ASIAN J. MATH.
Vol. 20, No. 4, pp. 695–708, September 2016

005

ON ASYMPTOTIC PLATEAU'S PROBLEM FOR CMC HYPERSURFACES ON RANK 1 SYMMETRIC SPACES OF NONCOMPACT TYPE*

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Abstract. Let M^n, $n \geq 3$, be a Hadamard manifold with strictly negative sectional curvature $K_M \leq -\alpha$, $\alpha > 0$. Assume that M satisfies the *strict convexity condition* at infinity according to [18] (see also the definition below) and, additionally, that M admits a *helicoidal* one parameter subgroup $\{\varphi_t\}$ of isometries (i.e. there exists a geodesic γ of M such that $\varphi_t(\gamma(s)) = \gamma(t+s)$ for all $s, t \in \mathbb{R}$). We then prove that, given a compact topological $\{\varphi_t\}$−starshaped hypersurface Γ in the asymptotic boundary $\partial_\infty M$ of M (that is, the orbits of the extended action of $\{\varphi_t\}$ to $\partial_\infty M$ intersect Γ at one and only one point), and given $H \in \mathbb{R}$, $|H| < \sqrt{\alpha}$, there exists a complete properly embedded constant mean curvature (CMC) H hypersurface S of M such that $\partial_\infty S = \Gamma$.

This result extends Theorem 1.8 of B. Guan and J. Spruck [11] to more general ambient spaces, as rank 1 symmetric spaces of noncompact type, and allows Γ to be starshaped with respect to more general one parameter subgroup of isometries of the ambient space. For example, in $\mathbb{H}^n$, Γ can be starshaped with respect to a family of *loxodromic* curves (that includes, in particular, the radial one parameter subgroup of conformal diffeormophisms of $\partial_\infty \mathbb{H}^n$ considered in [11]). A fundamental result used to prove our main theorem, which has interest on its own, is the extension of the interior gradient estimates for CMC Killing graphs proved in Theorem 1 of [7] to CMC graphs of Killing submersions.

Key words. Hadamard manifolds, Killing graphs, asymptotic Dirichlet problem, asymptotic Plateau problem.

AMS subject classifications. 53C20, 53C21, 58J32.

1. Introduction. Let M^n be a Cartan-Hadamard manifold (namely a simply connected, complete Riemannian manifold with nonpositive sectional curvature) of dimension $n \geq 3$.

The asymptotic boundary $\partial_\infty M$ of M is defined as the set of all equivalence classes of unit speed geodesic rays in M; two such rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are equivalent if $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$, where d is the Riemannian distance in M. The so called *geometric* compactification $\overline{M}$ of M is then given by $\overline{M} := M \cup \partial_\infty M$, endowed with the cone topology (see [9] or [19], Ch. 2). For any subset $S \subset M$, we define $\partial_\infty S = \partial_\infty M \cap \overline{S}$.

The asymptotic Plateau problem for k (≥ 2) dimensional area minimizing submanifolds in M consists in finding, for a given a $k-1$ dimensional, closed, topological submanifold Γ of $\partial_\infty M$, a locally area minimizing, complete submanifold S^k of M such that $\partial_\infty S = \Gamma$.

By using methods from the Geometric Measure Theory, this problem was first studied in the hyperbolic space by M.T. Anderson [2] and his results extended to Gromov hyperbolic manifolds by U. Lang and V. Bangert ([3], [13], [14]).

Within the framework of the classical Plateau problem, the second author of the present paper with F. Tomi [20] study the asymptotic problem for minimal disk type

*Received October 5, 2014; accepted for publication April 2, 2015.
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surfaces in a general Hadamard manifold M.

In codimension 1, given $H \in \mathbb{R}$, we may consider the asymptotic Plateau's problem for the constant mean curvature (CMC) H hypersurface in M, namely, given a compact topological hypersurface $\Gamma \subset \partial_\infty M$, find a complete CMC H hypersurface S of M ($H-$hypersurface, for short) such that $\partial_\infty S = \Gamma$. This problem has also attracted the attention of many mathematicians more recently. The results of M.T. Anderson [2] have been extended to the CMC case by Y. Tonegawa [21] and H. Alencar and H. Rosenberg in [1].

Both Geometric Measure theory and Plateau's technique are methods that lead, in general, to the existence of hypersurfaces with singularities. Thus, a natural question, raised by B. Guan and J. Spruck in [11], asks about the existence of a smooth constant mean curvature hypersurface asymptotic to Γ at infinity in $\mathbb{H}^n$. This problem in fact had already been studied earlier in the minimal case by F. H. Lin [15].

A way to obtain smooth solutions is by finding a suitable system of coordinates in order to write the hypersurface as a graph, and then to use standard elliptic PDE methods. In [15], F.H. Lin represented the hypersurfaces in the half space model of $\mathbb{H}^n$ as vertical graphs, that is, in the usual way of $\mathbb{R}^n_+$ when using the cartesian system of coordinates.

The results of F.H. Lin [15] were extended to the CMC case by B. Nelli and J. Spruck in [16] where they proved the existence of a smooth CMC $|H| < 1$ hypersurface in the hyperbolic space $\mathbb{H}^n$ with sectional curvature -1 if Γ is assumed to be convex and compact. Later, also using PDE's techniques, B. Guan and J. Spruck [11] (see also [8] for a different approach based on a variational method) improved the convexity condition by requiring a starshaped property of Γ. We refer the reader to the nice survey of B. Coskunuzer [5], where the references of many other closely related papers to this subject can be found.

In both papers [16] and [11] the authors used the underlying Euclidean structure of the half space model for $\mathbb{H}^n$ to state the convexity and starshaped properties of Γ. However, although the convexity is not an intrinsic notion of the hyperbolic geometry, the starshapness of Γ is. It can be formulated in intrinsic terms using the conformal structure of $\overline{\mathbb{H}}^n$ by requiring Γ to be "circle shaped", meaning that there are two points $p_1, p_2 \in \mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$ such that any arc of circle from p_1 to p_2 intersects Γ at one and only one point. A limit starshaped condition, where $p_1 = p_2$, was also introduced and used by the second author in [17] to ensure the existence of a smooth CMC hypersurface having Γ as asymptotic boundary (see the Introduction and Theorem 6 of [17] for a detailed description of this case).

In the present paper we extend Theorem 1.8 of [11] in two directions. First, we allow Γ to be "starshaped" with respect to a more general one parameter subgroup of conformal diffeomorphisms of $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$. Secondly, we allow the ambient space to be any rank 1 symmetric space of noncompact type. Both results are consequences of a more general theorem that holds in a Hadamard manifold endowed with some special Killing field.

As we shall see in the proof ahead, the Killing field allows to introduce a special system of coordinates which is quite suitable for using standard elliptic PDE techniques. To write down precise statements we first introduce some general notions and terminology.

Let $\gamma : (-\infty, \infty) \to M$ be an arc length geodesic. We say that a one parameter subgroup of isometries $\{\varphi_t^\gamma\}$ of M associated to γ is *helicoidal* if $\varphi_t^\gamma(\gamma(s)) = \gamma(t+s))$ for all $s, t \in \mathbb{R}$. In the sequel, since there is no possibility of confusion, we shall omit the dependance of $\{\varphi_t^\gamma\}$ with respect to the geodesic γ.

Let us illustrate the previous definition with a simple case that justifies this terminology: If $M = \mathbb{R}^3$ then any helicoidal one parameter subgroup of isometries, up to a conjugation, is of the form

$$\varphi_t(x, y, z) = \left(\begin{bmatrix} \cos at & \sin at \\ -\sin at & \cos at \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, z + t \right)$$

for some $a \in \mathbb{R}$. When $a = 0$, $\{\varphi_t\}$ is a one parameter subgroup of transvections along the $z-$axis. More generally, a one parameter subgroup of transvections along a geodesic in a symmetric space (see [12]) is a particular case of helicoidal one parameter subgroup of isometries.

Since the equivalence relation between geodesics and convergent sequences are preserved under isometries, the action of $\{\varphi_t\}$ on M extends to the compactification $\overline{M}$ of M and the extended action is continuous. The orbits of $\{\varphi_t\}$ are the curves $O(x) := \{\varphi_t(x) \mid t \in \mathbb{R}\}$ where $x \in \overline{M}$. Observe that $\{\varphi_t\}$ has two singular orbits in $\overline{M}$, namely, $O(\gamma(\pm\infty))$, where γ is the geodesic translated by $\{\varphi_t\}$.

Finally, we will also need to use the *Strictly Convexity Condition* ("SC condition") introduced in [20]. We say that M satisfies the SC condition if, given $x \in \partial_\infty M$ and a relatively open subset $W \subset \partial_\infty M$ containing x, there exists a C^2 open set $\Omega \subset \overline{M}$ such that $x \in \operatorname{Int}(\partial_\infty \Omega) \subset W$ and $M \backslash \Omega$ is convex, where $\operatorname{Int}(\partial_\infty \Omega)$ stands for the interior of $\partial_\infty \Omega$ in $\partial_\infty M$.

We are now in position to state our main result :

THEOREM 1. *Let M be a Hadamard manifold with sectional curvature $K_M \leq -\alpha$, for some $\alpha > 0$, satisfying the SC condition. Let $\{\varphi_t\}$ be a helicoidal one parameter subgroup of isometries of M. Let $\Gamma \subset \partial_\infty M$ be a compact topological embedded hypersurface of $\partial_\infty M$ and assume that any nonsingular orbit of $\{\varphi_t\}$ in $\partial_\infty M$ intersects Γ at one and only one point. Then, given $H \in \mathbb{R}$, $|H| < \sqrt{\alpha}$, there exists a complete, properly embedded $H-$hypersurface S of M such that $\partial_\infty S = \Gamma$. Moreover any orbit of $\{\varphi_t\}$ intersects S at one and only one point.*

We point out that the SC condition is satisfied by a large class of manifolds. For example, if the sectional curvature is bounded from above by a strictly negative constant and decreases at most exponentially (see Theorem 14 of [18]) then the SC condition is satisfied. In particular, it is satisfied by any rank 1 symmetric spaces of noncompact type. Therefore, as an immediate consequence of the previous theorem, we obtain:

COROLLARY 2. *Assume that M is a rank 1 symmetric space of noncompact type and assume that the sectional curvature of M is bounded by $-\alpha$, $\alpha > 0$. Let $\{\varphi_t\}$ be a one parameter of transvections of M. Let $\Gamma \subset \partial_\infty M$ be a compact embedded topological hypersurface of $\partial_\infty M$ that intersects any nonsingular orbit of $\{\varphi_t\}$ at one and only one point. Then, given $H \in \mathbb{R}$, $|H| < \sqrt{\alpha}$, there exists a complete, properly embedded $H-$hypersurface S of M such that $\partial_\infty S = \Gamma$. Moreover any orbit of $\{\varphi_t\}$ intersects S at one and only one point.*

Finally we point out an interesting corollary of Theorem 1 in the case where $M = \mathbb{H}^n$, the hyperbolic space of constant sectional curvature -1. Recalling that a *loxodromic curve* is a curve in $\mathbb{S}^{n-1}$ that intersects with a constant angle θ any arc of circle of $\mathbb{S}^{n-1}$ connecting two fixed points of $\mathbb{S}^{n-1}$ (see [22]). These curves are induced by one-parameter subgroups of isometries of $\mathbb{H}^n$ of helidoidal type. For example, in the half space model $z > 0$ of $\mathbb{H}^3$, up to conjugation, they are of the form

$$\varphi_t(x, y, z) = e^t \left(\begin{bmatrix} \cos\theta t & \sin\theta t \\ -\sin\theta t & \cos\theta t \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, z \right).$$

Corollary 3. *Let $0 \leq \theta < \pi/2$ and $p_1, p_2 \in \mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n$ be two distinct points of $\mathbb{S}^{n-1}$. Let L_θ be the family of loxodromic curves that intersects any arc of circle from p_1 to p_2 with a constant angle θ. Let $\Gamma \subset \mathbb{S}^{n-1}$ be a compact embedded topological hypersurface of $\mathbb{S}^{n-1}$ that intersects any curve of L_θ at one and only one point. Then, given $H \in \mathbb{R}$, $|H| < 1$, there exists a complete, properly embedded $H-$hypersurface S of $\mathbb{H}^n$ such that $\partial_\infty S = \Gamma$.*

We notice that, taking $\theta = 0$ in the previous corollary, we recover Theorem 1.8 of [11]. Theorem 1.8 also follows from Corollary 2 since radial graphs (considered in [11]) are transvections along a geodesic of $\mathbb{H}^n$.

A fundamental result for proving the above theorems, which has interest on its own, are the interior gradient estimates of the solutions of the CMC H graph PDE for Killing submersions (see Theorem 4 below). It extends Theorem 1 of [7].

2. Proofs of the results. In what follows we use most of the nomenclature and the results proved by M. Dajczer and J. H. de Lira in [6]. However, we introduce the notion of a Killing graph on a manner slightly different from the one considered in [6].

For the next result we allow M be any Riemannian manifold and Y a Killing field in M without singularities. Denote by $\mathcal{O}(x)$ the integral curve (which we also call orbit) of Y through a point $x \in M$. By a complete $Y-$ Killing section (we shall refer only to a Killing section because Y will be fixed throughout the text) we mean a complete up to the boundary (possibly empty) hypersurface P of M such that any orbit $\mathcal{O}(p)$ of Y through a point p of P intersects P only at p and the intersection is transversal. We call $\Omega := P \backslash \partial P$ a Killing domain. If $P = \overline{\Omega}$ is a hypersurface of class $C^{2,\alpha}$ in M then we say that Ω is a $C^{2,\alpha}$ Killing domain.

If u is a function defined on a subset T of P, the Killing graph of u is given by

$$\mathrm{Gr}(u) = \{\varphi(u(p), p) \mid p \in T\}$$

where $\varphi(s, x) = \varphi_s(x)$ is the flow of Y. In the sequel, s will stand for the flow parameter. We also set

$$\Gamma_T = \{\varphi(s, x) \mid x \in T \text{ and } s \in \mathbb{R}\}.$$

Next, we denote by $\Pi : M \to P$ the projection defined by $\Pi(x) = \mathcal{O}(x) \cap P$. In all the sequel, we endow P with the Riemannian metric $\langle\, ,\, \rangle_\Pi$ such that Π becomes a Riemannian submersion.

Assume that Ω is a $C^{2,\alpha}$ Killing domain. Given $H \in \mathbb{R}$, it is not difficult to show that $\mathrm{Gr}(u)$ has CMC H with respect to the unit normal vector field η to $\mathrm{Gr}(u)$ such that $\langle Y, \eta\rangle \leq 0$ if and only if u satisfies a certain second order quasi-linear elliptic PDE $Q_H[u] = 0$ on M in terms of the metric $\langle\, ,\, \rangle_\Pi$ in P (for details, including an explicit expression of Q_H, see Section 2.1 of [6] or the short revision done below).

We may then refer to the CMC H Dirichlet problem in a Killing domain $\Omega \subset M$ and for a given boundary data $\phi \in C^0(\partial\Omega)$ as the PDE boundary problem

$$\begin{cases} Q_H[u] = 0 \text{ in } \Omega \quad u \in C^{2,\alpha}(\Omega) \cap C^0\left(\overline{\Omega}\right) \\ u|_{\partial\Omega} = \phi. \end{cases} \tag{1}$$

We begin by obtaining interior gradient estimates for the solutions of (1). Our result generalizes Theorem 1 of [7] to the case of CMC H graph PDE of Killing submersions.

Fix a point $o \in \Omega$ and let $r > 0$ be such that $r < i(o)$, the injectivity radius of M at o. We obtain the following result :

THEOREM 4. *Let Ω be a Killing domain in M. Let $o \in \Omega$ and $r > 0$ such that the open geodesic ball $B_r(o)$ is contained in Ω. Let $u \in C^3(B_r(o))$ be a negative solution of $Q[u] = 0$ in $B_r(o)$. Then there is a constant L depending only on $u(o)$, r, $|Y|$ and H such that $|\nabla u(o)| \leq L$.*

Before proving the above theorem, we review the nomenclature and some facts of [6].

We fix a local reference frame $v_1, \ldots, v_n$ on $\overline{\Omega}$ and we set $\sigma_{ij} = \langle v_i, v_j \rangle_\Pi$. We will now define a local frame in M. We denote by $D_1, \ldots, D_n$ the basic vector fields Π-related to $v_1, \ldots, v_n$. The frame $D_0, \ldots, D_n$, we considered on M, is defined by $D_0 = f^{\frac{1}{2}} \partial_s$, where $f = \dfrac{1}{|Y|^2}$, $(\partial_s(q) = \varphi_*(s,p)\partial_s(p))$, and $D_i(q) = \varphi_*(s,p)D_i(p)$, where $q = \varphi(s,p)$, $p \in P$. We point out that the unit normal vector field to $Gr(u)$ pointing upward is given by

$$N = \frac{1}{W}(f^{\frac{1}{2}} D_0 - \hat{u}^j D_j), \tag{2}$$

where $\hat{u}^j = \sigma^{ij} D_i(u - s)$ and $W^2 = f + \hat{u}^i \hat{u}_i = f + \sigma_{ij} \hat{u}^i \hat{u}^j$. We notice that $\hat{u}_i$ and W are not depending on s and therefore can be seen as function defined on P. Finally, using the previous notation, the operator Q (defined in (1)) can be written as

$$Q_H[u] = \frac{1}{W}(A^{ij} \hat{u}_{j;i} - \frac{(f + W^2)}{W^2} \left\langle \Pi_* \bar{\nabla}_{D_0} D_0, Du \right\rangle) - nH,$$

where $\hat{u}_{i;j} = \left\langle \bar{\nabla}_{D_i} \bar{\nabla}(u - s), D_j \right\rangle$, $Du = \Pi_* \overline{\nabla}(u - s)$ and $A^{ik} = \sigma^{ik} - \dfrac{\hat{u}^i \hat{u}^k}{W^2}$.

Proof of Theorem 4. The proof will follow closely the one of Theorem 1 in [7]. Let $p \in B_r(o)$ be an interior point where $h = \eta W$ attains its maximum, where η is a smooth function with support in $B_r(o)$ which will be determined in the sequel. In all this section, the computations will be done at the point p. Let $v_1, \ldots, v_n$ be an orthonormal tangent frame at $p \in B_r(o)$. Then we have $h_i = 0$ (where the derivative is taken with respect to v_i). This implies that

$$\eta_i W = -\eta W_i. \tag{3}$$

We also have, since $\dfrac{A^{ij}}{W}$ is definite positive, that

$$0 \geq \frac{1}{W} A^{ij} h_{ij} = \frac{1}{W} A^{ij} (W \eta_{i;j} + 2\eta_i W_j + \eta W_{i;j}).$$

Using (3), the previous inequality can be rewritten as

$$A^{ij} \eta_{i;j} + \frac{\eta}{W^2} A^{ij} (W W_{i;j} - 2 W_i W_j) \leq 0. \tag{4}$$

From (2), we have

$$N^k = -\frac{\hat{u}^k}{W}. \tag{5}$$

Derivating W, we find

$$W_i = \frac{f_i}{2W} + \frac{\hat{u}^k \hat{u}_{k;i}}{W} = \frac{f_i}{2W} - N^k \hat{u}_{k;i}.$$

From (5), we get

$$N^k_{;j} = -\frac{\hat{u}^k_{;j}}{W} + \frac{\hat{u}^k W_j}{W^2}.$$

Using the previous inequalities, we have

$$\begin{aligned} W_{i;j} &= \frac{f_{i;j}}{2W} - \frac{f_i W_j}{2W^2} - N^k_{;j}\hat{u}_{k;i} - N^k \hat{u}_{k;ij} \\ &= \frac{f_{i;j}}{2W} + \frac{\hat{u}^k_{;j}\hat{u}_{k;i}}{W} - \frac{\hat{u}^k \hat{u}_{k;i} W_j}{W^2} - \frac{f_i W_j}{2W^2} - N^k \hat{u}_{k;ij} \\ &= \frac{f_{i;j}}{2W} + \frac{\hat{u}^k_{;j}\hat{u}_{k;i}}{W} - \frac{W_i W_j}{W} - N^k \hat{u}_{k;ij} \\ &= \frac{f_{i;j}}{2W} + \frac{A^{kl}}{W}\hat{u}_{l;j}\hat{u}_{k;i} + \frac{f_i f_j}{4W^3} - \frac{1}{2W^2}(W_i f_j + W_j f_i) - N^k \hat{u}_{k;ij}. \end{aligned}$$

Multiplying by A^{ij} the above equation and using (3), we find

$$A^{ij}W_{i;j} = \frac{A^{ij}f_{i;j}}{2W} + \frac{A^{ij}A^{kl}}{W}\hat{u}_{l;j}\hat{u}_{k;i} + \frac{A^{ij}f_i f_j}{4W^3} + \frac{1}{\eta W}A^{ij}\eta_i f_j - A^{ij}N^k \hat{u}_{k;ij}. \tag{6}$$

In order to get rid of the term involving three derivatives of u in (6), we want to find a commutation formula for $\hat{u}_{k;ij}$. We recall (see equation (11) of [6]) that

$$\hat{u}_{k;i} = u_{k;i} - s_{k;i} + \frac{1}{2}\gamma_{ki},$$

where $\gamma_{ki} = f^{\frac{1}{2}}\langle [D_k, D_i], D_0\rangle$. We deduce from the previous equality that

$$\begin{aligned} \hat{u}_{k;ij} &= u_{k;ij} - s_{k;ij} + \frac{1}{2}(\gamma_{ki})_j = u_{i;jk} + R^l_{kji}u_l - s_{k;ij} + \frac{1}{2}(\gamma_{ki})_j \\ &= (\hat{u}_{j;i} + s_{j;i} - \frac{1}{2}\gamma_{ji})_k + R^l_{kji}u_l - s_{k;ij} + \frac{1}{2}(\gamma_{ki})_j \\ &= \hat{u}_{j;ik} + s_{j;ik} - \frac{1}{2}(\gamma_{ji})_k + R^l_{kji}u_l - s_{k;ij} + \frac{1}{2}(\gamma_{ki})_j \\ &= \hat{u}_{j;ik} + R^l_{kji}\hat{u}_l + R^l_{kji}s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2}((\gamma_{ki})_j - (\gamma_{ji})_k) \\ &= \hat{u}_{j;ik} + R^l_{kji}\hat{u}_l + C_{ijk}, \end{aligned}$$

where $C_{ijk} = R^l_{kji}s_l + s_{j;ik} - s_{k;ij} + \frac{1}{2}((\gamma_{ki})_j - (\gamma_{ji})_k)$ is not depending on u. Using (1) and the commutation formula, the last term of (6) rewrites as

$$\begin{aligned} A^{ij}N^k\hat{u}_{k;ij} &= A^{ij}N^k\hat{u}_{j;ik} - \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W} - \frac{\hat{u}^k A^{ij}C_{ijk}}{W} \\ &= N^k(A^{ij}\hat{u}_{j;i})_k - N^k A^{ij}_{;k}\hat{u}_{i;j} - \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W} - \frac{\hat{u}^k A^{ij}C_{ijk}}{W} \\ &= nN^k(WH)_k + N^k\left(\frac{(f+W^2)}{W^2}\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle\right)_k \\ &\quad - N^k A^{ij}_{;k}\hat{u}_{j;i} - \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W} - \frac{\hat{u}^k A^{ij}C_{ijk}}{W}. \end{aligned}$$

Straightforward computations using (3) give

$$(WH)_k = W_k H + W H_k = \frac{W}{\eta}(-\eta_k H + \eta H_k),$$

and

$$(\frac{f+W^2}{W^2})_k = \frac{f_k}{W^2} - \frac{f}{W^4}(f_k + 2\hat{u}^l \hat{u}_{l;k}) = \frac{1}{W^2}(f_k + 2\frac{f\eta_k}{\eta}). \tag{7}$$

We also have, using (7),

$$\begin{aligned}\left(\frac{(f+W^2)}{W^2}\left\langle \Pi_* \bar{\nabla}_{D_0} D_0, Du\right\rangle\right)_k =& \frac{(f+W^2)}{2fW^2}[(\frac{f_l f_k}{f} - f_{k;l})WN^l + f^l \hat{u}_{l;k}] \\ &+ \left\langle \Pi_* \bar{\nabla}_{D_0} D_0, Du\right\rangle \frac{1}{W^2}(f_k + 2\frac{f\eta_k}{\eta}),\end{aligned}$$

and

$$\begin{aligned}A^{ij}_{;k} &= -\frac{1}{W^2}(\hat{u}^i_{;k}\hat{u}^j + \hat{u}^i \hat{u}^j_{;k}) + \frac{1}{W^4}(f_k - 2WN^l \hat{u}_{l;k})\hat{u}^i \hat{u}^j \\ &= \frac{1}{W}(\hat{u}^i_{;k} - N^i N^l \hat{u}_{l;k})N^j + \frac{1}{W}(\hat{u}^j_{;k} - N^i N^l \hat{u}_{l;k})N^i + \frac{1}{W^2} f_k N^i N^j \\ &= \frac{1}{W} A^{il}\hat{u}_{l;k} N^j + \frac{1}{W} A^{jl} \hat{u}_{l;k} N^i + \frac{1}{W^2} f_k N^i N^j.\end{aligned}$$

Multiplying the previous equality by $N^k \hat{u}_{j;i}$, we find

$$N^k A^{ij}_{;k} \hat{u}_{j;i} = \frac{1}{W} N^k \hat{u}_{j;i} \hat{u}_{l;k}(A^{il}N^j + A^{jl}N^i) + \frac{1}{W^2} f_k N^i N^j N^k \hat{u}_{j;i}.$$

Recalling that

$$N^k \hat{u}_{k;i} = \frac{f_i}{2W} + \frac{W\eta_i}{\eta},$$

and

$$\hat{u}_{i;j} = \hat{u}_{j;i} + \gamma_{ij},$$

we have

$$\begin{aligned}N^k A^{ij}_{;k} \hat{u}_{j;i} =& \frac{1}{W^2} f_k N^k N^i (\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) + \frac{1}{W} A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) N^k (\hat{u}_{k;l} + \gamma_{lk}) \\ &+ \frac{1}{W} A^{jl} N^k N^i (\hat{u}_{i;j} + \gamma_{ji})(\hat{u}_{k;l} + \gamma_{lk}) \\ =& \frac{1}{W^2} f_k N^k N^i (\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) + \frac{2}{W} A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta})(\frac{f_l}{2W} + \frac{W\eta_l}{\eta}) \\ &+ \frac{1}{W} A^{jl} N^k N^i \gamma_{ji}\gamma_{lk} + \frac{3}{W} A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) N^k \gamma_{lk},\end{aligned}$$

and

$$N^k f_l \hat{u}_{l;k} = N^k f_l (\hat{u}_{k;l} - \gamma_{kl}) = \frac{f}{2W}\sigma^{kl}\frac{f_k f_l}{f} + \frac{W}{\eta} f_l \eta^l - \gamma_{kl} N^k f_l.$$

Using the previous computations, we deduce that the last term of (6) can be rewritten as

$$\begin{aligned}
A^{ij}N^k\hat{u}_{k;ij} = nN^k\frac{W}{\eta}(-\eta_k H + \eta H_k) - \frac{2}{W}A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta})(\frac{f_l}{2W} + \frac{W\eta_l}{\eta})\\
- \frac{1}{W^2}f_kN^kN^i(\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) - \frac{1}{W}A^{jl}N^kN^i\gamma_{ji}\gamma_{lk}\\
- \frac{3}{W}A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta})N^k\gamma_{lk} - \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W}\\
- \frac{\hat{u}^kA^{ij}C_{ijk}}{W} + \left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle \frac{N_k}{W^2}(f_k + 2\frac{f\eta_k}{\eta})\\
+ \frac{(f+W^2)}{2fW^2}\left[\left(\frac{f}{2W}\sigma^{kl} + WN^kN^l\right)\frac{f_kf_l}{f} - WN^kN^lf_{k;l} + \frac{W}{\eta}f_l\eta^l - \gamma_{kl}N^kf_l\right].
\end{aligned}$$

Thus, from (6), we obtain

$$\begin{aligned}
&A^{ij}W_{ij} - \frac{2}{W}A^{ij}W_iW_j\\
&= \frac{3}{4W^3}A^{ij}f_if_j + \frac{1}{W}A^{ij}A^{kl}\hat{u}_{l;j}\hat{u}_{k;i} + \frac{3}{W\eta}A^{ij}f_i\eta_j + \frac{1}{2W}A^{ij}f_{i;j}\\
&\quad - nN^k\frac{W}{\eta}(-\eta_kH + \eta H_k) + \frac{1}{W^2}f_kN^kN^i(\frac{f_i}{2W} + \frac{W\eta_i}{\eta}) + \frac{1}{W}A^{jl}N^kN^i\gamma_{ji}\gamma_{lk}\\
&\quad + \frac{3}{W}A^{il}(\frac{f_i}{2W} + \frac{W\eta_i}{\eta})N^k\gamma_{lk} + \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W}\\
&\quad + \frac{\hat{u}^kA^{ij}C_{ijk}}{W} - \left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle \frac{N_k}{W^2}(f_k + 2\frac{f\eta_k}{\eta})\\
&\quad - \frac{(f+W^2)}{W^2}\frac{1}{2f}\left[\left(\frac{f}{2W}\sigma^{kl} + WN^kN^l\right)\frac{f_kf_l}{f} - WN^kN^lf_{k;l} + \frac{W}{\eta}f_l\eta^l - \gamma_{kl}N^kf_l\right].
\end{aligned}$$

Multiplying by $\frac{\eta}{W}$, we have

$$\begin{aligned}
&\frac{\eta}{W}(A^{ij}W_{ij} - \frac{2}{W}A^{ij}W_iW_j)\\
&\quad\geq \left[-nN^kH_k - \frac{f_k}{W^3}N^k\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle + \frac{1}{2W^2}A^{ij}f_{i;j}\right.\\
&\quad + \frac{1}{W^2}A^{jl}N^kN^i\gamma_{ji}\gamma_{lk} + \frac{3}{2W^3}A^{il}f_iN^k\gamma_{lk} + \frac{\hat{u}^kA^{ij}C_{ijk}}{W^2} + \frac{A^{ij}R^l_{kji}\hat{u}_l\hat{u}^k}{W^2}\\
&\quad \left. - \frac{(f+W^2)}{W^2}\frac{1}{2f}\left[\left(\frac{f}{2W^2}\sigma^{kl} + N^kN^l\right)\frac{f_kf_l}{f} - N^kN^lf_{k;l} - \frac{1}{W}\gamma_{kl}N^kf_l\right]\right]\eta\\
&\quad + \left[\left(nH + \frac{1}{W^2}N^kf_k - \frac{2f}{W^3}\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle\right)N^i\right.\\
&\quad \left. + \frac{3}{W}A^{jl}N^k\gamma_{lk} + \left(\frac{3}{W^2}A^{ij} - \frac{(f+W^2)}{W^2}\frac{1}{2f}\sigma^{ij}\right)f_j\right]\eta_i.
\end{aligned}$$

Thus it is easy to see that there exists a constant $M > 0$, not depending on u, such that

$$\frac{\eta}{W^2}(WA^{ij}W_{ij} - 2A^{ij}W_iW_j) \geq -M\eta - A^i\eta_i,$$

where A^i is the coefficient of η_i. From (4), we deduce that

$$A^{ij}\eta_{i;j} - M\eta - A^i\eta_i \leq 0. \tag{8}$$

We are now ready to choose an explicit η. We take

$$\eta(x) = g(\phi(x)) = e^{C_1\phi(x)} - 1 = e^{C_1(1 - \frac{d^2(x)}{r^2} + \frac{u(x)}{C})^+} - 1,$$

where $C = -\dfrac{1}{2u(o)}$. Straightforward computations give

$$\eta_i = g'(-r^{-2}(d^2)_i + C(u_i - s_i)) = g'(-r^{-2}(d^2)_i + C\hat{u}_i),$$

and

$$\eta_{i;j} = g'(-r^{-2}(d^2)_{i;j} + C\hat{u}_{i;j}) + g''(-r^{-2}(d^2)_i + C\hat{u}_i)(-r^{-2}(d^2)_j + C\hat{u}_j).$$

We deduce from the two previous lines that

$$A^{ij}(-r^{-2}(d^2)_i + C\hat{u}_i)(-r^{-2}(d^2)_j + C\hat{u}_j) \geq \frac{C^2 f}{W^2}\left(|Du|^2 - \frac{2}{Cr^2}\left\langle Du, \nabla d^2\right\rangle\right),$$

and

$$\begin{aligned} A^{ij}(-r^{-2}(d^2)_{i;j} + C\hat{u}_{i;j}) &= -r^{-2}A^{ij}(d^2)_{i;j} \\ &\quad + C\left(nWH + \frac{f+W^2}{W^2}\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle + A^{ij}\gamma_{ij}\right), \end{aligned}$$

where

$$A^{ij}(d^2)_{i;j} = \Delta(d^2) - \frac{1}{W^2}\left\langle \nabla_{Du}\nabla d^2, Du\right\rangle.$$

Inserting the previous expressions into (8), we have

$$\begin{aligned} &\frac{C^2 f}{W^2}\left(|Du|^2 - \frac{2}{Cr^2}\left\langle Du, \nabla d^2\right\rangle\right) g'' \\ &\quad + \Bigg[C\left(nWH + \frac{f+W^2}{W^2}\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle + A^{ij}\gamma_{ij}\right) \\ &\qquad\qquad - r^{-2}\left(\Delta(d^2) - \frac{1}{W^2}\left\langle \nabla_{Du}\nabla d^2, Du\right\rangle\right)\Bigg] g' \\ &\qquad\qquad\qquad \leq Mg + A^i(-r^2(d^2)_i + C\hat{u}_i)g'. \end{aligned}$$

Using the explicit expression of A^i, it is easy to see that $CA^i\hat{u}_i$ contains bounded terms and the term

$$C\left(nWH + \frac{f+W^2}{W^2}\left\langle \Pi_*\bar{\nabla}_{D_0}D_0, Du\right\rangle\right).$$

Therefore, we conclude that

$$\frac{C^2 f}{W^2}(|Du|^2 - \frac{2}{Cr^2}\left\langle Du, \nabla d^2\right\rangle)g'' + Pg' - Mg \leq 0,$$

where P and M do not depend on u. Finally, it is easy to check that the coefficient of g'' is strictly positive if we assume that $|Du| \geq \frac{16u_0}{r}$. It implies that

$$W(p) \leq C_2 = \sup_{B_r(o)} f + \frac{16u_0}{r}.$$

Since p is the maximum point of h, this implies that

$$(e^{\frac{C_1}{2}} - 1)W(0) \leq C_2 e^{C_1}.$$

□

For the proof of Theorem 1 we make use of the following lemma, which shows that the SC condition implies an explicit mean convexity condition. Precisely:

Lemma 5. *Assume M is a Hadamard manifold satisfying the strict convexity condition and such that $K_M \leq -\alpha$, for some constant $\alpha > 0$. Then M satisfies the h-mean convexity condition, for $h < \sqrt{\alpha}$, that is, given $x \in \partial_\infty M$, a relatively open subset $W \subset \partial_\infty M$ containing x and $h < \sqrt{\alpha}$, there exists a C^2 open set $\Lambda \subset \overline{M}$ such that $x \in \operatorname{Int}(\partial_\infty \Lambda) \subset W$ and the mean curvature of $M \backslash \Lambda$ with respect to the normal vector pointing to $M \backslash \Lambda$ is bigger than or equal to h.*

Proof. Given $x \in \partial_\infty M$ and a relatively open subset $W \subset \partial_\infty M$ containing x, let Ω be a convex unbounded domain in M, given by the SC condition such that $x \in \operatorname{Int}(\partial_\infty \Omega) \subset W$. Denote by $d : \Omega \to \mathbb{R}$ the distance function to $\partial\Omega$. Then the hessian comparison theorem (see [4]) yields

$$\Delta d \geq (n-1)\sqrt{\alpha} \tanh(\sqrt{\alpha} d),$$

i.e. the equidistant hypersurface Ω_d of Ω is $\sqrt{\alpha}\tanh(\sqrt{\alpha}d)$-convex. Since $\tanh(\sqrt{\alpha}d) \underset{d\to\infty}{\longrightarrow} 1$, we deduce that M also satisfies the h-mean convexity condition for $h < \sqrt{\alpha}$. □

Proof of Theorem 1. Let $\gamma : (-\infty, \infty) \to M$ be the geodesic translated by Y. Set $P = \exp_o \{Y(o)\}^{\perp}$ where $o = \gamma(0)$. Let $p \in P$ and $t \in \mathbb{R}$ be given. We may write $p = \exp_{\gamma(s)} u$ for some $s \in \mathbb{R}$ and $u \in \gamma'(s)^{\perp}$. Since $\tilde{\gamma}(r) = \exp_{\gamma(s)}(ru)$, $r \in [0,1]$, is a geodesic and φ_t an isometry, $\beta(r) := \varphi_t(\tilde{\gamma}(r))$ is also a geodesic which, moreover, satisfies the initial conditions

$$\begin{aligned}\beta(0) &= \varphi_t(\tilde{\gamma}(0)) = \varphi_t(\gamma(s)) = \gamma(s+t)\\ \beta'(0) &= d(\varphi_t)_{\gamma(s)} u =: v,\end{aligned}$$

we have $\beta(r) = \exp_{\gamma(s+t)}(rv)$ by uniqueness. It follows that

$$\varphi_t(p) = \beta(1) = \exp_{\gamma(s+t)} v.$$

Moreover, since

$$0 = \langle u, \gamma'(s) \rangle = \left\langle d(\varphi_t)_{\gamma(s)} u, d(\varphi_t)_{\gamma(s)} \gamma'(s) \right\rangle = \langle v, \gamma'(s+t) \rangle$$

we have $v \in \gamma'(s+t)^{\perp}$ and, as the normal exponential map of a geodesic in Hadamard manifold is a diffeomorphism from the normal bundle of the geodesic onto M, we have $\varphi_t(p) \cap P \neq \varnothing$ if and only if $t = 0$.

We now observe that Y is everywhere transversal to P. Indeed, assume by contradiction that Y is not transversal to P at some point $p \in P$. Let $d : N \to \mathbb{R}$ be the distance function to P. We set $f(t) = d(\varphi(t,p))$ and observe that $f(0) = 0$. Moreover, since $\varphi(t,\cdot)$ is an isometry of N, we have, for any fixed t,

$$\operatorname{grad} d(\varphi(t,p)) = d\varphi(t,p)_p (\operatorname{grad} d(p)).$$

Therefore, we obtain

$$\begin{aligned} f'(t) &= \left\langle \operatorname{grad} d, \left.\frac{\partial \varphi(s,p)}{\partial s}\right|_{s=t} \right\rangle = \langle \operatorname{grad} d(\varphi(t,p)), Y(\varphi(t,p))\rangle \\ &= \left\langle d\varphi(t,p)_p (\operatorname{grad} d(p)), d\varphi(t,p)_p (Y(p)) \right\rangle \\ &= \langle \operatorname{grad} d(p), Y(p)\rangle = 0. \end{aligned}$$

This implies that $f \equiv 0$ and, in return, that $\varphi(t,p) \in P$ for all t, which yields to a contradiction. This proves that P is a Killing section.

Since any orbit of φ at $\partial_\infty N$ intersects Γ at one and exactly one point, Γ is the Killing graph of a function $\phi \in C^0(\partial_\infty P)$. Let $F \in C^{2,\alpha}(P) \cap C^0(\bar{P})$ ($\bar{P} = P \cup \partial_\infty P$) be such that $F|_{\partial_\infty P} = \phi$.

Let ρ be the geodesic distance in P to a fixed point $o \in P$. We denote by B_k, for $k = 2, 3, \ldots$, the geodesic ball in P centered in o and of radius k. We first show that, for any $k = 2, 3, \ldots$, there is a solution $u_k \in C^{2,\alpha}(\bar{B}_k)$ of

$$\begin{cases} Q_H[u_k] = 0, & \text{on } B_k \\ u_k|_{\partial B_k} = F_k = F|_{\partial B_k}. \end{cases} \tag{9}$$

In order to prove the existence of the u_k's, we will need some a priori height estimate. More precisely, we claim that given some $k \geq 2$, there is a constant C_j depending only on j such that if u_k is a solution of (9) and $j \leq k$ then $\sup_{B_j} |u_k| \leq C_j$. Let us prove the claim. We choose two open subsets $U_\pm$ of $\gamma(\pm\infty)$ in $\partial_\infty M$. Using the SC condition, we obtain the existence of two C^2 convex subsets $W_\pm$ of M such that $\partial_\infty W_\pm \subset U_{K_\pm}$. Denote by $K_\pm$ the hypersurfaces $K_\pm = \partial W_\pm$. As observed in Lemma 5 and since $|H| < \sqrt{\alpha}$, we may assume that $K_\pm$ are H_0 mean convex with $H_0 \geq H$. We then choose C_j such that the orbit of $\{\varphi_t\}$ through a point of B_j intersects $W_\pm$ for some $t \geq C_j$. It is clear that we may assume that the Killing graph of F does not intersect $K_\pm$. The claim then follows from the tangency principle.

We have two important consequences of the previous height estimate. The first one is that problem (9) is solvable for any $k \geq 2$. In fact, the only missing hypothesis to apply Theorem 1 of [6] to guarantee the solvability of (9) are on the Ricci curvature of M and on the mean curvature of the Killing cylinder over the boundary of B_k (see [6], Theorem 1). Concerning the hypothesis on the mean curvature of the Killing cylinder over the boundary of B_k, we claim that it holds true in our setting. Indeed, since the orbits of φ_t are equidistant curves of γ, it follows that the Killing cylinder K_k over ∂B_k is an equidistant hypersurface of γ. Therefore the mean curvature H_{K_k} of K_k with respect to the inner normal vector field of K_k coincides with the Laplacian of the distance to γ. One may then apply the hessian comparison theorem to obtain

$$H_{K_k} \geq \sqrt{\alpha} \tanh\left(k\sqrt{\alpha}\right) \geq H$$

if k is large enough. Moreover, a direct inspection on the proof of Theorem 1 of [6] shows that the hypothesis on the Ricci curvature is only used to obtain a priori height estimates, which we just obtained above.

Secondly, the a priori height estimates we obtained above, Theorem 4 and classical Schauder estimate for linear elliptic PDE (see [10]) guarantee the compactness of the sequence of solutions $\{u_k\}$ on compact subsets of M. Then, by the diagonal method, the sequence $\{u_k\}$ contains a subsequence converging uniformally in C^2 norm on compact subsets of M to a global solution $u \in C^\infty(P)$ of $Q_H[u] = 0$, where $|H| < \sqrt{\alpha}$. It remains to show that u extends continuously to $\partial_\infty P$ and that $u|_{\partial_\infty P} = \phi$.

Let $(x_k)_k$ be a sequence of points of P converging to $x \in \partial_\infty P$. Since $\overline{P}$ is compact, there exists a subsequence $\varphi(u(x_{k_j}), x_{k_j})$ of $\varphi(u(x_k), x_k)$ which converges to $z \in \overline{P}$. Since x_k diverges and $\varphi(u(x_{k_j}), x_{k_j}) \in Gr(u)$, we have that $z \in \partial_\infty Gr(u)$. We claim that $z \in Gr(\phi)$. To prove the claim, we will show that if $z \in \partial_\infty M \backslash Gr(\phi)$ then $z \notin \partial_\infty Gr(u)$. Let $z \in \partial_\infty M \backslash Gr(\phi)$. Since $Gr(\phi)$ is compact and $z \notin Gr(\phi)$, using the SC condition, we can find an hypersurface $E \subset M$ such that $\partial_\infty E$ separates z and $Gr(\phi)$. Moreover, using Lemma 5, the mean curvature of E with respect to the unit normal vector field pointing to the connected component U of $M \backslash E$ whose asymptotic boundary contains $Gr(\phi)$, is larger or equal to h for $h < \sqrt{\alpha}$. Since $u_k|_{\partial B_k} \underset{k\to\infty}{\longrightarrow} \phi$, there exists k_0 such that, for all $k \geq k_0$, $\partial Gr(u_k) \subset U$ and $\partial Gr(u_k) \cap E = \varnothing$. By the tangency principle and using that $|H| < \sqrt{\alpha}$, we deduce that, for all $k \geq k_0$, $Gr(u_k) \subset U$. It follows that $z \notin \partial_\infty Gr(u)$. This proves the claim i.e. $z \in Gr(\phi)$. In particular, it follows that u is bounded.

Now, since $\partial_\infty Gr(u) \subset Gr(\phi)$, there exists $x_0 \in \partial_\infty P$ such that $z = \varphi(u(x_0), x_0)$. Using that u is bounded, we deduce there exists a subsequence $\{u(x_{k_{j_i}})\}$ which converges to some $t_0 \in \mathbb{R}$. It follows from the fact that the extension of φ_{t_0} to $\overline{P}$ is continuous that

$$z = \lim_{i\to\infty} \varphi(u(x_{k_{j_i}}), x_{k_{j_i}}) = \varphi(t_0, x).$$

Since $\varphi : \mathbb{R} \times \partial_\infty P \to \partial_\infty M$ is injective, we deduce that $t_0 = \phi(x_0)$ and $x_0 = x$. Since this last fact holds true for every converging subsequences, we have proved that $u(x_k) \underset{k\to\infty}{\longrightarrow} \phi(x)$. This concludes the proof of Theorem 1. ☐

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