VOLTERRA CALCULUS, LOCAL EQUIVARIANT FAMILY INDEX THEOREM AND EQUIVARIANT ETA FORMS*

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Abstract. In this paper, we give proofs of the family index formula and the equivariant family index formula by the Greiner's approach to heat kernel asymptotics. We compute equivariant family JLO characters. We also define the equivariant eta form and give a proof of its regularity.

Key words. Heat equation methods, family index formula, equivariant family index formula, equivariant family JLO characters, equivariant eta form.

AMS subject classifications. 58J20.

1. Introduction. The first success of proving the Atiyah-Singer index theorem directly by heat kernel method was achieved by Patodi [Pa], who carried out the "fantastic cancellation" (cf. [MS]) for the Laplace operators and who for the first time proved a local version of the Gauss-Bonnet-Chern theorem. After that several different direct heat kernel proofs of the Atiyah-Singer index theorem for Dirac operators appeared independently: Bismut [Bi1], Getzler [Ge1], [Ge2] and Yu [Yu], Ponge [Po1]. All the proofs have their own advantages. Motivated by the problem of generalizing the heat kernel proofs of the index theorem to prove a local index theorem for families of elliptic operators, Quillen [Qu] introduced the concept of superconnections, which was developed by Bismut to give a heat kernel representation for the Chern character of families of first order elliptic operators. Then using his probabilitistic method, Bismut [Bi2] obtained a proof of the local index theorem for families of Dirac operators. In [BV1],[Do1], they gave two different proofs of local index theorem for families of Dirac operators. Using the method of Yu, Zhang gave another proof of the local index theorem for families of Dirac operators in [Zh1]. The first purpose of this paper is to give another proof of the local index theorem for families of Dirac operators by the Ponge's method in [Po1].

The Atiyah-Bott-Segal-Singer index formula is a generalization of the Atiyah-Singer index theorem to the case with group actions. In [BV2], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. In [LYZ], Lafferty, Yu and Zhang presented a very simple and direct geometric proof for equivariant index of the Dirac operator. In [PW], Ponge and Wang gave a different proof of the equivariant index formula by the Greiner's approach of the heat kernel asymptotics. In [LM], in order to prove family rigidity theorems, Liu and Ma proved the equivariant family index formula. The second purpose of this paper is to give another proof of the local equivarint index theorem for families of Dirac operators by the Greiner's approach of the heat kernel asymptotics.

It is known, due to Connes [Co] and an equivalent but convenient version due to Jaffe, Lesniewski and Osterwalder, also known as a JLO formula [JLO], that the Chern character of a θ-summable Fredholm module (H, D) over a unital C^*-algebra A, takes value in the entire cyclic cohomology of A. JLO characters were computed in [CM1] and [BlF]. An explicit formula of an equivariant JLO character, associated to the invariant Dirac operator, in the presence of a countable discrete group action on

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a smooth compact spin Riemannian manifold, was given by Azmi, [Az1]. Moreover, in [Az1] it was shown that this equivariant cocycle is an element of the delocalized cohomology, and it can be paired with an equivariant K-theory idempotent. In the case when G is a compact Lie group, Chern and Hu [CH] gave an explicit formula of an equivariant Chern-Connes character, associated to a G-equivariant θ-summable Fredholm module. In [PW], Ponge and Wang computed equivariant Chern-Connes characters by the Greiner's approach of the heat kernel asymptotics.

On the other hand, Wu [Wu] constructed a bivariant Chern-Connes character for (a special class of) θ summable modules, by incorporating the JLO formula and the superconnectiom formalism of Quillen. Wu's bivariant character takes values in the bivariant cyclic theory described by Lott [Lo], who constructed it as a combination of entire cyclic (co)homology and noncommutative de Rham homology of graded differential algebra. Then by adopting Wu's method and employing Bismut's superconnection together with the canonical order calculus developed by Simon in [CFKB], Azmi [Az2] expressed the local formula for families in terms of differential forms on the base and the Chern roots of the fibration. In [Zh2], Zhang announced a different method to compute the family JLO characters by developing the family version of the method in [CH] and [Fe]. In the third part of this paper, we give details of the announcement in [Zh2] and compute the equivariant family JLO characters.

In [APS], Atiyah, Patodi and Singer proved the Atiyah-Patodi-Singer index theorem for manifolds with boundary and they introduced the eta invariants. Bismut and Freed gave a simple proof of the regularity of eta invariants in [BiF]. In [Po2], Ponge gave another proof of the regularity of eta invariants using the method in [Po1]. Bismut and Cheeger generalized the Atiyah-Patodi-Singer index theorem to the family case in [BC1,2]. They used the eta form for families of Dirac operators. The regularity of the eta form was proved by the probabilistic method in [BiGS]. Donnelly generalized the Atiyah-Patodi-Singer index theorem to the equivariant case and introduced the equivariant eta invariant in [Do2]. Zhang proved the regularity of the equivariant eta invariant by the Clifford asymptotics in [Zh3]. In this paper, we firstly prove the regularity of the equivariant eta invariant by a similar method in [PW] and by introducing the Grassmann variable. Then we define the equivariant eta form and prove its regularity.

Using the approach of Ponge and Ponge-Wang to give new proofs of the equivariant family index theorem and the regularity of equivariant eta forms has two advantages. One is using the Volterra pseudodifferential calculus to get the heat kernel asymtotic expansion instead of heat equation discussions as in [Yu], [Zh1], [BGV, Chapter 2]. In [LYZ] and [Zh3], for proving the equivariant local index theorem and the regularity of the equivariant eta invariants, transformed formulas between normal and tubular coordinates are needed. In [LM], Liu and Ma used the finite propagation speed method in [Bi3] to prove the equivariant family index theorem. The other advantage is that the transformed formulas between normal and tubular coordinates are the consequence of a standard change of variable formula for pseudodifferential symbols.

This paper is organized as follows: In Section 2, we give another proof of the local index theorem for families of Dirac operators by the Ponge's method. In Section 3, we give another proof of the local equivarint index theorem for families of Dirac operators by the Greiner's approach of the heat kernel asymptotics. In Section 4, we compute the equivariant family JLO characters. In Section 5, we define the equivariant eta form and prove its regularity.

2. The local index theorem for families of Dirac operators.

2.1. The Greiner's approach of heat kernel asymptotics with form co-efficients. In [Gr],[BGS], Greiner and Beals-Greiner-Stanton defined Volterra pseudodifferential operators and gave the heat kernel asymptotics by Volterra pseudodifferential calculus. A good summary on the Greiner's approach of the heat kernel asymptotics was given in [Po1]. In [Po1], Ponge gave a proof of the local index theorem by the Greiner's approach of the heat kernel asymptotics. In this section, we shall extend Volterra pseudodifferential calculus to the case with form coeffcients, i.e. the family case. A local family index theorem was proved originally by Bismut in [Bi2]. A superconnection due to Bismut played an important role in the proof of the local family index theorem.

Let us recall the definition of superconnection due to Bismut. Let M be an $n + \overline{q}$ dimensional compact connected manifold and B be a $\overline{q}$ dimensional compact connected manifold. We assume that $\pi : M \to B$ is a submersion of M onto B, which defines a fibration of M with fibre Z. For $y \in B$, $\pi^{-1}(y)$ is then a submanifolds M_y of M. Denote by TZ the n-dimensional vector bundle on M whose fibre $T_x M_{\pi x}$ is the tangent space at x to the fibre $M_{\pi(x)}$. We assume that M and B are oriented. We take a smooth horizontal subbundle $T^H M$ of TM. Vector fields $X \in \Gamma(B, TB)$ will be identified with their horizontal lifts $X \in \Gamma(M, T^H M)$. Moreover $T_x^H M$ is isomorphic to $T_{\pi(x)} B$ via π_*. We take a Riemannian metric on B and then lift the Euclidean scalar product g_B of TB to $T^H M$. We further assume that TZ is endowed with a scalar product g_Z. Thus we can introduce on TM a new scalar product $g_B \oplus g_Z$, and denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Set ∇^B denote the Levi-Civita connection on TB and we still denote by ∇^B the pullback connection on $T^H M$. Let $\nabla^Z = P_Z(\nabla^L)$ where P_Z denotes the orthogonal projection to TZ. Set $\nabla^\oplus = \nabla^B \oplus \nabla^Z$ and $S = \nabla^L - \nabla^\oplus$ and T be the torsion tensor of $\nabla^\oplus$. Denote by $SO(TZ)$ the $SO(n)$ bundle of oriented orthonormal frames in TZ. Now we assume that bundle TZ is spin. Denote by $S(TZ)$ the associated spinor bundle and ∇^Z can be lifted to a connection on $S(TZ)$. Let D be the Dirac operator in the tangent direction defined by $D = \sum_{i=1}^{n} c(e_i^*) \nabla_{e_i}^{S(TZ)}$ where $\nabla^{S(TZ)}$ is a spin connection on $S(TZ)$. Set E be the vector bundle $\pi^*(\wedge T^* B) \otimes S(TZ)$. This bundle carries a natural action m_0 of the degenerate Clifford module denoted as $Cl_0(M)$. The Clifford action of a horizontal cotangent vector $\alpha \in \Gamma(M, T_H^* M)$ is given by the exterior multiplication $m_0(\alpha) = \varepsilon(\alpha)$ acting on the first factor $\bigwedge T_H^* M$ in E, while the Clifford action of a vertical cotangent vector simply is its Clifford action on $S(TZ)$. Define the connection by ([BGV])

$$\nabla^{E,\oplus} := \pi^* \nabla^B \otimes 1 + 1 \otimes \nabla^S, \tag{2.1}$$

$$\omega(X)(Y, Z) := g(\nabla_X^L Y, Z) - g(\nabla_X^\oplus Y, Z), \tag{2.2}$$

$$\nabla_X^{E,0} := \nabla_X^{E,\oplus} + \frac{1}{2} m_0(\omega(X)), \tag{2.3}$$

for $X, Y, Z \in \Gamma(M, TM)$. Then the Bismut connection acting on $\Gamma(M, \wedge(\pi^* T^* B) \otimes S(TZ))$ is defined by

$$\mathcal{B} = \sum_{i=1}^{n} c(e_i^*) \nabla_{e_i}^{E,0} + \sum_{j=1}^{\overline{q}} c(f_j^*) \nabla_{f_j}^{E,0}. \tag{2.4}$$

Then by Proposition 10.15 in [BGV], $\mathcal{B} = D + A_{[+]}$, where $A_{[+]}$ is an operator with coefficients in $\Omega_{\geq 1}(B)$. By Theorem 10.17 in [BGV], we have

$$F = \mathcal{B}^2 = -\sum_{i=1}^{n}(\nabla_{e_i}^{E,0})^2 + \sum_{i=1}^{n}\nabla_{\nabla_{e_i}^{TZ}e_i}^{E,0} + \frac{1}{4}r = D^2 + F_{[+]}, \tag{2.5}$$

where r is the scalar curvature of fibres. Let $\triangle_k$ be the k-simplex defined by

$$\{(\sigma_0,\cdots,\sigma_k)|\sigma_0+\cdots\sigma_k = 1,\ 0\leq\sigma_j\leq 1\}.$$

For a fixed $t > 0$, define the operator e^{-tF} by

$$e^{-tF} = e^{-tD^2} + \sum_{k>0}(-t)^k I_k, \tag{2.6}$$

where

$$I_k = \int_{\triangle_k} e^{-\sigma_0 tD^2}F_{[+]}e^{-\sigma_1 tD^2}F_{[+]}\cdots e^{-\sigma_{k-1}tD^2}F_{[+]}e^{-\sigma_k tD^2}d\sigma, \tag{2.7}$$

and where the sum is finite (see [BGV, p. 312]). Then

$$(\frac{\partial}{\partial t} + F)e^{-tF} = 0, \quad Fe^{-tF} = e^{-tF}F. \tag{2.8}$$

In the following, we formulate the Greiner's approach of heat kernel asymptotics with coefficients in $\wedge T_z^* B$ as in [Gr] and [BeGS]. Let Q_0 be given by

$$(Q_0 u)(x,s) = \int_0^\infty e^{-sF}[u(x,t-s)]dt, \quad u\in\Gamma_c(M_z\times\mathbb{R}, S(TM_z))\otimes\wedge T_z^* B. \tag{2.9}$$

The operator Q_0 maps continuously from u to $D'(M_z\times\mathbb{R}, S(TM_z))\otimes\wedge T_z^* B$ which is the dual space of $\Gamma_c(M_z\times\mathbb{R}, S(TM_z))\otimes\wedge T_z^* B$. By (2.8), we have

$$(F+\frac{\partial}{\partial t})Q_0 u = Q_0(F+\frac{\partial}{\partial t})u = u, \quad u\in\Gamma_c(M_z\times\mathbb{R}, S(TM_z))\otimes\wedge T_z^* B, \tag{2.10}$$

We define the operator

$$Q = (F+\frac{\partial}{\partial t})^{-1} = (D^2+\frac{\partial}{\partial t})^{-1} + \sum_{k>0}(-1)^k(D^2+\frac{\partial}{\partial t})^{-1}[F_{[+]}(D^2+\frac{\partial}{\partial t})^{-1}]^k, \tag{2.11}$$

where $(D^2+\frac{\partial}{\partial t})^{-1}$ is the Volterra inverse of $D^2+\frac{\partial}{\partial t}$ as in [BeGS]. Then

$$(F+\frac{\partial}{\partial t})Q = I - R_1; \quad Q(F+\frac{\partial}{\partial t}) = 1 - R_2, \tag{2.22}$$

where R_1, R_2 are smoothing operators. Let $K_{Q_0}(x,y,t-s)$ be the distribution kernel of Q_0. That is

$$(Q_0 u)(x,t) = \int_{M_z\times\mathbb{R}} K_{Q_0}(x,y,t-s)u(y,s)dyds, \tag{2.23}$$

and let $k_t(x,y)$ be the heat kernel of e^{-tF}. Similar to the discussions in [BeGS p.363], we get

$$K_{Q_0}(x,y,t) = \begin{cases} k_t(x,y) & \text{when } t>0, \\ 0 & \text{when } t<0. \end{cases} \tag{2.24}$$

DEFINITION 2.1. *The operator P is called a Volterra ΨDO if (i) P has the Volterra property, i.e. it has a distribution kernel of the form $K_P(x, y, t - s)$ where $K_P(x, y, t)$ vanishes on the region $t < 0$.*
(ii) The heat operator $P + \frac{\partial}{\partial t}$ is parabolically homogeneous, i.e. the homogeneity with respect to the dilations of $\mathbb{R}^n \times \mathbb{R}^1$ is given by

$$\lambda \cdot (\xi, \tau) = (\lambda \xi, \lambda^2 \tau), \qquad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^1, \;\; \lambda \neq 0. \tag{2.25}$$

In the sequel for $g \in \mathcal{S}(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we let g_λ be the tempered distribution defined by

$$\langle g_\lambda(\xi, \tau), u(\xi, \tau) \rangle = |\lambda|^{-(n+2)} \langle g(\xi, \tau), u(\lambda^{-1}\xi, \lambda^{-2}\tau) \rangle, \;\; u \in \mathcal{S}(\mathbb{R}^{n+1}). \tag{2.26}$$

DEFINITION 2.2. *A distribution $g \in \mathcal{S}(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree m, where $m \in \mathbb{Z}$, if for any $\lambda \neq 0$, we have $g_\lambda = \lambda^m g$.*

Let $\mathbb{C}_-$ denote the complex halfplane $\{\text{Im}\,\tau < 0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

LEMMA 2.3 ([BeGS, Prop. 1.9]). *Let $q(\xi, \tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R})\backslash 0) \otimes \wedge T_z^* B$ be a parabolic homogeneous symbol of degree m such that:*
(i) q extends to a continuous function on $(\mathbb{R}^n \times \overline{\mathbb{C}_-})\backslash 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.
Then there is a unique $g \in \mathcal{S}(\mathbb{R}^{n+1}) \otimes \wedge T_z^* B$ agreeing with q on $\mathbb{R}^{n+1}\backslash 0$ so that:
(ii) g is homogeneous of degree m;
(iii) The inverse Fourier transform $\check{g}(x, t)$ vanishes for $t < 0$.

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U \times \mathbb{R}^{n+1}/0$ as follows.

DEFINITION 2.4. *The set $S_V^m(U \times \mathbb{R}^{n+1}) \otimes \wedge T_z^* B$, $m \in \mathbb{Z}$, consists of smooth functions $q(x, \xi, \tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j \geq 0} q_{m-j}$, where:*
- $q_l \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R})\backslash 0]) \otimes \wedge T_z^* B$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 2.3 with respect to the last $n + 1$ variables;
- The sign $\sim$ means that, for any integer N and any compact $K \subset U$, there is a constant $C_{NK\alpha\beta k} > 0$ such that for $x \in K$ and for $|\xi| + |\tau|^{\frac{1}{2}} > 1$ we have

$$||\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x, \xi, \tau)|| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}. \tag{2.27}$$

For $q = \sum_l q_l \omega^l$ where $q_l \in S_V^m(U \times \mathbb{R}^{n+1})$ and $\omega^l \in \wedge^l T_z^* B$, we define $||q|| = \sum_l |q_l| ||\omega^l||$ and $||\omega^l||$ is the norm of ω^l in $(\wedge^l T_z^* B, g_z^{TB})$.

DEFINITION 2.5. *The set $\Psi_V^m(U \times \mathbb{R}, \wedge T_z^* B)$, $m \in \mathbb{Z}$, consists of continuous operators Q from $C_c^\infty(U_x \times \mathbb{R}_t, \wedge T_z^* B)$ to $C^\infty(U_x \times \mathbb{R}_t, \wedge T_z^* B)$ such that:*
(i) Q has the Volterra property;
(ii) $Q = q(x, D_x, D_t) + R$ for some symbol q in $S_V^m(U \times \mathbb{R}, \wedge T_z^* B)$ and some smoothing operator R.

In the sequel if Q is a Volterra ΨDO, we let $K_Q(x, y, t - s)$ denote its distribution kernel, so that the distribution $K_Q(x, y, t)$ vanishes for $t < 0$.

DEFINITION 2.6. *Let* $q_m(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \backslash 0)) \otimes \wedge T_z^* B$ *be a homogeneous Volterra symbol of order* m *and let* $g_m \in C^\infty(U) \otimes \mathcal{S}'(\mathbb{R}^{n+1}) \otimes \wedge T_z^* B$ *denote its unique homogeneous extension given by Lemma 2.3. Then:*
- $\check{q}_m(x, y, t)$ *is the inverse Fourier transform of* $g_m(x, \xi, \tau)$ *in the last* $n + 1$ *variables;*
- $q_m(x, D_x, D_t)$ *is the operator with kernel* $\check{q}_m(x, y - x, t)$.

The composition of ordinary Volterra symbols naturally extends to a composition of Volterra symbols with form coefficients. Let $\mathbf{Q}$ and $\mathbf{Q}'$ be in $\Psi_V^{m_1}(U \times \mathbb{R}, \wedge^j T_z^* B)$ and $\Psi_V^{m_2}(U \times \mathbb{R}, \wedge^l T_z^* B)$ with the symbols $\mathbf{q}$ and $\mathbf{q}'$. Let the composition of $\mathbf{Q}$ and $\mathbf{Q}'$ have the symbol $\mathbf{q}\widetilde{\circ}\mathbf{q}'$, then

$$\mathbf{q}\widetilde{\circ}\mathbf{q}' = \omega_1 \wedge \omega_2 \otimes q \circ q' \tag{2.28}$$

where $\mathbf{q} = \omega_1 q$, $\mathbf{q}' = \omega_2 q'$ and $q \circ q'$ is the ordinary composition of symbols corresponding to the Volterra ΨDO algebra multiplication ([Gr, BeGs]). Thus we have

$$\Psi_V^{m_1}(U \times \mathbb{R}, \wedge^j T_z^* B) \times \Psi_V^{m_2}(U \times \mathbb{R}, \wedge^l T_z^* B) \to \Psi_V^{m_1+m_2}(U \times \mathbb{R}, \wedge^{j+l} T_z^* B). \tag{2.29}$$

PROPOSITION 2.7. *The following properties hold.*
1) Composition. Let $Q_j \in \Psi_V^{m_j}(U \times \mathbb{R}) \otimes \wedge T_z^* B$, $j = 1, 2$ *have symbol* q_j *and suppose that* Q_1 *or* Q_2 *is properly supported (see* [Ta, p. 43]*). Then* $Q_1 Q_2$ *is a Volterra* ΨDO *of order* $m_1 + m_2$ *with symbol* $q_1\widetilde{\circ}q_2 \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha q_1 D_x^\alpha q_2$.
2) Parametrices. Let $Q = Q^m + Q^{<m}$ *where* Q^m *is in* $\Psi_V^m(U \times \mathbb{R})$*(an order* m *Volterra* ΨDO *without form coefficients in* $\wedge T_z^* B$*) and* $Q^{<m}$ *is in* $\Psi_V^{<m}(U \times \mathbb{R}) \otimes \wedge T_z^* B$. *We assume that there is an operator* P *in* $\Psi_V^{-m}(U \times \mathbb{R})$ *such that*

$$Q^m P = 1 - R_1, \quad P Q^m = 1 - R_2 \tag{2.30}$$

where R_1, R_2 *are smoothing operators in* $\Psi_V^{-\infty}(U \times \mathbb{R})$. *let*

$$\widetilde{Q} = P + \sum_{k>0} (-1)^k P[Q^{<m}P]^k, \tag{2.31}$$

then

$$Q\widetilde{Q} = 1 - \widetilde{R_1}, \quad \widetilde{Q}Q = 1 - \widetilde{R_2} \tag{2.32}$$

where $\widetilde{R_1}$, $\widetilde{R_2}$ *are smoothing operators in* $\Psi_V^{-\infty}(U \times \mathbb{R}) \otimes \wedge T_z^* B$.

Proof. The claim 1) comes from (2.28) and 1) of Proposition 1 in [Po1]. By (2.30) and (2.31) and direct computations, we get (2.32). □

By (2.11) and the fact that $(D^2 + \partial_t)^{-1}$ is a Volterra ΨDO of order -2 and that $F_{[+]}$ is a first order Volterra ΨDO, we get

PROPOSITION 2.8. *The differential operator* $F + \partial_t$ *is invertible and its inverse* $(F + \partial_t)^{-1}$ *is a Volterra* ΨDO *of order* -2.

Let $Q \in \Psi_V^m(U \times \mathbb{R}, \wedge T_z^* B)$ have symbol $q \sim \sum q_{m-j}$ and $q_{m-j} = \sum_{s=1}^{2^{\overline{q}}} q_{m-j,s} \omega^s$ where $q_{m-j,s} \in S^{m-j}(U \times \mathbb{R})$ and $\omega^s \in \wedge^s T^* B$. We define the inverse Fourier transform of q_{m-j} by $\check{q}_{m-j} = \sum_{s=1}^{2^{\overline{q}}} \check{q}_{m-j,s} \omega^s$. Then we have

LEMMA 2.9 (Compare with Lemma 2 in [Po1]). *Let $Q \in \Psi_V^m(U \times \mathbb{R}, \wedge T_z^* B)$, we have in local coordinates*

$$K_Q(x, x, t) \sim t^{-(\frac{n}{2} + [\frac{m}{2}] + 1)} \sum_{l \geq 0} t^l \check{q}_{2[\frac{m}{2}] - 2l}(x, 0, 1). \qquad (2.33)$$

Proof. Let $Q = \sum_{r=1}^{2^{\overline{q}}} Q_r \omega_r$ where $Q_r \in \Psi_V^m(U \times \mathbb{R})$ and $\omega_r \in \wedge^r T^* B$. We note that the leading symbol of Q_r is probably zero. We set that the symbol $q^{[r]}$ of Q_r has an asymptotic expansion $q^{[r]} \sim \sum_{j \geq 0} q_{m-j}^{[r]}$. By Lemma 2 in [Po1], we have

$$K_{Q_r}(x, x, t) \sim t^{-(\frac{n}{2} + [\frac{m}{2}] + 1)} \sum_{l \geq 0} t^l \check{q}_{2[\frac{m}{2}] - 2l}^{[r]}(x, 0, 1). \qquad (2.34)$$

By the equality

$$K_Q(x, x, t) = \sum_{r=1}^{2^{\overline{q}}} K_{Q_r}(x, x, t) \omega_r, \qquad (2.35)$$

and the definition of the inverse Fourier transform of q_{m-j}, we get (2.33). $\square$

By (2.24) and Proposition 2.8 and Lemma 2.9, we get

THEOREM 2.10 (Compare with Theorem 1.6.1 in [Gr]). *In*

$$C^\infty(M_z, \mathrm{End}(S(T(M_z)))) \otimes \wedge T_z^* B,$$

we have

$$k_t(x, x) \sim t^{-\frac{n}{2}} \sum_{l \geq 0} t^l a_l(F)(x), \ t \to 0^+, \ \ a_l(F)(x) = \check{q}_{-2-2l}(x, 0, 1), \qquad (2.36)$$

where $q_{-2-2l}(x, \xi, \tau)$ is the $-2l - 2$ order symbol of $(\partial_t + F)^{-1}$ and the second equality in (2.36) holds in local coordinates.

By the same reason with Lemma 2.9 and Proposition 2 in [Po1], we have

PROPOSITION 2.11 (Compare with Proposition 2 in [Po1]). *Let $P : C^\infty(M_z, S(TM_z)) \to C^\infty(M_z, S(TM_z))$ be a differential operator of order m and let $h_t(x, y)$ denote the distribution kernel of Pe^{-tF}. Then in $C^\infty(M_z, \mathrm{End}(S(TM_z))) \otimes \wedge T_z^* B$, we have*

$$h_t(x, x) \sim t^{-([\frac{m}{2}] + \frac{n}{2})} \sum_{l \geq 0} t^l b_l(F)(x), \ t \to 0^+, \ \ b_l(x) = \check{q}_{2[\frac{m}{2}] - 2 - 2l}(x, 0, 1), \qquad (2.37)$$

where $q_{2[\frac{m}{2}] - 2 - 2l}(x, \xi, \tau)$ is the $2[\frac{m}{2}] - 2l - 2$ order symbol of $P(\partial_t + F)^{-1}$ and the second equality in (2.37) holds in local coordinates.

2.2. The local family index formula. In [Bi2],[BV1],[Do1],[Zh1], several different proofs of the local index theorem for families of Dirac operators were given. In this section, we shall give a new proof of the local family index formula by using the Greiner's approach of the heat kernel asymptotics with form coefficients. Comparing with previous proofs, we use the Volterra calculus with form coefficients instead of heat equation discussions to get the family heat kernel asymototics as in [Po1].

Let us introduce some notations. For $z \in B$, denote by D_z the restriction of D to the fiber acting on $\Gamma(M_z, S(TM_z))$. We set that the dimension n of fibre to be even. Then $\Gamma(M_z, S(TM_z))$ has a splitting as a sum of $\Gamma^+(M_z, S(TM_z))$ and $\Gamma^-(M_z, S(TM_z))$. The operator D_z interchanges $\Gamma^+(M_z, S(TM_z))$ and $\Gamma^-(M_z, S(TM_z))$. Let $D_{z,+}$, $D_{z,-}$ be the restrictions of D_z to $\Gamma^+(M_z, S(TM_z))$ and $\Gamma^-(M_z, S(TM_z))$ respectively. By Chapter 9 in [BGV], the difference bundle $[\ker D_{z,+}] - [\ker D_{z,-}]$ over B is well defined in the sense of K-theory. The family index theorem presents a calculation of the Chern character of the difference bundle as a differential form over B explicitly. We change the normalization constant in the definition of the Chern character. Namely, for a vector bundle V with connection form γ and curvature C, we set $\text{Ch}(V) = \text{Tr}(\exp(-C))$. Let $\widehat{A}(R^{TZ}) = \det^{\frac{1}{2}}\left(\frac{R^{TZ}/2}{\sinh(R^{TZ}/2)}\right)$ and $\int_{M_z}$ denote the integral along the fibre. Then we have

THEOREM 2.12 (Atiyah-Singer [AS]). *It holds that the form*

$$(2i\pi)^{-\frac{n}{2}} \int_{M_z} \widehat{A}(R^{TZ}) \qquad (2.38)$$

is a representative of $\text{Ch}([\ker D_{z,+}] - [\ker D_{z,-}])$.

Let $\overline{F}$ be a complex vector bundle on M and $D_z^{\overline{F}}$ be the twisted Dirac operator along the fibre. Denote by $\text{ch}(\overline{F})$ the Chern character of $\overline{F}$, we can get a twisted index bundle and the twisted family index theorem:

THEOREM 2.13 (Atiyah-Singer [AS]). *It holds that the form*

$$(2i\pi)^{-\frac{n}{2}} \int_{M_z} \widehat{A}(R^{TZ})\text{ch}(\overline{F}) \qquad (2.39)$$

is a representative of $\text{Ch}([\ker D_{z,+}^{\overline{F}}] - [\ker D_{z,-}^{\overline{F}}])$.

We shall prove Theorem 2.12 and Theorem 2.13 similarly as follows. For $t > 0$, we set that S_x is the spinor space on $T_x(M_z)$ and

$$\psi_t: \ \text{Hom}(S_x, S_x)\widehat{\otimes} \wedge (T_z^*B) \to \text{Hom}(S_x, S_x)\widehat{\otimes} \wedge (T_z^*B) \ \ hdy_\alpha \mapsto \frac{1}{\sqrt{t}}hdy_\alpha. \quad (2.40)$$

For $\omega \in \wedge(T_z^*B)$ and $A \in \text{Hom}(S_x, S_x)$, we define $\text{Tr}_s(\omega A) = \omega\text{Tr}_s(A)$. By Proposition 4.1 in [Z1], we get

PROPOSITION 2.14 ([Bi2, Z1]). For $t > 0$,

$$\psi_t \int_{M_z} \text{Tr}_s k_t^z(x, x)dx$$

is a representative of $\text{Ch}([\ker D_{z,+}] - [\ker D_{z,-}])$.

By Proposition 2.14, in order to prove Theorem 2.12, we only need to prove the following local family index formula:

THEOREM 2.15([AS]). *We have*

$$\lim_{t\to 0}\psi_t \int_{M_z} \text{Tr}_s k_t^z(x, x)dx = (2i\pi)^{-\frac{n}{2}} \int_{M_z} \widehat{A}(R^{TZ}). \qquad (2.41)$$

In order to prove Theorem 2.15, let us recall the symbol map and the Getzler order, then we compute the symbol map of the heat kernel. Let n be even and $\mathrm{End}(S(TM_z))$ is a bundle of algebras over M_z isomorphic to the Clifford bundle $\mathrm{Cl}(M_z)$, whose fiber $\mathrm{Cl}_x(M_z)$ at $x \in M_z$ is the complex algebra generated by 1 and elements of $T_x^* M_z$ with relations

$$\xi \cdot \eta + \eta \cdot \xi = -2 \langle \xi, \eta \rangle, \quad \xi, \eta \in T_x^* M_z. \tag{2.42}$$

Recall that the quantization map $c : \wedge T_{\mathbb{C}}^*(M_z) \to \mathrm{Cl}(M)$ and the symbol map $\sigma = c^{-1}$ satisfy

$$\sigma(c(\xi)c(\eta)) = \xi \wedge \eta - \langle \xi, \eta \rangle. \tag{2.43}$$

So, for ξ and η in $\wedge T_{\mathbb{C}}^*(M_z)$ we have

$$\sigma(c(\xi^{(i)})c(\eta^{(j)})) = \xi^{(i)} \wedge \eta^{(j)} \mod \wedge^{i+j-2} T_{\mathbb{C}}^*(M_z), \tag{2.44}$$

where $\xi^{(l)}$ denotes the component in $\wedge^l T_{\mathbb{C}}^*(M_z)$ of $\xi \in \wedge T_{\mathbb{C}}^*(M_z)$. Since we compute the local index in a fixed fibre M_z, so we use an observation due to Getzler as in the case of a single manifold. If $e_1, \cdots, e_n$ is an orthonormal frame of $T_x M_z$, then

$$\mathrm{Tr}_s[c(e^{i_1}) \cdots c(e^{i_k})] = \begin{cases} 0 & \text{if } k \neq n \\ (-2i)^{\frac{n}{2}} & \text{if } k = n. \end{cases} \tag{2.45}$$

We shall prove (2.41) at a fixed point $x_0 \in M_z$. Using normal coordinates centered at x_0 in M_z and paralleling ∂_i at x_0 along geodesics through x_0, we get the orthonormal frame $e_1, \cdots, e_n$. By

$$k_t^z(0,0) = K_Q^z(0,0,t) + O(t^\infty) \qquad \text{as } t \to 0^+,$$

and by (2.45), we get

$$\lim_{t \to 0} \psi_t \int_{M_z} \mathrm{Tr}_s k_t^z(x,x) dx = (-2i)^{\frac{n}{2}} \lim_{t \to 0} \int_{M_z} \sigma[\psi_t K_Q(0,0,t)]. \tag{2.46}$$

We define the Getzler order in [Po1] as follows:

$$\deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx_j) = \deg c(dy_j) = -\deg x^j = 1. \tag{2.47}$$

Let $Q \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S(TM_z) \otimes \wedge^* T_z^* B)$ have symbol

$$q(x, \xi, \tau) \sim \sum_{k \leq m'} \sum_{l=1}^{2^{\overline{q}}} q_{k,l}(x, \xi, \tau) \omega^{[l]}, \tag{2.48}$$

where $\omega^{[l]} \in \wedge^l T_z^* B$ and $q_{k,l}(x, \xi, \tau)$ is an order k symbol. Then taking components in each subspace $\wedge^j T^* M_z$ and using Taylor expansions at $x = 0$ give formal expansions

$$\sigma[q(x, \xi, \tau)] \sim \sum_{j,k,l} \sigma[q_{k,l}(x, \xi, \tau)]^{(j)} \omega^{[l]} \sim \sum_{j,k,l,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_{k,l}(0, \xi, \tau)]^{(j)} \omega^{[l]}. \tag{2.49}$$

The symbol $\frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_{k,l}(0, \xi, \tau)]^{(j)} \omega^{[l]}$ is Getzler homogeneous of order $k + j + l - |\alpha|$. So we can expand $\sigma[q(x, \xi, \tau)]$ as

$$\sigma[q(x, \xi, \tau)] \sim \sum_{j \geq 0} q_{(m-j)}(x, \xi, \tau), \qquad q_{(m)} \neq 0, \tag{2.50}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m - j$.

DEFINITION 2.16 ([Po1]). The integer m is called the Getzler order of Q. The symbol $q_{(m)}$ is the principle Getzler homogeneous symbol of Q. The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is called the model operator of Q.

Denote by $O(t^{\frac{k}{2}})$ a Laurant expansion about $t^{\frac{1}{2}}$ whose lowest degree about t is $\frac{k}{2}$.

LEMMA 2.17. Let $Q \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S(T(M_z)) \otimes \wedge^* T_z^* B)$ have Getzler order m and model operator $Q_{(m)}$. Then as $t \to 0^+$ we have:

$$\sigma[\psi_t K_Q(0,0,t)]^{(j)} = \omega^{\mathrm{odd}} O(t^{\frac{j-n-m-2}{2}}) + O(t^{\frac{j-n-m-1}{2}}), \quad if \ m - j \ \ is \ odd;$$

$$\sigma[\psi_t K_Q(0,0,t)]^{(j)} = t^{\frac{j-n-m-2}{2}} K_{Q_{(m)}}(0,0,1)^{(j)} + \omega_1^{\mathrm{odd}} O(t^{\frac{j-n-m-1}{2}}) + O(t^{\frac{j-n-m}{2}}),$$

if $m - j$ is even, where $\sigma[K_Q(0,0,t)]^{(j)}$ denotes the degree j form component in M_z and ω^{odd}, ω_1^{odd} are in $\wedge^{\mathrm{odd}}(T^* B) \otimes \wedge(T^*(M_z))$. In particular $m = -2$ and $j = n$ is even, we get

$$\sigma[\psi_t K_Q(0,0,t)]^{(n)} = K_{Q_{(-2)}}(0,0,1)^{(n)} + O(t). \tag{2.51}$$

Proof. By (1.7) in [Po1], we get

$$K_Q(0,0,t) \sim \sum_{m_0 - j_0 \ \mathrm{even}} \sum_l t^{\frac{j_0 - n - m_0 - 2}{2}} \breve{q}_{m_0 - j_0, l}(0,0,1) \omega^{[l]}, \tag{2.52}$$

where m_0 is the operator order of Q. And then

$$\sigma[\psi_t K_Q(0,0,t)]^{(j)} \sim \sum_{m_0 - j_0 \ \mathrm{even}} \sum_l t^{\frac{j_0 - n - m_0 - l - 2}{2}} \sigma[\breve{q}_{m_0 - j_0, l}(0,0,1)]^{(j)} \omega^{[l]}. \tag{2.53}$$

Let $L = m_0 - j_0 + j + l$. By Q having the Getzler order m, then $L \le m$. Then

$$\sigma[\psi_t K_Q(0,0,t)]^{(j)} \sim \sum_{m_0 - j_0 \ \mathrm{even}} \sum_l \sum_{L \le m} t^{\frac{j-n-L-2}{2}} \sigma[\breve{q}_{m_0 - j_0, l}(0,0,1)]^{(j)} \omega^{[l]}. \tag{2.54}$$

We note that the leading term degree is $L = m$ and $m_0 - j_0 + l = m - j$. When $m - j$ is odd, since $m_0 - j_0$ is even, then l is odd, then we get

$$\sigma[\psi_t K_Q(0,0,t)]^{(j)} = \omega^{\mathrm{odd}} O(t^{\frac{j-n-m-2}{2}}) + O(t^{\frac{j-n-m-1}{2}}). \tag{2.55}$$

When $L = m$ and $m - j$ are even, l is even. In this case, the leading coefficient is

$$\sigma[\breve{q}_{(m)}(0,0,1)]^{(j)} = \sum_l \sigma[\breve{q}_{m-j-l, l}(0,0,1)]^{(j)} \omega^{[l]} = K_{Q_{(m)}}(0,0,1)^{(j)}. \tag{2.56}$$

In the next term, $L = m - 1$ and $m - j$ are even and $m_0 - j_0 + l + j = m - 1$, then l is odd, so $m - j - l$ is odd and $\breve{q}_{m-j-l, l}(0,0,1) = 0$. Then the next term is $O(t^{\frac{j-n-m}{2}})$. ∎

In the sequel we say that a symbol or a ΨDO is $O_G(m)$ if it has Getzler order $\le m$. Similar to Lemma 4 in [Po1], we have:

LEMMA 2.18. *For $j = 1, 2$, let $Q_j \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S(T(M_z)) \otimes \wedge^* T_z^* B)$ have Getzler order m_j and model operator $Q_{(m_j)}$ and assume that either Q_1 or Q_2 is properly supported. Then we have:*

$$Q_1 Q_2 = c[Q_{(m_1)} Q_{(m_2)}] + O_G(m_1 + m_2 - 1). \tag{2.57}$$

Proof. Let $Q_j = \omega_j \widehat{Q}_j$ for $j = 1, 2$ where $\omega_j \in \wedge^{l_j}(T_z^* B)$ and $\widehat{Q}_j \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S(T(M_z)))$ has the Getzler order $m_j - l_j$. Then

$$Q_1 Q_2 = \omega_1 \wedge \omega_2 \widehat{Q}_1 \widehat{Q}_2. \tag{2.58}$$

By Lemma 4 in [Po1], we have

$$\widehat{Q}_1 \widehat{Q}_2 = c[\widehat{Q}_{(m_1 - l_1)} \widehat{Q}_{(m_2 - l_2)}] + O_G(m_1 + m_2 - l_1 - l_2 - 1). \tag{2.59}$$

By (2.58) and (2.59), we get (2.57). $\square$

In the following, we compute the model operator of F in (2.5). Let $x_0, x \in M_z$ and $\tau^E(x_0, x)$ be the parallel transport map in the bundle $\pi^* \wedge^* (T_z^* B) \otimes S(TM_z)$ along the geodesic from x to x_0, defined with respect to the Clifford connection $\nabla^{E,0}$. Using this map, we can trivialize the bundle $\pi^* \wedge^* (T_z^* B) \otimes S(TM_z)$. We note that $e_1, \cdots, e_n$ is the parallel transport frame with respect to ∇^{TZ} and so $m_0(e_i)$ is not a constant matrix under above trivialization. But we have

LEMMA 2.19. ([BGV Lemma 10.25]) *If c^i is the Clifford action of the cotangent vector dx_i acting on E at x_0 and ε^α is multiplication by f^α in the exterior algebra $\wedge^*(T_z^* B)$, we have*

$$m_0(e^i) = c^i + \sum u_\alpha^i \varepsilon^\alpha; \qquad m_0(f^\alpha) = \varepsilon^\alpha, \tag{2.60}$$

where u_α^i are smooth functions on U satisfying $u_\alpha^i(x) = O(|x|)$.

LEMMA 2.20 ([BGV Lemma 10.26]). *In the trivialization of E over U induced by the parallel transport map $\tau^E(x_0, x)$, the connection $\nabla^{E,0}$ equals $d + \Theta$, where*

$$\Theta(\partial_i) = -\frac{1}{4} \sum_{j, a < b} \langle R(\partial_i, \partial_j) e_a, e_b \rangle m^a m^b x^j + \sum_{a < b} f_{iab}(x) m^a m^b + g_i(x); \tag{2.61}$$

here m^a represents c^i or ε^α and $f_{iab} = O(|x|^2)$ and $g_i(x) = O(|x|)$.

By (2.5) and Lemma 2.19 and Lemma 2.20, we get

PROPOSITION 2.21. *In the trivialization of E over U induced by the parallel transport map $\tau^E(x_0, x)$ and the normal coordinates, the model operator of F is*

$$F_{(2)} = -\sum_{i=1}^n (\partial_i - \frac{1}{4} \sum_{j=1}^n a_{ij} x_j)^2, \qquad a_{ij} = \langle R^{TZ} \partial_i, \partial_j \rangle. \tag{2.62}$$

We note that $\lim_{t \to 0} \psi_t \int_{M_z} \operatorname{Tr}_s k_t^z(x, x) dx$ does not depend on the trivialization. Using the same discussions with Lemma 5 and Lemma 6 in [Po1], we get

LEMMA 2.22. *Let Q be a Volterra paramatrix for $F + \partial_t$. Then Q has the model operator $(F_{(2)} + \partial_t)^{-1}$ and*

$$K_{(F_{(2)} + \partial_t)^{-1}}(0, 0, 1) = (4\pi)^{-\frac{n}{2}} \widehat{A}(R^{TZ}). \tag{2.63}$$

By (2.46), (2.51) and Lemma 2.22, we proved Theorem 2.15.

3. The local equivariant family index theorem. In [LM], Liu and Ma gave a proof of the local equivariant family index theorem by the finite propagation speed method in [B3]. In [PW], a new proof of the local equivariant index theorem was given by the Volterra calculus. In this section, we shall give a new proof of the local equivariant family index theorem by the Volterra calculus.

Let us give the fundamental setup. We assume that M and TZ are oriented and G is a compact Lie group which is a fiberwise isometry on M and preserves the orientation of TZ. Then G acts as identity on B. We also assume that the action of G lifts to $S(TZ)$ and that the G-action commutes with D. By Proposition 1.1 in [LM], we know that $\mathrm{Ind}D^G \in K_G(B)$. Now let us calculate the equivariant Chern character $\mathrm{ch}_\phi(\mathrm{Ind}(D^G))$ in terms of the fixed point data of $\phi \in G$. Set $M^\phi = \{x \in M, \phi x = x\}$. Then $\pi : M^\phi \to B$ is a fibration with compact fibre M_z^ϕ. By [BGV, Proposition 6.14], TZ^ϕ is naturally oriented in M^ϕ. Let N denote the normal bundle of M^ϕ, then $N = TZ/TZ^\phi$. We fixed a fibre M_z and denote by $\widetilde{\phi}$ the lift of ϕ which maps $S(T(M_z)_x)$ to $S(T(M_z)_{\phi x})$. We denote by M_z^ϕ the fixed-point set of ϕ, and for $a = 0, \cdots, n$, we let $M_z^\phi = \bigcup_{0 \leq a \leq n} M_{z,a}^\phi$, where $M_{z,a}^\phi$ is an a-dimensional submanifold. Given a fixed-point x_0 in a component $M_{z,a}^\phi$, consider some local coordinates $x = (x^1, \cdots, x^a)$ around x_0. Setting $b = n-a$, we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame $e_1(x), \cdots, e_b(x)$ of N_z^ϕ. This defines fiber coordinates $v = (v_1, \cdots, v_b)$. Composing with the map $(x, v) \in N_z^\phi(\varepsilon_0) \to \exp_x(v)$ we then get local coordinates $x^1, \cdots, x^a, v^1, \cdots, v^b$ for M_z near the fixed point x_0. We shall refer to this type of coordinates as *tubular coordinates*. Then $N_z^\phi(\varepsilon_0)$ is homeomorphic with a tubular neighborhood of M_z^ϕ. We have

THEOREM 3.1 ([LM]). *For any $t > 0$, the form $\mathrm{Str}[\widetilde{\phi}\psi_t \exp(-tF)]$ is closed and its de-Rham cohomology class in B is independent of t and represents $\mathrm{ch}_\phi(\mathrm{Ind}(D^G))$ in the de-Rham cohomology of B.*

We shall use Theorem 3.1 to find a local index formula for $\mathrm{ch}_\phi(\mathrm{Ind}(D^G))$ by estimating $\mathrm{Str}[\widetilde{\phi}\psi_t \exp(-tF)]$. By the Mckean-Singer formula, we have

$$\mathrm{Str}[\widetilde{\phi}\psi_t \exp(-tF)] = \psi_t \int_{M_z} \mathrm{Str}[\widetilde{\phi}k_t^z(x, \phi(x))]dx$$

$$= \psi_t \int_{M_z} \mathrm{Str}[\widetilde{\phi}K_{(F+\partial_t)^{-1}}^z(x, \phi(x), t)]dx. \qquad (3.1)$$

Let $Q = (F + \partial_t)^{-2}$. For $x \in M_z^\phi$ and $t > 0$ set

$$I_Q(x, t) := \widetilde{\phi}(x)^{-1} \int_{N_x^\phi(\varepsilon)} \widetilde{\phi}(\exp_x v)K_Q(\exp_x v, \exp_x(\phi'(x)v), t)dv. \qquad (3.2)$$

Here we use the trivialization of $S(TM_z)$ about the tubular coordinates. Using the tubular coordinates, then

$$I_Q(x, t) = \int_{|v|<\varepsilon} \widetilde{\phi}(x, 0)^{-1}\widetilde{\phi}(x, v)K_Q(x, v; x, \phi'(x)v; t)dv. \qquad (3.3)$$

Let

$$q_{m-j}^E(x, v; \xi, \nu; \tau) := \widetilde{\phi}(x, 0)^{-1}\widetilde{\phi}(x, v)q_{m-j}(x, v; \xi, \nu; \tau). \qquad (3.4)$$

Using the same proof as Lemma 2.9, we obtain the following proposition by Lemma 8.2 and Lemma 8.3 in [PW]

PROPOSITION 3.2. *Let* $Q \in \Psi_V^m(M_z \times \mathbb{R}, E)$, $m \in \mathbb{Z}$. *Uniformly on each component* $M_{z,a}^\phi$

$$I_Q(x,t) \sim \sum_{j \geq 0} t^{-(\frac{a}{2}+[\frac{m}{2}]+1)} I_Q^j(x) \qquad \text{as} \ \ t \to 0^+, \qquad (3.5)$$

where $I_Q^j(x)$ *is defined in tubular coordinates by*

$$I_Q^{(j)}(x) := \sum_{|\alpha| \leq m-[\frac{m}{2}]+2j} \int \frac{v^\alpha}{\alpha!} \left(\partial_v^\alpha q_{2[\frac{m}{2}]-2j+|\alpha|}^E \right)^\vee (x,0;0,(1-\phi'(x))v;1)dv. \quad (3.6)$$

By Proposition 8.7 in [PW] and Proposition 3.2 and the definition of ψ_t, we have

PROPOSITION 3.3. *Let* $P : C^\infty(M_z, \pi^* \wedge T_z^*B \otimes S(T(M_z))) \to C^\infty(M_z, \pi^* \wedge T_z^*B \otimes S(T(M_z)))$ *be a differential operator of order* m.
(1)*Uniformly on each component* $M_{z,a}^\phi$,

$$I_{P(F+\partial_t)^{-1}}(x,t) \sim \sum_{j \geq 0} t^{-(\frac{n}{2}+[\frac{m}{2}])+j} I_{P(F+\partial_t)^{-1}}^{(j)}(x) \qquad \text{as} \ \ t \to 0^+, \qquad (3.7)$$

(2)*As* $t \to 0^+$, *we have*

$$\psi_t \mathrm{Str}[\widetilde{\phi} P e^{-tF}] \sim \sum_{0 \leq a \leq n} \sum_{j \geq 0} \sum_{l=1}^{2^{\overline{q}}} t^{-(\frac{a}{2}+[\frac{m}{2}]+\frac{1}{2})+j} \int_{M_{z,a}^\phi} \mathrm{Str}[\widetilde{\phi} I_{P(F+\partial_t)^{-1}}^{(j),l}(x)]dx, \quad (3.8)$$

where $I_{P(F+\partial_t)^{-1}}^{(j),l}(x)$ *denotes the* l*-degree component of* $I_{P(F+\partial_t)^{-1}}^{(j)}(x)$ *in* $\wedge T_z^*B$.

Let

$$\widehat{A}(R^{TZ^\phi}) = \det^{\frac{1}{2}} \left(\frac{R^{TZ^\phi}/2}{\sinh(R^{TZ^\phi}/2)} \right); \ \ \nu_\phi(R^{N^\phi}) := \det^{-\frac{1}{2}}(1-\phi^N e^{-R^{N^\phi}}). \quad (3.9)$$

For a top degree form $\omega \in C^\infty(M_{z,a}, \wedge T_z^*(B) \otimes \wedge^a T^* M_{z,a})$ in $M_{z,a}$ with coefficients in $\wedge T_z^*(B)$, we denote by $|\omega|^{(a)}$ the Berezin integral which in $\wedge T_z^*(B)$, i.e. its inner-product with the volume form of $M_{z,a}$.

THEOREM 3.4 (Local equivariant family index theorem). *Let* $x_0 \in M^\phi$, *then*

$$\lim_{t \to 0} \mathrm{Str} \left[\widetilde{\phi}(x_0) \psi_t I_{(F+\partial_t)^{-1}}(x_0,t) \right] = (-i)^{\frac{n}{2}} (2\pi)^{-\frac{a}{2}} \left| \widehat{A}(R^{TZ^\phi}) \nu_\phi(R^{N^\phi}) \right|^{(a)}. \quad (3.10)$$

REMARK. It is easy to generalize Theorem 3.4 to the twisted ϕ-complex vector bundle case.

By (3.1) and Lemma 8.1 in [PW], as $t \to 0^+$

$$\psi_t \int_{M_z} \mathrm{Str}[\widetilde{\phi} K_{(F+\partial_t)^{-1}}(x,\phi(x),t)]dx = \psi_t \int_{M_z} \mathrm{Str}[\widetilde{\phi} I_Q(x,\phi(x),t)]dx, \qquad (3.11)$$

By Theorem 3.1, Theorem 3.4 and (3.1), (3.11), we get a representative of $\mathrm{ch}_\phi(\mathrm{Ind}D^G)$.

In order to prove Theorem 3.4, we shall compute $\widetilde{\phi}$ in tubular coordinates. Let

$e_1, \ldots, e_n$ be an oriented orthonormal basis of $T_{x_0} M_z$ such that $e_1, \cdots, e_a$ span $T_{x_0} M_z^\phi$ and $e_{a+1}, \cdots, e_n$ span $N_{x_0}^\phi$. This provides us with normal coordinates $(x_1, \cdots, x_n) \rightarrow \exp_{x_0}(x^1 e_1 + \cdots + x^n e_n)$. Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame $e_1(x), \ldots, e_n(x)$ such that $e_1(x), \cdots, e_a(x)$ form an oriented frame of $TM_{z,a}^\phi$ and $e_{a+1}(x), \cdots, e_n(x)$ form an (oriented) frame N^ϕ (when both frames are restricted to M_z^ϕ). This gives rise to trivializations of the tangent and spinor bundles. Write

$$\phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix}.$$

Let $\wedge(n) = \wedge_{\mathbb{C}}^* \mathbb{R}^n$ be the complexified exterior algebra of $\mathbb{R}^n$. We shall use the following gradings on $\wedge(n)$,

$$\wedge(n) = \bigoplus_{1 \le j \le n} \wedge^j(n) = \bigoplus_{\substack{1 \le k \le a \\ 1 \le \bar{l} \le b}} \wedge^{k,\bar{l}}(n),$$

where $\wedge^j(n)$ is the space of forms of degree j and $\wedge^{k,\bar{l}}(n)$ is the space of forms $dx^{i_1} \wedge \cdots \wedge dx^{i_{k+\bar{l}}}$ with $1 \le i_1 < \cdots < i_k \le a$ and $a+1 \le i_{k+1} < \cdots < i_{k+\bar{l}} \le n$. Given a form $\omega \in \wedge(n)$ we shall denote by $\omega^{(j)}$ (resp., $\omega^{(k,\bar{l})}$) its component in $\wedge^j(n)$ (resp., $\wedge^{k,\bar{l}}(n)$). We denote by $|\omega|^{(a,0)}$ the Berezin integral $|\omega^{(*,0)}|^{(a,0)}$ of its component $\omega^{(*,0)}$ in $\wedge^{(*,0)}(n)$. Then we have

LEMMA 3.5 ([PW,Lemma 9.5]). *Let S_n be the spinor space associated to $\mathbb{R}^n$ and $A \in \mathrm{End}(S_n)$, then*

$$\mathrm{Str}[\widetilde{\phi}A] = (-2i)^{\frac{n}{2}} 2^{-\frac{b}{2}} \det^{\frac{1}{2}}(1 - \phi^N) |\sigma(A)|^{(a,0)}$$
$$+ (-2i)^{\frac{n}{2}} \sum_{0 \le b' < b} |\sigma(\widetilde{\phi})^{(0,b')} \sigma(A)^{(a,b-b')}|^{(n)}. \tag{3.12}$$

Similar to Proposition 2.21, we get the same expression of the model operator of F in the trivialization of E over U induced by the parallel transport map $\tau(x_0, x)$ about ∇^{TZ} and the normal coordinate. We will compute the local index in this trivialization.

LEMMA 3.6. *Let the operator $Q \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S) \otimes \wedge(T_z^* B)$ have the Getzler order m and model operator $Q_{(m)}$. Then as $t \to 0^+$*
  (1) $\sigma[\psi_t I_Q(0,t)]^{(j)} = \omega^{\mathrm{odd}} O(t^{\frac{j-m-a-2}{2}}) + O(t^{\frac{j-m-a-1}{2}})$ *if $m - j$ is odd.*
  (2) $\sigma[\psi_t I_Q(0,t)]^{(j)} = O(t^{\frac{j-m-a-2}{2}}) I_{Q_{(m)}}(0,1)^{(j)} + \omega_1^{\mathrm{odd}} O(t^{\frac{j-m-a-1}{2}}) + O(t^{\frac{j-m-a}{2}})$
      if $m - j$ is even.
In particular, for $m = -2$ and $j = a$ we get

$$\sigma[\psi_t I_Q(0,t)]^{(a,0)} = I_{Q(-2)}(0,1)^{(a,0)} + O(t^{\frac{1}{2}}). \tag{3.13}$$

Proof. By the change of variable formula for symbol, similar to (9.23) in [PW], we have

$$\sigma[\psi_t I_Q(0,t)]^{(j)} \sim \sum_{\substack{|\alpha| + |\beta| - |\gamma| - \bar{l} \text{ even} \\ \bar{1} \le m', \ 2|\gamma| < |\beta|}} \sum_{l=1}^{2^{\overline{q}}} t^{\frac{|\alpha|+|\beta|-|\gamma|-l-\bar{l}-(a+2)}{2}} I_{\bar{l}\alpha\beta\gamma,l}^{(j)}, \tag{3.14}$$

where m' is the operator order of Q and

$$I^{(j)}_{\bar{l}\alpha\beta\gamma,l} = \int_{\mathbb{R}^b} a_{\beta\gamma,\bar{l}}(0,v)\frac{v^\alpha}{\alpha!}\left(\partial_v^\alpha\sigma[\xi^\gamma D_\xi^\beta q_{\bar{l},l}]^{(j)}\right)^\vee (0,0;0,(1-\phi^N(0))v;1)dv\omega^{[l]}, \quad (3.15)$$

where $a_{\beta\gamma,\bar{l}}(x',v)$ are smooth functions such that $a_{\beta\gamma,\bar{l}}(x) = 1$ when $\beta = \gamma = 0$. Since Q has the Getzler order m, then all the coefficients $I^{(j)}_{\bar{l}\alpha\beta\gamma,l}$ with $\bar{l}+j+l-|\alpha| > m$ are zero. So, if $\bar{l}+j+l-|\alpha| \leq m$ and $2|\gamma| \leq |\beta|$, then $t^{\frac{|\alpha|+|\beta|-|\gamma|-l-\bar{l}-(a+2)}{2}}$ is $O(t^{\frac{j-m-a-2}{2}})$ and even is $o(t^{\frac{j-m-a-2}{2}})$ if we have $\bar{l}+j+l-|\alpha| < m$ or $(\beta,\gamma) \neq (0,0)$.

Observe that terms in asymptotic (3.14) containing integer powers (resp. half integer powers) of t when l is even (resp. odd) since a is even. When $m-j$ is odd, the leading term is $O(t^{\frac{j-m-a-2}{2}})$ which is a half integer, so its coefficient is in $\Omega^{\text{odd}}(B)$ and (1) is verified. When $m-j$ is even, similar discussions show that

$$\sigma[\psi_t I_Q(0,t)]^{(j)} = \omega^{\text{even}}O(t^{\frac{j-m-a-2}{2}}) + \omega^{\text{odd}}O(t^{\frac{j-m-a-1}{2}}) + O(t^{\frac{j-m-a}{2}}). \quad (3.16)$$

Then similar to the discussions in [PW], we prove (2). ∎

By (3.12) and Lemma 3.6 (2) and (3.13) and Lemma 9.13 in [PW], we get Theorem 3.4.

4. The equivariant JLO character for a family of the Dirac operators. In [CH], Chern and Hu computed the equivariant JLO characters for invariant Dirac operators. In [Az2], Azmi computed the JLO characters for a family of Dirac operators. In [Az3], Azmi constructed an equivariant bivariant cyclic theory, as a combination of equivariant cyclic and noncommutative de Rham theories for unital G-Banach algebras, where G is a compact Lie group. By incorporating the JLO formula and the superconnection formalism of Quillen, an equivariant bivariant JLO character of Kasparov's G-bimodule is defined, with values in the bivariant cyclic theory. In this section, following the ideas in [CH], [Zh2] and [Fe], we compute the equivariant JLO characters for a family of the Dirac operators.

Let $C^1(M)$ be the Banach algebra which is the completion of $C^\infty(M)$ with respect to the norm $|f| := ||f|| + ||[D,f]||$, for $f \in C^\infty(M)$. The commutator $[D,f]$ extends to a bounded operator on $\mathcal{H} = L^2(M,S(TZ))$. The algebra $C^1(M)$ acts on $L(\mathcal{H})$
(bounded operators on $\mathcal{H}$) by multiplication. Denote by Δ the projective tensor product of the Banach algebras $C^1(M)$ and $C^\infty(B)$, i.e $\Delta = C^\infty(B)\hat{\otimes}C^1(M)$, with the projective tensor product norm.

Let $\mathcal{M} = \wedge B \otimes \mathcal{H}$ be a $\Delta - C^\infty(B)$ bimodule, where Δ acts on the left of $\mathcal{M}$ by letting $C^\infty(B)$ act on $\wedge(B)$ by multiplication by left and $C^1(M)$ acts on $\mathcal{H}$ while $C^\infty(B)$ acts on $\mathcal{M}$ by multiplication from right. There is a continuous $C^\infty(B)$-valued inner product on $\mathcal{M}$. There is an obvious continuous action of $\phi \in G$ on $\mathcal{M}$, by letting ϕ act on $\mathcal{H}$ via $\tilde{\phi}$ and on $\wedge B$ via the identity map. Let $\mathcal{B}_t = \sqrt{t}\psi_t(\mathcal{B})$ be the rescaled Bismut' superconnection with the rescaled curvature $F_t = \mathcal{B}_t^2 = t\psi_t(F)$. If $h_i \otimes f_i \in \Delta$, $0 \leq i \leq 2k$ are operators on $\mathcal{M}$, we define the equivariant bivariant JLO character by:

$$\text{Ch}_{2k}(\mathcal{B}_t)(\phi)(h_0 \otimes f_0, \cdots, h_{2k} \otimes f_{2k})$$
$$= t^k \int_{\triangle_{2k}} \text{Str}\left[\psi_t\tilde{\phi}h_0 \otimes f_0 e^{-ts_1 F}\right.$$
$$\left.\cdot[\mathcal{B}, h_1 \otimes f_1]e^{-t(s_2-s_1)F}\cdots[\mathcal{B}, h_{2k} \otimes f_{2k}]e^{-t(1-s_{2k})F}\right]ds, \quad (4.1)$$

where $\triangle_{2k} = \{(s_1, \cdots, s_{2k})| \ 0 \le s_1 \le \cdots \le s_{2k} \le 1\}$ is the simplex in $\mathbb{R}^{2k}$. We will compute

$$\lim_{t \to 0} \mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi).$$

In the following, we give some estimates about $\mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi)$. Since B is compact, we can fix a fibre M_z and estimate $\mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi)$. Let H be a Hilbert space. For $q \ge 0$, denote by $||.||_q$ the Schatten p-norm on the Schatten ideal L^p. Denote by $L(H)$ the Banach algebra of bounded operators on H.

LEMMA 4.1 ([Si]). *(i)* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, *for A, $B \in L(H)$ and AB, $BA \in L^1$.*
(ii) For $A \in L^1$, we have $|\mathrm{Tr}(A)| \le ||A||_1$, $||A|| \le ||A||_1$.
(iii) For $A \in L^q$ and $B \in L(H)$, we have: $||AB||_q \le ||B||\,||A||_q$, $||BA||_q \le ||B||\,||A||_q$.
(iv) (Hölder Inequality) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $p, q, r > 0$, $A \in L^p$, $B \in L^q$, then $AB \in L^r$ and $||AB||_r \le ||A||_p ||B||_q$.

LEMMA 4.2. *For any $u > 0$, $t > 0$ and t is small and any order l fibrewise differential operator $\overline{B}$ with form coefficients, we have:*

$$||e^{-utF}\overline{B}||_{u^{-1}} \le C_l u^{-\frac{l}{2}} t^{-\frac{l}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{2}}])^u. \tag{4.2}$$

Proof. By (2.7), we have

$$||e^{-utF}\overline{B}||_{u^{-1}} = || \sum_{m \ge 0} (-ut)^m \int_{\triangle_m} e^{-v_0 utD^2} F_{[+]} e^{-v_1 utD^2}$$

$$\cdot F_{[+]} \cdots e^{-v_{m-1}utD^2} F_{[+]} e^{-v_m utD^2} \overline{B} dv||_{u^{-1}}, \tag{4.3}$$

We estimate the term of $m = 2$ in the right hand of (4.3), other terms are similar. We split $\triangle(2) = J_0 \cup J_1 \cup J_2$ where $J_i = \{(v_0, v_1, v_2) \in \Delta(2)| v_i \ge \frac{1}{3}\}$.

$$(ut)^2 || \int_{J_0} e^{-v_0 utD^2} F_{[+]} e^{-v_1 utD^2} F_{[+]} e^{-v_2 utD^2} \overline{B} dv||_{u^{-1}}$$

$$\le (ut)^2 \int_{J_0} ||e^{-\frac{v_0 ut}{2} D^2}||_{(uv_0)^{-1}} ||e^{-\frac{v_0 ut}{2} D^2} (1 + D^2)^{\frac{l+2}{2}}|| \, ||(1 + D^2)^{-\frac{l+2}{2}} F_{[+]} (1 + D^2)^{\frac{l+1}{2}}||$$

$$\cdot ||e^{-v_1 utD^2}||_{(uv_1)^{-1}} ||(1 + D^2)^{-\frac{l+1}{2}} F_{[+]} (1 + D^2)^{\frac{l}{2}}|| \, ||e^{-v_2 utD^2}||_{(uv_2)^{-1}} ||1 + D^2)^{-\frac{l}{2}} \overline{B}|| dv$$

$$\le (ut)^2 \int_{J_0} \left(\mathrm{Tr} e^{-\frac{t}{2} D^2} \right)^{uv_0} \left(\mathrm{Tr} e^{-tD^2} \right)^{u(v_1+v_2)} (uv_0 t)^{-\frac{l+2}{2}}$$

$$\cdot ||(1 + D^2)^{-\frac{l+2}{2}} F_{[+]} (1 + D^2)^{\frac{l+1}{2}}|| \, ||(1 + D^2)^{-\frac{l+1}{2}} F_{[+]} (1 + D^2)^{\frac{l}{2}}|| \, ||1 + D^2)^{-\frac{l}{2}} \overline{B}|| dv$$

$$\le C_2 \left(\mathrm{Tr} e^{-\frac{t}{2} D^2} \right)^u (ut)^{-\frac{l}{2}+1} \tag{4.4}$$

where we use $F_{[+]}$ is a fibrewise first order differential operator and the equality

$$\sup\{(1 + x)^{\frac{l}{2}} e^{-\frac{utx}{2}}\} = (ut)^{-\frac{l}{2}} e^{-\frac{l-ut}{2}}. \tag{4.5}$$

In the above estimate, we omit the norm of coefficient forms since B is compact. For J_1 and J_2 we have similar estimates. For the general m, we get

$$||(-ut)^m \int_{\triangle_m} e^{-v_0 utD^2} F_{[+]} e^{-v_1 utD^2} F_{[+]} \cdots e^{-v_{m-1}utD^2}$$

$$\cdot F_{[+]} e^{-v_m utD^2} \overline{B} dv||_{u^{-1}} \le C_2 \left(\mathrm{Tr} e^{-\frac{t}{2} D^2} \right)^u (ut)^{-\frac{l}{2}+\frac{m}{2}}. \tag{4.6}$$

By (4.3) and (4.6), we get (4.2). ∎

LEMMA 4.3. *Let $\overline{B}_1$, $\overline{B}_2$ be positive order p, q fibrewise pseudodifferential operators with form coefficients respectively, then for any s, $t > 0$, $0 \leq u \leq 1$, we have the following estimate:*

$$||\overline{B}_1 e^{-ustF} \overline{B}_2 e^{-(1-u)stF}||_{s^{-1}} \leq C_{p,q} s^{-\frac{p+q}{2}} t^{-\frac{p+q}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{4}}])^s. \tag{4.7}$$

Proof. Similar to the proof of Lemma 4.2, we have when t is small and $u \geq \frac{1}{2}$,

$$||(1+D^2)^{-\frac{q}{2}} e^{-(u-\frac{1}{2})stF} (1+D^2)^{\frac{q}{2}}|| \leq C_0; \quad ||e^{-(1-u)stF}|| \leq C_1, \tag{4.8}$$

where C_0, C_1 are constants which do not depend on t, u. When $u \geq \frac{1}{2}$, by Lemma 4.2 and (4.8), we have:

$$\begin{aligned}
&||\overline{B}_1 e^{-ustF} \overline{B}_2 e^{-(1-u)stF}||_{s^{-1}} \\
&\leq ||\overline{B}_1 (1+D^2)^{-\frac{p}{2}}|| ||(1+D^2)^{\frac{p}{2}} e^{-\frac{1}{2}stF} (1+D^2)^{\frac{q}{2}}||_{s^{-1}} \\
&\quad \cdot ||(1+D^2)^{-\frac{q}{2}} e^{-(u-\frac{1}{2})stF} (1+D^2)^{\frac{q}{2}}|| ||(1+D^2)^{-\frac{q}{2}} \overline{B}_2|| ||e^{-(1-u)stF}|| \\
&\leq C_{p,q} s^{-\frac{p+q}{2}} t^{-\frac{p+q}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{4}}])^s, \tag{4.9}
\end{aligned}$$

where we use

$$|(1+D^2)^{\frac{p}{2}} e^{-\frac{1}{2}stF} (1+D^2)^{\frac{q}{2}}||_{s^{-1}} = ||e^{-\frac{1}{2}stF} (1+D^2)^{\frac{p+q}{2}}||_{s^{-1}}. \tag{4.10}$$

When $u \leq \frac{1}{2}$, by Lemma 4.2, we have:

$$\begin{aligned}
&||\overline{B}_1 e^{-ustF} \overline{B}_2 e^{-(1-u)stF}||_{s^{-1}} \\
&\leq ||\overline{B}_1 e^{-ustF} (1+D^2)^{-\frac{p}{2}}|| ||(1+D^2)^{\frac{p}{2}} \overline{B}_2 e^{-\frac{1}{2}stF}||_{s^{-1}} ||e^{-(1-u-\frac{1}{2})stF}|| \\
&\leq C_{p,q} s^{-\frac{p+q}{2}} t^{-\frac{p+q}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{4}}])^s. \tag{4.11}
\end{aligned}$$

By (4.9) and (4.11), we have proved this lemma. ∎

Let $\overline{B}$ be a fibrewise operator with form coefficients and l be a positive interger. Write

$$\overline{B}^{[l]} = [F, \overline{B}^{[l-1]}], \ \overline{B}^{[0]} = \overline{B}.$$

LEMMA 4.4. *Let $\overline{B}$ a finite order fibrewise differential operator with form coefficients, then for any $s > 0$, we have:*

$$e^{-sF} \overline{B} = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l \overline{B}^{[l]} e^{-sF} + (-1)^N s^N \overline{B}^{[N]}(s), \tag{4.12}$$

where $\overline{B}^{[N]}(s)$ is given by

$$\overline{B}^{[N]}(s) = \int_{\triangle_N} e^{-u_1 sF} \overline{B}^{[N]} e^{-(1-u_1)sF} du_1 du_2 \cdots du_N. \tag{4.13}$$

Proof. Here we use Lemma 1.9 in [BeC],

$$[A, e^{-F}] = -\int_0^1 e^{-sF}[A, F]e^{-(1-s)F}ds, \tag{4.14}$$

then similar to Lemma 3 in [Fe], we prove this lemma. $\square$

For $h_j \otimes f_j$, we have

$$[\mathcal{B}, h_j \otimes f_j] = d_B(h_j \otimes f_j) \wedge + h \otimes [D, f]. \tag{4.15}$$

Write T_j is $d_B(h_j \otimes f_j)\wedge$ or $h \otimes [D, f], 1 \le j \le 2k$ and $T_0 = h_0 \otimes f_0$. By Lemma 4.4, we have:

$$T_0 e^{-s_1 tF} T_1 e^{-(s_2-s_1)tF} T_2 \cdots e^{-(s_{2k}-s_{2k-1})tF} T_{2k} e^{-(1-s_{2k})tF}$$

$$= \sum_{\lambda_1,\cdots,\lambda_{2k}=0}^{N-1} \frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}} s_1^{\lambda_1} \cdots s_{2k}^{\lambda_{2k}} t^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1!\cdots\lambda_{2k}!} T_0[T_1]^{[\lambda_1]} \cdots [T_{2k}]^{[\lambda_{2k}]} e^{-tF}$$

$$+ \sum_{1 \le q \le 2k} \sum_{\lambda_1,\cdots,\lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1+\cdots+\lambda_{q-1}+N} s_1^{\lambda_1} \cdots s_{q-1}^{\lambda_{q-1}} s_q^N t^{\lambda_1+\cdots+\lambda_{q-1}+N}}{\lambda_1!\cdots\lambda_{q-1}!} T_0[T_1]^{[\lambda_1]}$$

$$\cdots [T_{q-1}]^{[\lambda_{q-1}]}\{[T_q]^{[N]}(s_q t)\} e^{-(s_{q+1}-s_q)tD^2} \cdots T_{2k} e^{-(1-s_{2k})tF}. \tag{4.16}$$

Since $T_0[T_1]^{[\lambda_1]} \cdots [T_{q-1}]^{[\lambda_{q-1}]}$ is a $\lambda_1 + \cdots + \lambda_{q-1}$ order differential operator and by Lemma 4.2 and Lemma 4.3, we get (see pp. 61-62 in [Fe])

$$\|\psi_t \int_{\Delta_{2k}} t^k \sum_{1 \le q \le 2k}$$

$$\times \sum_{\lambda_1,\cdots,\lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1+\cdots+\lambda_{q-1}+N} s_1^{\lambda_1} \cdots s_{q-1}^{\lambda_{q-1}} s_q^N t^{\lambda_1+\cdots+\lambda_{q-1}+N}}{\lambda_1!\cdots\lambda_{q-1}!} T_0[T_1]^{[\lambda_1]}$$

$$\cdots [T_{q-1}]^{[\lambda_{q-1}]}\{[T_q]^{[N]}(s_q t)\} e^{-(s_{q+1}-s_q)tD^2} \cdots T_{2k} e^{-(1-s_{2k})tF} dv\|$$

$$\sim O(t^{\frac{2k+N+\lambda_1+\cdots+\lambda_{2q}-\dim M}{2}}). \tag{4.17}$$

So we get

THEOREM 4.5. (1) *if* $2k \le \dim M$, *then*

$$\mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi)(h_0 \otimes f_0, \cdots, h_{2k} \otimes f_{2k})$$

$$= \psi_t \sum_{\lambda_1,\ldots,\lambda_{2k}=0}^{\dim M - 2k} \frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1!\cdots\lambda_{2k}!} Ct^{|\lambda|+k}\mathrm{Str}[\widetilde{\phi}T_0[T_1]^{[\lambda_1]} \cdots [T_{2k}]^{[\lambda_{2k}]} e^{-tF}] + O(\sqrt{t}), \tag{4.18}$$

with the constant

$$C = \frac{1}{\lambda_1 + 1} \frac{1}{\lambda_1 + \lambda_2 + 2} \cdots \frac{1}{\lambda_1 + \cdots + \lambda_{2k} + 2k}. \tag{4.19}$$

(2) *if* $2k > \dim M$, *then*

$$\mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi)(h_0 \otimes f_0, \cdots, h_{2k} \otimes f_{2k}) = 0. \tag{4.20}$$

Write $P_{k,\lambda} := T_0[T_1]^{[\lambda_1]} \cdots [T_{2k}]^{[\lambda_{2k}]}$. We will compute

$$\lim_{t\to 0^+} \psi_t t^{|\lambda|+k} \mathrm{Str}[\widetilde{\phi} P_{k,\lambda} e^{-tF}].$$

By Lemma 2.18 and Proposition 2.21, we have

$$\sigma(P_{k,\lambda}) = T_0[F_{(2)}, \sigma(T_1)]^{[\lambda_1]} \cdots [F_{(2)}, \sigma(T_{2k})]^{[\lambda_{2k}]} + O_G(2|\lambda| + 2k - 1). \qquad (4.21)$$

Direct computations show that

$$[F_{(2)}, \sigma(T_j)] = O_G(2), \quad O_G([F_{(2)}, \sigma(T_j)]^{[\lambda_j]}) < 2\lambda_j + 1. \qquad (4.22)$$

Then when $(\lambda_1, \cdots, \lambda_{2k}) \neq (0, \cdots, 0)$, then

$$O_G(P_{k,\lambda}) = O_G(2|\lambda| + 2k - 1); \quad O_G(P_{k,\lambda}(F + \partial_t)^{-1}) = O_G(2|\lambda| + 2k - 3). \qquad (4.23)$$

By Lemma 3.5 and Lemma 3.6 (1), we have

$$\sigma[\psi_t I_{P_{k,\lambda}(F+\partial_t)^{-1}}(0,t)]^{(a,0)} = \omega^{\mathrm{odd}} O(t^{-|\lambda|-k+\frac{1}{2}}) + O(t^{-|\lambda|-k+1}), \qquad (4.24)$$

where $\omega^{\mathrm{odd}} \in \Omega^{\mathrm{odd}}(B)$. So in this case,

$$\lim_{t\to 0^+} \psi_t t^{|\lambda|+k} \mathrm{Str}[\widetilde{\phi} P_{k,\lambda} e^{-tF}] = 0. \qquad (4.25)$$

When $(\lambda_1, \cdots, \lambda_{2k}) = (0, \cdots, 0)$. Then $O_G(T_0 T_1 \cdots T_{2k}) = 2k$ and $O_G(P_{k,\lambda}(F + \partial_t)^{-1}) = O_G(2k-2)$. The model operator of $P_{k,\lambda}(F+\partial_t)^{-1}$ is $T_0 \sigma(T_1) \cdots \sigma(T_{2k})(F_{(2)} + \partial_t)^{-1}$. By Lemma 3.6(2), we get

$$\lim_{t\to 0^+} t^k \sigma[\psi_t I_{P_{k,\lambda}(F+\partial_t)^{-1}}(0,t)]^{(a,0)} = T_0 \sigma(T_1) \cdots \sigma(T_{2k}) I_{(F_{(2)}+\partial_t)^{-1}}(0,1)^{(a,0)}. \qquad (4.26)$$

By (4.25) and (4.26) and the lemma 9.13 in [PW], we get

THEOREM 4.6. *The following equality holds*

$$\lim_{t\to 0^+} \mathrm{Ch}_{2k}(\mathcal{B}_t)(\phi)(h_0 \otimes f_0, \cdots, h_{2k} \otimes f_{2k})$$
$$= \frac{1}{(2k)!}(-i)^{\frac{n}{2}} \sum_a (2\pi)^{-\frac{a}{2}} \int_{M_a^\phi/B} T_0 \sigma(T_1) \cdots \sigma(T_{2k}) \widehat{A}(R^{G^\phi}) \nu_\phi(R^{N^\phi}). \qquad (4.27)$$

5. The regularity of the equivariant eta form.

5.1. The regularity of the equivariant eta invariant. In [BiF], Bismut and Freed gave a simple proof of the regularity of eta invariants. In [Po2], Ponge gave another proof of the regularity of eta invariants using the method in [Po1]. Zhang proved the regularity of the equivariant eta invariant by the Clifford asymptotics in [Zh3]. In this section, we shall extend the approach of [Po2] to the equivariant setting and prove the regularity of the equivariant eta invariant by this approach. Then we define the equivariant eta form and prove its regularity.

We shall give some notations. Let X be a compact oriented odd dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinor on X. Denote by D the associated Dirac operator on $H = L^2(X; S)$, the Hilbert space of L^2-sections of the bundle X. Suppose that ϕ acts on X by orientation-preserving isometries and ϕ has a lift $\widetilde{\phi} \colon \Gamma(S) \to \Gamma(S)$ (see [LYZ]), then

we have $\widetilde{\phi}$ commutes with the Dirac operator and $\widetilde{\phi}$ is a bounded operator. Then the equivariant eta invariant is defined by

$$\eta_\phi(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{t^{\frac{1}{2}}} \mathrm{Tr}[\widetilde{\phi} D e^{-tD^2}] dt. \tag{5.1}$$

We have

THEOREM 5.1 ([Zh3]). *As* $t \to 0^+$

$$\mathrm{Tr}[\widetilde{\phi} D e^{-tD^2}] \sim O(t^{\frac{1}{2}}). \tag{5.2}$$

In order to prove Theorem 5.1, we introduce an auxiliary Grassmann variable z as in [BiF], i. e. $z^2 = 0$. The auxiliary Grassmann variable z may be considered as one form on the base S^1 of the fibre bundle $X \times S^1$ and set $O_G(z) = 1$. By the Duhamel principle, we have

$$\exp(-t(D^2 - zD)) = \exp(-tD^2) + ztD\exp(-tD^2). \tag{5.3}$$

Set

$$h(x) = 1 + \frac{1}{2}z \sum_{i=1}^{n} x_i c(e_i), \tag{5.4}$$

where $(x_1, \cdots, x_n)$ is the normal coordinates under the parallel frame $e_1, \cdots, e_n$ and we consider h as $h\chi$ where χ is a cut function about $(x_1, \cdots, x_n)$. By [Zh3], we have

$$hc(e_i)h^{-1} = c(e_i) + O_G(0); \quad h(D^2 - zD)h^{-1} = D^2 + zu, \tag{5.5}$$

where $O_G(u) \leq 0$, u contains no z and the equality

$$ztD\exp(-tD^2)(x,y) = h^{-1}(x)\exp(-t(D^2 + zu)(x,y))h(y) - \exp(-tD^2)(x,y). \tag{5.6}$$

Let

$$(D^2 + zu)^{-1} = D^{-2} - zD^{-2}uD^{-2}. \tag{5.7}$$

We may consider $D^2 + zu$ as the operator with 1-form coefficients on $X \times S^1$. Then the Greiner's approach of the heat kernel asymptotics with form coefficients in Section 2.1 works for $D^2 + zu$. By

$$(D^2 + zu + \partial_t)^{-1} = (D^2 + \partial_t)^{-1} - z(D^2 + \partial_t)^{-1}u(D^2 + \partial_t)^{-1}. \tag{5.8}$$

$$K_{(D^2+zu+\partial_t)^{-1}}(x,y,t) = K_{(D^2+\partial_t)^{-1}}(x,y,t) - zK_{(D^2+\partial_t)^{-1}u(D^2+\partial_t)^{-1}}(x,y,t). \tag{5.9}$$

Then

$$\begin{aligned}
&\widetilde{\phi}(x)h^{-1}(x)K_{(D^2+zu+\partial_t)^{-1}}(x,\phi(x),t)h(\phi(x)) \\
&= \widetilde{\phi}(x)h^{-1}(x)K_{(D^2+\partial_t)^{-1}}(x,\phi(x),t)h(\phi(x)) \\
&\quad - z\widetilde{\phi}(x)K_{(D^2+\partial_t)^{-1}u(D^2+\partial_t)^{-1}}(x,\phi(x),t).
\end{aligned} \tag{5.10}$$

Recall since the dimension n is odd, we have

$$\mathrm{Tr}[c(e^{i_1})\cdots c(e^{i_k})] = \begin{cases} 2^{[n/2]} & \text{if } k = 0 \\ 0 & \text{if } 0 < k < n \\ (-i)^{[n/2]+1}2^{[n/2]} & \text{if } k = n. \end{cases} \tag{5.11}$$

Similar to Lemma 3.5, we have:

LEMMA 5.2. *Let* $A \in \mathrm{Cl}^{\mathrm{odd}}(n) \subset \mathrm{End}(S_n)$, *then*

$$\mathrm{Tr}[\widetilde{\phi}A] = (-i)^{\frac{n+1}{2}}2^{\frac{n-1}{2}}2^{-\frac{b}{2}}\det^{\frac{1}{2}}(1-\phi^N)|\sigma(A)|^{(a,0)}$$
$$+ (-i)^{\frac{n+1}{2}}2^{\frac{n-1}{2}}\sum_{0\le b'<b}|\sigma(\widetilde{\phi})^{(0,b')}\sigma(A)^{(a,b-b')}|^{(n)}. \tag{5.12}$$

We know that Lemma 9.12 in [PW] also holds for the odd dimensional manifold X and when a is odd. We know that $(D^2 + \partial_t)^{-1}u(D^2 + \partial_t)^{-1}$ has the Getzler order -4 and odd Clifford elements. By Lemma 8.1 and Lemma 9.12 in [PW] and Lemma 5.2 for $(D^2 + \partial_t)^{-1}u(D^2 + \partial_t)^{-1}$ and $j \ge a$, so

$$\int_X \mathrm{Tr}[\widetilde{\phi}K_{(D^2+\partial_t)^{-1}u(D^2+\partial_t)^{-1}}(x,\phi(x),t)]dx = O(t^{\frac{3}{2}}). \tag{5.13}$$

By equalities (5.10) and (5.13), in order to prove Theorem 5.1, we need to prove

$$\widetilde{\phi}(x)h^{-1}(x)K_{(D^2+\partial_t)^{-1}}(x,\phi(x),t)h(\phi(x)) = \widetilde{\phi}(x)K_{(D^2+\partial_t)^{-1}}(x,\phi(x),t) + O(t^{\frac{3}{2}}). \tag{5.14}$$

This comes from the equality

$$h^{-1}(x',v)c(e_i)h(x',\phi'(x)v) = c(e_i) + \frac{1}{2}zc(e_i)\sum_{j=a+1}^{n}[(\phi'(x)-1)v]_jc(e_j) - zx_i, \tag{5.15}$$

and $\widetilde{\phi}$ only contains the Clifford elements $c(e_{a+1}),\cdots,c(e_n)$.

5.2. The regularity of the equivariant eta form. In this section, the fundamental setup is the same as Section 2.1. But we assume the dimension n of the fibre to be odd. Let ϕ be an isometry which acts fiberwise on M. We will consider that ϕ acts as identity on B. Let $\mathrm{Tr}^{\mathrm{even}}$ be the trace on the coefficients of even forms on B. We assume that $\ker D$ is a complex vector bundle. Then the equivariant eta form is defined by

$$\widehat{\eta}(\phi) := \int_0^\infty \mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}\frac{d\mathcal{B}_t}{dt}e^{-\mathcal{B}_t^2}]dt. \tag{5.16}$$

Let T be the torsion tensor of $\nabla^\oplus$ and $c(T) = \sum_{1\le\alpha<\beta\le m}dy_\alpha dy_\beta c(T(\frac{\partial}{\partial y_\alpha},\frac{\partial}{\partial y_\beta}))$. Then

$$\widehat{\eta}(\phi) = \int_0^\infty \frac{1}{2\sqrt{t}}\mathrm{Tr}^{\mathrm{even}}[\psi_t\widetilde{\phi}(D + \frac{c(T)}{4})e^{-tF}]dt. \tag{5.17}$$

LEMMA 5.3. *We assume that* $\ker D$ *is a complex vector bundle. As* $t \to +\infty$, *we have*

$$\frac{1}{2\sqrt{t}}\mathrm{Tr}^{\mathrm{even}}[\psi_t\widetilde{\phi}(D + \frac{c(T)}{4})e^{-tF}] \sim O(t^{-\frac{3}{2}}). \qquad (5.18)$$

Proof. Since $\widetilde{\phi}$ is a bounded operator, so the proof of Lemma 5.3 is the same as the proof of Theorem 9.7 in [BGV]. □

Let $e_1(x), \cdots e_n(x)$ denote the orthonormal frame of TZ. If $A(Y)$ is any 0 order operator depending linearly on $Y \in \Gamma(M, TM)$, we define the operator $(\nabla_{e_i} + A(e_i))^2$ as follows

$$(\nabla_{e_i} + A(e_i))^2 = \sum_1^n (\nabla_{e_i(x)} + A(e_i(x)))^2 - \nabla_{\sum_j \nabla_{e_j}e_j} - A(\sum_j \nabla_{e_j}e_j). \qquad (5.21)$$

Then by [Bi2], we have

$$F = -(\nabla_{e_i}^{TZ} + \frac{1}{2} < S(e_i)e_j, f_\alpha > e_j dy_\alpha + \frac{1}{4} < S(e_i)f_\alpha, f_\beta > dy_\alpha dy_\beta)^2 + \frac{r}{4}. \qquad (5.22)$$

Set

$$h(x) = 1 + \frac{1}{2}dt\sum_{i=1}^n x_i e_i. \qquad (5.23)$$

Then we have

$$he_ih^{-1} = e_i + dtL_1; \quad h(\frac{1}{2}e_idt)h^{-1} = \frac{1}{2}e_idt, \qquad (5.24)$$

$$h\nabla_{e_i}^G h^{-1} = \nabla_{e_i}^G - \frac{1}{2}dte_i + dtL_2, \qquad (5.25)$$

$$h(\frac{1}{2} < S(e_i)e_j, f_\alpha > e_j dy_\alpha)h^{-1} = \frac{1}{2} < S(e_i)e_j, f_\alpha > e_j dy_\alpha + dtL_3, \qquad (5.26)$$

$$h(\frac{1}{4} < S(e_i)f_\alpha, f_\beta > dy_\alpha dy_\beta)h^{-1} = \frac{1}{4} < S(e_i)f_\alpha, f_\beta > dy_\alpha dy_\beta, \qquad (5.27)$$

where $L_1, L_2, L_3 \in O_G(-1)$. By Proposition 2.10 in [BeGS], we have

$$F + dt(D + \frac{c(T)}{4}) = -(\nabla_{e_i}^{TZ} + \frac{1}{2}\langle S(e_i)e_j, f_\alpha\rangle e_j dy_\alpha$$
$$+ \frac{1}{4}\langle S(e_i)f_\alpha, f_\beta\rangle dy_\alpha dy_\beta - \frac{1}{2}e_idt)^2 + \frac{r}{4}, \qquad (5.28)$$

By (5.24)-(5.28), we have

$$h[F + dt(D + \frac{c(T)}{4})]h^{-1} = F + dtu, \qquad (5.29)$$

where $O_G(u) \leq 0$. By the Duhamel principle, we can get

$$tdt\mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}(D + \frac{c(T)}{4})\exp(-tF)]$$
$$= \mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}\exp(-tF)] - \mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}h^{-1}\exp(-t(F + dtu))h]. \qquad (5.30)$$

Then

$$t\frac{dt}{\sqrt{t}}\mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}(D + \frac{c(T)}{4t})\psi_t\exp(-tF)]$$
$$= \mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}\psi_t\exp(-tF)] - \mathrm{Tr}^{\mathrm{even}}[\widetilde{\phi}h^{-1}\psi_t\exp(-t(F + dtu))h]. \qquad (5.31)$$

We know that Lemma 3.6 is still correct. When we take $\mathrm{Tr}^{\mathrm{even}}$, the terms having coefficients in Ω^{odd} vanish. Similar to Section 5.1, we can get

THEOREM 5.4. *As $t \to 0^+$, we have*

$$\mathrm{Tr}^{\mathrm{even}}[\psi_t\widetilde{\phi}(D + \frac{c(T)}{4})e^{-tF}] \sim O(t^{\frac{1}{2}}). \qquad (5.32)$$

By Lemma 5.3 and Theorem 5.4, the equivariant eta form is well defined. When n is even, we define

$$\widehat{\eta}(\phi) := \int_0^\infty \mathrm{Str}[\widetilde{\phi}\frac{d\mathcal{B}_t}{dt}e^{-\mathcal{B}_t^2}]dt. \qquad (5.33)$$

Similarly, we may prove the regularity in this case.

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