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KÄHLER-RICCI SOLITON AND H-FUNCTIONAL*

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Abstract. We consider Kähler-Ricci soliton on a Fano manifold M. We introduce an H-functional on M; we show that its critical point has to be a Kähler-Ricci soliton and the Kähler-Ricci flow can be viewed as its reduced gradient flow. We then obtain a natural lower bound of H-functional in terms of an invariant of holomorphic vector fields on M. As an application, we prove that a Kähler-Ricci soliton, if exists, maximizes Perelman's μ-functional. Second we consider a conjecture proposed by S. K. Donaldson regarding the existence of Kähler metrics with constant scalar curvature in terms of $\mathcal{K}$-energy; a simple observation is that on Fano manifolds, one can consider Donaldson's conjecture in terms of Ding's $\mathcal{F}$-functional. We then state geodesic stability conjecture on Fano manifolds in terms of $\mathcal{F}$-functional. Similar pictures can be naturally extended to a Kähler-Ricci soliton and modified $\mathcal{F}$-functional.

Key words. Kähler-Ricci soliton, H-functional, geodesic stability.

AMS subject classifications. 53C55 (32Q15, 58E11).

1. Introduction. In this paper we consider Kähler-Ricci soliton on a Fano manifold M. Ricci solitons were defined by R. Hamilton in the study of Ricci flow [23] and it plays a significant role in the theory of Ricci flow. A Kähler-Ricci soliton on a Fano manifold is Kähler-Einstein precisely when the Futaki invariant vanishes and hence Kähler-Ricci soliton is a natural generalization of Kähler-Einstein metrics. In his study of Hamilton's Ricci flow, Perelman [27] introduced many revolutionary ideas, including the well-known entropy functionals, which lead him to the solution of the Poincare conjecture and Thurston's geometrization conjecture. Ricci flow is the gradient flow of Perelman's μ-functional (modulo diffeomorphisms). On Fano manifolds, Kähler-Ricci solitons are critical points of Perelman's μ-functional. A Kähler-Einstein metric, if exists, maximizes μ-functional. Kähler-Ricci soliton is also expected to maximize μ-functional. This is actually the case [37], for example, if we consider only metrics which are invariant with respect to the imaginary part of extremal vector field studied in Tian-Zhu [36].

Our main observation is to consider a functional (this quantity has been studied along the Kähler-Ricci flow in literature) on Fano manifolds. We show that a Kähler-Ricci soliton is a critical point of this functional. We then consider a lower bound of this functional and this turns out to be rather straightforward for invariant metrics. In general, we need to consider a natural geometric structure of the space of Kähler potentials, studied by Mabuchi [24], Semmes [30] and Donaldson [17]. The key notion is the geodesic segments and the geodesic rays, in the manner of X.-X. Chen [11, 12]. We then obtain a natural lower bound, called H-invariant which was studied in [37] and this result relies on the key technique fact of the convexity of Ding's $\mathcal{F}$-functional [15], established by Berndtsson [5, 7]. As a direct application, we prove that Kähler-Ricci soliton, if exists, maximizes Perelman's μ-functional.

Donaldson [17] formulated a conjecture relating existence of constant scalar curvature with the geometric structure of the space of Kähler potentials, in particular with the limit behavior of (derivative of) Mabuchi's $\mathcal{K}$-energy (defined in [24]) along geodesic rays. It is very intuitive to understand Donaldson's conjecture in terms of

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critical points of the (formally) convex functional, Mabuchi's $\mathcal{K}$-energy. The constant scalar curvature metric is a critical point of $\mathcal{K}$-energy (actually minimizer) and $\mathcal{K}$-energy is (formally) convex in terms of geodesics in the space of Kähler potentials. Donaldson's conjecture naturally leads to the geodesic stability which was first introduced by Chen [12] in terms of ρ-invariant along geodesic rays (see Phong-Sturm [28] also for some related work). Chen [13] made one further step to study the lower bound of $\mathcal{K}$-energy and partially confirmed Donaldson's conjecture.

On Fano manifold, it is natural to consider the geodesic stability in terms of $\mathcal{F}$-functional. The key advantage of $\mathcal{F}$-functional is that it is actually convex for geodesic rays with very weak regularity by Berndtsson [7]. This key feature has already been explored by Berndtsson [7] and others (see Berman [3, 4] for example). For us this removes the main technical obstacle caused by the rather weak regularity of geodesic segments and geodesic rays. Our discussion can naturally be extended to the Kähler-Ricci solitons, using the modified $\mathcal{F}$-functional and the modified Futaki invariant, studied by Tian-Zhu [35, 36]. We then formulate a version of geodesic stability for Kähler-Ricci solitons on Fano manifolds.

We organize the paper as follows. In Section 2 we study the H-functional and its relation with Kähler-Ricci solitons. In Section 3 we discuss the geodesic stability in terms of $\mathcal{F}$-functional and modified $\mathcal{F}$-functional for Kähler-Einstein metrics and Kähler-Ricci solitons.

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2. Kähler-Ricci soliton and H-functional.

2.1. H-functional on Fano manifolds. Let $(M, [\omega_0])$ be a compact Fano manifold. For any Kähler metric $\omega \in [\omega_0]$, the Ricci potential h of ω is defined to be

$$Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h, \tag{2.1}$$

with the normalization condition

$$\int_M e^h \omega^n = \int_M \omega^n = V, \tag{2.2}$$

where $V = (2\pi)^n [c_1(M)]^n$. We define H *functional* as follows,

$$H(\omega) = \int_M h e^h \omega^n. \tag{2.3}$$

It is worthwhile to point out that H-functional can be viewed as the H-entropy in probability theory applied to the measure $V^{-1}e^h\omega^n$ with respect to the measure $V^{-1}\omega^n$. The celebrated Csiszár-Kullback-Pinsker inequality asserts that, on a complete metric and separate space $\mathcal{X}$, for any two probability measures ν, μ, we have,

$$\|\nu - \mu\|_{TV} \le \sqrt{2H(\nu|\mu)}, \tag{2.4}$$

where $\|\nu - \mu\|_{TV}$ is the total variation and

$$H(\nu|\mu) = \int_{\mathcal{X}} \log \frac{d\nu}{d\mu} d\nu$$

is the so-called H-entropy. We refer the readers to a survey paper [22] for example, for a nice proof of (2.4). Apply (2.4) with $\mu = V^{-1}\omega^n, \nu = V^{-1}e^h\omega^n$, we get

$$\frac{1}{2V}\left(\int_M |1-e^h|\omega^n\right)^2 \leq \int_M he^h\omega^n = H(\omega). \tag{2.5}$$

Hence $H(\omega)$ is a norm-like functional and it is zero precisely when $h = 0$, namely when ω is a Kähler-Einstein metric.

Recall a Kähler-Ricci soliton satisfies that ∇h is the real part of a holomorphic vector field X; equivalently, the metric satisfies

$$Ric(g) = g + L_X g.$$

When $X = 0$ the metric is then a Kähler-Einstein metric with positive scalar curvature. In [35, 36] Tian-Zhu proved the uniqueness of a Kähler-Ricci soliton modulo automorphisms of M, extending Bando-Mabuchi's uniqueness theorem [2] on Kähler-Einstein metrics on Fano manifolds. In particular they proved that X is determined *a priori* by $(M, [\omega_0])$ and it is unique up to automorphism. In general we will show that there is a close relation of H-functional with a Kähler-Ricci soliton. The main result of the paper is the following,

THEOREM 1. *For any metric $\omega \in [\omega_0]$, there exists a nonnegative numerical invariant N_X of $(M, [\omega_0])$ such that*

$$H(\omega) \geq N_X, \tag{2.6}$$

where the equality holds if and only of ω is a Kähler-Ricci soliton in $[\omega_0]$.

The invariant N_X in terms of the holomorphic vector field X appears in a recent paper of Tian-Zhang-Zhang-Zhu [37] and the relevant definitions will be recalled below.

Before we prove Theorem 1, we shall first explore some interesting properties of H-functional and its relation with a Kähler-Ricci soliton. We will need the following result due to A. Futaki [21].

PROPOSITION 2.1 (Futaki). *Let $(M, [\omega_0])$ be a Fano manifold. Suppose h is the normalized Ricci potential of $\omega \in [\omega_0]$, then*

$$L_h u = -(\Delta u + \nabla u \nabla h + u)$$

is a self-adjoint positive operator with respect to $e^h\omega^n$. In particular, the modified Poincare inequality holds

$$\int_M u(L_h u)e^h\omega^n = \int_M (|\nabla u|^2 - u^2)e^h\omega^n \geq 0. \tag{2.7}$$

where u satisfies the normalized condition

$$\int_M ue^h\omega^n = 0.$$

The equality holds if and only if $L_h u = 0$ and it is equivalent to that ∇u is a real holomorphic vector field.

First we have the following,

Proposition 2.2. *The Euler-Lagrangian equation of $H(\omega)$ is given by*

$$\Delta h + |\nabla h|^2 + h = constant.$$

Equivalently ∇h is a real holomorphic vector field. Hence a critical point of $H(\omega)$ is a Kähler-Ricci soliton.

Proof. We compute the first variation of $H(\omega)$ directly as follows. We write ω in terms of its Kähler potential,

$$\omega = \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi.$$

Suppose the variation of ϕ is given by $\delta\phi$. By the definition of Ricci-potential (2.1), we compute,

$$\partial\bar{\partial}(\delta h) = \partial\bar{\partial}(-\Delta\delta\phi - \delta\phi).$$

Since $H(\omega)$ does not depend on the choice of normalization Kähler potential, we can choose a normalization of Kähler potential such that

$$\delta h = -\Delta\delta\phi - \delta\phi.$$

The normalization condition (2.2) then implies that $\delta\phi$ satisfies

$$\int_M \delta\phi e^h \omega^n = 0.$$

We compute

$$\begin{aligned} \delta H(\omega) &= \int_M \left(-(\Delta\delta\phi + \delta\phi)e^h - he^h(\Delta\delta\phi + \delta\phi) + he^h\Delta\delta\phi\right)\omega^n \\ &= -\int_M \delta\phi(\Delta h + |\nabla h|^2 + h)e^h\omega^n. \end{aligned} \tag{2.8}$$

The Euler-Lagrangian equation is given by

$$\Delta h + |\nabla h|^2 + h = constant. \tag{2.9}$$

Integrating (2.9) with respect to $e^h\omega^n$, the constant above is given exactly by $V^{-1}H$. Applying Proposition 2.1 to $h - V^{-1}H$, (2.9) implies that ∇h is a real holomorphic vector field and hence a critical point of $H(\omega)$ is a Kähler-Ricci soliton. □

Along the Kähler-Ricci flow

$$\frac{\partial\omega}{\partial t} = \omega - Ric(\omega),$$

one can compute directly that

$$\frac{\partial H}{\partial t} = -\int_M \left(|\nabla h|^2 - (h - V^{-1}H)^2\right) e^h\omega^n \leq 0. \tag{2.10}$$

The equality holds exactly when ω is a Kähler-Ricci soliton. By (2.8) the gradient flow of $H(\omega)$ is given by

$$\frac{\partial\phi}{\partial t} = \Delta h + |\nabla h|^2 + h$$

and this is a fourth-order equation. However, since the operator

$$L_h u = -(\Delta u + \nabla u \nabla h + u)$$

is a self-adjoint positive operator with respect to $e^h\omega^n$, we can choose naturally $\delta\phi = -h$ to decrease H-functional, with the corresponding flow by

$$\frac{\partial \phi}{\partial t} = -h,$$

which is exactly the Kähler-Ricci flow on potential level. The advantage is that this reduces the flow equation by two orders (compared with the gradient flow). This formal picture resembles for a general Kähler class that of the Calabi energy, extremal metrics and the Calabi flow [8]; $H(\omega)$ plays the similar role as the Calabi energy.

Next we introduce some numerical invariants of $(M, [\omega_0])$ in terms of holomorphic vector fields, in particular we recall the definition of N_X. Let $Aut_{\mathbb{C}}(M)$ be the automorphism group of M and G be a maximal compact subgroup. Suppose further that ω is a G-invariant metric. Then there is a Lie algebra homomorphism from $Lie(G)$ to the functions on M, under Poisson bracket. Let $\xi \in Lie(G)$ and let θ_ξ be the corresponding Hamiltonian; namely $d\theta_\xi = \iota_\xi \omega$ with a normalization condition for θ_ξ,

$$\int_M e^{\theta_\xi}\omega^n = \int_M \omega^n = V.$$

Define the integral

$$H_0(\xi, \omega) = \int_M \theta_\xi e^h \omega^n. \tag{2.11}$$

Tian-Zhang-Zhang-Zhu proved that ([37] Section 5) that $H_0(\xi,\omega)$ is independent the choice of ω, and hence it defines a numerical invariant on $Lie(G)$. We denote it simply by $H_0(\xi)$ for $\xi \in G$; moreover, Tian-Zhang-Zhang-Zhu proved that H_0 is a concave function in $Lie(G)$ with a unique maximizer $\xi_0 \in Lie(G)$, where ξ_0 is the imaginary part of the extremal vector field X. The invariant N_X is then defined by

$$N_X = H_0(\xi_0) = \max_{\xi \in Lie(G)} H_0(\xi).$$

It is also proved that in [37] that $N_X \geq 0$ and it is zero precisely when $X = 0$, or equivalently, the Futaki invariant is zero.

Suppose we only consider G-invariant metrics, then one can easily prove that $H(\omega) \geq N_X$, as a special case of Theorem 1. Since the proof is straightforward, we include the argument here.

Proposition 2.3. *Let ω be a G-invariant metric, then*

$$H(\omega) \geq N_X = \max_{\xi \in Lie(G)} H_0(\xi) = H_0(\xi_0).$$

Proof. Suppose ω is G-invariant and let $\xi \in Lie(G)$. Let h, θ_ξ be Ricci potential and Hamiltonian of ξ with respect to ω, satisfying the normalization

$$\int_M e^h \omega^n = \int_M e^{\theta_\xi}\omega^n = V.$$

We only need to show

$$\int_M (h-\theta_\xi)e^h\omega^n \geq 0$$

This follows from Proposition 2.4 below. □

The following elementary property will be important for us.

PROPOSITION 2.4. *Let f, g be two continuous functions, then we have*

$$\int_M e^g(f-g)\omega^n \leq \int_M e^f\omega^n - \int_M e^g\omega^n \leq \int_M e^f(f-g)\omega^n \tag{2.12}$$

Proof. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a (smooth) convex function, then for any $f, g : M \to \mathbb{R}$, we define $F : [0,1] \to \mathbb{R}$ by

$$F(t) = \int_M \varphi(tf + (1-t)g)\omega^n.$$

Then F is convex since

$$F^{''}(t) = \int_M \varphi^{''}(tf + (1-t)g)(f-g)^2\omega^n \geq 0.$$

Hence by convexity,

$$F^{'}(0) \leq F(1) - F(0) \leq F^{'}(1).$$

We have $F(0) = \int_M \varphi(g)\omega^n, F(1) = \int_M \varphi(f)\omega^n,\ F^{'}(0) = \int_M \varphi^{'}(g)(f-g), F^{'}(1) = \int_M \varphi^{'}(f)(f-g)$. Taking $\varphi(t) = e^t$ we get (2.12). □

2.2. Proof of Theorem 1. X.-X. Chen [12] and S. Donaldson [19] proved that the Calabi energy is bounded below by a natural invariant; an extremal metric, if exists, realizes such a lower bound. Chen uses deep estimates of homogeneous complex Monge-Ampere equations in the space of Kähler potentials [11, 14] and S. Donaldson [19] uses finite dimensional approximations (for projective manifolds). In Chen's argument, the geometric structure of the space of Kähler potentials plays an important role. We will mimic Chen's approach to prove Theorem 1.

We first recall the space of Kähler potentials $\mathcal{H}$,

$$\mathcal{H} = \{\phi \in C^\infty : \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

The Mabuchi metric [25] on $\mathcal{H}$ is defined as, for $\psi_1, \psi_2 \in T_\phi\mathcal{H}$,

$$\langle\psi_1, \psi_2\rangle_\phi = \int_M \psi_1\psi_2\omega_\phi^n.$$

For any path $\phi(t) \in \mathcal{H}$, the geodesic equation is given by

$$\ddot{\phi} - |\nabla\dot{\phi}|^2_{\omega_\phi} = 0, \tag{2.13}$$

where we use the complex notation of gradient and Laplacian etc. For any interval I in $\mathbb{R}$, denote $U = I \times S^1$. We use (z, w) to denote points on $M \times U$. Then the geodesic equation is equivalent to the homogeneous complex Monge-Ampere equation [30, 17] (assuming for each w, ϕ defines a strictly positive Kähler metric),

$$\Omega_\phi^{n+1} = 0, \tag{2.14}$$

where $\Omega_\phi = \pi^*\omega_0 + \partial\bar{\partial}_{w,z}\phi$, $\pi : M \times U \to M$ is the projection onto M and ϕ is regarded as a S^1 invariant function on $M \times U$. A fundamental result of Chen [11] asserts that for $I = [0, 1]$ and $\phi_0, \phi_1 \in \mathcal{H}$, there exists a unique $C^{1,1}$ solution of (2.14) in the sense that

$$\|\phi\|_{1,1} := \|\phi\|_{C^1} + \max\{|\partial\bar{\partial}_{w,z}\phi|\} \leq C. \tag{2.15}$$

Here $\phi(w, z)$ is regarded as a function on $M\times U$. We emphasize that Chen's estimates rely on the fact that the two end points are actually smooth Kähler potentials in $\mathcal{H}$. It is also useful to consider generalized Kähler potentials. The minimal requirement is that $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ defines a closed positive $(1, 1)$ current. If $\phi \in L^\infty$, then ω_ϕ^n is a well-defined volume form such that

$$\int_M \omega_\phi^n = \int_M \omega_0^n = V.$$

In this case we call ϕ a bounded Kähler potential and we denote the set of all bounded Kähler potentials as $\mathcal{H}_\infty$. We also consider generalized $C^{1,1}$ Kähler potentials defined as

$$\mathcal{H}_{1,1} = \{\phi : \omega_\phi \geq 0, \|\phi\|_{C^1} < \infty, 0 \leq n + \Delta\phi < \infty\}.$$

We also define the (weak) $C^{1,1}$ norm on M as follows (ϕ is a function on M), fixing a background metric,

$$\|\phi\|_{1,1}^w = \|\phi\|_{C^1} + \max|\Delta\phi|.$$

If $I = [0,\infty)$ and $\phi(t)$ satisfies (2.14), then $\phi(t)$ is called a geodesic ray. It is called a bounded geodesic ray if $\phi(t) \in \mathcal{H}_\infty$ for each t and it is called a $C^{1,1}$ geodesic ray if $\phi(t) \in \mathcal{H}_{1,1}$ for each t. Note that we do not specify any condition on ϕ_{tt} (or $\phi_{w\bar{w}}$).

First we state an interesting result proved by Berndtsson [6],

Proposition 2.5 (Berndtsson). *Let $\phi(t), t \in [0, T]$ be the unique geodesic such that the end points $\phi(0), \phi(T) \in \mathcal{H}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that*

$$\int_M f(\dot{\phi})\omega_\phi^n$$

is integrable. Then the above integral is a constant for $t \in [0, T]$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. To prove (2.6), we only need to show that for any ω and $\xi \in G$,

$$H(\omega) = \int_M he^h\omega^n \geq H_0(\xi). \tag{2.16}$$

Fix a background metric ω_0 and we assume ω_0 is a G-invariant metric. For any smooth Kähler metric ω, we write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$. The Ricci potentials are related by

$$h = h_0 - \log\frac{\omega^n}{\omega_0^n} - \phi + constant.$$

Hence we get,

$$e^h\omega^n = \lambda e^{h_0-\phi}\omega_0^n \tag{2.17}$$

for some positive constant λ; by the normalization condition (2.2), we have

$$\lambda = V\left(\int_M e^{h_0-\phi}\omega_0^n\right)^{-1}.$$

The relation (2.17) is fundamental for us. We will understand $e^h\omega^n$ as in (2.17) for any closed positive $(1,1)$ current $\omega \in [\omega_0]$ provided that ϕ is only assumed to be bounded. For any $\xi \in Lie(G)$, we denote θ_ξ to be its Hamiltonian with respect to ω_0; namely,

$$d\theta_\xi = \iota_\xi\omega_0, \int_M e^{\theta_\xi}\omega_0^n = V.$$

By the definition of $H_0(\xi)$, we then need to show

$$\int_M he^h\omega^n \geq H_0(\xi) = \int_M \theta_\xi e^{h_0}\omega_0^n.$$

Now let $Y = -J\xi - i\xi$ be the holomorphic vector field and σ_t be a one-parameter holomorphisms generated by Y ($\sigma_0 = id$). Denote

$$\omega_{\rho(t)} = \sigma_t^*\omega_0 = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho(t).$$

We choose $\rho(0) = 0$ and $\rho(t), t \in (-\infty, \infty)$ is a smooth geodesic line such that

$$\dot{\rho} = -\sigma_t^*\theta_\xi. \tag{2.18}$$

We pick up a geodesic segment $\phi(t)$ starting at ϕ (the geodesic will be specified later). By (2.12), we know that

$$\int_M (h-f)e^h\omega^n \geq \int_M e^h\omega^n - \int_M e^f\omega^n.$$

By choosing $f = -\dot{\phi}(0)$, we get (note that $\omega = \omega_{\phi(0)}$)

$$H(\omega) \geq -\int_M \dot{\phi}(0)e^h\omega^n + \int_M e^h\omega^n - \int_M e^{-\dot{\phi}(0)}\omega^n. \tag{2.19}$$

If ω is also G-invariant, then as above we can choose $\phi(t)$ (a geodesic line) starting at ϕ such that $\dot{\phi}(t) = -\sigma_t^*\tilde{\theta}_\xi$ (in particular, $\dot{\phi}(0) = -\tilde{\theta}_\xi$), where $\tilde{\theta}_\xi$ is the normalized Hamiltonian of ξ with respect to ω. Then by (2.19), we get $H(\omega) \geq \int_M \tilde{\theta}_\xi e^h\omega^n = H_0(\xi)$, as shown in Proposition 2.3.

If ω is not G-invariant, the argument above does not apply. However given the geodesic line $\rho(t)$ generated by $Y = -J\xi - \sqrt{-1}\xi$ through an invariant metric ω_0, we construct the unique geodesic segment $\phi(t), t \in [0,T]$ such that $\phi(0) = \phi, \phi(T) = \rho(T)$ for any fixed T. We want to understand the asymptotic behavior when T is large. Roughly speaking, such geodesic segments converge to a geodesic ray starting at ϕ which is *parallel* to $\rho(t)$ when $T \to \infty$. This intuitive statement can be made precise and strict as in Chen [11]. We will only consider those geodesic segments here since it is sufficient for our purpose.

Along the geodesic segment $\phi(t)$ connecting ϕ and $\rho(T)$, we consider the function

$$A_\phi(t) = \int_M \dot{\phi} e^{h(t)} \omega_t^n.$$

An essential point for our argument is that $A_\phi(t)$ is monotone. First we note that $A_\phi(t)$ is related to Ding's $\mathcal{F}$-functional defined in [15]. Fix a background metric ω_0 and we write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$. Recall Ding's $\mathcal{F}$-functional is defined by

$$\mathcal{F}_{\omega_0}(\omega) = \mathcal{E}_0(\phi) + \mathcal{F}_0(\phi), \tag{2.20}$$

where $\mathcal{E}_0(\phi)$ is the Aubin-Yau functional with the form

$$\mathcal{E}_0(\phi) = -\frac{1}{(n+1)} \sum_{j=1}^{n} \int_M \phi \omega_0^j \wedge \omega_\phi^{n-j},$$

and $\mathcal{F}_0(\phi)$ takes the form

$$\mathcal{F}_0(\phi) = -V \log \left(\int_M e^{h_0 - \phi} \omega_0^n \right).$$

The Aubin-Yau functional $\mathcal{E}_0(\phi)$ can be characterized by its derivative along any (C^1 for example) path $\phi(t)$

$$\frac{d\mathcal{E}_0(\phi(t))}{dt} = -\int_M \dot{\phi} \omega_\phi^n.$$

We can also compute the derivative of $\mathcal{F}_0(\phi)$ as follows,

$$\frac{d}{dt}\mathcal{F}_0(\phi) = V \left(\int_M e^{h_0 - \phi} \omega_0^n \right)^{-1} \int_M \dot{\phi} e^{h_0 - \phi} \omega_0^n. \tag{2.21}$$

By (2.17), we obtain

$$\frac{d}{dt}\mathcal{F}_0(\phi) = \int_M \dot{\phi} e^{h_\phi} \omega_\phi^n \tag{2.22}$$

and it agrees with $A_\phi(t)$ we have defined above. We can compute directly that (along a smooth curve $\phi(t)$), as in (2.26) below,

$$\frac{d^2}{dt^2}\mathcal{F}_0(\phi) = \int_M (\ddot{\phi} - |\nabla \dot{\phi}|^2) e^{h_\phi} \omega_\phi^n + \int_M (|\nabla \dot{\phi}|^2 - (\dot{\phi} + a(t))^2) e^{h_\phi} \omega_\phi^n. \tag{2.23}$$

By (2.1), we compute

$$\partial\bar{\partial}\dot{h} = \partial\bar{\partial}(-\Delta\dot{\phi} - \dot{\phi}).$$

Hence there exists a time-dependent constant $a(t)$ such that

$$\dot{h} = -\Delta\dot{\phi} - \dot{\phi} - a(t). \tag{2.24}$$

By the normalization condition (2.2), we can then get

$$\int_M (\dot{\phi} + a(t)) e^{h(t)} \omega_t^n = 0. \tag{2.25}$$

Now we can compute directly, using (2.24) and (2.25), that

$$\begin{aligned} \frac{d}{dt} A_\phi(t) &= \int_M \ddot{\phi} e^{h(t)} \omega_t^n + \int_M \dot{\phi} e^{h(t)} \dot{h} \omega_t^n + \int_M \dot{\phi} e^{h(t)} \Delta_t \dot{\phi} \omega_t^n \\ &= \int_M (\ddot{\phi} - \dot{\phi}^2 - a(t)\dot{\phi}) e^{h(t)} \omega_t^n \\ &= \int_M (\ddot{\phi} - |\nabla \dot{\phi}|_t^2) e^{h(t)} \omega_t^n + \int_M (|\nabla \dot{\phi}|^2 - (\dot{\phi} + a(t))^2) e^{h(t)} \omega_t^n. \end{aligned} \tag{2.26}$$

By the modified Poincare inequality (2.7), we know that $\mathcal{F}_0(\phi)$ is convex along (smooth) geodesics. The convexity of $\mathcal{F}_0(\phi)$ and $\mathcal{F}$-functional is established by Berndtsson [5, 7] with only bounded Kähler potential, using his curvature formulas. We state Berndtsson's results for convenience (for more details including notations, we refer to [7]). Let X be a projective manifold with semi-negative canonical line bundle ($-K_X \geq 0$) of dimension n and let U be a domain in $\mathbb{C}$. We use w to denote the coordinate in U.

Theorem 2 (Berndtsson). *Assume that $-K_X \geq 0$ and let ϕ_w be a curve of metrics on $-K_X$ such that*

$$\sqrt{-1}\partial\bar{\partial}_{w,X}\phi_w \geq 0$$

in the sense of current on $X \times U$, Then

$$F(w) := -\log \int_X e^{-\phi_w}$$

is subharmonic in U. If ϕ_w depends only on t, the real part of w ($w = t + is$), then F is convex in t. Moreover, assume $H^{0,1}(X) = 0$ and ϕ_t is uniformly bounded in the sense that there is a smooth metric ψ on $-K_X$ such that $|\psi - \phi_t| \leq C$. Then if $F(t)$ is a linear function of t in the neighborhood of $0 \in U$, then there exists a holomorphic vector field (possibly t-dependent) V on X with flow σ_t such that

$$\sigma_t^*(\partial\bar{\partial}\phi_t) = \partial\bar{\partial}\phi_0.$$

Given the interpretation of A_ϕ as the derivative of $\mathcal{F}_0$, one can give a direct proof of the following fact.

Lemma 2.1. *We have the following estimate*

$$A_\phi(0) = \int_M \dot{\phi}(0) e^h \omega^n \leq -H_0(\xi) + O(T^{-1}). \tag{2.27}$$

Proof. We know that $A_\phi(t) = \frac{d}{dt}\mathcal{F}_0$. By the convexity of $\mathcal{F}_0$, we know that

$$A_\phi(0) \leq \frac{\mathcal{F}_0(\phi(T)) - \mathcal{F}_0(\phi_0)}{T}$$

Since $\phi(T) = \rho(T)$ and we know that $A_\rho(t) = -H_0(\xi)$ for any t (equivalently, $\mathcal{F}_0(\rho(t))$ is an affine function), we then have

$$A_\phi(0) \leq \frac{\mathcal{F}_0(\rho(T)) - \mathcal{F}_0(\rho_0)}{T} + O(T^{-1}) = -H_0(\xi) + O(T^{-1}).$$

□

REMARK 2.1. This argument was pointed out by the referee which simplifies our original arguments.

Hence by (2.19) and (2.27) we obtain,

$$H(\omega) \geq H_0(\xi) + V - \int_M e^{-\dot{\phi}(0)}\omega^n + O(T^{-1}). \tag{2.28}$$

By Proposition 2.5, $\int_M e^{-\dot{\phi}}\omega_t^n$ is constant. Hence we obtain,

$$H(\omega) \geq H_0(\xi) + V - \int_M e^{-\dot{\phi}(T)}\omega_{\phi(T)}^n. \tag{2.29}$$

When T is large enough, we want to show $\dot{\phi}(T) \approx \dot{\rho}(T)$ in an effective way. To be more precise, we will show in Lemma 2.2 that

$$\int_M (\dot{\phi}(T) - \dot{\rho}(T))^2 \omega_{\rho(T)}^n = O(T^{-2}). \tag{2.30}$$

With this approximation, we will establish the following approximation in Lemma 2.3 below,

$$\left| \int_M e^{-\dot{\rho}(T)}\omega_{\rho(T)}^n - \int_M e^{-\dot{\phi}(T)}\omega_{\phi(T)}^n \right| = O(T^{-1}). \tag{2.31}$$

We then obtain, by (2.29) and (2.31),

$$H(\omega) \geq H_0(\xi) + V - \int_M e^{-\dot{\rho}(T)}\omega_{\rho(T)}^n + O(T^{-1}). \tag{2.32}$$

By the definition of $\rho(t)$, in particular (2.18), we have,

$$\int_M e^{-\dot{\rho}(T)}\omega_{\rho(T)}^n = V.$$

By letting $T \to \infty$, (2.32) then implies

$$H(\omega) \geq H_0(\xi).$$

This completes the proof. □

Next we establish (2.30) and (2.31) in the proof of Theorem 1. First we follow the comparison geometric argument in Chen [12] to establish (2.30). The argument relies on the results established in Chen [11] and Calabi-Chen [10] asserting that the space of Kähler potentials with Mabuchi metric is actually an Alexanderov space of nonpositive curvature.

LEMMA 2.2. *We have the estimate* (2.30).

Proof. Consider the triangle in $\mathcal{H}$ with vertices $B = 0, C = \phi_0, D = \phi(T) = \rho(T)$. Denote the distance $|BC| = d$ and along geodesic segment $\phi(t)$, $\int_M \dot{\phi}^2 \omega_\phi$ is constant for any $t \in [0, T]$ and we assume both are nonzero for BD, CD. Then we have

$$|BD|^2 = T^2 \int_M \dot{\rho}^2 \omega_\rho^n, |CD|^2 = T^2 \int_M \dot{\phi}^2 \omega_\phi^n.$$

Let $\tilde{B}, \tilde{C}, \tilde{D}$ be the vertices of an Euclidean triangle with the same length as BCD and denote the angle at $\tilde{D}$ by $\tilde{\theta}$. Then

$$\cos \tilde{\theta} = (|BD|^2 + |CD|^2 - d^2)/2|BD||CD|.$$

The tangent vector at D for BD is given by $\dot{\rho}(T)$, for CD by $\dot{\phi}(T)$. The inner product of two vectors is then

$$(\dot{\phi}(T), \dot{\rho}(T)) = \int_M \dot{\phi}\dot{\rho}\omega_{\phi(T)}^n.$$

Hence the angle θ formed at D of the triangle BCD is given by

$$\cos \theta = \frac{\int_M \dot{\phi}\dot{\rho}\omega_{\phi(T)}^n}{\left(\int_M \dot{\phi}^2 \omega_{\phi(T)}^n \int_M \dot{\rho}^2 \omega_{\phi(T)}^n\right)^{1/2}}.$$

Since $\mathcal{H}$ has nonpositive curvature, we know $\theta \leq \tilde{\theta}$, hence $0 < \cos \tilde{\theta} \leq \cos \theta \leq 1$ (note that we consider T really large, hence $\tilde{\theta}$ is small). It follows that

$$T^2 \int_M (\dot{\phi}(T) - \dot{\rho}(T))^2 \omega_{\phi(T)}^n \leq d^2. \tag{2.33}$$

This proves (2.30). □

LEMMA 2.3. *The approximation* (2.31) *holds.*

Proof. First we apply Proposition 2.4 to get

$$\int_M e^{-\dot{\phi}}(\dot{\phi} - \dot{\rho})\omega_\rho^n \leq \int_M (e^{-\dot{\rho}} - e^{-\dot{\phi}})\omega_\rho^n \leq \int_M e^{-\dot{\rho}}(\dot{\phi} - \dot{\rho})\omega_\rho^n. \tag{2.34}$$

We omit the subscript T in above for simplicity. By Hölder inequality, we have

$$\left|\int_M e^{-\dot{\rho}}(\dot{\phi} - \dot{\rho})\omega_\rho^n\right|^2 \leq \int_M e^{-2\dot{\rho}}\omega_\rho^n \int_M (\dot{\phi} - \dot{\rho})^2 \omega_\rho^n.$$

Observe that $\dot{\rho} = -\sigma_t^* \theta_\xi = -\theta_\xi \circ \sigma_t$ and hence it is uniformly bounded. It then follows from Lemma 2.2 that

$$\left|\int_M e^{-\dot{\rho}}(\dot{\phi} - \dot{\rho})\omega_\rho^n\right| \leq CT^{-1},$$

where C depends only on ω_0, ξ. Similarly we have

$$\left|\int_M e^{-\dot{\phi}}(\dot{\phi} - \dot{\rho})\omega_\rho^n\right|^2 \leq \int_M e^{-2\dot{\phi}}\omega_\rho^n \int_M (\dot{\phi} - \dot{\rho})^2 \omega_\rho^n. \tag{2.35}$$

Along the geodesic segment we know that $\ddot{\phi}$ is nonnegative (ϕ is convex in t), hence

$$\dot{\phi}(T) \geq (\phi(T) - \phi(0))\, T^{-1} = \rho(T)T^{-1} - \phi_0 T^{-1}.$$

It follows that

$$-\dot{\phi}(T) \leq -\rho(T)T^{-1} + \phi_0 T^{-1}.$$

Since $\dot{\rho} = -\theta_\xi \circ \sigma_t$ is uniformly bounded, we have that $|\rho(t)| \leq Ct$ for any t. It follows that $-\dot{\phi}(T) \leq C$ for some uniform constant C (we assume $T \geq 1$), depending only on ϕ_0 and θ_ξ. Hence by (2.35) and (2.30), we have

$$\left| \int_M e^{-\dot{\phi}}(\dot{\phi} - \dot{\rho})\omega_\rho^n \right| \leq CT^{-1}.$$

This completes the proof of (2.31). □

We include a proof of the monotonicity of $A_\phi(t)$ based on the modified Poincare inequality (see (2.26)) along $C^{1,1}$ geodesics. We hope this alternative proof is interesting in its own right.

Lemma 2.4. *The function $A_\phi(t)$ is monotone increasing along a geodesic segment.*

Proof. By the observation in (2.17) above $e^h\omega^n$ is well-defined for any bounded generalized Kähler potential. First suppose $\phi(t)$ is a smooth geodesic path of Kähler potentials, then by (2.22) and (2.23), $A_\phi(t)$ is monotone increasing by Proposition 2.1 ((2.7)). Now suppose ϕ_0, ϕ_T are two smooth Kähler potentials in $[\omega_0]$ and we consider the $C^{1,1}$ geodesic segment $\phi(t)$ connecting the given two Kähler potentials. We use an approximation argument. For any $\epsilon > 0$, there exists an approximating smooth geodesic $\phi_\epsilon(t)$ and we can associate such a path

$$A_\epsilon(t) = \int_M \dot{\phi}_\epsilon(t) e^{h_\epsilon(t)} \omega_\epsilon^n(t) = \lambda_\epsilon(t) \int_M \dot{\phi}_\epsilon(t) e^{h_0 - \phi_\epsilon(t)} \omega_0^n, \tag{2.36}$$

where $\lambda_\epsilon(t)$ is a time dependent constant such that

$$\lambda_\epsilon(t) \int_M e^{h_0 - \phi_\epsilon(t)} \omega_0^n = V.$$

A direct computation as above shows that

$$\frac{dA_\epsilon}{dt} = \int_M (\ddot{\phi}_\epsilon - |\nabla \dot{\phi}_\epsilon|^2) e^{h_\epsilon} \omega_\epsilon^n + \int_M \left(|\nabla \dot{\phi}_\epsilon|^2 - (\dot{\phi}_\epsilon + a_\epsilon(t))^2 \right) e^{h_\epsilon} \omega_\epsilon^n > 0.$$

It then follows that

$$A_\epsilon(0) < A_\epsilon(T).$$

Since $\phi_\epsilon(t) \to \phi(t)$ in $C^{1,\alpha}$ when $\epsilon \to 0$, we have

$$A_\epsilon(t) = A_\phi(t) + o(\epsilon).$$

The desired monotonicity follows by letting $\epsilon \to 0$. □

In general, one can associate an invariant for any geodesic rays and obtain a lower bound of $H(\omega)$ in terms of such an invariant with some extra efforts. Suppose $\rho(t)$ is a geodesic ray with $\rho(t) \in \mathcal{H}_\infty$. By an observation of Berndtsson [7], $\rho(t)$ is Lipschitz in t-direction, and hence $\dot{\rho}$ is a L^∞ function in t.

DEFINITION 2.1. Let $\rho(t)$ be a geodesic ray with $\rho(t) \in \mathcal{H}_\infty$, define the function

$$Y_\rho(t) = -\int_M \dot{\rho} e^{h_\rho} \omega_\rho^n - V \log \left(V^{-1} \int_M e^{-\dot{\rho}} \omega_\rho^n \right),$$

where we understand the notion as follows,

$$e^{h_\rho} \omega_\rho^n = \lambda e^{h_0 - \rho} \omega_0^n, \text{ with } \lambda = V \left(\log \int_M e^{h_0 - \phi} \omega_0^n \right)^{-1}.$$

By the discussion above, we know that

$$\frac{d\mathcal{F}_0(\rho(t))}{dt} = \int_M \dot{\rho} e^{h_\rho} \omega_\rho^n$$

and hence it is increasing by Berndtsson's result. Berndtsson's result (Proposition 2.5) holds in a more general setting. Hence $Y_\rho(t)$ is a decreasing function and we can define

$$Y_\rho = \lim_{t \to \infty} Y_\rho(t).$$

Using the very similar ideas as above, one can obtain that, for any geodesic ray ρ (with some mild assumption, say $|\rho(t)| \leq Ct$ and $\rho(t)$ is C^1),

$$H(\omega) \geq Y_\rho.$$

We shall not pursue this generality here.

2.3. Modified $\mathcal{F}$-functional. When the Futaki invariant is not zero, or equivalently, the extremal holomorphic vector field X is not zero, Tian-Zhu [35] introduced the notion of modified $\mathcal{F}_X$ functional. This functional is important for the study of Kähler-Ricci solitons. Recall that the $\mathcal{F}_X$ functional is defined by, given any path $\phi(t)$ connecting ω and $\omega_\phi = \omega + \partial\bar{\partial}\phi$,

$$\mathcal{F}_X(\phi) = -\int_0^1 \int_M \dot{\phi}_t e^{\theta_X(\phi_t)} \omega_{\phi_t}^n dt - V \log \left(\frac{1}{V} \int_M e^{h_g - \phi} \omega^n \right), \tag{2.37}$$

where h is the Ricci potential of ω, and $\theta_X(\omega)$ is the potential of X with respect to ω ($\iota_X \omega = \sqrt{-1}\bar{\partial}\theta_X$), both satisfying the normalization,

$$\int_M e^h \omega^n = \int_M e^{\theta_X} \omega^n = \int_M \omega^n = V.$$

Tian-Zhu proved that $\mathcal{F}_X$ is independent of the path and its first variation formula of $\mathcal{F}_X$ is given by,

$$\frac{d\mathcal{F}_X}{dt} = -\int_M \dot{\phi} e^{\theta_X(\phi)} \omega_\phi^n + V \left(\int_M e^{h-\phi} \omega^n \right)^{-1} \int_M \dot{\phi} e^{h-\phi} \omega^n. \tag{2.38}$$

The Euler-Lagragian equation is given by

$$e^{\theta_X(\phi)}\omega_\phi^n = Ve^{h-\phi}\omega^n\left(\int_M e^{h-\phi}\omega^n\right)^{-1},$$

which is exactly the equation for a Kähler-Ricci soliton with extremal vector field X. By (2.17), we can write (2.38) as

$$\frac{d\mathcal{F}_X}{dt} = -\int_M \dot\phi e^{\theta_X(\phi)}\omega_\phi^n + \int_M \dot\phi e^{h_\phi}\omega_\phi^n. \tag{2.39}$$

We can then compute the second derivative of $\mathcal{F}_X$,

Proposition 2.6. *The second variation of $\mathcal{F}_X$ is given by*

$$\begin{aligned}\frac{d^2\mathcal{F}_X}{dt^2} = &-\int_M(\ddot\phi - |\nabla\dot\phi|_\phi^2)\left(e^{\theta_X(\phi)}\omega_\phi^n - Ve^{h-\phi}\omega^n\right)\\ &+\int_M(|\nabla\dot\phi|_\phi^2 - |\dot\phi + a(t)|^2)e^{h_\phi}\omega_\phi^n.\end{aligned} \tag{2.40}$$

Proof. We compute

$$\begin{aligned}\frac{d}{dt}\int_M \dot\phi e^{\theta_X(\phi)}\omega_\phi^n &= \int_M(\ddot\phi + \dot\phi X(\dot\phi) + \dot\phi\Delta\dot\phi)e^{\theta_X(\phi)}\omega_\phi^n\\ &= \int_M(\ddot\phi - |\nabla\dot\phi|^2)e^{\theta_X(\phi)}\omega_\phi^n,\end{aligned}$$

where we use integration by parts and the fact that $\nabla_\phi\theta_X(\phi) = X$. The other part is exactly the same as the computation of $\mathcal{F}$-functional. □

Invoking Berndtsson [7], we have the following,

Proposition 2.7. *The $\mathcal{F}_X$ functional is convex along any $C^{1,1}$ geodesic $\phi(t) \in \mathcal{H}_{1,1}$. If $\mathcal{F}_X(\phi(t))$ is a linear function in t, then there exists a one-parameter holomorphism $\sigma_t : M \to M$ such that*

$$\sigma_{\phi(t)} = \sigma_t^*\omega_{\phi_0}.$$

Proof. The only thing we need in addition is that $\theta_X(\phi)$ is uniformly bounded. This follows from [39] and the proof can be extended directly to $C^{1,1}$ potentials. The statement then follows Berndtsson [7]. □

Remark 2.2. As a consequence of this convexity one can give a proof of Tian-Zhu's result [35] on the uniqueness of Kähler-Ricci soliton in $(M, [\omega_0])$. The only additional fact we need is that the extremal vector field for Kähler-Ricci soliton is unique up to automorphisms [36]. In Kähler-Einstein case, Berman [3] and Berndtsson [7] have already given a new proof of Bando-Mabuchi's uniqueness theorem [2] using such convexity directly. We learned from Song Sun that Berndtsson can extend his results in [7] to give a new proof of Tian-Zhu's uniqueness theorem.

2.4. Perelman's μ-functional. As a direct consequence of Theorem 1, we can obtain an upper bound for Perelman's μ-functional and Kähler Ricci soliton, if exists, maximizes the μ functional. Recall Perelman's W-functional [27] is defined as

$$W(g,\tau,f)=\frac{1}{(4\pi\tau)^{-n/2}}\int_M(\tau(R+|\nabla f|^2)-n+f)e^{-f}dv_g,$$

where R is the scalar curvature of the metric g. On Fano manifolds, however, it is more convenient to let $\tau=1/2$ and it is also convenient using (complex) geometric quantities which differ only by multiple of a constant. Hence we consider W-functional on $(M,[\omega_0])$ by

$$W(\omega,f)=\int_M(R+|\nabla f|^2+f)e^{-f}\omega^n, \tag{2.41}$$

with the normalization condition

$$\int_M e^{-f}\omega^n=\int_M\omega^n=V.$$

And the μ-functional is defined to be

$$\mu(\omega)=\inf_f W(\omega,f).$$

By Rothaus [29], there always exists some smooth function f to minimize $W(\omega,\cdot)$ and such a minimizer f satisfies the equation

$$2\Delta f+f+R-|\nabla f|^2=V^{-1}\mu. \tag{2.42}$$

When ω_0 is a Kähler-Ricci soliton, then $f=-h_0$. An easy way to see this fact is to use a result of Sun-Wang [32] (Lemma 3.4), where they prove that there exists a unique solution of (2.42) if g is a gradient shrinking Ricci-soliton. Suppose ω_0 is a Kähler-Ricci soliton. Then

$$\Delta h_0+|\nabla h_0|^2+h_0=constant.$$

We can write the scalar curvature as $R=n+\Delta h_0$. It follows that $-h_0+c$ is a solution of (2.42) for an appropriate constant c. By the uniqueness and the normalization condition, we know $f=-h_0$. In particular,

$$\mu(\omega_0)=W(\omega_0,f)=W(\omega,-h_0).$$

COROLLARY 2.1. *For any $\omega\in[\omega_0]$,*

$$\mu(\omega)+H(\omega)\le nV.$$

Hence Perelman's μ-functional is bounded above by,

$$\mu(\omega)\le nV-N_X.$$

In particular, if ω_0 is a Kähler-Ricci soliton, then for any $\omega\in[\omega_0]$,

$$\mu(\omega)\le\mu(\omega_0)=nV-N_X.$$

Proof. We observe that, since $R = n + \Delta h$,

$$\mu(\omega) \leq W(\omega, -h) = \int_M \left((n + \Delta h) + |\nabla h|^2 - h\right) e^h \omega^n = nV - H(\omega).$$

When ω_0 is a Kähler-Ricci soliton, then h_0 is the Hamiltonian of ξ_0 (the imaginary part of the extremal vector field),

$$H(\omega_0) = \int_M h_0 e^{h_0} \omega_0^n = \int_M \theta_{\xi_0} e^{h_0} \omega_0^n = H_0(\xi_0) = N_X.$$

By the discussion above, we know

$$\mu(\omega_0) = W(\omega, -h_0) = nV - N_X.$$

□

For metrics which is invariant with respect to $Im(X)$, such an upper bound was proved in a recent paper by Tian-Zhang-Zhang-Zhu [37] and the authors proved that the Kähler-Ricci flow with any invariant initial metric actually maximizes μ functional asymptotically provided that the modified Mabuchi functional is bounded below, and it is used to give an alternative proof of convergence of the Kähler-Ricci flow for invariant metric assuming the existence of Kähler-Ricci soliton. It is a belief that if Kähler-Ricci soliton exists, then the Kähler-Ricci flow would actually converge to the soliton, without invariant assumption on initial metric. In [37] the authors proposed a conjecture (Conjecture 3.3), which suggests how one might prove such a convergent result. Corollary 2.1 seems to strengthen this belief.

3. Kähler-Ricci soliton and geodesic stability. First we recall the following Donaldson's conjecture [17],

Conjecture 3 (Donaldson). *The following are equivalent:*

1. *There is no constant scalar metric in* $(M, [\omega_0])$.
2. *There is a geodesic ray* $\phi(t), t \in [0, \infty)$ *such that the derivative of Mabuchi's K-energy, for all* $t \in [0, \infty)$,

$$\frac{d\mathcal{K}}{dt} = -\int_M \dot{\phi}(R - \underline{R})\omega_\phi^n < 0.$$

3. *For any Kähler potential* ϕ, *there exists a geodesic ray as in (2) starting at* ϕ.

The conjecture remains open in both directions (see [13] for partial results). One technical difficulty is that the geodesic rays do not have enough regularity to talk about the derivative of $\mathcal{K}$-energy in general. But $\mathcal{F}$-functional and its derivative make sense even for geodesics with only bounded potential. When $(M, [\omega_0])$ is Fano, it is very natural to formulate the conjecture in terms of $\mathcal{F}$-functional.

Conjecture 4. *Let* $(M, [\omega_0])$ *be a Fano manifold. Then the following are equivalent*

1. *There exists no Kähler-Einstein metric in* $(M, [\omega_0])$.
2. *There exists a geodesic ray* $\phi(t)$ *such that for all* $t \in [0, \infty)$ $\frac{d\mathcal{F}}{dt} < 0$.
3. *For every point* $\phi \in \mathcal{H}$, *there exists a geodesic ray as in (2) starting at* ϕ.

From analytic point of view, Kähler-Ricci soliton is a natural generalization of Kähler-Einstein metric. We extend the discussion for Kähler-Einstein metrics to Kähler-Ricci solitons. We formulate a version of geodesic stability for Kähler-Ricci soliton, with the hope that it will motivate an algebro-geometric notion of stability for Kähler-Ricci solitons. The key notion in our formulation is the modified $\mathcal{F}$ functional introduced by Tian-Zhu [35].

CONJECTURE 5. *Let $(M, [\omega_0])$ be a Fano manifold. Recall X, the extremal holomorphic vector field, is determined* a priori. *We consider $Im(X)$ invariant metrics in $\mathcal{H}$. The following are equivalent,*

1. *There exists no Kähler-Ricci soliton in $(M, [\omega_0])$.*
2. *There exists a geodesic ray $\phi(t)$ such that for all $t \in [0, \infty)$ $\frac{d\mathcal{F}_X}{dt} < 0$.*
3. *For every point $\phi \in \mathcal{H}$, there exists a geodesic ray as in (2) starting at ϕ.*

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