi.$$

Let $D \subset M$ be a closed convex domain with boundary and

$$f(x) = e^{\frac{\pi}{2} - \mathrm{d}(x, \partial D)}$$

be a strictly convex function on $D \setminus \partial D$ according to Proposition 1.3. We extend f to a function $\tilde{f}$ on $\mathbb{R}_+ \times N$ by $\tilde{f}(t, x) = f(x)$ for $x \in N, t \in \mathbb{R}_+$. The following proposition is crucial for our later detection of overtwistedness.

Proposition 2.4 (Weak Maximum Principle). *Using the notation from above, for $e^{\pi/2} > c > \min f$, let $\Omega^c := \mathbb{R}_+ \times f^{-1}((-\infty, c])$ and $\Sigma^c := \mathbb{R}_+ \times f^{-1}(c) = \partial\Omega^c$. Then the interior of any J-holomorphic curve C in Ω^c is disjoint from Σ^c.*

Proof. Since $c < e^{\pi/2}$, Proposition 1.3 implies that f is τ-convex for some $\tau > 0$ which depends only on c. Suppose $\Sigma^c \cap int(C)$ is nonempty, it contains a point, say, p. By the proof of Proposition 1.3, there exists a supporting function g of f at p. Denote by Σ' the hypersurface $g^{-1}(c) \times \mathbb{R}_+$ in W. The following calculation of the Levi form $L_{\tilde{g}}$ of $\tilde{g}$ is obtained in [EKM12, Proposition 3.7], where $\tilde{g}$ is the usual extension of g on W. For any unit vector $v \in T\Sigma' \cap JT\Sigma'$, we can write $v = a\partial_t + bR_\alpha + ev_0$, where $v_0 \perp \text{span}\,(\partial_t, R_\alpha)$, is a unit vector. Hence $a^2 + b^2 + e^2 = 1$. We calculate:

$$L_{\tilde{g}}(v,v) = D^2\tilde{g}(v,v) + D^2\tilde{g}(Jv,Jv) \geq \tau(2 - a^2 - b^2) \geq \tau > 0$$

since $\tilde{g}$ is constant in the $\mathbb{R}_+$-direction and h is τ-convex by construction. In particular Σ' is strictly pseudoconvex. On the other hand, it is easy to see that C is tangent to the smooth hypersurface Σ' at p, which contradicts the pseudoconvexity of Σ'. □

Remark 2.5. Note that all the level surfaces Σ^c, $e^{\pi/2} > c > \min f$, are topologically homeomorphic to the 2-sphere, thanks to the strict convexity of f.

An important consequence is

Corollary 2.6. *For any $e^{\pi/2} > c > \min f$, $f^{-1}((-\infty, c])$ is a tight ball.*

Proof. It is easy to see that $f^{-1}((-\infty, c])$ is homeomorphic to a ball. Arguing by contradiction, suppose there exists an overtwisted disk $D_{OT} \subset f^{-1}((-\infty, c])$. Then Proposition 2.4 guarantees that Hofer's proof of the Weinstein conjecture for overtwisted contact structures [Hof93] carries over in our situation to produce a closed Reeb orbit γ. Consider the (trivial) J-holomorphic cylinder $C_\gamma := \mathbb{R}_+ \times \gamma$ contained in the interior of Ω^c. Define

$$T = \inf\{c \geq s > \min f \mid C_\gamma \subset \Omega^s, C_\gamma \cap \Sigma^s = \emptyset\}.$$

Then clearly Ω^T intersects the interior of C_γ nontrivially, which contradicts Proposition 2.4. □

3. Proof of Theorem 0.1. In this section we assume that (M, ξ, g) satisfies the assumptions in Theorem 0.1. Passing to the universal cover if necessary, we may further assume that M is simply connected. Note that the compactness condition is preserved due to the positivity of $\sec(g)$. We start by a slight refinement of Lemma 1.1 as follows.

Lemma 3.1. *Given $\sec(g) > 1/4$, there exists $\delta > 0$ such that the set $M \setminus B(p, (1-\delta)\pi)$ is convex.*

Proof. Since M is compact, there exists $\epsilon > 0$ such that $\sec(g) \geq \frac{1}{4} + \epsilon$. Rescaling the metric to have lower curvature bound 1 and applying Lemma 1.1 gives the desired convexity. In fact one can take $\delta > 0$ such that

$$(1-\delta)\pi = \frac{\pi}{2\sqrt{\frac{1}{4} + \epsilon}}.$$

□

Now let us assume the curvature is 1/4-pinched and let $N = B(p, (1-\delta)\pi)$. Consider the distance function $h(x) = \mathrm{d}(x, \partial N)$. Let

$$B_1 = B(p, \pi) \quad \text{and} \quad B_2 = \{x \in N \mid h(x) \geq \frac{\delta}{2}\pi\}$$

Then by Proposition 1.3 and Corollary 2.6, B_2 is a tight ball. Now M is coved by two balls:

$$M = B_1 \cup B_2.$$

Moreover we note that ∂B_2 is contained in the interior of B_1 and also ∂B_1 is contained in B_2. See Figure 1.

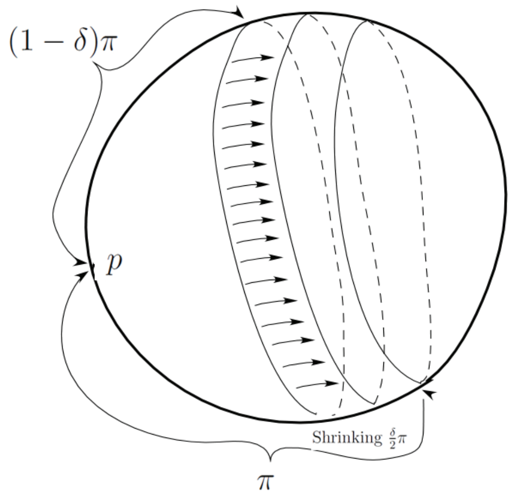

FIG. 1. *Covering M by two balls*

Before complete the proof of Theorem 0.1, we need the following theorem from [EKM12], which tells us where to look for overtwisted disks.

THEOREM 3.2 ([EKM12] Theorem 1.2). *Let (M, ξ, g) be a contact metric 3-manifold and $\mathrm{inj}(g)$ be the injective radius of g. Fix a point $p \in M$. Suppose $B(p, r)$ is overtwisted for some $r < \mathrm{inj}(g)$. Then for any $r \leq R < \mathrm{inj}(g)$, the distance sphere $\partial B(p, R)$ contains an overtwisted disk.*

Now we are ready to finish the proof of our contact sphere theorem.

Proof of Theorem 0.1. Recall $M = B_1 \cup B_2$ and B_2 is tight. Arguing by contradiction, suppose there exists an overtwisted disk $D_{OT} \subset M$. By Eliashberg's uniqueness theorem of tight contact structures on the 3-ball [Eli92], there exists a radial contact vector field on B_2 whose flow induces a contact isotopy $\phi_t : M \to M$, $t \in [0, 1]$, such that

$$\phi_0 = \mathrm{id}, \ \phi_t|_{M \setminus B_2} = \mathrm{id}, \text{ and } \phi_1(D_{OT}) \subset B_1.$$

The last assertion follows from the fact that a small neighborhood of ∂B_2 is contained in the interior of B_1. Now Theorem 3.2, applied to B_1, implies that for sufficiently small $\epsilon > 0$ and any $r_0 \in (\pi - \epsilon, \pi)$, the distance sphere $\partial B(p, r_0)$ contains an overtwisted disk. However $\partial B(p, r_0) \subset B_2$ for small ϵ, which contradicts the fact the B_2 is tight.

Now the classical 1/4-pinched sphere theorem implies that M is homeomorphic to S^3, and since we are in dimension 3, it is diffeomorphic to S^3. Again, Eliashberg's uniqueness theorem of tight contact structures on S^3 implies that (M, ξ) must be contactomorphic to (S^3, ξ_{std}). □

4. Proof of Theorem 0.3 and Theorem 0.4. In this section we prove Theorem 0.3 and sketch a proof of Theorem 0.4. Although Theorem 0.4 implies Theorem 0.3, we would like to emphasize the proof of the positively curved case, which shows how the strict convexity of the Busemann function played a role. The idea is to construct a strictly convex exhaustion function. Define the Busemann function $b : M \to \mathbb{R}$ by

$$b(x) = \lim_{t\to\infty} (t - \mathrm{d}(x, \partial B(p, t))),$$

where $p \in M$ is a fixed point and $B(p, t)$ is the geodesic ball of radius t as before. It was shown in [CG72] (cf. also [Wu79]) that under the assumption that the M is of nonnegative curvature, b satisfies the following properties:

1. b is a strictly convex Lipschitz function bounded from below by $a_0 > -\infty$;
2. b is an exhaustion function, i.e., if we denote $C^s := b^{-1}((-\infty, s])$, then for all $c \geq a_0$, C^t is compact and $M = \cup_{t\geq a_0} C^s$;
3. for $a_0 \leq r < s$, if $x \in \partial C^r$, then $\mathrm{d}(x, \partial C^s) = s - r$; in other words, b measures the distance to the boundary of C^s up to a constant, i.e., $\mathrm{d}(\partial C^r, \partial C^s) = |r - s|$.

The third property shows that Busemann function can be viewed as a certain distance function from infinity and this is exactly why our convexity estimate also works for b. In fact the compactness of C^t implies the sectional curvature is bounded on C^t. Namely,

$$0 < \delta \leq \sec(g)|_{C^t} \leq \Delta,$$

for some positive constants δ and Δ which depend on t. Hence we can rescale the metric such that the sectional curvatures of the rescaled metric are ≥ 1. Then we can apply Proposition 1.3 and Corollary 2.6 to C^t. Therefore, C^t is tight. Since $M = \cup_{t\geq a_0} C^t$, M itself is tight. This finishes the proof of Theorem 0.3.

Finally, we sketch the proof of Theorem 0.4, which is along the same lines as in the previous proof. The only difficulty is that the sectional curvature is only nonnegative and the Proposition 1.3 does not apply. However Theorem C(a) in [Wu79] shows that b is an essentially convex function, i.e., there exists $\epsilon > 0$ such that $\kappa \circ b$ is ϵ-convex for smooth $\kappa : \mathbb{R} \to \mathbb{R}$ with $\kappa > 0, \kappa' > 0$ and $\kappa'' > 0$. Hence by Corollary 2.6, C^t is tight for all large t. This completes the proof of Theorem 0.4.

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