WEAK HOPF QUASIGROUPS[*]

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Abstract. In this paper we introduce the notion of weak Hopf quasigroup as a generalization of weak Hopf algebras and Hopf quasigroups. We obtain its main properties and we prove the fundamental theorem of Hopf modules for these algebraic structures.

Key words. Weak Hopf algebra, Hopf quasigroup, bigroupoid, Hopf module.

AMS subject classifications. 18D10, 16T05, 17A30, 20N05.

1. introduction. The notion of Hopf algebra and its generalizations appeared as useful tools in relation with many branch of mathematics such as algebraic geometry, number theory, Lie theory, Galois theory, quantum group theory and so on. A common principle for obtaining generalizations of the original notion of Hopf algebra is to weaken some of its defining axioms. For example, if one does not force the coalgebra structure to respect the unit of the algebra structure, one is lead to weak Hopf algebras. In a different way, the weakening of the associativity leads to Hopf quasigroups and quasi-Hopf algebras.

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [12]) were introduced by Böhm, Nill and Szlachányi [4] as a new generalization of Hopf algebras and groupoid algebras. A weak Hopf algebra H in a braided monoidal category [2] is an object that has both, monoid and comonoid structure, with some relations between them. The main difference with other Hopf algebraic constructions is that weak Hopf algebras are coassociative but the coproduct is not required to preserve the unit, equivalently, the counit is not a monoid morphism. Some motivations to study weak Hopf algebras come from the following facts: firstly, as group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras and, secondly, these algebraic structures have a remarkable connection with the theory of algebra extensions, important applications in the study of dynamical twists of Hopf algebras and a deep link with quantum field theories and operator algebras [12], as well as being useful tools in the study of fusion categories in characteristic zero [6]. Moreover, Hayashi's face algebras (see [7]) are particular instances of weak Hopf algebras, whose counital subalgebras are commutative, and Yamanouchi's generalized Kac algebras [17] are exactly C^*-weak Hopf algebras with involutive antipode.

On the other hand, Hopf quasigroups are a generalization of Hopf algebras in the context of non associative algebra. Like in the quasi-Hopf setting, Hopf quasigroups are not associative but the lack of this property is compensated by some axioms involving the antipode. The concept of Hopf quasigroup is a particular instance of the notion of unital coassociative H-bialgebra introduced in [15]. It includes the

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example of an enveloping algebra of a Malcev algebra (see [11] and [14]) when the base ring has characteristic not equal to 2 nor 3, and in this sense Hopf quasigroups extend the notion of Hopf algebra in a parallel way that Malcev algebras extend the one of Lie algebra. On the other hand, it also contains as an example the notion of quasigroup algebra of an I.P. loop. Therefore, Hopf quasigroups unify I.P. loops and Malcev algebras in the same way that Hopf algebras unify groups and Lie algebras. Actually, Hopf quasigroups in a category of vector spaces were introduced by Klim and Majid in [11] in order to understand the structure and relevant properties of the algebraic 7-sphere.

The main purposes of this paper are to introduce the notion of weak Hopf quasigroup as a new Hopf algebra generalization that encompass weak Hopf algebras and Hopf quasigroups and to prove that the more relevant properties of these algebraic structures can be obtained under a unified approach, that is, we show that the fundamental assertions proved in [4] and [2] about weak Hopf algebras and in [11] for Hopf quasigroups can be obtained in this new setting. Also, we construct a family of examples working with bigroupoids, i.e. bicategories where every 1-cell is an equivalence and every 2-cell is an isomorphism. The organization of the paper is the following. After this introduction, in Section 2 we introduce weak Hopf quasigroups and we explain in detail how the first non-trivial examples of this algebraic structures can be obtained considering bigroupoids. In Section 3 we discuss the consequences of the definition of weak Hopf quasigroups obtaining the first relevant properties of this objects. Finally, Section 4 is devoted to prove the fundamental theorem of Hopf modules associated to a weak Hopf quasigroups.

2. Definitions and examples. Throughout this paper $\mathcal{C}$ denotes a strict monoidal category with tensor product $\otimes$ and unit object K. For each object M in $\mathcal{C}$, we denote the identity morphism by $id_M : M \to M$ and, for simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

From now on we assume that $\mathcal{C}$ admits split idempotents, i.e. for every morphism $\nabla_Y : Y \to Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$ there exist an object Z and morphisms $i_Y : Z \to Y$ and $p_Y : Y \to Z$ such that $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$. There is no loss of generality in assuming that $\mathcal{C}$ admits split idempotents, taking into account that, for a given category $\mathcal{C}$, there exists an universal embedding $\mathcal{C} \to \hat{\mathcal{C}}$ such that $\hat{\mathcal{C}}$ admits split idempotents, as was proved in [9].

Also we assume that $\mathcal{C}$ is braided, that is: for all M and N objects in $\mathcal{C}$, there is a natural isomorphism $c_{M,N} : M \otimes N \to N \otimes M$, called the braiding, satisfying the Hexagon Axiom (see [8] for generalities). If the braiding satisfies $c_{N,M} \circ c_{M,N} = id_{M \otimes N}$, the category $\mathcal{C}$ will be called symmetric.

DEFINITION 2.1. By a unital magma in $\mathcal{C}$ we understand a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. If μ_A is associative, that is, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, the unital magma will be called a monoid in $\mathcal{C}$. Given two unital magmas (monoids) $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is a morphism of unital magmas (monoids) if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

By duality, a counital comagma in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. If δ_D is coassociative, that is, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, the counital comagma will be called a comonoid. If

$D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are counital comagmas (comonoids), $f : D \to E$ is a morphism of counital comagmas (comonoids) if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

If A, B are unital magmas (monoids) in $\mathcal{C}$, the object $A \otimes B$ is a unital magma (monoid) in $\mathcal{C}$ where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. In a dual way, if D, E are counital comagmas (comonoids) in $\mathcal{C}$, $D \otimes E$ is a counital comagma (comonoid) in $\mathcal{C}$ where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Finally, if D is a comagma and A a magma, for two morphisms $f, g : D \to A$ with $f * g$ we will denote its convolution product in $\mathcal{C}$, that is

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D.$$

DEFINITION 2.2. A weak Hopf quasigroup H in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:
- (a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}$.
- (a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$
    $= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
    $= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H}^{-1} \circ \delta_H) \otimes H)$.
- (a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
    $= (H \otimes (\mu_H \circ c_{H,H}^{-1}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.
- (a4) There exists $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) such that, if we denote the morphisms $id_H * \lambda_H$ by Π_H^L (target morphism) and $\lambda_H * id_H$ by Π_H^R (source morphism):
  - (a4-1) $\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
  - (a4-2) $\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
  - (a4-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
  - (a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
  - (a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
  - (a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
  - (a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

Note that, if in the previous definition the triple (H, η_H, μ_H) is a monoid, we obtain the notion of weak Hopf algebra in a braided category introduced in [1] (see also [2]). Under this assumption, if $\mathcal{C}$ is symmetric, we have the monoidal version of the original definition of weak Hopf algebra introduced by Böhm, Nill and Szlachányi in [4]. On the other hand, if ε_H and δ_H are morphisms of unital magmas, $\Pi_H^L = \Pi_H^R = \eta_H \otimes \varepsilon_H$ and, as a consequence, we have the notion of Hopf quasigroup defined by Klim and Majid in [11] (note that in this case there is not difference between the definitions for the symmetric and the braided settings).

EXAMPLE 2.3. In this example we will show that it is possible to obtain non-trivial examples of weak Hopf quasigroups working with bicategories in the sense of Bénabou [3]. A bicategory $\mathcal{B}$ consists of :
- (b1) A set $\mathcal{B}_0$, whose elements x are called 0-cells.
- (b2) For each x, $y \in \mathcal{B}_0$, a category $\mathcal{B}(x, y)$ whose objects $f : x \to y$ are called 1-cells and whose morphisms $\alpha : f \Rightarrow g$ are called 2-cells. The composition of 2-cells is called the vertical composition of 2-cells and if f is a 1-cell in $\mathcal{B}(x, y)$, x is called the source of f, represented by $s(f)$, and y is called the target of f, denoted by $t(f)$.

(b3) For each $x \in \mathcal{B}_0$, an object $1_x \in \mathcal{B}(x,x)$, called the identity of x; and for each $x, y, z \in \mathcal{B}_0$, a functor

$$\mathcal{B}(y,z) \times \mathcal{B}(x,y) \to \mathcal{B}(x,z)$$

which in objects is called the 1-cell composition $(g, f) \mapsto g \circ f$, and on arrows is called horizontal composition of 2-cells:

$$f, f' \in \mathcal{B}(x,y), \ \ g, g' \in \mathcal{B}(y,z), \ \alpha : f \Rightarrow f', \ \beta : g \Rightarrow g'$$

$$(\beta, \alpha) \mapsto \beta \bullet \alpha : g \circ f \Rightarrow g' \circ f'$$

(b4) For each $f \in \mathcal{B}(x,y)$, $g \in \mathcal{B}(y,z)$, $h \in \mathcal{B}(z,w)$, an associative isomorphisms $\xi_{h,g,f} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$; and for each 1-cell f, unit isomorphisms $l_f : 1_{t(f)} \circ f \Rightarrow f$, $r_f : f \circ 1_{s(f)} \Rightarrow f$, satisfying the following coherence axioms:

　(b4-1) The morphism $\xi_{h,g,f}$ is natural in h, f and g and l_f, r_f are natural in f.

　(b4-2) Pentagon axiom: $\xi_{k,h,g \circ f} \circ \xi_{k \circ h,g,f} = (id_k \bullet \xi_{h,g,f}) \circ \xi_{k,h \circ g,f} \circ (\xi_{k,h,g} \bullet id_f)$.

　(b4-3) Triangle axiom: $r_g \bullet id_f = (id_g \bullet l_f) \circ \xi_{g,1_{t(f)},f}$.

A bicategory is normal if the unit isomorphisms are identities. Every bicategory is biequivalent to a normal one. A 1-cell f is called an equivalence if there exists a 1-cell $g : t(f) \to s(f)$ and two isomorphisms $g \circ f \Rightarrow 1_{s(f)}$, $f \circ g \Rightarrow 1_{t(f)}$. In this case we will say that $g \in Inv(f)$ and, equivalently, $f \in Inv(g)$.

A bigroupoid is a bicategory where every 1-cell is an equivalence and every 2-cell is an isomorphism. We will say that a bigroupoid $\mathcal{B}$ is finite if $\mathcal{B}_0$ is finite and $\mathcal{B}(x,y)$ is small for all x, y. Note that if $\mathcal{B}$ is a bigroupoid where $\mathcal{B}(x,y)$ is small for all x, y and we pick a finite number of 0-cells, considering the full sub-bicategory generated by these 0-cells, we have an example of finite bigroupoid.

Let $\mathcal{B}$ be a finite normal bigroupoid and denote by $\mathcal{B}_1$ the set of 1-cells. Let $\mathbb{F}$ be a field and $\mathbb{F}\mathcal{B}$ the direct product

$$\mathbb{F}\mathcal{B} = \bigoplus_{f \in \mathcal{B}_1} \mathbb{F}f.$$

The vector space $\mathbb{F}\mathcal{B}$ is a unital nonassociative algebra where the product of two 1-cells is equal to their 1-cell composition if the latter is defined and 0 otherwise, i.e. $g.f = g \circ f$ if $s(g) = t(f)$ and $g.f = 0$ if $s(g) \neq t(f)$. The unit element is

$$1_{\mathbb{F}\mathcal{B}} = \sum_{x \in \mathcal{B}_0} 1_x.$$

Let $H = \mathbb{F}\mathcal{B}/I(\mathcal{B})$ be the quotient algebra where $I(\mathcal{B})$ is the ideal of $\mathbb{F}\mathcal{B}$ generated by

$$h - g \circ (f \circ h), \ p - (p \circ f) \circ g,$$

with $f \in \mathcal{B}_1$, $g \in Inv(f)$, and $h, p \in \mathcal{B}_1$ such that $t(h) = s(f)$, $t(f) = s(p)$. In what follows, for any 1-cell f we denote its class in H by $[f]$.

If there exists a 1-cell f in $\mathcal{B}(x,y)$ such that $[f] = 0$ and we pick $g \in Inv(f)$ we have that $[1_x] = [g.f] = [g].[f] = 0$ and $[1_y] = [f.g] = [f].[g] = 0$. Conversely, if

$[1_x] = 0$ or $[1_y] = 0$ we have that $[f] = 0$ because $f.1_x = 1_y.f = f$. Therefore, the following assertion holds: There exists a 1-cell f in $\mathcal{B}(x, y)$ such that $[f] = 0$ if and only if $[h] = 0$ for all 1-cell h in $\mathcal{B}(x, y)$. Moreover, let x, y, z, w be 0-cells. If there exists a 1-cell $f \in \mathcal{B}(x, y)$ satisfying that $[f] = 0$ we have that $[1_y] = 0$ and then $[h] = 0$ for all 1-cell h in $\mathcal{B}(y, z)$. As a consequence, $[1_z] = 0$ and this clearly implies that $[p] = 0$ for all 1-cell p in $\mathcal{B}(z, w)$. Thus, if there exists a 1-cell f such that $[f] = 0$ we obtain that $[h] = 0$ for all $h \in \mathcal{B}_1$. According to this reasoning, there exists a 1-cell f such that $[f] = 0$ if and only if $I(\mathcal{B}) = \mathbb{F}\mathcal{B}$. Equivalently, H is not null, if and only if $[f] \neq 0$ for all $f \in \mathcal{B}_1$.

Then, in the remainder of this section, we assume that $I(\mathcal{B})$ is a proper ideal. Under this condition if $f \in \mathcal{B}_1$ and $g, h \in Inv(f)$ we have

$$[g] = [g.(f.g)] = [g.1_y] = [1_x.g] = [(h.f).g] = [h].$$

Moreover, for all $f, f' \in \mathcal{B}_1$ such that $[f] = [f']$, the following holds: if $s(f) \neq s(f')$ we have

$$[f] = [f.1_{s(f)}] = [f].[1_{s(f)}] = [f'].[1_{s(f)}] = [f'.1_{s(f)}] = 0.$$

In a similar way, if $t(f) \neq t(f')$ we obtain that $[f] = 0$. Thus, $[f] = [f']$, clearly forces that f and f' are 1-cells in $\mathcal{B}(s(f), t(f))$. Moreover, if f, f' are 1-cells in $\mathcal{B}(x, y)$ such that $[f] = [f']$ and $g \in Inv(f)$, $g' \in Inv(f')$ we have

$$[g'] = [1_x.g'] = [(g.f).g'] = ([g].[f]).[g'] = ([g].[f']).[g'] = [(g.f').g'] = [g].$$

Then, for a 1-cell f we denote by $[f]^{-1}$ the class of any $g \in Inv(f)$. Note that, in the previous equalities, we proved that $[f]^{-1}$ is independent of the choices of $g \in Inv(f)$ and f' such that $[f] = [f']$.

Therefore, the vector space H with the product $\mu_H([g] \otimes [f]) = [g.f]$ and the unit

$$\eta_H(1_{\mathbb{F}}) = [1_{\mathbb{F}\mathcal{B}}] = \sum_{x \in \mathcal{B}_0} [1_x]$$

is a unital magma. Also, it is easy to show that H is a comonoid with coproduct $\delta_H([f]) = [f] \otimes [f]$ and counit $\varepsilon_H([f]) = 1_{\mathbb{F}}$. Moreover, the morphism $\lambda_H : H \to H$, $\lambda_H([f]) = [f]^{-1}$ is well-defined and $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ is a weak Hopf quasigroup. Indeed: First note that, for all 1-cells f, g we have

$$(\delta_H \circ \mu_H)([g] \otimes [f]) = [g.f] \otimes [g.f]$$

if $s(g) = t(f)$ and 0 otherwise. On the other hand,

$$((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H})([g] \otimes [f]) = (\mu_H([g] \otimes [f]) \otimes \mu_H([g] \otimes [f])) = [g.f] \otimes [g.f]$$

if $s(g) = t(f)$ and 0 otherwise because $c_{H,H}([g] \otimes [f]) = [f] \otimes [g]$. Therefore, (a1) of Definition 2.2 holds.

If f, g, h are 1-cells we have the following equalities:

$$(\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H))([h] \otimes [g] \otimes [f]) = 1_{\mathbb{F}} = (\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H))([h] \otimes [g] \otimes [f])$$

when $s(h) = t(g)$, $s(g) = t(f)$ and

$$(\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H))([h] \otimes [g] \otimes [f]) = 0 = (\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H))([h] \otimes [g] \otimes [f])$$

otherwise. Also,

$$(((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H))([h] \otimes [g] \otimes [f])$$
$$= ((\varepsilon_H \circ \mu_H)([h] \otimes [g])) \otimes ((\varepsilon_H \circ \mu_H))([g] \otimes [f])) = 1_{\mathbb{F}}$$

if $s(h) = t(g)$, $s(g) = t(f)$ and

$$(((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H))([h] \otimes [g] \otimes [f]) = 0$$

otherwise. Then (a2) of Definition 2.2 holds because in this case $c_{H,H} = c_{H,H}^{-1}$ and $\delta_H \circ c_{H,H} = \delta_H$ (i.e. H is cocommutative).

To prove (a3) first note that

$$((\delta_H \otimes H) \circ \delta_H \circ \eta_H)(1_{\mathbb{F}}) = \sum_{x \in \mathcal{B}_0} [1_x] \otimes [1_x] \otimes [1_x].$$

Then (a3) holds because:

$$((H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)))(1_{\mathbb{F}} \otimes 1_{\mathbb{F}})$$
$$= (H \otimes \mu_H \otimes H)(\sum_{x \in \mathcal{B}_0} [1_x] \otimes [1_x] \otimes \sum_{y \in \mathcal{B}_0} [1_y] \otimes [1_y])$$
$$= \sum_{x,y \in \mathcal{B}_0} [1_x] \otimes [1_x.1_y] \otimes [1_y] = \sum_{x \in \mathcal{B}_0} [1_x] \otimes [1_x] \otimes [1_x].$$

To prove the antipode identities first note that

$$\Pi_H^L([f]) = [1_{t(f)}], \quad \Pi_H^R([f]) = [1_{t(s)}] \tag{1}$$

for all 1-cell f.

Then, (a4-1) and (a4-2) hold because, for all 1-cell f,

$$(((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H))([f])$$
$$= (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}))(\sum_{x \in \mathcal{B}_0} [1_x] \otimes [1_x] \otimes [f])$$
$$= \sum_{x \in \mathcal{B}_0} \varepsilon_H([1_x.f]) \otimes [1_x] = [1_{t(f)}]$$

and, by a similar calculus,

$$((H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)))([f]) = [1_{s(f)}].$$

Also, if $f \in \mathcal{B}_1$, by (1),

$$(\lambda_H * \Pi_H^L)([f]) = [f]^{-1}.[1_{t(f)}] = [f]^{-1} = \lambda_H([f]),$$

$$(\Pi_H^R * \lambda_H)([f]) = [1_{s(f)}].[f]^{-1} = [f]^{-1} = \lambda_H([f])$$

and then (a4-3) holds.

The proof for (a4-4) is the following: It follows easily that for two 1-cells f, h we have that

$$(\mu_H \circ (\Pi_H^R \otimes H))([h] \otimes [f]) = [f]$$

if $s(h) = t(f)$ and 0 otherwise. On the other hand,

$$(\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H))([h] \otimes [f]) = \mu_H([h]^{-1} \otimes [h.f])$$

if $s(h) = t(f)$ and 0 otherwise. Therefore, if $m \in Inv(h)$ and $s(h) = t(f)$ the equality

$$\mu_H([h]^{-1} \otimes [h.f]) = [m.(h.f)] = [f]$$

holds and thus (a4-4) holds.

If f, h are 1-cells we have

$$(\mu_H \circ (\Pi_H^L \otimes H))([h] \otimes [f]) = [f]$$

if $t(h) = t(f)$ and 0 otherwise. Moreover, let $m \in Inv(h)$, then

$$(\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H))([h] \otimes [f])$$
$$= \mu_H([h] \otimes [m.f]) = [h.(m.f)] = [f]$$

if $t(h) = t(f)$ and 0 otherwise. Therefore, (a4-5) holds.

The proofs for (a4-6) and (a4-7) are similar and the details are left to the reader.

Note that, in this example, if $\mathcal{B}_0 = \{x\}$ we obtain that H is a Hopf quasigroup. Moreover, if $|\mathcal{B}_0| > 1$ and the product defined in H is associative we have an example of weak Hopf algebra.

3. Basic properties for weak Hopf quasigroups. In this section we will show the main properties of weak Hopf quasigroups. First, note that by the naturality of the braiding, for the morphisms target and source the following equalities hold:

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((c_{H,H}^{-1} \circ \delta_H \circ \eta_H) \otimes H), \tag{2}$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1})) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (c_{H,H}^{-1} \circ \delta_H \circ \eta_H)). \tag{3}$$

PROPOSITION 3.1. *Let H be a weak Hopf quasigroup. The following equalities hold:*

$$\Pi_H^L * id_H = id_H * \Pi_H^R = id_H, \tag{4}$$

$$\Pi_H^L \circ \eta_H = \eta_H = \Pi_H^R \circ \eta_H, \tag{5}$$

$$\varepsilon_H \circ \Pi_H^L = \varepsilon_H = \varepsilon_H \circ \Pi_H^R. \tag{6}$$

Proof. By the definition of Π_H^L and (a1) of Definition 2.2 we have

$$\Pi_H^L * id_H = (\varepsilon_H \otimes H) \circ \mu_{H \otimes H} \circ ((\delta_H \circ \eta_H) \otimes \delta_H) = (\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H \circ (\eta_H \otimes H) = id_H.$$

We can now proceed analogously to the proof of $id_H * \Pi_H^R = id_H$. Finally (5) and (6) follow easily from the definitions of Π_H^L and Π_H^R. $\square$

PROPOSITION 3.2. *The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.*

Proof. Let $\lambda_H, s_H : H \to H$ two morphisms satisfying (a4) of Definition 2.2. Then,

$$s_H = (s_H * H) * s_H = (\lambda_H * H) * s_H = \mu_H \circ (\mu_H \otimes s_H) \circ (\lambda_H \otimes \delta_H) \circ \delta_H = \lambda_H * \Pi_H^L = \lambda_H,$$

where the first and the last equalities follow by (a4-3) of Definition 2.2, the second one by (a4-2) of Definition 2.2, the third one by the coassociativity of δ_H and the fourth one by (a4-6) of Definition 2.2.

On the other hand, by (a4-3), (a3) of Definition 2.2, the naturality of the braiding and (5), we have

$$\begin{aligned}
\lambda_H \circ \eta_H &= (\Pi_H^R * \lambda_H) \circ \eta_H \\
&= \mu_H \circ (H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes \lambda_H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\
&= \mu_H \circ (((H \otimes \varepsilon_H) \circ \delta_H) \otimes \lambda_H) \circ \delta_H \circ \eta_H \\
&= \Pi_H^L \circ \eta_H \\
&= \eta_H.
\end{aligned}$$

The proof for the equalities involving the counit follows a similar pattern but using (a2) of Definition 2.2 and (6) instead of (a3) and (5) respectively. □

DEFINITION 3.3. Let H be a weak Hopf quasigroup. We define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

PROPOSITION 3.4. *Let H be a weak Hopf quasigroup. The morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotents.*

Proof. First, by (2) and (a3) of Definition 2.2 we have that

$$\begin{aligned}
&\Pi_H^L \circ \Pi_H^L \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes \varepsilon_H \otimes H) \circ (H \otimes ((\delta_H \otimes H) \circ \delta_H \circ \eta_H)) \\
&= \Pi_H^L.
\end{aligned}$$

With the same reasoning but using (3) instead of (2) we prove that Π_H^R is an idempotent morphism. Finally, by (a3) of Definition 2.2, $\overline{\Pi}_H^L \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L$ and $\overline{\Pi}_H^R \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R$. □

PROPOSITION 3.5. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \qquad (7)$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \tag{8}$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{9}$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H). \tag{10}$$

Proof. We first prove (7).

$$\mu_H \circ (H \otimes \Pi_H^L)$$
$$= (\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H \circ (H \otimes \Pi_H^L)$$
$$= (\varepsilon_H \otimes H) \circ \mu_{H \otimes H} \circ (\delta_H \otimes \delta_H) \circ (H \otimes (((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))))$$
$$= (((((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H}^{-1} \circ \delta_H) \otimes H)) \otimes \mu_H)$$
$$\quad \circ (H \otimes H \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H)$$
$$\quad \circ (\delta_H \otimes c_{H,H}^{-1} \otimes H) \circ (H \otimes H \otimes (\delta_H \circ \eta_H))$$
$$= ((\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H)) \otimes \mu_H) \circ (H \otimes H \otimes c_{H,H} \otimes H)$$
$$\quad \circ (H \otimes c_{H,H} \otimes c_{H,H}) \circ (\delta_H \otimes (\delta_H \circ \eta_H) \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\mu_{H \otimes H} \circ (\delta_H \otimes \delta_H)) \otimes H) \circ (H \otimes \eta_H \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

In the last identities, the first one follows by the properties of the counit, the second one follows by (2) and (a1) of Definition 2.2. The third and the fifth ones rely on the naturality of c. The fourth equality is a consequence of (a2) of Definition 2.2 and finally the last one follows by (a1) of Definition 2.2 and the properties of the unit.

The proof for (8) is similar but in the second step we must use (3) instead of (2). To finish the proof we show that (9) holds. The proof for (10) is similar.

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L)$$
$$= (\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H \circ (H \otimes \overline{\Pi}_H^L)$$
$$= (\mu_H \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_{H \otimes H} \otimes H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H) \otimes H)$$
$$= (\mu_H \otimes (((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H))) \circ (\delta_{H \otimes H} \otimes H) \circ (H \otimes \eta_H \otimes H)$$
$$= (\mu_H \otimes (\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H))) \circ (\delta_{H \otimes H} \otimes H) \circ (H \otimes \eta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (((\delta_H \circ \mu_H) \circ (H \otimes \eta_H)) \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H).$$

The first equality follows by the counit properties, the second and the fifth ones by (a1) of Definition 2.2, the third one follows from the coassociativity of δ_H, the fourth one by (a2) of Definition 2.2, and the sixth one by the properties of the unit. □

REMARK 3.6. Note that if we compose with ε_H in the equalities (7), (8), (9) and (10) we obtain

$$\varepsilon_H \circ \mu_H \circ (H \otimes \Pi_H^L) = \varepsilon_H \circ \mu_H, \tag{11}$$

$$\varepsilon_H \circ \mu_H \circ (\Pi_H^R \otimes H) = \varepsilon_H \circ \mu_H, \tag{12}$$

$$\varepsilon_H \circ \mu_H \circ (H \otimes \overline{\Pi}_H^L) = \varepsilon_H \circ \mu_H, \tag{13}$$

$$\varepsilon_H \circ \mu_H \circ (\overline{\Pi}_H^R \otimes H) = \varepsilon_H \circ \mu_H. \tag{14}$$

PROPOSITION 3.7. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{15}$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{16}$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{17}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \tag{18}$$

Proof. The proof for (15) is the following:

$$(\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= (((H \otimes \varepsilon_H) \circ \delta_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= (\mu_H \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H}) \circ (\delta_H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes \delta_H)$$
$$= (H \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ (((\mu_H \otimes H)$$
$$\quad \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)) \otimes H) \circ \delta_H$$
$$= ((\mu_H \circ c_{H,H}^{-1}) \otimes (((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}))) \circ (H \otimes ((H \otimes \delta_H)$$
$$\quad \circ \delta_H \circ \eta_H) \otimes H) \circ \delta_H$$
$$= ((\mu_H \circ c_{H,H}^{-1}) \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes H \otimes (\mu_H \circ c_{H,H}^{-1}) \otimes c_{H,H})$$
$$\quad \circ (H \otimes (\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H) \circ \delta_H$$
$$= (\mu_H \otimes (((\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H)) \otimes H) \circ (H \otimes H \otimes c_{H,H})$$
$$\quad \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes H \otimes H))) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H)$$
$$= (\mu_H \otimes (((\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)) \otimes H) \circ (H \otimes H \otimes c_{H,H})$$
$$\quad \circ (H \otimes c_{H,H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes H \otimes H))) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H)$$
$$= (H \otimes \Pi_H^L) \circ (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H)$$
$$= (H \otimes \Pi_H^L) \circ \delta_H.$$

In the last equalities the first one follows by the counit properties, the second one by (a1) of Definition 2.2 and the naturality of c. In the third one we used the naturality of c and the coassociativity of δ_H. The fourth and the sixth ones are consequence of the equality

$$(\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= ((\mu_H \circ c_{H,H}^{-1}) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) \tag{19}$$

and the fifth one follows by (a3) of Definition 2.2. The seventh one relies on (a2) of Definition 2.2, the eight one follows by the naturality of c and the last one by the unit properties.

The proof for (16) is similar but using

$$(H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$$
$$= (H \otimes (\mu_H \circ c_{H,H}^{-1})) \circ ((\delta_H \circ \eta_H) \otimes H) \tag{20}$$

instead of (19).

Moreover, (17) holds because

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H$$
$$= (\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \mu_H \circ (\eta_H \otimes H)$$
$$= (\overline{\Pi}_H^L \otimes H) \circ (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\eta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)) \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_{H \otimes H} \circ (\eta_H \otimes H)))$$
$$= (H \otimes (((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)) \otimes \mu_H)$$
$$\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_{H \otimes H} \circ (\eta_H \otimes H)))$$
$$= (H \otimes (\varepsilon_H \circ \mu_H) \otimes ((\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H)$$
$$= (H \otimes \varepsilon_H \otimes \mu_H) \circ (((\delta_H \otimes H) \circ \delta_H \circ \eta_H) \otimes H)$$
$$= (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H).$$

The first equality follows by the unit properties, the second one by (a1) of Definition 2.2 and the third one by the definition of $\overline{\Pi}_H^L$. The fourth equality relies on (a2) of Definition 2.2. The fifth one is a consequence of the coassociativity of δ_H and the sixth one follows by (a3) of Definition 2.2. Finally the last one holds by the properties of the counit.

The proof of (18) is similar to the developed for (17) and we leave it to the reader. □

REMARK 3.8. Note that if we compose with η_H in the equalities (15), (16), (17) and (18) we obtain

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H, \tag{21}$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H, \tag{22}$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H, \tag{23}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H. \tag{24}$$

As a consequence of Propositions 3.5 and 3.7 we can get other useful identities.

PROPOSITION 3.9. *Let H be a wek Hopf quasigroup. The following identities hold:*

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H, \tag{25}$$

$$\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H, \tag{26}$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \tag{27}$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R. \tag{28}$$

Proof. The equality (25) holds because:

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (((H \otimes \Pi_H^L) \circ \delta_H) \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H)) \otimes H) \circ (H \otimes H \otimes \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H)$$
$$\quad \circ ((\delta_H \circ \eta_H) \otimes H \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)) \otimes H) \circ (H \otimes H \otimes \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H)$$
$$\quad \circ ((\delta_H \circ \eta_H) \otimes H \otimes H)$$
$$= \Pi_H^L \circ \mu_H.$$

In the previous equalities, the first one follows by (7), the second and the fifth ones by the naturality of c, the third one by (15) and the fourth one by (a2) of Definition 2.2.

The proof for (25) is similar. To finish we will show that (27) holds (using the same reasoning we obtain (28)). Indeed:

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L$$
$$= (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes \Pi_H^L)$$
$$= ((\mu_H \circ (H \otimes \Pi_H^L)) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= ((((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$$
$$= \delta_H \circ \Pi_H^L.$$

The first and the third equalities follow by (15) and the second one by the naturality of c. Finally, the last one relies on the naturality of c and the coassociativity of δ_H. $\square$

REMARK 3.10. By the equalities contained in Remark 3.8 and using similar arguments to the ones utilized in the previous Proposition we have that

$$\overline{\Pi}_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \overline{\Pi}_H^L \circ \mu_H, \tag{29}$$

$$\overline{\Pi}_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \overline{\Pi}_H^R \circ \mu_H, \tag{30}$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \overline{\Pi}_H^L = \delta_H \circ \overline{\Pi}_H^L, \tag{31}$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \overline{\Pi}_H^R = \delta_H \circ \overline{\Pi}_H^R. \tag{32}$$

PROPOSITION 3.11. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \tag{33}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \tag{34}$$

$$\Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{35}$$

$$\overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{36}$$

Proof. We only check (33) and (34). The proof for the other equalities can be verified in a similar way. Taking into account the equalities (2) and (a2) of Definition 2.2 we have

$$
\begin{aligned}
&\Pi_H^L \circ \overline{\Pi}_H^L \\
&= (H \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ ((c_{H,H}^{-1} \circ \delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H) \otimes H) \\
&= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((c_{H,H}^{-1} \circ \delta_H \circ \eta_H) \otimes (\mu_H \circ (\eta_H \otimes H))) \\
&= \Pi_H^L,
\end{aligned}
$$

and composing with $\varepsilon_H \otimes H$ in (32) we obtain $\Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R$. Also, composing with $\eta_H \otimes H$ in (29) we have the equality $\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L$.

Finally, by the usual arguments

$$
\begin{aligned}
&\overline{\Pi}_H^R \circ \Pi_H^L \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes ((\varepsilon_H \otimes H) \circ \delta_H)) \circ (H \otimes (\delta_H \circ \eta_H)) \\
&= \Pi_H^L.
\end{aligned}
$$

$\square$

PROPOSITION 3.12. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \tag{37}$$

$$\Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L, \tag{38}$$

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \tag{39}$$

$$\Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R. \tag{40}$$

Proof. As in the previous Proposition, it is sufficient to check (37) and (39). The proof for the other equalities can be verified in a similar way.

The equalities of (37) hold because by (a4-3) of Definition 2.2 and (25) we have

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ (\lambda_H * \Pi_H^L) = \Pi_H^L \circ \Pi_H^R$$

and by (28)

$$\lambda_H \circ \Pi_H^R = (\Pi_H^R * \lambda_H) \circ \Pi_H^R = \Pi_H^L \circ \Pi_H^R.$$

On the other hand, by (a4-3) of Definition 2.2, (30) and (34) we obtain

$$\overline{\Pi}_H^R \circ \lambda_H = \overline{\Pi}_H^R \circ (\Pi_H^R * \lambda_H) = \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L.$$

Moreover, by (31) and (33)

$$\lambda_H \circ \overline{\Pi}_H^L = (\Pi_H^R * \lambda_H) \circ \overline{\Pi}_H^L = \Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L,$$

and then (39) holds. ∎

PROPOSITION 3.13. *Let H be a weak Hopf quasigroup. Put $H_L = Im(\Pi_H^L)$ and let $p_L : H \to H_L$ and $i_L : H_L \to H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$. Then,*

$$H_L \xrightarrow{\quad i_L \quad} H \underset{(H \otimes \Pi_H^L) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram and

$$H \otimes H \underset{\mu_H \circ (H \otimes \Pi_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{\quad p_L \quad} H_L$$

is a coequalizer diagram. As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in $\mathcal{C}$ and $(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$ is a comonoid in $\mathcal{C}$.

Proof. Composing with i_L in the equality (27), we have that i_L equalizes δ_H and $(H \otimes \Pi_H^L) \circ \delta_H$. Now, let $t : B \to H$ be a morphism such that $(H \otimes \Pi_H^L) \circ \delta_H \circ t = \delta_H \circ t$. If $v = p_L \circ t$, since $\Pi_H^L \circ t = t$ we have $i_L \circ v = t$. Trivially the morphism v is unique and therefore, the diagram is an equalizer diagram. In a similar way we can prove that the second diagram is a coaqualizer diagram using (25) instead of (27). Finally, note that the morphisms η_{H_L} and μ_{H_L} are the factorizations, through the equalizer i_L, of the morphisms η_H and $\mu_H \circ (i_L \otimes i_L)$ and then it is an easy exercise to show that $(H_L, \eta_{H_L}, \mu_{H_L})$ is a unital magma in $\mathcal{C}$. The proof for the comonoid structure it is similar and we leave it to the reader. ∎

EXAMPLE 3.14. If H is the weak Hopf quasigroup defined in Example 2.3 note that $H_L = \langle [1_x], \ x \in \mathcal{B}_0 \rangle$. Then, in this case we have that H_L is a monoid because its induced product μ_{H_L} is associative because $[1_x].([1_y].[1_z])$ and $([1_x].[1_y]).[1_z]$ are equal to $[1_x]$ if $x = y = z$ and 0 otherwise.

Note that if we denote by $H_R = Im(\Pi_H^R)$ and $p_R : H \to H_R$ and $i_R : H_R \to H$ are the morphisms such that $\Pi_H^R = i_R \circ p_R$ and $p_R \circ i_R = id_{H_R}$, in this example $H_L = H_R$.

REMARK 3.15. By the second equality of (34) it is easy to show that

$$H_L \xrightarrow{\quad i_L \quad} H \underset{(H \otimes \overline{\Pi}_H^R) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram in $\mathcal{C}$. Analogously, by (33),

$$H \otimes H \underset{\mu_H \circ (H \otimes \overline{\Pi}_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{p_L} H_L$$

is a coequalizer diagram in $\mathcal{C}$.

Also, by a similar proof to the one used in Proposition 3.13, we obtain that $H_R = Im(\Pi_H^R)$ is a unital magma in $\mathcal{C}$ with structure $(H_R, \eta_{H_R} = p_R \circ \eta_H, \mu_{H_R} = p_R \circ \mu_H \circ (i_R \otimes i_R))$ and it is a comonoid in $\mathcal{C}$ where $\varepsilon_{H_R} = \varepsilon_H \circ i_R$ and $\delta_{H_R} = (p_R \otimes p_R) \circ \delta_H \circ i_R$. Moreover,

$$H_R \xrightarrow{i_R} H \underset{(\Pi_H^R \otimes H) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

$$H_R \xrightarrow{i_R} H \underset{(\overline{\Pi}_H^L \otimes H) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

are equalizer diagrams and

$$H \otimes H \underset{\mu_H \circ (\Pi_H^R \otimes H)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{p_R} H_R$$

$$H \otimes H \underset{\mu_H \circ (\overline{\Pi}_H^R \otimes H)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{p_R} H_R$$

are coequalizer diagrams.

PROPOSITION 3.16. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\Pi_H^L \otimes H) \circ \delta_H)) \\ = \mu_H = \mu_H \circ (\mu_H \otimes \Pi_H^R) \circ (H \otimes \delta_H), \tag{41}$$

$$\mu_H \circ (\Pi_H^L \otimes \mu_H) \circ (\delta_H \otimes H) \\ = \mu_H = \mu_H \circ (H \otimes (\mu_H \circ (\Pi_H^R \otimes H))) \circ (\delta_H \otimes H), \tag{42}$$

$$\mu_H \circ (\lambda_H \otimes (\mu_H \circ (\Pi_H^L \otimes H))) \circ (\delta_H \otimes H) \\ = \mu_H \circ (\lambda_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes (\mu_H \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H), \tag{43}$$

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\lambda_H \otimes \Pi_H^L) \circ \delta_H)) \\ = \mu_H \circ (H \otimes \lambda_H) = \mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\Pi_H^R \otimes \lambda_H) \circ \delta_H)). \tag{44}$$

Proof. Let us first prove (41). By (a1) of Definition 2.2 and (7) we have

$$\mu_H = (\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H = ((\varepsilon_H \circ \mu_H) \otimes \mu_H) \circ \delta_{H \otimes H}$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\Pi_H^L \otimes H) \circ \delta_H)).$$

On the other hand, by the coassociativity of δ_H, (a4-6) and (a4-7) of Definition 2.2,

$$\mu_H \circ (\mu_H \otimes \Pi_H^R) \circ (H \otimes \delta_H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (\mu_H \otimes \lambda_H \otimes H) \circ (H \otimes \otimes H \otimes \delta_H) \circ (H \otimes \delta_H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (\mu_H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes \delta_H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (H \otimes ((\Pi_H^L \otimes H) \circ \delta_H)).$$

The proof for (42) is similar but we must use (8), (a4-5) and (a4-4) of Definition 2.2 instead of (7), (a4-6) and (a4-7) respectively.

To prove (43) first note that by (a4-5), (a4-4) of Definition 2.2 and the coassociativity of δ_H we have:

$$\mu_H \circ (\lambda_H \otimes (\mu_H \circ (\Pi_H^L \otimes H))) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\lambda_H \otimes (\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) \circ (H \otimes (\mu_H \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\Pi_H^R \otimes (\mu_H \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H).$$

On the other hand, by (15), the naturality of c, the coassociativity of δ_H and (a1) of Definition 2.2 we also have the following identity:

$$(\mu_H \otimes (\mu_H \circ (H \otimes \Pi_H^L))) \circ \delta_{H \otimes H} = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H). \qquad (45)$$

Therefore,

$$\mu_H \circ (\Pi_H^R \otimes (\mu_H \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes ((\delta_H \circ \mu_H) \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H})$$
$$\quad \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (((\mu_H \otimes (\Pi_H^L \circ \mu_H)) \circ \delta_{H \otimes H})$$
$$\quad \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (((\mu_H \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L)))$$
$$\quad \circ \delta_{H \otimes H}) \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (((\mu_H \otimes (\mu_H \circ (H \otimes \Pi_H^L)))$$
$$\quad \circ \delta_{H \otimes H}) \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (((\mu_H \otimes H) \circ (H \otimes c_{H,H})$$
$$\quad \circ (\delta_H \otimes H)) \circ (\lambda_H \otimes H))) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (((H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes H)$$
$$\quad \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\Pi_H^R * \lambda_H) \otimes H)$$
$$= \mu_H \circ (\lambda_H \otimes H).$$

In the last calculus the first and the eight equalities are consequence of (8). In the second one we used (a1) of Definition 2.2. The third and the fifth ones follow by (11), the fourth one by (25) and the sixth one by (45). The seventh one relies on the naturality of c and the last one follows by (a4-3) of Definition 2.2.

By a similar reasoning but using that

$$((\mu_H \circ (\Pi_H^R \otimes H)) \otimes \mu_H) \circ \delta_{H \otimes H} = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \qquad (46)$$

instead of (45) we obtain that (44) holds. □

REMARK 3.17. Note that as a consequence of (41) or (42) we have

$$\Pi_H^L * id_H = id_H = id_H * \Pi_H^R. \qquad (47)$$

PROPOSITION 3.18. *Let H be a weak Hopf quasigroup. The following identities hold:*

$$\mu_H \circ (\Pi_H^L \otimes \Pi_H^R) = \mu_H \circ c_{H,H}^{-1} \circ (\Pi_H^L \otimes \Pi_H^R), \qquad (48)$$

$$(\Pi_H^L \otimes \Pi_H^R) \circ \delta_H = (\Pi_H^L \otimes \Pi_H^R) \circ c_{H,H}^{-1} \circ \delta_H, \qquad (49)$$

$$\mu_H \circ (\Pi_H^R \otimes \Pi_H^L) = \mu_H \circ c_{H,H} \circ (\Pi_H^R \otimes \Pi_H^L), \qquad (50)$$

$$(\Pi_H^R \otimes \Pi_H^L) \circ \delta_H = (\Pi_H^R \otimes \Pi_H^L) \circ c_{H,H} \circ \delta_H, \qquad (51)$$

Proof. The equalities (50) and (51) can be obtained from (48) and (49) composing with $c_{H,H}$. Then we only need to prove (48) and (49). Note that, by (2), (3) and (a3) of Definition 2.2 we have:

$$\begin{aligned}
&\mu_H \circ (\Pi_H^L \otimes \Pi_H^R) \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1})) \circ (H \otimes ((H \otimes \mu_H \otimes H) \\
&\quad \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))) \otimes H) \\
&= ((\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1}) \otimes H \otimes (\varepsilon_H \circ \mu_H \circ c_{H,H}^{-1})) \\
&\quad \circ (H \otimes ((H \otimes (\mu_H \circ c_{H,H}^{-1}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))) \otimes H) \\
&= \mu_H \circ c_{H,H}^{-1} \circ (\Pi_H^L \otimes \Pi_H^R).
\end{aligned}$$

Therefore, (48) holds. The proof for (49) is similar using (a2) of Definition 2.2 instead of (a3). □

THEOREM 3.19. *Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:*

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H), \qquad (52)$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H, \qquad (53)$$

Proof. We will prove (52). The proof for (53) is similar and we leave the details to the reader.

$$\lambda_H \circ \mu_H$$
$$= (\lambda_H * \Pi_H^L) \circ \mu_H$$
$$= \mu_H \circ ((\lambda_H \circ \mu_H) \otimes (\Pi_H^L \circ \mu_H)) \circ \delta_{H \otimes H}$$
$$= \mu_H \circ ((\lambda_H \circ \mu_H) \otimes (\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L))) \circ \delta_{H \otimes H}$$
$$= \mu_H \circ (\lambda_H \otimes \Pi_H^L) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes \lambda_H) \circ (\lambda_H \otimes \delta_H) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\lambda_H \otimes H)) \otimes \lambda_H) \circ (((\mu_H \otimes H) \circ (H \otimes c_{H,H})$$
$$\quad \circ (\delta_H \otimes H)) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\lambda_H \otimes H)) \otimes \lambda_H) \circ (((\mu_H \otimes (\mu_H \circ (H \otimes \Pi_H^L))))$$
$$\quad \circ \delta_{H \otimes H}) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\lambda_H \otimes H)) \otimes \lambda_H) \circ (((\mu_H \otimes (\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H))))$$
$$\quad \circ \delta_{H \otimes H}) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H)) \otimes H) \circ (\mu_H \otimes \lambda_H \otimes H)$$
$$\quad \circ (H \otimes \delta_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\Pi_H^R \otimes H)) \otimes H) \circ (\mu_H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes \lambda_H)$$
$$\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\Pi_H^R \otimes H)) \otimes H) \circ ((\mu_H \circ (\Pi_H^R \otimes H)) \otimes \lambda_H \otimes H)$$
$$\quad \circ (H \otimes \delta_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ ((\mu_H \circ (\Pi_H^R \otimes H)) \otimes H) \circ (((H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H)$$
$$\quad \circ (H \otimes \delta_H)) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (((\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H)) \otimes H) \circ (((H \otimes (\varepsilon_H \circ \mu_H))$$
$$\quad \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \mu_H \otimes H) \circ (\lambda_H \otimes ((H \otimes (\varepsilon_H \circ \mu_H))$$
$$\quad \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)) \otimes \lambda_H \otimes \lambda_H) \circ (c_{H,H} \otimes \delta_H \otimes H)$$
$$\quad \circ (H \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \mu_H \otimes H) \circ (\lambda_H \otimes ((\mu_H \circ (\Pi_H^R \otimes H))) \otimes \lambda_H$$
$$\quad \otimes \lambda_H) \circ (c_{H,H} \otimes \delta_H \otimes H) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes ((\mu_H \circ (\Pi_H^R \otimes \Pi_H^L))) \otimes \lambda_H) \circ (c_{H,H} \otimes H \otimes H)$$
$$\quad \circ (H \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes ((\mu_H \circ c_{H,H} \circ (\Pi_H^R \otimes \Pi_H^L))) \otimes \lambda_H)$$
$$\quad \circ (c_{H,H} \otimes H \otimes H) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \mu_H \otimes H) \circ (((\lambda_H \otimes \Pi_H^L) \circ \delta_H) \otimes ((\Pi_H^R \otimes \lambda_H) \circ \delta_H)) \circ c_{H,H}$$
$$= \mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes ((\Pi_H^R \otimes \lambda_H) \circ \delta_H)) \circ c_{H,H}$$
$$= \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H).$$

The first equality follows by (a4-3) of Definition 2.2, the second one by (a1) of Definition 2.2 and the third one by (25). The fourth and seventh ones relies on (45). The fifth, eighth and sixteenth ones are consequence of (a4-6) of Definition 2.2. In the sixth, ninth and eighteenth equalities we used the naturality of c and the equalities tenth and thirteenth follow by (a4-4) of Definition 2.2. By (26) we obtain the eleventh

equality and the twelfth and fifteenth ones are consequence of (8). The naturality of c and the coassociativity of δ_H imply the fourteenth equality and the seventeenth one follows by (48). Finally, the nineteenth equality relies on (43) and the last one on (44). ▫

A general notion of dyslexia was introduced by Pareigis in [13]. The following definition is the weak Hopf quasigroup version of dyslexia introduced by us in the weak Hopf algebra setting [2]. As application of (52) and (53) we obtain a generalization of the main result about (co)dyslexia contained in [2].

DEFINITION 3.20. Let H be a weak Hopf quasigroup. We will say that H is n-dyslectic if $\mu_H \circ c^n_{H,H} = \mu_H$ where $c^n_{H,H} = c_{H,H} \circ \overset{n)}{\cdots} \circ c_{H,H}$. When $c^n_{H,H} \circ \delta_H = \delta_H$ we will say that H is n-codyslectic.

PROPOSITION 3.21. *Let H be a weak Hopf quasigroup. If $\lambda^n_H = id_H$, then H is n-dyslectic and n-codyslectic.*

Proof. By (52), we have that

$$
\begin{aligned}
\mu_H &= \lambda^n_H \circ \mu_H = \lambda^{n-1}_H \circ \lambda_H \circ \mu_H = \lambda^{n-1}_H \circ \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H} \\
&= \lambda^{n-2}_H \circ \lambda_H \circ \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H} = \lambda^{n-2}_H \circ \mu_H \circ (\lambda^2_H \otimes \lambda^2_H) \circ c^2_{H,H} \\
&= \cdots = \mu_H \circ (\lambda^n_H \otimes \lambda^n_H) \circ c^n_{H,H} = \mu_H \circ c^n_{H,H}.
\end{aligned}
$$

Analogously, if we use (53), with a similar calculation, we obtain that H is n-codyslectic. ▫

THEOREM 3.22. *If H is a weak Hopf quasigroup (co)commutative then the antipode λ_H satisfies $\lambda^2_H = id_H$.*

Proof. If we assume that H is commutative ($\mu_H \circ c_{H,H} = \mu_H$, equivalently, $\mu_H \circ c^{-1}_{H,H} = \mu_H$), by (2) $\Pi^L_H = \overline{\Pi}^R_H$. Analogously, by (3), $\Pi^R_H = \overline{\Pi}^L_H$. Then

$$
\begin{aligned}
\lambda_H \circ \lambda_H &= \lambda_H \circ (\lambda_H * \Pi^L_H) \\
&= (\lambda_H \circ \lambda_H) * (\lambda_H \circ \Pi^L_H) \\
&= (\lambda_H \circ \lambda_H) * (\lambda_H \circ \overline{\Pi}^R_H) \\
&= (\lambda_H \circ \lambda_H) * \Pi^R_H \\
&= \mu_H \circ (\mu_H \otimes H) \circ ((\lambda_H \circ \lambda_H) \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ \delta_H \\
&= ((\lambda_H \circ \lambda_H) * \lambda_H) * id_H \\
&= (\lambda_H \circ \Pi^R_H) * id_H \\
&= (\lambda_H \circ \overline{\Pi}^L_H) * id_H \\
&= \Pi^L_H * id_H = id_H.
\end{aligned}
$$

The first equality follows by (a4-3) of Definition 2.2, the second and seventh ones by (52) and the commutativity of μ_H, the third and eight ones by the commutativity of μ_H and the fourth one by (40). The fifth equality relies on (a4-7) of Definition 2.2 and the sixth one follows by the coassociativity of δ_H. In the ninth one we used (39) and finally the last one follows by (47).

The proof for a cocommutative weak Hopf quasigroup is similar using that $\Pi^L_H = \overline{\Pi}^L_H$ and $\Pi^R_H = \overline{\Pi}^R_H$. ▫

4. The fundamental theorem of Hopf modules. In the following definition we introduce the notion of right-right H-Hopf module for a weak Hopf quasigroup H. Note that if H is a Hopf quasigroup and $\mathcal{C}$ is the symmetric monoidal category $\mathbb{F} - Vect$, we get the notion defined by Brzeziński in [5].

DEFINITION 4.1. Let H be a weak Hopf quasigroup and M an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a right-right H-Hopf module if the following axioms hold:
  (c1) The pair (M, ρ_M) is a right H-comodule, i.e. $\rho_M : M \to M \otimes H$ is a morphism such that $(M \otimes \varepsilon_H) \circ \rho_M = id_M$ and $(\rho_M \otimes H) \circ \rho_M = (M \otimes \delta_H) \circ \rho_M$.
  (c2) The morphism $\phi_M : M \otimes H \to M$ satisfies:
    (c2-1) $\phi_M \circ (M \otimes \eta_H) = id_M$.
    (c2-2) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$, i.e. ϕ_M is a morphism of right H-comodules with the codiagonal coaction on $M \otimes H$.
  (c3) $\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi_H^L)$.
  (c4) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \lambda_H \otimes H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi_H^R)$.
  (c5) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes \delta_H) = \phi_M$.

REMARK 4.2. Obviously, if H is a weak Hopf quasigroup, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module. Moreover, if (M, ϕ_M, ρ_M) is a right-right H-Hopf module, the axiom (c5) is equivalent to

$$\phi_M \circ (\phi_M \otimes \Pi_H^R) \circ (M \otimes \delta_H) = \phi_M. \tag{54}$$

because by (c3) and (c4) of Definition 4.1 we have that

$$\begin{aligned} &\phi_M \circ (\phi_M \otimes \Pi_H^R) \circ (M \otimes \delta_H) \\ &= \phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes \delta_H). \end{aligned} \tag{55}$$

Also, composing in (c2-2) with $M \otimes \eta_H$ and $M \otimes \varepsilon_H$ we have that

$$\phi_M \circ (M \otimes \Pi_H^R) \circ \rho_M = id_M. \tag{56}$$

Finally, by (c5) and (54) we obtain

$$\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes (\delta_H \circ \eta_H)) = id_M, \tag{57}$$

$$\phi_M \circ (\phi_M \otimes \Pi_H^R) \circ (M \otimes (\delta_H \circ \eta_H)) = id_M. \tag{58}$$

PROPOSITION 4.3. *Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism $q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$ satisfies*

$$\rho_M \circ q_M = (M \otimes \Pi_H^L) \circ \rho_M \circ q_M \tag{59}$$

and, as a consequence, is an idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$M^{coH} \xrightarrow{\ i_M\ } M \quad \begin{array}{c} \xrightarrow{\rho_M} \\[-2pt] \xrightarrow{\hspace{3cm}} \\[-4pt] \underset{(M \otimes \overline{\Pi}_H^R) \circ \rho_M}{} \end{array} \quad M \otimes H$$

is an equalizer diagram.

   Proof. The equality (59) holds because

$$\rho_M \circ q_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes (\delta_H \circ \lambda_H)) \circ \rho_M$$
$$= (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ \rho_M$$
$$= ((\phi_M \circ (M \otimes \lambda_H)) \otimes \Pi_H^L) \circ (M \otimes (c_{H,H} \circ \delta_H)) \circ \rho_M$$
$$= ((\phi_M \circ (M \otimes \lambda_H)) \otimes (\Pi_H^L \circ \Pi_H^L)) \circ (M \otimes (c_{H,H} \circ \delta_H)) \circ \rho_M$$
$$= (M \otimes \Pi_H^L) \circ \rho_M \circ q_M$$

where the first equality follows by (c2-2) of Definition 4.1, the second one by (53), the third one relies on (c1) of Definition 4.1 as well as the naturality of the braiding, the fourth one is a consequence of the properties of Π_H^L and the last one uses the arguments of the three first identities but in the inverse order.

   On the other hand, q_M is an idempotent. Indeed,

$$q_M \circ q_M = \phi_M \circ (M \otimes \lambda_H) \circ \rho_M \circ q_M$$
$$= \phi_M \circ (M \otimes (\lambda_H \circ \Pi_H^L)) \circ \rho_M \circ q_M$$
$$= \phi_M \circ (M \otimes (\lambda_H \circ \overline{\Pi}_H^R \circ \Pi_H^L)) \circ \rho_M \circ q_M$$
$$= \phi_M \circ (M \otimes \Pi_H^R) \circ \rho_M \circ q_M$$
$$= q_M.$$

In the last equalities, the first one follows by definition, the second one by (59), the third one by (34), the fourth one by (40) and (59) and the last one by (56).

   Finally, by (40) and (56)

$$\phi_M \circ (M \otimes (\lambda_H \circ \overline{\Pi}_H^R)) \circ \rho_M = \phi_M \circ (M \otimes \Pi_H^R) \circ \rho_M = id_M.$$

Then,

$$M^{coH} \xrightarrow{\ i_M\ } M \begin{array}{c} \xrightarrow{\quad\quad \rho_M \quad\quad} \\[-6pt] \xrightarrow[\quad (M \otimes \overline{\Pi}_H^R) \circ \rho_M \quad]{} \end{array} M \otimes H$$

is a split cofork [10] and thus an equalizer diagram. $\square$

   REMARK 4.4. Note that, in the conditions of Proposition (4.3), by (33) and (34), we obtain that

$$M^{coH} \xrightarrow{\ i_M\ } M \begin{array}{c} \xrightarrow{\quad\quad \rho_M \quad\quad} \\[-6pt] \xrightarrow[\quad (M \otimes \Pi_H^L) \circ \rho_M \quad]{} \end{array} M \otimes H$$

is also an equalizer diagram.

   Moreover, by the comodule condition and (c4) of Definition 4.1 we have

$$\phi_M \circ (q_M \otimes H) \circ \rho_M = id_M. \tag{60}$$

   Finally, the following identities hold:

$$\rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H), \tag{61}$$

$$p_M \circ \phi_M \circ (i_M \otimes H) = p_M \circ \phi_M \circ (i_M \otimes \Pi_H^L), \tag{62}$$

$$p_M \circ \phi_M \circ (i_M \otimes H) = p_M \circ \phi_M \circ (i_M \otimes \overline{\Pi}_H^L). \tag{63}$$

Indeed:

$$
\begin{aligned}
&\rho_M \circ \phi_M \circ (i_M \otimes H) \\
&= (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ ((\rho_M \circ i_M) \otimes \delta_H) \\
&= (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (((M \otimes \overline{\Pi}_H^R) \circ \rho_M \circ i_M) \otimes \delta_H) \\
&= (\phi_M \otimes (\mu_H \circ (\overline{\Pi}_H^R \otimes H))) \circ (M \otimes c_{H,H} \otimes H) \circ ((\rho_M \circ i_M) \otimes \delta_H) \\
&= (\phi_M \otimes (\varepsilon_H \circ \mu_H) \otimes H) \circ (M \otimes c_{H,H} \otimes \delta_H) \circ ((\rho_M \circ i_M) \otimes \delta_H) \\
&= (((M \otimes \varepsilon_H) \circ \rho_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H) \\
&= (\phi_M \otimes H) \circ (i_M \otimes \delta_H).
\end{aligned}
$$

The first equality follows from (c2-2) of Definition 4.1, the second one by Proposition 4.3 and the third one by the naturality of the braiding. The fourth equality is a consequence of (10). In the fifth one we used the coassociativity of δ_H and (c2-2) of Definition 4.1. The last one follows by (c1) of Definition 4.1.

On the other hand, by (61) and (a4-6) of Definition 2.2 we have

$$
\begin{aligned}
p_M \circ \phi_M \circ (i_M \otimes H) &= p_M \circ q_M \circ \phi_M \circ (i_M \otimes H) \\
&= p_M \circ \phi_M \circ (M \otimes \lambda_H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) \\
&= p_M \circ \phi_M \circ (\phi_M \otimes \lambda_H) \circ (i_M \otimes \delta_H) \\
&= p_M \circ \phi_M \circ (i_M \otimes \Pi_H^L).
\end{aligned}
$$

Finally, composing with $M^{coH} \otimes \overline{\Pi}_H^L$ in (62) and using (33) we obtain (63).

PROPOSITION 4.5. *Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism*

$$\nabla_M := (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) : M^{coH} \otimes H \to M^{coH} \otimes H$$

is an idempotent and the equalities

$$\nabla_M = ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H), \tag{64}$$

$$(M^{coH} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{coH} \otimes \delta_H). \tag{65}$$

$$\nabla_M = (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (M^{coH} \otimes \eta_H)) \otimes H). \tag{66}$$

hold.

Proof. Trivially, by (60) we have that

$$\nabla_M \circ \nabla_M = (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (q_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) = \nabla_M.$$

The equality (64) follows from (61) and (65) is a consequence of (64) and the coassociativity of δ_H. Finally, (66) holds because, by (64), (17) and (63) we have

$$(M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (M^{coH} \otimes \eta_H)) \otimes H)$$
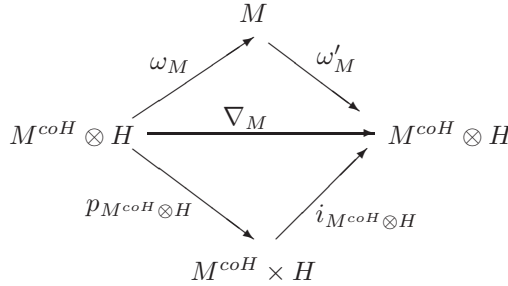
$$= ((p_M \circ \phi_M \circ (M \otimes \overline{\Pi}_H^L)) \otimes H) \circ (i_M \otimes \delta_H) = \nabla_M.$$

<div style="text-align: right;">□</div>

REMARK 4.6. In the conditions of Proposition 4.5 we define the morphisms

$$\omega_M : M^{coH} \otimes H \to M, \quad \omega'_M : M \to M^{coH} \otimes H$$

by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\nabla_M = \omega'_M \circ \omega_M$. Also, we have a commutative diagram

where $M^{coH} \times H$ denotes the image of ∇_M and $p_{M^{coH} \otimes H}, i_{M^{coH} \otimes H}$ are the morphisms such that $p_{M^{coH} \otimes H} \circ i_{M^{coH} \otimes H} = id_{M^{coH} \times H}$ and $i_{M^{coH} \otimes H} \circ p_{M^{coH} \otimes H} = \nabla_M$. Therefore, the morphism $\alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M$ is an isomorphism of right H-modules (i.e. $\rho_{M^{coH} \times H} \circ \alpha_M = (\alpha_M \otimes H) \circ \rho_M$) with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$. The comodule structure of $M^{coH} \times H$ is the one induced by the isomorphism α_M and it is equal to

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H}.$$

PROPOSITION 4.7. *Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M), (N, ϕ_N, ρ_N) right-right H-Hopf modules. If there exists a right H-comodule isomorphism $\alpha : M \to N$, the triple $(M, \phi_M^\alpha = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$ is a right-right H-Hopf module.*

Proof. The proof follows easily because, if α is a right H-comodule isomorphism, $\rho_M = (\alpha^{-1} \otimes H) \circ \rho_N \circ \alpha$ holds. □

PROPOSITION 4.8. *Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The triple $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ where*

$$\phi_{M^{coH} \times H} = p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H),$$

and $\rho_{M^{coH} \times H}$ is the coaction defined in Remark 4.6, is a right-right H-Hopf module.

Proof. By Remark 4.6 we have that $(M^{coH} \times H, \rho_{M^{coH} \times H})$ is a right H-comodule and it is clear that (c2-1) of Definition 4.1 holds. On the other hand, by (65),

$$\rho_{M^{coH} \times H} \circ \phi_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes (\delta_H \circ \mu_H)) \circ (i_{M^{coH} \otimes H} \otimes H).$$

Moreover, by (65), the properties of ∇_M and (a1) of Definition 2.2 we obtain

$$(\phi_{M^{coH} \times H} \otimes \mu_H) \circ (M^{coH} \times H \otimes c_{H,H} \otimes H) \circ (\rho_{M^{coH} \times H} \otimes \delta_H)$$

$$= ((p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H)) \otimes H) \circ (\nabla_M \otimes H \otimes \mu_H)$$
$$\circ (M^{coH} \otimes H \otimes c_{H,H} \otimes H) \circ (M^{coH} \otimes \delta_H \otimes \delta_H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H})) \circ ((\nabla_M \circ i_{M^{coH} \otimes H}) \otimes H)$$
$$= (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes (\delta_H \circ \mu_H)) \circ (i_{M^{coH} \otimes H} \otimes H)$$

and then (c2-2) of Definition 4.1 holds.

The proof for (c3) of Definition 4.1 is the following:

$$\phi_{M^{coH} \times H} \circ (\phi_{M^{coH} \times H} \otimes \lambda_H) \circ (M^{coH} \times H \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes (\delta_H \circ \mu_H) \otimes \lambda_H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes \lambda_H)$$
$$\circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M \circ (i_M \otimes \mu_H)) \otimes (\mu_H \circ (H \otimes \Pi_H^L)))$$
$$\circ (M^{coH} \otimes \delta_{H \otimes H}) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M \circ (i_M \otimes \mu_H)) \otimes H) \circ (M^{coH} \otimes H \otimes c_{H,H})$$
$$\circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M \circ (i_M \otimes (\Pi_H^L \circ \mu_H))) \otimes H) \circ (M^{coH} \otimes H \otimes c_{H,H})$$
$$\circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M \circ (i_M \otimes \mu_H)) \otimes H) \circ (M^{coH} \otimes H \otimes c_{H,H})$$
$$\circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \Pi_H^L)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes (((\varepsilon_H \circ \mu_H) \otimes H \otimes H)$$
$$\circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes (((\varepsilon_H \circ \mu_H) \otimes \delta_H)$$
$$\circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes (\delta_H \circ \mu_H \circ (H \otimes \Pi_H^L))) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ \nabla_M \circ (M^{coH} \otimes (\mu_H \circ (H \otimes \Pi_H^L))) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= \phi_{M^{coH} \times H} \circ (M^{coH} \times H \otimes \Pi_H^L).$$

The first and tenth equalities follow by (64), the second one by (a1) of Definition 2.2 and the third one by the coassociativity of δ_H and (a4-6) of Definition 2.2. In the fourth one we used (45). The fifth equality relies on (62) and the sixth one follows by (25) and (62). The seventh one is a consequence of the naturality of the braiding and (7). The eighth equality follows by the naturality of the braiding and the coassociativity of δ_H. Finally, the ninth equality follows by (7) and the last one relies on the properties of ∇_M.

We continue in this fashion proving (c4) of Definition 4.1. Indeed:

$$\phi_{M^{coH} \times H} \circ (\phi_{M^{coH} \times H} \otimes H) \circ (M^{coH} \times H \otimes \lambda_H \otimes H) \circ (M^{coH} \times H \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes (\delta_H \circ \mu_H \circ (H \otimes \lambda_H)) \otimes H)$$
$$\circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}$$
$$\circ (H \otimes \lambda_H)) \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M \circ (M \otimes \Pi_H^L)) \otimes \mu_H) \circ (i_M \otimes ((\mu_H \otimes \mu_H)$$
$$\circ \delta_{H \otimes H} \circ (H \otimes \lambda_H)) \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$

$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes [((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H) \otimes H$$
$$\otimes \mu_H) \circ (H \otimes \delta_H \otimes c_{H,H} \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)$$
$$\circ ((\delta_H \circ \eta_H) \otimes \delta_H \otimes (\delta_H \circ \lambda_H))] \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes (\Pi_H^L \otimes ((\varepsilon_H \otimes H) \circ \delta_H \circ \mu_H))$$
$$\otimes H) \circ (M^{coH} \otimes \delta_H \otimes \lambda_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes H \otimes \mu_H \otimes H)$$
$$\circ (M^{coH} \otimes \delta_H \otimes \lambda_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes \delta_H \otimes \Pi_H^R) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ i_{M^{coH} \otimes H}) \otimes \Pi_H^R)$$
$$= \phi_{M^{coH} \times H} \circ (M^{coH} \times H \otimes \Pi_H^R).$$

The first and eighth equalities follow by (64), the second one by (a1) of Definition 2.2 and the third one by (62). In the fourth one we used the naturality of the braiding and (a2) of Definition 2.2. The fifth one is a consequence of the naturality of the braiding, the coassociativity of δ_H and (a1) of Definition 2.2. The sixth equality follows from the counit properties and (62) and the seventh one by (a4-7) of Definition 2.2. Finally, the last equality is a consequence of the properties of ∇_M.

The only point remaining is (c5) of Definition 4.1. This equality holds because:

$$\phi_{M^{coH} \times H} \circ (\phi_{M^{coH} \times H} \otimes H) \circ (M^{coH} \times H \otimes \Pi_H^L \otimes H) \circ (M^{coH} \times H \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes (\delta_H \circ \mu_H \circ (H \otimes \Pi_H^L)) \otimes H)$$
$$\circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes ((\mu_H \otimes \mu_H) \circ \delta_{H \otimes H} \circ (H \otimes \Pi_H^L))$$
$$\otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes \mu_H) \circ (i_M \otimes ((\mu_H \otimes (\mu_H \circ (H \otimes \Pi_H^L)))$$
$$\circ \delta_{H \otimes H} \circ (H \otimes \Pi_H^L)) \otimes H) \circ (i_{M^{coH} \otimes H} \otimes \delta_H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes ((\Pi_H^L \otimes H) \circ \mu_{H \otimes H}$$
$$\circ (\delta_H \otimes ((\Pi_H^L \otimes H) \circ \delta_H)))) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes ((\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L)) \otimes \mu_H)$$
$$\circ \delta_{H \otimes H}) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes ((\Pi_H^L \otimes H) \circ \delta_H \circ \mu_H))$$
$$\circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes (\delta_H \circ \mu_H)) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ \nabla_M \circ (M^{coH} \otimes \mu_H) \circ \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= \phi_{M^{coH} \times H}.$$

The first equalitiy follows by (64), the second one by (a1) of Definition 2.2 and the third one by (27). In the fourth one we used (45) as well as (62). The fifth one relies on the naturality of the braiding and the sixth one is a consequence of (25) and (a1) of Definition 2.2. The seventh one follows by (62) and the eighth one by 64. Finally, the last one follows by the properties of ∇_M. $\square$

PROPOSITION 4.9. *Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) be a right-right H-Hopf module and $\alpha_M : M \to M^{coH} \times H$ be the isomorphism of right H-comodules defined in Remark 4.6. Then, for the action $\phi_M^{\alpha_M}$ introduced in Proposition*

4.7, the triple $(M, \phi_M^{\alpha_M}, \rho_M)$ *is a right-right H-Hopf module with the same object of coinvariants of* (M, ϕ_M, ρ_M). *Moreover, the identity*

$$\phi_M^{\alpha_M} = \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H) \qquad (67)$$

holds and $q_M^{\alpha_M} = q_M$ *where* $q_M^{\alpha_M} = \phi_M^{\alpha_M} \circ (M \otimes \lambda_H) \circ \rho_M$ *is the idempotent morphism associated to the Hopf module* $(M, \phi_M^{\alpha_M}, \rho_M)$. *Finally, if for the triple* $(M, \phi_M^{\alpha_M}, \rho_M)$, $\nabla_M^{\alpha_M}$ *denotes the idempotent morphism defined in Proposition 4.5, we have that*

$$\nabla_M^{\alpha_M} = \nabla_M \qquad (68)$$

and then, for $(M, \phi_M^{\alpha_M}, \rho_M)$, *the associated isomorphism between M and $M^{coH} \times H$ defined in Remark 4.6 is* α_M. *Finally,*

$$(\phi_M^{\alpha_M})^{\alpha_M} = \phi_M^{\alpha_M} \qquad (69)$$

holds.

    Proof. By Proposition 4.7 we obtain that $(M, \phi_M^{\alpha_M}, \rho_M)$ is a right-right H-Hopf module and by the equalizer diagram of Proposition 4.3 the object of coinvariants of $(M, \phi_M^{\alpha_M}, \rho_M)$ is equal to the one of (M, ϕ_M, ρ_M). Also, by (60) we have

$$\phi_M \circ (i_M \otimes H) \circ \nabla_M = \phi_M \circ (i_M \otimes H) \qquad (70)$$

and

$$\nabla_M \circ (p_M \otimes H) \circ \rho_M = (p_M \otimes H) \circ \rho_M. \qquad (71)$$

Then, (67) holds because

$$\begin{aligned}
\phi_M^{\alpha_M} &= \alpha_M^{-1} \circ \phi_{M^{coH} \times H} \circ (\alpha_M \otimes H) \\
&= \phi_M \circ (i_M \otimes H) \circ \nabla_M \circ (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (p_M \otimes H) \circ \rho_M) \otimes H) \\
&= \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H).
\end{aligned}$$

    On the other hand, by (67), the coassociativity of δ_H, (c1), (c4) of Definition 4.1 and (54) we obtain:

$$\begin{aligned}
q_M^{\alpha_M} &= \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes \lambda_H) \circ \rho_M \\
&= \phi_M \circ (\phi_M \otimes H) \circ ((((\phi_M \circ (M \otimes \lambda_H)) \otimes H) \circ (M \otimes \delta_H)) \otimes \lambda_H) \\
&\quad \circ (M \otimes \delta_H) \circ \rho_M \\
&= \phi_M \circ ((\phi_M \circ (M \otimes \Pi_H^R)) \otimes \lambda_H) \circ (M \otimes \delta_H) \circ \rho_M \\
&= \phi_M \circ ((\phi_M \circ (M \otimes \Pi_H^R) \circ \rho_M) \otimes \lambda_H) \circ \rho_M \\
&= q_M.
\end{aligned}$$

    Then, $i_M = i_M^{\alpha_M}$ and $p_M = p_M^{\alpha_M}$ and, as a consequence, $\omega_M^{\alpha_M \prime} = (p_M \otimes H) \circ \rho_M = \omega_M'$. Moreover, by (c2-1) of Definition 4.1, (66) and (70)

$$\begin{aligned}
\omega_M^{\alpha_M} &= \phi_M^{\alpha_M} \circ (i_M \otimes H) \\
&= \phi_M \circ (q_M \otimes \mu_H) \circ ((\rho_M \circ i_M) \otimes H) \\
&= \phi_M \circ (i_M \otimes \mu_H) \circ (\nabla_M \otimes H) \circ (M^{coH} \otimes \eta_H \otimes H) \\
&= \phi_M \circ (i_M \otimes H) \circ \nabla_M
\end{aligned}$$

$$= \omega_M.$$

Therefore, $\nabla_M^{\alpha_M} = \nabla_M$ and then, for $(M, \phi_M^{\alpha_M}, \rho_M)$, the associated isomorphism between M and $M^{coH} \times H$ is α_M.

Finally, by (66)

$$
\begin{aligned}
(\phi_M^{\alpha_M})^{\alpha_M} &= \phi_M \circ (q_M \otimes \mu_H) \circ ((\rho_M \circ q_M) \otimes \mu_H) \circ (\rho_M \otimes H) \\
&= \phi_M \circ (i_M \otimes \mu_H) \circ ((\nabla_M \circ (p_M \otimes \eta_H)) \otimes \mu_H) \circ (\rho_M \otimes H) \\
&= \phi_M \circ (i_M \otimes H) \circ \nabla_M \circ (p_M \otimes \mu_H) \circ (\rho_M \otimes H) \\
&= \phi_M^{\alpha_M}
\end{aligned}
$$

and (69) holds. $\square$

REMARK 4.10. Let H be a weak Hopf quasigroup. The triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module and $\phi_H^{\alpha_H} = \phi_H$ because by (42),

$$
\phi_H^{\alpha_H} = \mu_H \circ (\Pi_H^L \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H = \phi_H.
$$

DEFINITION 4.11. Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) and (N, ϕ_N, ρ_N) be right-right H-Hopf modules. A morphism in $\mathcal{C}$ $f : M \to N$ is said to be H-quasilineal if the following identity holds

$$
\phi_N^{\alpha_N} \circ (f \otimes H) = f \circ \phi_M^{\alpha_M}. \tag{72}
$$

A morphism of right-right H-Hopf modules between M and N is a morphism $f : M \to N$ in $\mathcal{C}$ such that is both a morphism of right H-comodules and H-quasilineal. The collection of all right H-Hopf modules with their morphisms forms a category which will be denoted by $\mathcal{M}_H^H$.

PROPOSITION 4.12. *Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, for $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ the identity*

$$
\phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} = \phi_{M^{coH} \times H} \tag{73}
$$

holds.

Proof. First note that, by (65) we have

$$
q_{M^{coH} \times H} = p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \Pi_H^L) \circ i_{M^{coH} \otimes H} \tag{74}
$$

and, as a consequence, by (67), the equality

$$
\begin{aligned}
\phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} &= p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (M^{coH} \otimes \Pi_H^L)) \otimes \mu_H) \\
&\quad \circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H).
\end{aligned} \tag{75}
$$

holds.

Then,

$$
\begin{aligned}
&\phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} \\
&= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (i_M \otimes \Pi_H^L)) \otimes \mu_H) \\
&\quad \circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H)
\end{aligned}
$$

$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (\phi_M \otimes \lambda_H) \circ (i_M \otimes \delta_H)) \otimes \mu_H)$$
$$\circ (M^{coH} \otimes \delta_H \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (\phi_M \otimes \lambda_H)) \otimes \mu_H)$$
$$\circ (i_M \otimes ((H \otimes \delta_H) \circ \delta_H) \otimes H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (M \otimes \lambda_H)) \otimes \mu_H) \circ (((M \otimes \delta_H)$$
$$\circ \rho_M \circ \phi_M \circ (i_M \otimes H) \circ i_{M^{coH} \otimes H}) \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ q_M) \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)$$
$$\circ i_{M^{coH} \otimes H}) \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ i_M) \otimes \mu_H) \circ ((\nabla_M \circ i_{M^{coH} \otimes H}) \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (p_M \otimes \mu_H) \circ ((\rho_M \circ i_M) \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (M^{coH} \otimes \eta_H)) \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= p_{M^{coH} \otimes H} \circ \nabla_M \circ (M^{coH} \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H)$$
$$= \phi_{M^{coH} \times H}$$

where the first equality follows by (75) and the definition of ∇_M, the second one by (c3) of Definition 4.1, the third one by the coassociativity of δ_H and the fourth one by (61). In the fifth equality we used (c1) of Definition 4.1 and the sixth and eighth ones are consequence of the definition of ∇_M. Finally, the seventh and the tenth one rely on the properties of ∇_M and the ninth one follows by (66). $\square$

THEOREM 4.13. *(Fundamental Theorem of Hopf modules) Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, the right-right H-Hopf modules (M, ϕ_M, ρ_M) and $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ are isomorphic in $\mathcal{M}_H^H$.*

Proof. By Remark 4.6 $\alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M$ is an isomorphism of right H-comodules with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$. To finish the proof we only need to show that (72) holds. Indeed, by (73), (70) and (71) we have

$$\alpha_M^{-1} \circ \phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} \circ (\alpha_M \otimes H)$$
$$= \phi_M \circ (i_M \otimes H) \circ \nabla_M \circ (M^{coH} \otimes \mu_H) \circ ((\nabla_M \circ (p_M \otimes H) \circ \rho_M) \otimes H)$$
$$= \phi_M^{\alpha_M}.$$

$\square$

REFERENCES

[1] J. N. ALONSO ÁLVAREZ, J. M. FERNÁNDEZ VILABOA, AND R. GONZÁLEZ RODRÍGUEZ, *Weak Hopf algebras and weak Yang-Baxter operators*, J. Algebra, 320 (2008), pp. 2101–2143.
[2] J. N. ALONSO ÁLVAREZ, J. M. FERNÁNDEZ VILABOA, AND R. GONZÁLEZ RODRÍGUEZ, *Weak braided Hopf algebras*, Indiana Univ. Math. J., 57 (2008), pp. 2423–2458.
[3] J. BÉNABOU, *Introduction to bicategories*, in Reports of the Midwest Categorical Seminar, LNM 47, Springer, 1967, pp. 1–77.
[4] G. BÖHM, F. NILL, AND K. SZLACHÁNYI, *Weak Hopf algebras, I. Integral theory and C^*-structure*, J. Algebra, 221 (1999), pp. 385–438.

[5] T. Brzeziński, *Hopf modules and the fundamental theorem for Hopf (co)quasigroups*, Internat. Elec. J. Algebra, 8 (2010), pp. 114–128.

[6] P. Etingof, D. Nikshych, and V. Ostrick, *On fusion categories*, Ann. Math., 162 (2005), pp. 581–642.

[7] T. Hayashi, *Face algebras I. A generalization of quantum group theory*, J. Math. Soc. Japan, 50 (1998), pp. 293–315.

[8] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math., 102 (1993), pp. 20–78.

[9] M. Karoubi, *K-théorie*, Les Presses de l'Université de Montréal, Montréal, 1971.

[10] S. Mac Lane, *Categories for the Working Mathematician*, GTM 5, Springer-Verlag, New-York, 1971.

[11] J. Klim and S. Majid, *Hopf quasigroups and the algebraic 7-sphere*, J. Algebra, 323 (2010), pp. 3067–3110.

[12] D. Nikshych and L. Vainerman, *Finite quantum groupoids and their applications*, New Directions in Hopf Algebras, MSRI Publications, 43 (2002), pp. 211–262.

[13] B. Pareigis, *On braiding and Dyslexia*, J. Algebra, 171 (1995), pp. 413–425.

[14] J. M. Pérez-Izquierdo and I. P. Shestakov, *An envelope for Malcev algebras*, J. Algebra, 272 (2004), pp. 379–393.

[15] J. M. Pérez-Izquierdo, *Algebras, hyperalgebras, nonassociative bialgebras and loops*, Adv. Math., 208 (2007), pp. 834–876.

[16] M. P. López and E. Villanueva, *The antipode and the (co)invariants of a finite Hopf (co)quasigroup*, Appl. Cat. Struct., 21 (2013), pp. 237–247.

[17] Y. Yamanouchi, *Duality for generalized Kac algebras and characterization of finite groupoid algebras*, J. Algebra, 163 (1994), pp. 9–50.