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GAGE'S ORIGINAL NORMALIZED CSF CAN ALSO YIELD THE GRAYSON THEOREM*

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Abstract. Mimicking Andrews-Bryan's argument, it is proved in this note that Gage's original normalized curve shortening flow can also yield the Grayson theorem.

Key words. Normalized curve shortening flow, distance comparison, Grayson's theorem.

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1. Introduction. Let X_0 be a simple, closed and smooth curve in a Euclidean plane $\mathbb{R}^2$. In the 1980s, Gage ([4, 5]) and Gage-Hamilton ([7]) investigated the following curve shortening flow (CSF):

$$\begin{cases} \frac{\partial \widetilde{X}}{\partial \tau}(\varphi,\tau) = \widetilde{\kappa}(\varphi,\tau)N(\varphi,\tau), \quad (\varphi,\tau) \in S^1 \times (0,\widetilde{\omega}), \\ \widetilde{X}(\varphi,0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases} \tag{1.1}$$

where $\widetilde{\kappa}(\varphi,\tau)$ is the curvature of the evolving curve and $N(\varphi,\tau)$ its unit normal vector field. They proved that if the initial curve is strictly convex then the evolving curve shrinks to a "round" point in a finite time. In 1987, Grayson showed in [8] that an embedded curve evolving under CSF can be deformed into a convex one that ultimately shrinks to a point.

In the 1990s, Hamilton ([9]) and Huisken ([10]) gave estimates of some geometric quantities of the evolving curve and showed that the type II singularities will not occur if a simple curve evolves under the CSF. By quoting the classification of self-similar solutions of the CSF given by Abresch-Langer ([1]), both Huisken and Hamilton gave new proofs of Grayson's theorem. In 2011, Andrews-Bryan ([2]) proposed an original proof of Grayson's theorem which is based on a refinement of isoperimetric arguments of Huisken. They considered a normalized CSF so that the evolving curve has total length 2π. They gave a new isoperimetric estimate and deduced that the curvature converges exponentially to 1, which gives a self-contained proof of Grayson's theorem.

In this note, we will imitate Andrews-Bryan's argument to reconsider Gage's original normalization (see [5]) of the CSF so the bounded area equals π and give a new proof of Grayson's theorem. Comparing with Andrews-Bryan's argument, the asymptotic analysis of curvature is a bit more difficult. That is to say that the CSF (1.1) is normalized by setting $X : S^1 \times [0,\omega) \to \mathbb{R}^2$ as follows:

$$X(\varphi,t) = \sqrt{\frac{\pi}{\widetilde{A}(\tau)}}\widetilde{X}(\varphi,\tau), \qquad t = \int_0^\tau \frac{\pi}{\widetilde{A}(\tau')}d\tau', \tag{1.2}$$

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where $\widetilde{A}(\tau)$ is the area enclosed by the curve $\widetilde{X}(\cdot,\tau)$. Denote by $A(t)$ the area enclosed by $X(\cdot,t)$, then $A(t)=\pi$ for every t. Under the normalization (1.2), $X(\varphi,t)$ satisfies

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t)=\kappa(\varphi,t)N(\varphi,t)+X(\varphi,t), \quad (\varphi,t)\in S^1\times(0,\omega), \\ \\ X(\varphi,0)=\sqrt{\frac{\pi}{\widetilde{A}(0)}}X_0(\varphi), \quad \varphi\in S^1. \end{cases} \tag{1.3}$$

Direct calculations can imply that the curvature κ evolves as

$$\begin{cases} \frac{\partial \kappa}{\partial t}=\frac{\partial^2\kappa}{\partial s^2}+\kappa(\kappa^2-1), \quad (s,t)\in[0,L(t)]\times(0,\omega), \\ \\ \kappa(s,0)=\kappa_0(s), \quad s\in[0,L(0)], \end{cases} \tag{1.4}$$

here s stands for the arc length of the evolving curve $X(\cdot,t)$ and $L(t)$ its length.

This paper is organized as follows. In section 2, Andrews-Bryan's method of distance comparison will be applied to study the flow (1.3). It is shown that $\kappa(\varphi,t)$ is bounded on $S^1\times[0,+\infty)$. And in Section 3 it is proved that κ converges to 1 as time tends to infinity. It follows that the evolving curve converges modulo translations to a unit circle. So there exists a positive number T_0 such that the evolving curve $X(\cdot,t)$ is strictly convex for $t>T_0$.

2. The bound of the curvature. Denote by κ and $\widetilde{\kappa}$ the relative curvature of $X(\cdot,t)$ and $\widetilde{X}(\cdot,\tau)$, respectively. Under the normalization (1.2), one has $ds=\sqrt{\frac{\pi}{\widetilde{A}(\tau)}}d\widetilde{s}$, $\kappa(\varphi,t)=\sqrt{\frac{\widetilde{A}(\tau)}{\pi}}\widetilde{\kappa}(\varphi,\tau)$ and

$$\begin{aligned} \frac{\partial X}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}\left(\sqrt{\frac{\pi}{\widetilde{A}(\tau)}}\widetilde{X}(\varphi,\tau)\right) \\ &= \frac{\widetilde{A}(\tau)}{\pi}\sqrt{\pi}\left(-\frac{1}{2}\right)\left(\widetilde{A}(\tau)\right)^{-\frac{3}{2}}(-2\pi)\widetilde{X}(\varphi,\tau)+\frac{\widetilde{A}(\tau)}{\pi}\sqrt{\frac{\pi}{\widetilde{A}(\tau)}}\widetilde{\kappa}N \\ &= \sqrt{\frac{\pi}{\widetilde{A}(\tau)}}\widetilde{X}(\varphi,\tau)+\sqrt{\frac{\widetilde{A}(\tau)}{\pi}}\widetilde{\kappa}N=X(\varphi,t)+\kappa N, \end{aligned}$$

which is (1.3). The following two lemmata 2.1 and 2.2 will be used in the present note.

LEMMA 2.1 (Theorem 3.2.1 of Gage-Hamilton [7]). *If a smooth and simple curve evolves according to the CSF and curvature of the evolving curve is bounded in the time interval $[0,\tau_0]$ then the evolving curve is embedded for each $\tau<\tau_0$.*

Denote by $L(t)$ the length of the evolving curve $X(\cdot,t)$. From Lemma 2.1 and the classical isoperimetric inequality, it follows that

$$L(t)\geq 2\pi, \quad t\geq 0. \tag{2.1}$$

As done by Hamilton ([9]), Huisken ([10]) and Andrews-Byan ([2]), define the chord length by $d(p,q,t)=\|X(q,t)-X(p,t)\|$ and the arc length along the curve $X(\cdot,t)$ by $l(p,q,t)=\int_p^q ds$, here p,q are the parameters of the arc length. Following Andrews-Bryan's idea, we use their mysterious function

$$f(x,t)=2e^t\arctan\left(e^{-t}\sin\frac{x}{2}\right),\ x\in[0,\pi],\ t\in\mathbb{R}.$$

It is easy to compute that

$$\lim_{t\to+\infty} f(x,t) = 2\sin\frac{x}{2}, \quad \lim_{t\to-\infty} f(x,t) = 0, \quad \frac{\partial f}{\partial x} = \frac{\cos\frac{x}{2}}{1+e^{-2t}\sin^2\frac{x}{2}} \leq 1,$$

and

$$\frac{\partial f}{\partial t} = 2e^t\left[\arctan\left(e^{-t}\sin\frac{x}{2}\right) - \frac{e^{-t}\sin\frac{x}{2}}{1+e^{-2t}\sin^2\frac{x}{2}}\right] \triangleq 2e^t g(e^{-t}\sin\frac{x}{2}),$$

here

$$g(z) = \arctan z - \frac{z}{1+z^2}.$$

Noticing that

$$g(0) = 0, \qquad g'(z) = \frac{2z^2}{(1+z^2)^2} > 0,$$

we have $\frac{\partial f}{\partial t} > 0$ for $x > 0$. The following function $b(p,q,t)$ is also given by Andrews-Bryan ([2]):

$$b(p,q,t) = \inf\left\{e^{\tilde{t}}\middle| d(p,q,t) \geq f(l(p,q,t), -\tilde{t}), \quad 0 < l(p,q,t) \leq \pi\right\}.$$

By Lemma 2.1, one obtains that the evolving curve is embedded if the flow has no singularity. So the above infimum in the definition of $b(p,q,t)$ is always attained. Since f and d are smooth, where $0 < l(p,q,t) < \pi$, $b(p,q,t)$ is also smooth and can be defined by

$$d(p,q,t) = f(l(p,q,t), -\ln(b)).$$

Lemma 2.2. (Lemma 2 of Andrews-Bryan [2]) *For the function $b(p,q,t)$ one has*

$$\lim_{q\to p} b(p,q,t) = \sqrt{\frac{1}{2}\max\left\{\kappa(p,t)^2 - 1, 0\right\}}\ .$$

If one defines $b(p,p,t) \triangleq \sqrt{\frac{1}{2}\max\{\kappa(p,t)^2-1,0\}}$ then, for each t, one gets a continuous function $b(p,q,\cdot)$ in the domain

$$\{(p,q,\cdot)|0 \leq l(p,q,\cdot) \leq \pi\}. \tag{2.2}$$

By the continuity of b, the function $\overline{b}(t) \triangleq \sup\{b(p,q,t)|0 \leq l(p,q,t) \leq \pi\}$ is bounded if the normalised flow exists in the time interval $[0,t]$.

From Lemma 2.2 it follows that

$$d(p,q,t) \geq f(l(p,q,t), -\ln b(p,q,t)) \geq f(l(p,q,t), t-B) \tag{2.3}$$

holds in the domain (2.2) for $t = 0$, here $B = \ln\overline{b}(0)$. One can prove the inequality (2.3) holds for all $t > 0$.

Lemma 2.3. *Let $X(\cdot)$ be a closed, embedded smooth curve in the plane. Denote by e_1, e_2 the unit tangential vectors of the curve at the points $X(p_0) \triangleq P$ and $X(q_0) \triangleq Q$, separately. If $p_0 < q_0$, $e_1 = e_2$ and e_1 makes an acute angle with $\overrightarrow{PQ}$, then the chord $\mathfrak{L}$ from P to Q has at least one other intersection $X(s)(p_0 < s < q_0)$ with the curve.*

Proof. This lemma is an understanding of the proof of Theorem 1 in Andrews-Bryan's paper [2]. Let us choose $X(p_0)$ as the origin, e_1 as the positive direction of the x-axis and the normal vector at $X(p_0)$ as the positive direction of the y-axis. In this Cartesian coordinate system, one has a graph presentation

$$\{f(x)|x \in (-\varepsilon_1, \varepsilon_1)\}$$

of the curve near the origin:

$$\{X(p)|p \in (p_0 - \varepsilon_0, p_0 + \varepsilon_0)\},$$

here $\varepsilon_0, \varepsilon_1 > 0$. By the assumption, one obtains that $f(0) = 0, f'(0) = 0$. Now, define

$$g(x) \triangleq r - \sqrt{r^2 - x^2} - f(x),$$

here r is a positive constant. If $|x|$ is small then

$$g(x) = \frac{1}{2}\left(\frac{1}{r} - f''(0)\right)x^2 + O(x^3).$$

Thus one can choose a proper r such that $g(x) \geq 0,\ x \in (-r, r)$ and $g(x) = 0$ if and only if $x = 0$. Since the positive direction of the y-axis is the normal vector at $X(p_0)$, one can choose r small enough such that the interior of the circle $\{(x, y)|x^2+(y-r)^2 = r^2\}$ is contained in the interior of the domain enclosed by the curve X. Therefore, there is a point (near the origin $X(p_0)$) on the chord $\mathfrak{L}$ lying in the interior of the domain enclosed by X.

Noticing that the angle from $\overrightarrow{Q_0P_0}$ to e_2 is obtuse, one can show that there is a point (near the point $X(p_0)$) on the chord $\mathfrak{L}$ lying in the exterior of the domain enclosed by X. By the Jordan Theorem of simple closed curves, there exists at least one other intersection $X(s)(p_0 < s < q_0)$ where the chord $\mathfrak{L}$ meets the curve X. $\square$

Define a function by

$$Z(p, q, t) = d(p, q, t) - f(l(p, q, t), t - B),$$

here $0 \leq l(p, q, t) \leq \pi$. $Z(p, q, t)$ is continuous in the domain

$$\{(p, q, t)|0 \leq l(p, q, \cdot) \leq \pi, t \in [0, T), T > 0\}.$$

And Z is smooth where $p \neq q$. We will prove that $Z_\varepsilon = Z + \varepsilon e^{2t}$ is positive in its domain of definition for all $\varepsilon > 0$. If $t = 0$ then $Z_\varepsilon \geq \varepsilon > 0$ by Lemma 2.2. If there exists a $t_0 \in (0, t_1]$ and $p_0 \neq q_0, l(p_0, q_0, t_0) \in (0, \frac{L(t_0)}{2}]$ such that

$$Z_\varepsilon(p_0, q_0, t_0) = 0 = \inf\left\{Z_\varepsilon(p, q, t)\middle| l(p, q, t) \in \left(0, \frac{L(t)}{2}\right]\right\},$$

then $\frac{\partial Z_\varepsilon}{\partial t} = \frac{\partial Z}{\partial t} + \varepsilon e^{2t} \leq 0$ at the point (p_0, q_0, t_0). If $Z_\varepsilon(p, q, t_0) > 0$ for $0 \leq l(p, q, t_0) \leq \pi$ then we have done. Otherwise, we will consider the case $0 \leq l(p_0, q_0, t_0) \leq \pi$.

For arbitrary $\xi, \eta \in \mathbb{R}$, let $\sigma(u) = (p_0 + \xi u, q_0 + \eta u, t_0)$. Then one has

$$\frac{\partial}{\partial u} Z(\sigma(u)) = \xi(-\langle w, e_1\rangle + f') + \eta(-\langle w, e_2\rangle - f'),$$

where

$$e_1 = \frac{\partial X}{\partial s}(p_0, t_0), \quad e_2 = \frac{\partial X}{\partial s}(q_0, t_0), \quad f' = \frac{\partial f}{\partial x}, \quad w(p, q, t) = \frac{X(q, t) - X(p, t)}{d(p, q, t)}.$$

Since the first derivative of Z_ε vanishes at (p_0, q_0, t_0), one gets that $f' = \langle w, e_1\rangle = \langle w, e_2\rangle$. So there are two possibilities: either $e_1 = e_2$ or w bisects e_1 and e_2.

By using Lemma 2.3, one can exclude the first case, i.e. $e_1 = e_2$ as Andrews-Bryan did in [2]. Also one can mimic the proof in Andrews-Bryan's paper to exclude the second case. Thus one can conclude the following theorem.

THEOREM 2.4. *The inequality* (2.3) *holds in the domain* (2.2) *for all* $t \geq 0$.

3. Proof of the Grayson theorem. In this section, we will first give a uniform curvature bound of the evolving curve for $t \in [0, \infty)$. Then we will show that the curvature converges to 1 as time tends to infinity.

THEOREM 3.1. *The flow* (1.3) *has a smooth solution in the time interval* $[0, +\infty)$.

Proof. By the inequality (2.3), one gets $d(p, q, t) \geq f(l(p, q, t), t - B)$ for $t \geq 0$, here B is a positive constant dependent on the initial curve. If $-\widetilde{t} \leq t - B$ then $d(p, q, t) \geq f(l(p, q, t), -\widetilde{t})$. Therefore, if $0 < l \leq \pi$, then

$$\begin{aligned} b(p, q, t) &= \inf\left\{ e^{\tilde{t}} \,\middle|\, d(p, q, t) \geq f(l(p, q, t), -\tilde{t}) \right\} \\ &\leq \inf\left\{ e^{\tilde{t}} \,\middle|\, -\widetilde{t} \leq t - B \right\} = e^{B - t}. \end{aligned}$$

Now letting q tends to p (i.e., $l \to 0$) can give us (by Lemma 2.2)

$$\sqrt{\frac{1}{2} \max\{\kappa(p, t)^2 - 1, 0\}} \leq e^{B - t},$$

which yields

$$\kappa(p, t)^2 \leq 1 + 2e^{2(B - t)}, \qquad (p, t) \in [0, L(t)] \times [0, \infty). \tag{3.1}$$

By the standard bootstrapping argument, one can show that all derivatives of curvature are bounded in any finite time interval. So we can draw our conclusion. ∎

In order to prove Grayson's theorem, it suffices to show that the curvature κ becomes a positive function as times goes. If we can show that κ tends to a constant 1 as $t \to \infty$, then we have done.

Denote by g, L, $\{T, N\}$ and κ by the metric (i.e., $g = \langle \frac{\partial X}{\partial \varphi}, \frac{\partial X}{\partial \varphi}\rangle^{\frac{1}{2}}$), the length, the Frenet frame and the curvature of the evolving curve, respectively. By direct calculations one has the following equations under the flow (1.3):

$$\begin{aligned}
&\frac{\partial g}{\partial t} = \left(1-\kappa^2\right) g, \qquad \frac{\partial L}{\partial t} = L - \int_0^L \kappa^2 ds,\\
&\frac{\partial}{\partial t}\frac{\partial}{\partial s} = (\kappa^2-1)\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t},\\
&\frac{\partial T}{\partial t} = \frac{\partial \kappa}{\partial s}N, \qquad \frac{\partial N}{\partial t} = -\frac{\partial \kappa}{\partial s}T,\\
&\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa(\kappa^2-1). \qquad (3.2)
\end{aligned}$$

REMARK 1. *If the initial curve X_0 is strictly convex then Gage's inequality (see Inequality B of [4]) implies that the length $L(t)$ is monotonically decreasing. However, Gage's inequality does not always hold for general simple closed plane curves (one can find a counterexample given by Jacobowitz in [4]).*

Since $L(t)$ is not monotonic, its boundedness needs more careful work. On this occasion, since we do not know whether $L(t)$ converges to 2π or not, the trick used in [2] to prove the convergence of Andrews-Bryan's normalized flow does not work for the flow (1.3). In the following, we will borrow techniques from PDEs to show that the curvature of the evolving curve indeed converges to 1. And this convergence property of κ immediately implies Grayson's convergence result of the CSF.

From now on, we use the subindex of a function to stand for its derivative. For example, $\kappa_t = \frac{\partial\kappa}{\partial t}, \kappa_s = \frac{\partial\kappa}{\partial s}, \kappa_{ss} = \frac{\partial^2\kappa}{\partial s^2}, \ldots$. By the evolution equation (3.2), one obtains that

$$\begin{aligned}
(\kappa^2)_t &= 2\kappa(\kappa_{ss}+\kappa^3-\kappa) = (\kappa^2)_{ss} - 2(\kappa_s)^2 + 2\kappa^2(\kappa^2-1),\\
(\kappa_s)_t &= (\kappa^2-1)\kappa_s + (\kappa_t)_s = (\kappa^2-1)\kappa_s + \kappa_{sss} + 3\kappa^2\kappa_s - \kappa_s\\
&= \kappa_{sss} + 4\kappa^2\kappa_s - 2\kappa_s.
\end{aligned}$$

Let $f = (\kappa_s)^2 + 4\kappa^2 + 8(1+e^{2B})e^{2(B-t)} - \varepsilon t$, then from (3.1) we get

$$\begin{aligned}
f_t &= f_{ss} - 2(\kappa_{ss})^2 + 8\kappa^2(\kappa_s)^2 - 4(\kappa_s)^2 - 8(\kappa_s)^2 + 8\kappa^2(\kappa^2-1) - 16(1+e^{2A})e^{2(A-t)} - \varepsilon\\
&\le f_{ss} + 8(\kappa_s)^2\left(\kappa^2 - \frac{3}{2}\right) + 8\kappa^2(\kappa^2-1) - 16(1+e^{2B})e^{2(B-t)} - \varepsilon\\
&\le f_{ss} + 8(\kappa_s)^2\left(\kappa^2 - \frac{3}{2}\right) - \varepsilon.
\end{aligned}$$

And also from (3.1) it follows that there exists a $T_1 > 0$ such that $\kappa^2 - \frac{3}{2} < 0$ for $t > T_1$. Thus

$$f_t \le f_{ss} - \varepsilon.$$

The maximum principle implies that $f \le \max\{f(s,t)|0 \le t \le T_1\}$, i.e., f has an upper bound which is independent of the time t. Since the constant ε can be arbitrarily chosen, $(\kappa_s)^2$ has an upper bound independent of t.

Compute that

$$\begin{aligned}
\frac{d}{dt}\int_0^L \kappa^2 ds &= 2\int_0^L \kappa(\kappa_{ss}+\kappa^3-\kappa)ds + \int_0^L \kappa^2(1-\kappa^2)ds\\
&= -2\int_0^L (\kappa_s)^2 ds + \int_0^L \kappa^2(\kappa^2-1)ds, \qquad (3.3)
\end{aligned}$$

which, together with (3.1), gives us

$$\frac{d}{dt}\int_0^L \kappa^2 ds \le 2e^{2(B-t)}\int_0^L \kappa^2 ds.$$

So one has an upper bound of $\int_0^L \kappa^2 ds$ as

$$\int_0^L \kappa^2 ds \le \exp(e^{2B})\int_0^{L_0} (\kappa_0)^2 ds. \tag{3.4}$$

Because

$$\frac{d}{dt}\int_0^L (\kappa_s)^2 ds = -2\int_0^L (\kappa_{ss})^2 ds + 7\int_0^L \kappa^2(\kappa_s)^2 ds - 3\int_0^L (\kappa_s)^2 ds,$$

one gets, for $t > T_1$,

$$\begin{aligned}\frac{d}{dt}\left(5\int_0^L \kappa^2 ds + \int_0^L (\kappa_s)^2 ds\right) &\le 7\int_0^L (\kappa^2 - \frac{13}{7})(\kappa_s)^2 ds + 5\int_0^L \kappa^2(\kappa^2-1)ds \\ &\le 10\exp(e^{2B})\int_0^{L_0}(\kappa_0)^2 ds e^{2(B-t)}.\end{aligned}$$

Integrating this inequality with respect to t implies that $\int_0^L (\kappa_s)^2 ds$ has an upper bound independent of t and so is $\frac{d}{dt}\int_0^L (\kappa_s)^2 ds$. Since

$$\begin{aligned}\frac{d}{dt}\int_0^L \kappa^4 ds &= 4\int_0^L \kappa^3(\kappa_{ss} + \kappa^3 - \kappa)ds + \int_0^L \kappa^4(1-\kappa^2)ds \\ &= -12\int_0^L \kappa^2(\kappa_s)^2 ds + \int_0^L \kappa^4(\kappa^2-1)ds \le 2e^{2(B-t)}\int_0^L \kappa^4 ds,\end{aligned}$$

one gets

$$\int_0^L \kappa^4 ds \le \exp(e^{2B})\int_0^{L_0}(\kappa_0)^4 ds.$$

So $\frac{d}{dt}\int_0^L \kappa^4 ds$ has an upper bound independent of t. Therefore we get

LEMMA 3.2. *Under the normalized flow* (1.3), *each of the following geometric quantities has an upper bound independent of the time t,*

$$(\kappa_s)^2,\ \int_0^L \kappa^2 ds,\ \frac{d}{dt}\int_0^L \kappa^2 ds,\ \int_0^L (\kappa_s)^2 ds,\ \frac{d}{dt}\int_0^L (\kappa_s)^2 ds,\ \int_0^L \kappa^4 ds,\ \frac{d}{dt}\int_0^L \kappa^4 ds.$$

In fact, one can use the estimate (3.1) to obtain a better result.

LEMMA 3.3. *The integral $\int_0^t\int_0^L (\kappa_s)^2 dsdt$ is bounded on the time interval $[0,\infty)$ and furthermore*

$$\lim_{t\to\infty}\int_0^L (\kappa_s)^2 ds = 0. \tag{3.5}$$

Proof. Integrating the both sides of (3.3) with respect to t can give us

$$\int_0^L \kappa^2 ds - \int_0^L \kappa_0^2 ds = -2\int_0^t \int_0^L (\kappa_s)^2 ds dt + \int_0^t \int_0^L \kappa^2(\kappa^2 - 1) ds dt,$$

which can yield

$$\begin{aligned} 2\int_0^t \int_0^L (\kappa_s)^2 ds dt &\le \int_0^{L_0} \kappa_0^2 ds + \int_0^t \left(e^{2(B-t)} \int_0^L \kappa^2 ds \right) dt \\ &\le \int_0^L \kappa_0^2 ds + \exp(e^{2B}) \int_0^{L_0} (\kappa_0)^2 ds \int_0^t e^{2(B-t)} dt. \end{aligned}$$

Therefore $\int_0^t \int_0^L (\kappa_s)^2 ds dt$ is bounded on the time interval $[0, \infty)$. Since $\int_0^L (\kappa_s)^2$ is non-negative and $\frac{d}{dt}\int_0^L (\kappa_s)^2 ds$ has an upper bound for $t \in [0, \infty)$, one gets the limit in (3.5) immediately. ∎

Now we give the convergence of the curvature.

THEOREM 3.4. *Under the normalized flow* (1.3), *the curvature* κ *tends to 1 as* t *goes to infinity.*

Proof. By Theorem 3.1 and Lemma 3.2, one knows that both κ^2 and $(\kappa_s)^2$ have upper bound on the time interval $[0, \infty)$. So $\kappa(\cdot, t)$ has convergent subsequences. One needs to show that any convergent subsequence of κ must converge to 1.

Since $L(t) \ge 2\pi$, it suffices to show that $L(t)$ has an upper bound for all $t > 0$. Otherwise, there is an increasing sequence, denoted by $\{L(t_j)\}_{j=1}^{\infty}$, running to infinity as $t_j \to \infty$. Since $\{t_j\}$ has a subsequence $\{t_p\}$ such that $k(s, t_p)$ is convergent as $t_p \to \infty$. Denote by κ_∞ the limit of $k(s, t_p)$. By (3.5), one obtains that

$$\lim_{t\to\infty} \int_0^\infty \left(\frac{\partial \kappa_\infty}{\partial s} \right)^2 ds = 0.$$

So κ_∞ is a constant function. Noticing that $\int_0^L \kappa^2 ds$ is bounded, one has

$$\int_0^\infty (\kappa_\infty)^2 ds < \infty,$$

which gives us $\kappa_\infty \equiv 0$. So the limit curve is a part of a line. Since the evolving curve becomes thinner and thinner and converges to a part of a line, there exist two points $X(s_1, t)$ and $X(s_2, t)$ such that the distance between these two points tends to 0 as $t \to \infty$ and $|s_1 - s_2|$ always larger than a positive constant. This is impossible, because the isoperimetric ratio d/l is invariable under rescarling and Huisken [10] has proven that this ratio increases at its minimal value as time goes. Therefore, there exists a positive constant C such that $L(t) < C$ for all $t > 0$.

Combining the evolution equation (3.3) and that of L can give us

$$\begin{aligned} \int_0^L (\kappa^2 - 1)^2 ds &= \int_0^L \kappa^4 ds - 2\int_0^L \kappa^2 ds + L \\ &= \frac{d}{dt}\int_0^L \kappa^2 ds + 2\int_0^L (\kappa_s)^2 ds + \int_0^L \kappa^2 ds - 2\int_0^L \kappa^2 ds + L \\ &= \frac{d}{dt}\int_0^L \kappa^2 ds + 2\int_0^L (\kappa_s)^2 ds + \frac{\partial L}{\partial t}. \end{aligned}$$

Thus, we get

$$\int_0^t \int_0^L (\kappa^2-1)^2 ds dt = \int_0^L \kappa^2 ds - \int_0^{L_0} \kappa_0^2 ds + \int_0^t 2\int_0^L (\kappa_s)^2 ds dt + L - L_0.$$

By Lemmata 3.2 and 3.3 and the above discussion, each of $\int_0^L \kappa^2 ds$, $\int_0^t \int_0^L (\kappa_s)^2 ds dt$ and L has an upper bound independent of the time t. So is $\int_0^t \int_0^L (\kappa^2-1)^2 ds dt$. Noticing that

$$\begin{aligned}
\frac{d}{dt}\int_0^L (\kappa^2-1)^2 ds &= \frac{d}{dt}\int_0^L \kappa^4 ds - 2\frac{d}{dt}\int_0^L \kappa^2 ds + \frac{\partial L}{\partial t} \\
&= -12\int_0^L \kappa^2(\kappa_s)^2 ds + \int_0^L \kappa^4(\kappa^2-1)ds \\
&\quad +2\int_0^L (\kappa_s)^2 ds - \int_0^L \kappa^2(\kappa^2-1)ds + L - \int_0^L \kappa^2 ds \\
&\le 2e^{2(B-t)}\exp(e^{2B})\int_0^{L_0} (\kappa_0)^4 ds + 2\int_0^L (\kappa_s)^2 ds + L,
\end{aligned}$$

one obtains that $\frac{d}{dt}\int_0^L (\kappa^2-1)^2 ds$ has an upper bound independent of t. Since the integral $\int_0^\infty \int_0^L (\kappa^2-1)^2 ds dt < \infty$, one can get

$$\lim_{t\to\infty} \int_0^L (\kappa^2-1)^2 ds = 0. \tag{3.6}$$

Now let us choose any convergent subsequence of curvature, denoted by $k(s,t_i)$, here $t_i \to \infty$. Denote by κ_∞ the limit of $k(s,t_i)$. Since $L(t_i)$ is bounded, there exists a convergent subsequence, denoted by $L(t_p), p = 1, 2, \ldots, t_p \to \infty$. Let p tend to infinity. Since $k(s,t_i)$ converges to κ_∞ uniformly, by (3.6), one has

$$\int_0^{L_\infty} (\kappa_\infty^2 - 1)^2 ds = 0,$$

here L_∞ is the limit of $L(t_p)$. Because κ_∞ is smooth and the curve is positively oriented, one can conclude that $\kappa_\infty \equiv 1$. Hence the limit of the curvature is 1 as time goes to infinity. □

REMARK 2. *The above Theorem 3.4 implies that there exists a positive T_0 such that the evolving curve $X(\varphi,t)$ is strictly convex for $t > T_0$. This gives a proof of the Grayson theorem.*

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