

SADDLE TOWERS IN HEISENBERG SPACE*

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Abstract. We construct most symmetric Saddle towers in Heisenberg space i.e. periodic minimal surfaces that can be seen as the desingularization of vertical planes intersecting equiangularly. The key point is the construction of a suitable barrier to ensure the convergence of a family of bounded minimal disks. Such a barrier is actually a periodic deformation of a minimal plane with prescribed asymptotic behavior. A consequence of the barrier construction is that the number of disjoint minimal graphs supported on domains is not bounded in Heisenberg space.

Key words. Heisenberg space, minimal surfaces, saddle tower, Scherk surface, symmetric surfaces, deformation of surfaces, barrier construction, supported minimal surfaces.

AMS subject classifications. 53A10, 53C42.

1. Introduction. The study of periodic surfaces has recently encountered new developments in homogeneous 3-spaces. In $\mathbb{H}^2 \times \mathbb{R}$, new examples have been constructed such as for instance doubly periodic minimal surfaces by Mazet, Rodríguez and Rosenberg [12] or genus one constant mean curvature $1/2$ surfaces by Plehnert [19]. Other examples are involved in important results such as the resolution of Alexandrov problem in a quotient space of $\mathbb{H}^2 \times \mathbb{R}$ by Menezes [15] or the fact the Calabi-Yau conjectures do not hold for embedded minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ by Rodríguez and Tinaglia [21]. In Heisenberg space, new examples are mostly of Scherk type like the Jenkins-Serrin theorem for compact domains obtained by Pinheiro [18].

The present paper deals with the construction of another kind of Scherk type surfaces, called *most symmetric Saddle towers*, in Heisenberg space. A Saddle tower is a minimal surface that can be thought of as the desingularization of n vertical planes, $n \geq 2$, intersecting along a vertical geodesic; in particular it is a complete embedded minimal surface with $2n$ planar ends. Historically, they were first found by Scherk [23] for $n = 2$ in Euclidean space \mathbb{R}^3 — a Saddle tower with $n = 2$ is usually called a *singly periodic Scherk surface* — and (more than) a century and half later Karcher [8] generalized the construction to any $n \geq 2$. Another couple of decades later, Morabito and Rodríguez [17] and Pyo [20] constructed Saddle towers in $\mathbb{H}^2 \times \mathbb{R}$. Recently, Menezes [16] constructed the singly (and doubly) periodic Scherk surfaces in semi-direct product spaces, including Heisenberg space.

We use a classical method to construct Saddle towers in Heisenberg space (see Section 4). We first construct a sequence of embedded minimal disks — by solving Plateau problems on suitable Jordan curves — such that it converges to an embedded minimal surface bordered by horizontal geodesic arcs, which we call the *fundamental piece*. Reflecting the fundamental piece along the geodesics in its boundary, we get the desired Saddle tower:

THEOREM 4.1. *For any $n \geq 2$ and $a > 0$, there exists a properly embedded singly periodic minimal surface of genus zero with $2n$ planar ends distributed at constant angle π/n . We call it a most symmetric Saddle tower.*

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Since the Saddle tower is composed of reflections of the fundamental piece, its ends must be equiangularly distributed to ensure the embeddedness of the surface. Note that Morabito and Rodríguez [17] constructed non equiangular Saddle towers with a conjugate Plateau technique, which does not apply in Heisenberg space.

The key step is to prove the convergence of the sequence of minimal disks, which is done by using a suitable barrier. We construct that barrier in Section 3 by deforming a minimal plane so that we control the asymptotic behavior of the deformed minimal surfaces. The deformation technique is based on the existence of an elliptic operator, called the *compactified mean curvature operator*, containing information on both the asymptotic behavior and the mean curvature. This technique has already been used by the author and Hauswirth [3] for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ of constant mean curvature $1/2$.

It is an immediate consequence of Weierstrass representation that the construction of such surfaces is not possible in Euclidean space since — under reasonable assumptions on the total curvature — minimal ends are asymptotically rotational. The present barrier construction highlights another difference with Euclidean case:

THEOREM 3.9. *In Heisenberg space, the number of disjoint domains supporting minimal vertical graphs is unbounded.*

2. Preliminaries. We recall basic definitions and properties on Heisenberg space and fix classic notations.

2.1. A model for Heisenberg space. 3-dimensional Heisenberg space Nil_3 is a nilpotent Lie group which is usually represented in $\text{GL}_3(\mathbb{R})$ by:

$$\text{Nil}_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\},$$

and its Lie algebra \mathfrak{nil}_3 can be seen as the subset of $\mathcal{M}_3(\mathbb{R})$:

$$\mathfrak{nil}_3 = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

In this paper we use the parametrization of Nil_3 induced by the exponential map:

$$\exp : \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{nil}_3 \mapsto \begin{pmatrix} 1 & x_1 & x_3 + \frac{x_1 x_2}{2} \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Nil}_3.$$

With these notations, Nil_3 identifies with \mathbb{R}^3 and the group law is:

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{x_1 y_2 - x_2 y_1}{2} \right).$$

We call *canonical frame* the frame (E_1, E_2, E_3) which is the extension by left translation of the canonical frame of \mathbb{R}^3 at the origin i.e.:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3}.$$

Endowing Nil_3 with the following left-invariant metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2 dx_1 - x_1 dx_2) + dx_3 \right)^2,$$

makes the canonical frame orthonormal.

Heisenberg space is a riemannian fibration for the projection on the first two coordinates $\pi : (x_1, x_2, x_3) \in \text{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$, which means that the isomorphism $d\pi|_{(\ker d\pi)^\perp}$ onto the Euclidean plane is an isometry. Given this structure, vector fields spanned by (E_1, E_2) are referred to as *horizontal* and the direction E_3 is said *vertical*. In particular, a *vertical graph* in Nil_3 is a complete immersion transverse to E_3 .

In the sequel, we mostly work in cylindrical coordinates (ρ, θ, x_3) with $\rho \geq 0$ and $\theta \in \mathbb{R}$ such that:

$$x_1 = \rho \cos \theta \quad \text{and} \quad x_2 = \rho \sin \theta,$$

and we consider the *cylindrical frame* (E_ρ, E_θ, E_3) with:

$$\begin{aligned} E_\rho &= \cos \theta E_1 + \sin \theta E_2 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} \\ \text{and} \quad E_\theta &= -\sin \theta E_1 + \cos \theta E_2 = -\sin \theta \frac{\partial}{\partial x_1} + \cos \theta \frac{\partial}{\partial x_2} + \frac{\rho}{2} \frac{\partial}{\partial x_3}. \end{aligned}$$

The cylindrical frame is also orthonormal and the Levi-Civita connection ∇ on Nil_3 in terms of (E_ρ, E_θ, E_3) is given by:

$$\begin{aligned} \nabla_{E_\rho} E_\rho &= 0 & \nabla_{E_\theta} E_\rho &= \frac{1}{\rho} E_\theta - \frac{1}{2} E_3 & \nabla_{E_3} E_\rho &= -\frac{1}{2} E_\theta \\ \nabla_{E_\rho} E_\theta &= \frac{1}{2} E_3 & \nabla_{E_\theta} E_\theta &= -\frac{1}{\rho} E_\rho & \nabla_{E_3} E_\theta &= \frac{1}{2} E_\rho \\ \nabla_{E_\rho} E_3 &= -\frac{1}{2} E_\theta & \nabla_{E_\theta} E_3 &= \frac{1}{2} E_\rho & \nabla_{E_3} E_3 &= 0 \end{aligned} .$$

2.2. Model surface. The barrier constructed in Section 3 comes from a (suitable) deformation of a surface, labeled S^0 , to which we refer as the *model surface*. In our model of Nil_3 , S^0 is nothing but the entire graph $\{x_3 = 0\}$. It is a complete embedded minimal surface, its tangent plane is spanned by the coordinate vector fields:

$$\frac{\partial}{\partial x_1} = E_1 + \frac{x_2}{2} E_3 \quad \text{and} \quad \frac{\partial}{\partial x_2} = E_2 - \frac{x_1}{2} E_3,$$

and has unit normal:

$$N^0 = \frac{1}{\sqrt{4 + \rho^2}}(\rho E_\theta + 2E_3).$$

The end of S^0 is of annulus type and is vertical in the sense that the normal vector N^0 is asymptotically horizontal: $\langle N^0, E_3 \rangle \rightarrow 0$ when $\rho \rightarrow +\infty$.

Consider a point $x^0 = (x_1^0, x_2^0, x_3^0)$ in Nil_3 and a unit vector V in the tangent space $T_{x^0}\text{Nil}_3$ at x^0 , written $V = R \cos \varphi E_1 + R \sin \varphi E_2 + \gamma E_3$ with $R \geq 0$, $\varphi, \gamma \in \mathbb{R}$ and

$R^2 + \gamma^2 = 1$. The geodesic passing through x^0 and directed by V at x^0 admits the parametrization $t \in \mathbb{R} \mapsto (x_1(t), x_2(t), x_3(t))$ by arc length with (see [11] for details):

$$\begin{cases} x_1(t) = x_1^0 + \frac{R}{2\gamma}(\sin(2\gamma t + \varphi) - \sin \varphi) \\ x_2(t) = x_2^0 - \frac{R}{2\gamma}(\cos(2\gamma t + \varphi) - \cos \varphi) \\ x_3(t) = x_3^0 - \frac{R}{4\gamma} \left[x_1^0(\cos(2\gamma t + \varphi) - \cos \varphi) + x_2^0(\sin(2\gamma t + \varphi) - \sin \varphi) \right] \\ \qquad \qquad \qquad + \frac{1 + \gamma^2}{2\gamma}t - \frac{R^2}{4\gamma^2} \sin(2\gamma t) \end{cases}$$

If $x^0 \in S^0$ and $V = N^0$, meaning:

$$R = \frac{\rho}{\sqrt{4 + \rho^2}}, \quad \varphi = \theta + \frac{\pi}{2} \quad \text{and} \quad \gamma = \frac{2}{\sqrt{4 + \rho^2}},$$

with $x_1^0 = \rho \cos \theta$, $x_2^0 = \rho \sin \theta$ and $x_3^0 = 0$, the parametrization writes:

$$\begin{cases} x_1(t) = \rho \cos \theta + \frac{\rho}{4} \left[\cos \left(\frac{4t}{\sqrt{4 + \rho^2}} + \theta \right) - \cos \theta \right] \\ x_2(t) = \rho \sin \theta + \frac{\rho}{4} \left[\sin \left(\frac{4t}{\sqrt{4 + \rho^2}} + \theta \right) - \sin \theta \right] \\ x_3(t) = \frac{\rho^2}{16} \sin \left(\frac{4t}{\sqrt{4 + \rho^2}} \right) + \frac{8 + \rho^2}{16} \frac{4t}{\sqrt{4 + \rho^2}} \end{cases} \quad (1)$$

Fixing $t \neq 0$ and letting (ρ, θ) vary, System (1) becomes a parametrization of the equidistant surface to S^0 at signed (normal) distance t . And making $\rho \rightarrow +\infty$, we see that asymptotically the equidistant surface grows linearly with respect to ρ — namely, the equidistant surface asymptotically behaves like the quadric:

$$t^2 x_1^2 + t^2 x_2^2 - 4x_3^2 = t^2 \left(4 - \frac{t^2}{3} \right).$$

The growth of equidistant surfaces to the model surface is important in Section 3. Indeed, the information we have on the asymptotic behavior of the deformations of S^0 is precisely the asymptotic normal signed distance to S^0 .

DEFINITION 2.1. Consider a surface S which can be parametrized by:

$$(\rho, \theta) \mapsto (\rho \cos \theta, \rho \sin \theta, h(\rho, \theta)),$$

at least for ρ big enough and some $\theta \in \mathbb{R}$. When it exists, the *asymptotic horizontal signed distance* of S to the model surface S^0 in the direction θ is the quantity:

$$d_\infty(S, S^0)(\theta) = \lim_{\rho \rightarrow +\infty} \frac{2}{\rho} h(\rho, \theta),$$

the qualification “horizontal” coming from the fact that N^0 is asymptotically horizontal.

2.3. Schwarz symmetrization. The classical Schwarz reflection for minimal surfaces in Euclidean space states that if a minimal immersion contains a straight line in its boundary, it can be smoothly extended across that line by the π -rotation about the line. This result applies to Heisenberg space — and in general to homogeneous 3-spaces with 4-dimensional isometry group — if restricted to extensions across horizontal or vertical geodesics (see [1] and details in [22, 10]). Moreover, in Heisenberg space the geodesic reflection along a horizontal geodesic passing through the vertical axis $\{x_1 = x_2 = 0\}$ is exactly the euclidean π -rotation about the geodesic.

Consider a minimal surface Σ bordered only by horizontal and vertical geodesic arcs. Then Σ can be extended to a smooth minimal surface $\hat{\Sigma}$ by geodesic reflections along each geodesic in the boundary of Σ . The boundary of $\hat{\Sigma}$ also only contains horizontal and vertical geodesic arcs and the extension process can be iterated. Note that a meeting point of several geodesic arcs of the boundary of Σ is a removable singularity of the extended surface $\hat{\Sigma}$ and thus $\hat{\Sigma}$ is smooth at this point (see [4, 7]).

We use Schwarz symmetrization in Section 4 to extend the fundamental piece to a complete embedded smooth minimal surface with the desired symmetry properties.

2.4. Some notations. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk, $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ its closure and (r, θ) the polar coordinates on $\overline{\mathbb{D}}$. The boundary $\partial\mathbb{D}$ of \mathbb{D} is identified with \mathbb{S}^1 .

The space $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, with $k \geq 0$ and $0 < \alpha < 1$, is the usual Hölder space over $\overline{\mathbb{D}}$ and $\mathcal{C}_0^{k,\alpha}(\overline{\mathbb{D}})$ is the subspace of $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ of functions that are zero on the boundary of \mathbb{D} .

Finally, we consider the spaces $L^2(\cdot)$ endowed with the natural scalar product denoted $\langle \cdot, \cdot \rangle_{L^2(\cdot)}$ and Hilbert norm $|\cdot|_{L^2(\cdot)}$.

3. Barrier construction. The model surface S^0 is an embedded disk. We deform it by mean of a differential operator taking into account both the asymptotic behavior and the mean curvature. The construction of this operator is called a *compactification* of the mean curvature since it is based on a conformal change of the induced metric to extend it on the boundary of the parameterizing disk. To do so, we need a conformal parametrization of S^0 which writes (in polar coordinates):

$$X^0 : (r, \theta) \in \mathbb{D} \mapsto \left(\frac{4r}{1-r^2} \cos \theta, \frac{4r}{1-r^2} \sin \theta, 0 \right).$$

We parametrize entire vertical graphs of Nil_3 by:

$$X^\eta : (r, \theta) \in \mathbb{D} \mapsto \left(\frac{4r}{1-r^2} \cos \theta, \frac{4r}{1-r^2} \sin \theta, \eta(r, \theta) \frac{1+r^2}{1-r^2} \right),$$

for some map $\eta : \mathbb{D} \rightarrow \mathbb{R}$. We call such a parametrization *graph coordinates at infinity*.

In the sequel, we are interested in graphs such that $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ and we use graph coordinates at infinity to compactify surfaces and quantify their asymptotic behavior. Indeed, using Definition 2.1 a graph X^η with $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ is at asymptotic horizontal signed distance to the model surface $d_\infty(X^\eta, S^0)(\theta) = \eta(1, \theta)$. Surfaces are thus considered as *compact surfaces with boundary* and we can apply the method first developed by White [26]. Also note that the value $\eta|_{\partial\mathbb{D}}$ is invariant under vertical translations.

3.1. Compactification of the mean curvature. From now on, to ease the notations, we denote with indexes 1, 2 quantities related to coordinates r, θ respectively.

THEOREM 3.1. *For any entire graph admitting graph coordinates X^η with $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$, the mean curvature $H(\eta)$ verifies:*

$$\frac{2}{r} \sqrt{|g(0)|} H(\eta) = \sum_{i,j} A_{ij}(r, \eta, D\eta) \eta_{ij} + B(r, \eta, D\eta), \tag{2}$$

where $|g(0)|$ is the determinant of the metric induced by X^0 , A_{ij} and B are $\mathcal{C}^{0,\alpha}$ functions on $\overline{\mathbb{D}}$ which are real-analytic in their variables and $A = (A_{ij})$ is a coercive matrix on $\overline{\mathbb{D}}$.

Proof (See Appendix A for computation details). The definition of the mean curvature of X^η is the following:

$$2H(\eta) = \sum_{i,j} g^{ij}(\eta) \langle \nabla_{X_i^\eta} X_j^\eta, N^\eta \rangle,$$

where X_1^η (resp. X_2^η) is the derivative of X^η with respect to r (resp. θ), N^η is the unit normal to X^η and $(g^{ij}(\eta))$ is the inverse matrix of the metric $g(\eta) = (g_{ij}(\eta))$ induced by X^η . Namely, we have:

$$\begin{aligned} X_1^\eta(r, \theta) &= \frac{4(1+r^2)}{(1-r^2)^2} E_\rho + \frac{4r}{(1-r^2)^2} \left(\eta + \frac{(1+r^2)\eta_1}{4r} (1-r^2) \right) E_3 \\ \text{and } X_2^\eta(r, \theta) &= \frac{4r}{1-r^2} E_\theta - \frac{8r^2}{(1-r^2)^2} \left(1 - \frac{(1+r^2)\eta_2}{8r^2} (1-r^2) \right) E_3, \end{aligned}$$

with (E_ρ, E_θ, E_3) the cylindrical frame. The first fundamental form $g(\eta)$ is:

$$\begin{aligned} g_{11}(\eta) &= \frac{16(1+r^2)^2}{(1-r^2)^4} \left[\left(1 + \frac{r^2\eta^2}{(1+r^2)^2} \right) + \frac{r\eta\eta_1}{2(1+r^2)} (1-r^2) + \frac{\eta_1^2}{16} (1-r^2)^2 \right], \\ g_{12}(\eta) &= -\frac{32r^3}{(1-r^2)^4} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2) - \frac{(1+r^2)^2\eta_1\eta_2}{32r^3} (1-r^2)^2 \right] \\ \text{and } g_{22}(\eta) &= \frac{16r^2(1+r^2)^2}{(1-r^2)^4} \left[1 - \frac{\eta_2}{1+r^2} (1-r^2) + \frac{\eta_2^2}{16r^2} (1-r^2)^2 \right], \end{aligned}$$

and its determinant $|g(\eta)|$ writes:

$$|g(\eta)| = \left(\frac{16r(1+r^2)^2}{(1-r^2)^4} w(\eta) \right)^2 = |g(0)| w^2(\eta),$$

with the following expression of $w(\eta)$:

$$\begin{aligned} w(\eta) = \left[1 - \frac{\eta_2}{1+r^2} (1-r^2) + \left(\frac{r^2\eta^2}{(1+r^2)^4} + \frac{\eta_2^2}{16} \right) (1-r^2)^2 + \frac{r\eta\eta_1}{2(1+r^2)^3} (1-r^2)^3 \right. \\ \left. + \frac{\eta_1^2}{16(1+r^2)^2} (1-r^2)^4 \right]^{1/2}. \end{aligned}$$

Also the expression of the unit normal N^η is:

$$\begin{aligned} N^\eta = \frac{1}{\sqrt{|g(\eta)|}} X_1^\eta \times X_2^\eta = \frac{1}{w(\eta)} \left[-\frac{r(1-r^2)}{(1+r^2)^2} \left(\eta + \frac{(1+r^2)\eta_1}{4r} (1-r^2) \right) E_\rho \right. \\ \left. + \frac{2r}{1+r^2} \left(1 - \frac{(1+r^2)\eta_2}{8r^2} (1-r^2) \right) E_\theta + \frac{1-r^2}{1+r^2} E_3 \right]. \end{aligned}$$

The computation detailed in Appendix A gives the expression (2) with the desired regularity and:

$$A_{11} = 1 + O(1 - r^2), \quad A_{12} = A_{21} = \frac{\eta}{2} + O(1 - r^2) \quad \text{and} \quad A_{22} = 1 + \frac{\eta^2}{4} + O(1 - r^2),$$

which shows that A is coercive on $\overline{\mathbb{D}}$. \square

We call such a process a *compactification* of the mean curvature since the quantity $r^{-1}\sqrt{g(0)}H(\eta)$ can be extended to the boundary $\partial\mathbb{D}$. It is also strongly linked with the compactification of the induced metric $g(\eta)$ by the following equality:

$$A^{-1} = \begin{pmatrix} 1 + \eta^2/4 & \eta/2 \\ \eta/2 & 1 \end{pmatrix} + O(1 - r^2) = \frac{1}{\sqrt{|g(0)|}}g(\eta) + O(1 - r^2).$$

From now on, we denote H the operator:

$$H : \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \mapsto H(\eta) \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}),$$

where $H(\eta)$ is the mean curvature of X^η , and we call it the *mean curvature operator*. Using Theorem 3.1, we define the *compactified mean curvature operator* to be:

$$\overline{H} : \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \mapsto \frac{2}{r}\sqrt{|g(0)|}H(\eta) \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}).$$

The *compactified Jacobi operator* is $\overline{L} = D\overline{H}(0) : \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \rightarrow \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$. We get that:

$$\overline{L} = \frac{\sqrt{|g(0)|}}{r}L,$$

where L is the Jacobi operator of S^0 , since it is a standard fact that:

$$\forall \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}), \quad DH(0) \cdot \eta = \frac{1}{2}L\eta.$$

Furthermore, conducting the computation in Appendix A more carefully, we get an explicit expression of \overline{L} :

$$\forall \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}), \quad \overline{L}\eta = \Delta\eta + \frac{8\eta}{(1 + r^2)^2},$$

where Δ stands for the usual flat laplacian.

We highlight two immediate consequences. First, the vertical coordinate φ^0 of N^0 — meaning $\varphi^0 = \langle N^0, E_3 \rangle$ — is in the kernel of \overline{L} i.e. $\overline{L}\varphi^0 = 0$. And second, \overline{L} verifies the following Green identity:

$$\forall u, v \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}), \quad \int_{\overline{\mathbb{D}}} (u\overline{L}v - v\overline{L}u) d\overline{A} = \int_0^{2\pi} \left(u \frac{\partial v}{\partial r} - v \frac{\partial u}{\partial r} \right) \Big|_{r=1} d\theta, \quad (3)$$

with $d\overline{A}$ the Lebesgue measure on $\overline{\mathbb{D}}$. We deduce the following lemma:

LEMMA 3.2. *There is no solution $u \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ to the equation:*

$$\begin{cases} \overline{L}u = 0 & \text{on } \overline{\mathbb{D}} \\ u|_{\partial\mathbb{D}} = 1 \end{cases}.$$

Proof. By contradiction, suppose such a u exist and apply the Green identity (3) to φ^0 and u :

$$\begin{aligned} 0 &= \int_{\mathbb{D}} (\varphi^0 \overline{L}u - u \overline{L}\varphi^0) d\overline{A} = \int_0^{2\pi} \left(\varphi^0 \frac{\partial u}{\partial r} - u \frac{\partial \varphi^0}{\partial r} \right) \Big|_{r=1} d\theta \\ &= \int_0^{2\pi} d\theta = 2\pi, \end{aligned}$$

since:

$$\varphi^0|_{r=1} = \frac{1 - r^2}{1 + r^2} \Big|_{r=1} = 0 \quad \text{and} \quad \frac{\partial \varphi^0}{\partial r} \Big|_{r=1} = -\frac{4r}{(1 + r^2)^2} \Big|_{r=1} = -1.$$

This is impossible. \square

Consider the restriction \overline{L}_0 of \overline{L} to $\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})$ and $K = \ker \overline{L}_0$. We use the inclusions $\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}}) \subset \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}) \subset L^2(\mathbb{D})$ and denote K^\perp the orthogonal to K in $\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$ for the natural scalar product of $L^2(\mathbb{D})$ and $K_0^\perp = K^\perp \cap \mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})$.

A standard result states that the restriction \overline{L}_0 is a Fredholm operator with index zero (see [6]) and furthermore $K = \mathbb{R}\varphi^0$ and $\overline{L}_0(\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})) = K^\perp$.

3.2. Deformations of the model surface. Let $\mu : \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \rightarrow \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ be the operator such that $\mu(\gamma)$ is the harmonic function on $\overline{\mathbb{D}}$ (for the flat laplacian) with value γ on the boundary $\partial\mathbb{D}$. In the sequel, we decompose $\mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ into $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R} \times K_0^\perp$, meaning that any $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ is in one-to-one correspondence with a triple $(\gamma, \lambda, \sigma) \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R} \times K_0^\perp$ such that:

$$\eta = \mu(\gamma) + \lambda\varphi^0 + \sigma.$$

Consider Π_K and Π_{K^\perp} , the orthogonal projections on K and K^\perp respectively. We follow White [26] and define a suitable map to apply the Implicit Function Theorem:

LEMMA 3.3. *Let $\Phi : \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R} \times K_0^\perp \rightarrow K^\perp$ be the map defined by:*

$$\Phi(\gamma, \lambda, \sigma) = \Pi_{K^\perp} \circ \overline{H}(\mu_a(\gamma) + \lambda\varphi^a + \sigma).$$

Then $D_3\Phi(\gamma^a, 0, 0) : K_0^\perp \rightarrow K^\perp$ is an isomorphism.

Proof. We compute that $D_3\Phi(\gamma^a, 0, 0) = \Pi_{K^\perp} \circ \overline{L}_0|_{K_0^\perp}$ and we know K^\perp is the range of \overline{L}_0 , which means $D_3\Phi(\gamma^a, 0, 0) : K_0^\perp \rightarrow K^\perp$ is an isomorphism. \square

Therefore, we can apply the Implicit Function Theorem to Φ , which states that there exist an open neighborhood U of $(0, 0)$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R}$ and a unique smooth map $\sigma : U \rightarrow K_0^\perp$ such that:

$$\forall (\gamma, \lambda) \in U, \quad \Phi(\gamma, \lambda, \sigma(\gamma, \lambda)) = 0.$$

Consequently, define smooth maps $\eta : U \rightarrow \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ and $\kappa : U \rightarrow K$ by:

$$\eta(\gamma, \lambda) = \mu(\gamma) + \lambda\varphi^0 + \sigma(\gamma, \lambda) \quad \text{and} \quad \kappa(\gamma, \lambda) = \Pi_K \circ \overline{H}(\eta(\gamma, \lambda)).$$

If an entire graph admits $X^{\eta(\gamma,\lambda)}$ as graph coordinates at infinity, we say that $\{\gamma, \lambda\}$ are the *data* of the surface and that γ is the *value at infinity*.

LEMMA 3.4. *The map η has the following properties:*

1. $\eta(0, 0) = 0$.
2. $\forall (\gamma, \lambda) \in U, \eta(\gamma, \lambda)|_{\partial\mathbb{D}} = \gamma$.
3. $D_2\eta(0, 0) : \lambda \in \mathbb{R} \mapsto \lambda\varphi^0 \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$.

Proof. Point 1 is a direct consequence of the uniqueness in the Implicit Function Theorem. For Point 2 compute:

$$\eta(\gamma, \lambda)|_{\partial\mathbb{D}} = \mu(\gamma)|_{\partial\mathbb{D}} + \lambda\varphi^0|_{\partial\mathbb{D}} + \sigma(\gamma, \lambda)|_{\partial\mathbb{D}} = \gamma.$$

And for Point 3, we only have to show $D_2\sigma(\gamma^a, 0) = 0$. To do so, compute:

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \Phi(0, t, \sigma(0, t)) = \Pi_{K^\perp} \circ \overline{L}(\varphi^0 + D_2\sigma(0, 0) \cdot 1) \\ &= \Pi_{K^\perp} \circ \overline{L}_0(\varphi^0 + D_2\sigma(0, 0) \cdot 1) = \Pi_{K^\perp} \circ \overline{L}_0(D_2\sigma(0, 0) \cdot 1) \\ &= \overline{L}_0(D_2\sigma(0, 0) \cdot 1). \end{aligned}$$

Thus, $D_2\sigma(0, 0) \cdot 1 \in K \cap K_0^\perp = \{0\}$ i.e. $D_2\sigma(0, 0) = 0$. \square

REMARK 3.5. Lemma 3.4 Point 2 shows that the value at infinity of a surface $X^{\eta(\gamma,\lambda)}$ does not depend on λ , meaning that given a value at infinity γ there exists a 1-parameter family of surfaces all with value at infinity equals to γ . Moreover, it can be shown that any two surfaces in this family are congruent only up to a vertical translation. Indeed, from the half-space theorem of Daniel, Meeks and Rosenberg [5], we know that the difference of heights of two entire minimal graphs diverges unless the graphs differ from each other by a vertical translation. In our case, if $(\gamma, \lambda), (\gamma, \lambda') \in U$ then:

$$\eta(\gamma, \lambda) - \eta(\gamma, \lambda') = O(1 - r^2),$$

and the difference of heights is bounded.

The values of the mean curvature of deformations $X^{\eta(\gamma,\lambda)}$ of S^0 are determined by κ . Indeed, for $(\gamma, \lambda) \in U$, we have $\Phi(\gamma, \lambda, \sigma(\gamma, \lambda)) = 0$ and:

$$\overline{H}(\eta(\gamma, \lambda)) = \kappa(\gamma, \lambda) + \Phi(\gamma, \lambda, \sigma(\gamma, \lambda)) = \kappa(\gamma, \lambda). \tag{4}$$

Consider $\mathcal{U} = \kappa^{-1}(\{0\}) \cap U$. The fact that the parameter λ is associated to vertical translations means that we can write $\mathcal{U} = \Gamma \times \mathbb{R}$, with Γ a subset of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$. Furthermore, since the construction is local, we can suppose Γ connected.

PROPOSITION 3.6. Γ is a codimension 1 smooth submanifold of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ and is a subset of:

$$\left\{ \gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \mid \int_0^{2\pi} \gamma d\theta = 0 \right\}.$$

Proof. We show that κ is a submersion at $(0, 0)$. From Equation (4) compute:

$$\begin{aligned} D_2\kappa(0, 0) \cdot 1 &= \left. \frac{d}{dt} \right|_{t=0} \kappa(0, t) = \left. \frac{d}{dt} \right|_{t=0} \overline{H}(\eta(0, t)) \\ &= \overline{L}(D_2\eta(0, 0) \cdot 1) = \overline{L}_0(\varphi^0) = 0, \end{aligned}$$

using Lemma 3.4 Point 3 and that $\varphi^0 \in K$. Moreover, $D_1\kappa(0, 0) \cdot 1$ is not identically zero. Indeed, using (4):

$$\begin{aligned} D_1\kappa(0, 0) \cdot 1 &= \left. \frac{d}{dt} \right|_{t=0} \kappa(t, 0) = \left. \frac{d}{dt} \right|_{t=0} \overline{H}(\eta(t, 0)) \\ &= \overline{L}(D_1\eta(0, 0) \cdot 1) \neq 0, \end{aligned}$$

using Corollary 3.2 with $(D_1\eta(0, 0) \cdot 1)|_{\partial\mathbb{D}} = 1$ deduced from Lemma 3.4 Point 2. Since $D\kappa$ is continuous and non zero at $(0, 0)$, there exists an open neighborhood of $(0, 0)$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R}$ on which κ is a submersion. Therefore, up to a restriction on Γ , we can suppose κ is a submersion on $\Gamma \times \{0\}$, which implies Γ is a submanifold of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ of codimension 1.

The inclusion for Γ is actually equivalent to the nullity of the vertical flux of an entire graph in Nil_3 . Consider a minimal surface admitting graph coordinates at infinity X^η with $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$. As shown in [2], the vertical flux f_3 of X^η is for any $R \in (0, 1)$:

$$\begin{aligned} f_3 &= \int_0^{2\pi} \left\langle \frac{g_{22}(\eta)}{\sqrt{|g(\eta)|}} X_1^\eta - \frac{g_{12}(\eta)}{\sqrt{|g(\eta)|}} X_2^\eta, E_3 \right\rangle \Big|_{r=R} d\theta \\ &= \frac{4R^2}{(1 + R^2)^2} \int_0^{2\pi} \eta(R, \theta) d\theta + O(1 - R^2) \\ &= \int_0^{2\pi} \eta|_{r=1} d\theta, \end{aligned}$$

when taking the limit when $R \rightarrow 1$. Hence, if X^η is an entire graph such that $\eta = \eta(\gamma, \lambda)$ for some data $\{\gamma, \lambda\}$ with $\gamma \in \Gamma$, the flux f_3 is zero and so is the mean of γ . \square

Treibergs [25] showed that given a \mathcal{C}^2 curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}$, there exists a constant mean curvature complete entire vertical graph in 3-dimensional Minkowski space which is asymptotically at signed distance γ from the light cone. Proposition 3.6 is actually a $\mathcal{C}^{2,\alpha}$ local version of this result in Nil_3 :

THEOREM 3.7. *Consider $\gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ small enough for the $\mathcal{C}^{2,\alpha}$ -norm and with zero mean. Then there exists a minimal complete entire vertical graph at asymptotic horizontal signed distance γ from S^0 . Moreover, such a surface is unique up to vertical translations.*

Proof. For any γ sufficiently small in the $\mathcal{C}^{2,\alpha}$ norm, we have $\gamma \in \Gamma$ and $X^{\eta(\gamma,0)}$ is a minimal entire graph admitting γ as value at infinity. And as in Remark 3.5, uniqueness comes from Daniel, Meeks and Rosenberg half-space theorem [5]. \square

3.3. Periodic deformations. In the sequel, we fix a natural $n \geq 2$ and a parameter $a > 0$. To ease the writing, we denote $\theta_n = \pi/n$ and $\gamma_{k,u}$ the horizontal geodesic directed by $\cos(k\theta_n)E_1 + \sin(k\theta_n)E_2$ and passing through $(0, 0, u)$ for any $k \in \{0, \dots, n - 1\}$ and $u \in \mathbb{R}$. We are interested in rotationally symmetric solutions to act as barriers in Section 4. They are constructed by the following result, which a corollary of Theorem 3.7:

PROPOSITION 3.8. *There exist a minimal entire graph S_n such that:*

1. *For any $k \in \{0, \dots, n - 1\}$, S_n contains the horizontal geodesic $\gamma_{k,0}$ and is invariant for the geodesic reflection along that geodesic.*

2. *The height of S_n is nonnegative on the (open) angular sector $\{0 < \theta < \theta_n\}$.*

Proof. For Point 1, fix $\varepsilon > 0$ small enough such that $s_n : \theta \mapsto \varepsilon \sin(n\theta)$ is in Γ . From Theorem 3.7, we know there exists a minimal entire graph S_n with $X^{\eta(s_n, \lambda)}$ as graph coordinates at infinity, where the translation parameter λ is chosen so that S_n contain the origin. Consider the map $\eta' \in \mathcal{C}^{2,\alpha}(\overline{D})$ defined for some $k \in \{0, \dots, n - 1\}$ by:

$$\eta'(r, \theta) = -\eta(s_n, \lambda)(r, 2k\theta_n - \theta).$$

The surface $X^{\eta'}$ is the image of $X^{\eta(s_n, \lambda)}$ under the geodesic reflection along $\gamma_{k,0}$. The value at infinity of η' is:

$$\eta'|_{r=1}(\theta) = -\eta(\gamma^n, \lambda)|_{r=1}(2k\theta_n - \theta) = -\sin(2\pi - n\theta) = \sin(n\theta) = s_n(\theta),$$

Using Remark 3.5, we know that $X^{\eta'}$ and $X^{\eta(s_n, \lambda)}$ differ by a vertical translation and since $X^{\eta(s_n, \lambda)}$ contains the origin — which is fixed by the geodesic reflection along $\gamma_{k,0}$ —, we get $X^{\eta'} = X^{\eta(s_n, \lambda)}$.

For Point 2, consider the restriction of S_n to the angular sector $\{0 \leq \theta \leq \theta_n\}$, which we also denote S'_n in the sequel. For any $\theta \in [0, \theta_n]$, we have $s_n(\theta) \geq 0$ and s_n is positive in the interior. Hence, the height of S_n is bounded from below and by vertical translation T_h with positive h , we can make $T_h(S_n)$ and S^0 disjoint on $\{0 \leq \theta \leq \theta_n\}$. Now, suppose there exist a point inside the open angular sector at which the height of S_n is negative. Translating back, we obtain a first interior contact point between $T_{h_0}(S_n)$ and S^0 for some $h_0 > 0$, which is impossible by maximum principle. \square

Given an open subset G of \mathbb{R}^p , $p \geq 2$, a minimal hypersurface in Euclidean space \mathbb{R}^{p+1} is said to be *supported* on G if it is the graph of a function that does not change sign over G and is zero on the boundary ∂G . A question of Meeks and Rosenberg [13] is to know if the number of disjoint domains supporting minimal graphs is bounded. Li and Wang [9] proved it to be true in Euclidean space for any dimension and Tkachev [24] refined the bounds.

In Nil_3 , we consider open subsets of \mathbb{R}^2 — where \mathbb{R}^2 is seen as the range of the projection π — and vertical minimal graphs. The construction of surfaces S_n shows:

THEOREM 3.9. *In Heisenberg space, the number of disjoint domains supporting minimal vertical graphs is unbounded.*

4. Saddle towers in Heisenberg space. In this section, we use notations introduced in Section 3.3. We build a Saddle tower with $2n$ ends distributed at constant angle θ_n .

For any $b > 0$, consider the polygonal Jordan curve Γ_b which is the reunion of the following geodesic segments:

$$\begin{aligned} h_1(b) &= \{(t, 0, 0) \mid 0 \leq t \leq b\}, & h_2(b) &= \{(t \cos \theta_n, t \sin \theta_n, 0) \mid 0 \leq t \leq b\}, \\ \tilde{h}_1(b) &= \{(t, 0, a) \mid 0 \leq t \leq b\}, & \tilde{h}_2(b) &= \{(t \cos \theta_n, t \sin \theta_n, a) \mid 0 \leq t \leq b\}, \\ v_1(b) &= \{(b, 0, t) \mid 0 \leq t \leq a\} & \text{and } v_2(b) &= \{(b \cos \theta_n, b \sin \theta_n, t) \mid 0 \leq t \leq a\}. \end{aligned}$$

Note that $\tilde{h}_1(b), \tilde{h}_2(b)$ are horizontal lifts of $h_1(b), h_2(b)$ respectively.

Since the (euclidean) convex hull H_b of Γ_b is mean-convex, we know from Meeks and Yau [14] that the Plateau problem with boundary Γ_b is solvable, meaning there exists an embedded minimal disk $\Sigma_b \subset H_b$ bordered by Γ_b (see Figure 1).

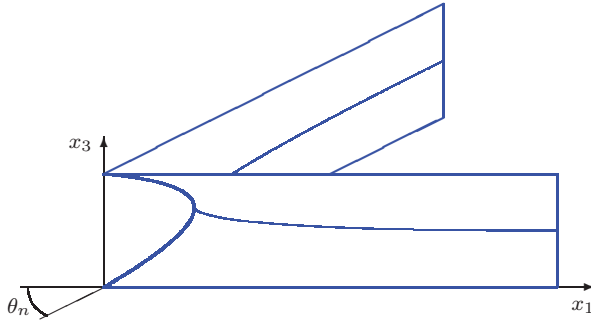


FIG. 1. Plateau solution Σ_b bordered by Γ_b for some $b > 0$.

To ensure the convergence of a subsequence of the family (Σ_b) when $b \rightarrow +\infty$, we only need barriers from below and above and such barriers are the surfaces S_n and $a - S_n$ respectively as constructed in Proposition 3.8. Hence, there exists an embedded minimal surface Σ_∞ bordered by Γ_∞ which is the reunion of the horizontal geodesic rays:

$$h_1 = \{(t, 0, 0) \mid t \geq 0\}, \quad h_2 = \{(t \cos \theta_n, t \sin \theta_n, 0) \mid t \geq 0\},$$

$$\tilde{h}_1 = \{(t, 0, a) \mid t \geq 0\} \quad \text{and} \quad \tilde{h}_2 = \{(t \cos \theta_n, t \sin \theta_n, a) \mid t \geq 0\}.$$

The surface Σ_∞ is the fundamental piece we are looking for. Extending Σ_∞ by recursive geodesic reflections along the geodesics in its boundary.

THEOREM 4.1. *For any natural $n \geq 2$ and any $a > 0$, there exists a properly embedded minimal surface $\mathcal{S}(a, n)$ in Nil_3 of genus zero, invariant by the rotation of angle $2\theta_n$ and axis $\{x_1 = x_2 = 0\}$ and by the vertical translation of parameter $2a$. Moreover, for any $k \in \{0, \dots, n - 1\}$, $\mathcal{S}(a, n)$ contains the geodesics $\gamma_{k,0}$ and $\gamma_{k,a}$ and is asymptotic — away from $\{x_1 = x_2 = 0\}$ — to the vertical plane containing $\gamma_{k,0}$. We call $\mathcal{S}(a, n)$ a most symmetric Saddle tower.*

Appendix A. Proof of Theorem 3.1. Consider a map $\eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ for some $\alpha \in (0, 1)$ — where $\mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ denotes the usual Hölder space over $\overline{\mathbb{D}}$ — and let $X^\eta : \mathbb{D} \rightarrow \text{Nil}_3$ be the immersion given by:

$$X^\eta(r, \theta) = \left(\frac{4r}{1-r^2} \cos \theta, \frac{4r}{1-r^2} \sin \theta, \eta \frac{1+r^2}{1-r^2} \right),$$

where (r, θ) are the polar coordinates on \mathbb{D} . Denoting by indexes 1, 2 the derivatives with respect to r, θ respectively, the first derivatives of X^η are:

$$X_1^\eta(r, \theta) = \frac{4(1+r^2)}{(1-r^2)^2} E_\rho + \frac{4r}{(1-r^2)^2} \left(\eta + \frac{(1+r^2)\eta_1}{4r}(1-r^2) \right) E_3$$

and

$$X_2^\eta(r, \theta) = \frac{4r}{1-r^2} E_\theta - \frac{8r^2}{(1-r^2)^2} \left(1 - \frac{(1+r^2)\eta_2}{8r^2}(1-r^2) \right) E_3,$$

with (E_ρ, E_θ, E_3) denoting the cylindrical orthonormal frame in Nil_3 and defined at generic point $(x_1 = \rho \cos \theta, x_2 = \rho \sin \theta, x_3)$ by:

$$E_\rho = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2}, \quad E_\theta = -\sin \theta \frac{\partial}{\partial x_1} + \cos \theta \frac{\partial}{\partial x_2} + \frac{\rho}{2} \frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3}.$$

We then compute the first fundamental form $g(\eta) = (g_{ij}(\eta))$:

$$\begin{aligned}
 g_{11}(\eta) &= \frac{16(1+r^2)^2}{(1-r^2)^4} \left[\left(1 + \frac{r^2\eta^2}{(1+r^2)^2} \right) + \frac{r\eta\eta_1}{2(1+r^2)}(1-r^2) + \frac{\eta_1^2}{16}(1-r^2)^2 \right], \\
 g_{12}(\eta) &= -\frac{32r^3}{(1-r^2)^4} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2) - \frac{(1+r^2)^2\eta_1\eta_2}{32r^3}(1-r^2)^2 \right] \\
 \text{and } g_{22}(\eta) &= \frac{16r^2(1+r^2)^2}{(1-r^2)^4} \left[1 - \frac{\eta_2}{1+r^2}(1-r^2) + \frac{\eta_2^2}{16r^2}(1-r^2)^2 \right].
 \end{aligned}$$

And the determinant $|g(\eta)|$ of the first fundamental form writes:

$$|g(\eta)| = \left(\frac{16r(1+r^2)^2}{(1-r^2)^4} w(\eta) \right)^2 = |g(0)|w^2(\eta),$$

with the following expression of $w(\eta)$:

$$\begin{aligned}
 w(\eta) = \left[1 - \frac{\eta_2}{1+r^2}(1-r^2) + \left(\frac{r^2\eta^2}{(1+r^2)^4} + \frac{\eta_2^2}{16} \right) (1-r^2)^2 + \frac{r\eta\eta_1}{2(1+r^2)^3}(1-r^2)^3 \right. \\
 \left. + \frac{\eta_1^2}{16(1+r^2)^2}(1-r^2)^4 \right]^{1/2}.
 \end{aligned}$$

We also get the unit normal N^η to X^η :

$$\begin{aligned}
 N^\eta = \frac{1}{\sqrt{|g(\eta)|}} X_1^\eta \times X_2^\eta = \frac{1}{w(\eta)} \left[-\frac{r(1-r^2)}{(1+r^2)^2} \left(\eta + \frac{(1+r^2)\eta_1}{4r}(1-r^2) \right) E_\rho \right. \\
 \left. + \frac{2r}{1+r^2} \left(1 - \frac{(1+r^2)\eta_2}{8r^2}(1-r^2) \right) E_\theta + \frac{1-r^2}{1+r^2} E_3 \right].
 \end{aligned}$$

Recall that the Levi-Civita connection ∇ on Nil_3 is given in the cylindrical frame (E_ρ, E_θ, E_3) by:

$$\begin{aligned}
 \nabla_{E_\rho} E_\rho &= 0 & \nabla_{E_\theta} E_\rho &= \frac{1}{\rho} E_\theta - \frac{1}{2} E_3 & \nabla_{E_3} E_\rho &= -\frac{1}{2} E_\theta \\
 \nabla_{E_\rho} E_\theta &= \frac{1}{2} E_3 & \nabla_{E_\theta} E_\theta &= -\frac{1}{\rho} E_\rho & \nabla_{E_3} E_\theta &= \frac{1}{2} E_\rho \\
 \nabla_{E_\rho} E_3 &= -\frac{1}{2} E_\theta & \nabla_{E_\theta} E_3 &= \frac{1}{2} E_\rho & \nabla_{E_3} E_3 &= 0
 \end{aligned}$$

We compute the conormal derivatives:

$$\begin{aligned} \nabla_{X_1^\eta} X_1^\eta &= \frac{8r(3+r^2)}{(1-r^2)^3} E_\rho - \frac{16r(1+r^2)}{(1-r^2)^4} \left(\eta + \frac{(1+r^2)\eta_1}{4r} (1-r^2) \right) E_\theta \\ &\quad + \frac{4}{(1-r^2)^3} ((1+3r^2)\eta + 2r\eta_1(1-r^2)) E_3 + \frac{(1+r^2)\eta_{11}}{1-r^2} E_3, \\ \nabla_{X_1^\eta} X_2^\eta &= \frac{8r^2}{(1-r^2)^3} \left(\eta + \frac{(1+r^2)\eta_1}{4r} (1-r^2) \right) E_\rho \\ &\quad + \frac{4(1+r^2)^3}{(1-r^2)^4} \left(1 - \frac{\eta_2}{2(1+r^2)} (1-r^2) \right) E_\theta \\ &\quad - \frac{8r(1+r^2)}{(1-r^2)^3} \left(1 - \frac{\eta_2}{2(1+r^2)} (1-r^2) \right) E_3 + \frac{(1+r^2)\eta_{12}}{1-r^2} E_3 \\ \text{and } \nabla_{X_2^\eta} X_2^\eta &= -\frac{4r}{(1-r^2)^3} ((1+6r^2+r^4) - (1+r^2)\eta_2(1-r^2)) E_\rho + \frac{(1+r^2)\eta_{22}}{1-r^2} E_3. \end{aligned}$$

The Weingarten operator of X^η is then determined by the following quantities:

$$\begin{aligned} \langle \nabla_{X_1^\eta} X_1^\eta, N^\eta \rangle &= -\frac{32r^2}{w(\eta)(1-r^2)^4} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2) - \frac{\eta_1\eta_2}{8r} (1-r^2)^2 \right. \\ &\quad \left. + R'_{11}(1-r^2)^4 \right] + \frac{\eta_{11}}{w(\eta)}, \end{aligned}$$

with $R'_{11} = R'_{11}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, identically zero if $\eta = 0$ and real-analytic in its variables. Also:

$$\begin{aligned} \langle \nabla_{X_1^\eta} X_2^\eta, N^\eta \rangle &= \frac{32r^3}{w(\eta)(1-r^2)^4} \left[1 - \frac{(1+r^2)\eta_2}{4r^2} (1-r^2) \right. \\ &\quad \left. - \frac{1}{4} \left(\frac{\eta^2}{(1+r^2)^2} - \frac{\eta_2^2}{4r^2} \right) (1-r^2)^2 - \frac{\eta\eta_1}{8r(1+r^2)} (1-r^2)^3 + R'_{12}(1-r^2)^4 \right] + \frac{\eta_{12}}{w(\eta)}, \end{aligned}$$

again with $R'_{12} = R'_{12}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, zero if $\eta = 0$ and real-analytic in its variables, and:

$$\begin{aligned} \langle \nabla_{X_2^\eta} X_2^\eta, N^\eta \rangle &= \frac{32r^4}{(1+r^2)^2 w(\eta)(1-r^2)^2} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2) \right. \\ &\quad \left. + R'_{22}(1-r^2)^4 \right] + \frac{\eta_{22}}{w(\eta)}, \end{aligned}$$

with $R'_{22} = R'_{22}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, identically zero if $\eta = 0$ and real-analytic in its variables.

We shall now compute the mean curvature itself. Denote:

$$H_{ij}(\eta) = g^{ij}(\eta) \langle \nabla_{X_i^\eta} X_j^\eta, N^\eta \rangle,$$

where $(g^{ij}(\eta))$ is the inverse matrix of the metric $g(\eta)$. We get:

$$\begin{aligned}
 H_{11}(\eta) = & -\frac{2r^2}{(1+r^2)^2 w^3(\eta)} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{(1+10r^2+r^4)\eta\eta_2}{2r(1+r^2)^2} \right) (1-r^2) \right. \\
 & - \frac{3\eta_2}{8r} \left(\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2)^2 + \frac{\eta_2^2}{32r} \left(3\eta_1 - \frac{\eta\eta_2}{2r} \right) (1-r^2)^3 \\
 & \left. + R_{11}(1-r^2)^4 \right] + \frac{g^{11}(\eta)}{w(\eta)} \eta_{11},
 \end{aligned}$$

with $R_{11} = R_{11}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, identically zero if $\eta = 0$ and real-analytic in its variables, also:

$$\begin{aligned}
 H_{12}(\eta) = & \frac{4r^4}{(1+r^2)^4 w^3(\eta)} \left[\eta + \frac{1+r^2}{4r} \left(\eta_1 - \frac{3\eta\eta_2}{2r} \right) (1-r^2) \right. \\
 & - \frac{1}{4} \left(\frac{\eta^3}{(1+r^2)^2} + \frac{3\eta_1\eta_2}{2r} - \frac{3\eta\eta_2^2}{4r^2} \right) (1-r^2)^2 \\
 & \left. - \frac{3}{16r} \left(\frac{\eta^2\eta_1}{1+r^2} - \frac{\eta_1\eta_2^2}{2} - \frac{\eta^3\eta_2}{6r(1+r^2)} + \frac{\eta\eta_2^3}{12r} \right) (1-r^2)^3 + R_{12}(1-r^2)^4 \right] + \frac{g^{12}(\eta)}{w(\eta)} \eta_{12},
 \end{aligned}$$

with $R_{12} = R_{12}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, zero if $\eta = 0$ and real-analytic in its variables, and:

$$\begin{aligned}
 H_{22}(\eta) = & \frac{2r^2(1-r^2)^2}{(1+r^2)^4 w^3(\eta)} \left[\eta \left(1 + \frac{r^2\eta^2}{(1+r^2)^2} \right) \right. \\
 & \left. + \frac{r}{2} \left(\eta_1 - \frac{\eta\eta_2}{2r} + \frac{3\eta^2\eta_1}{2(1+r^2)} - \frac{\eta^3\eta_2}{4r(1+r^2)} \right) (1-r^2) + R_{22}(1-r^2)^2 \right] + \frac{g^{22}(\eta)}{w(\eta)} \eta_{22},
 \end{aligned}$$

with $R_{22} = R_{22}(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, zero if $\eta = 0$ and real-analytic in its variables. Finally, we obtain a Taylor expansion of the mean curvature:

$$\begin{aligned}
 H(\eta) &= \frac{1}{2} (H_{11}(\eta) + 2H_{12}(\eta) + H_{22}(\eta)) \\
 &= \frac{1}{2w(\eta)|g(\eta)|} (g_{22}(\eta)\eta_{11} - 2g_{12}(\eta)\eta_{12} + g_{11}(\eta)\eta_{22}) + R(1-r^2)^4, \tag{5}
 \end{aligned}$$

as before with $R = R(r, \eta, D\eta)$ defined on $\overline{\mathbb{D}}$, identically zero if $\eta = 0$ and real-analytic in its variables.

Equation (5) can be written:

$$\begin{aligned}
 H(\eta) &= \frac{r}{\sqrt{|g(0)|}} \sum_{i,j} A_{ij} + \frac{r}{\sqrt{|g(0)|}} B, \\
 \text{with } A_{11} &= \frac{1}{2rw^3(\eta)} \frac{g_{22}(\eta)}{\sqrt{|g(0)|}} = \frac{1}{2} + O(1-r^2), \\
 A_{12} = A_{21} &= -\frac{1}{2rw^3(\eta)} \frac{g_{12}(\eta)}{\sqrt{|g(0)|}} = \frac{\eta}{4} + O(1-r^2) \\
 \text{and } A_{22} &= \frac{1}{2rw^3(\eta)} \frac{g_{11}(\eta)}{\sqrt{|g(0)|}} = \frac{1}{2} \left(1 + \frac{\eta^2}{4} \right) + O(1-r^2).
 \end{aligned}$$

Moreover, $A_{ij} = A_{ij}(r, \eta, D\eta)$ and $B = B(r, \eta, D\eta)$ are defined on $\overline{\mathbb{D}}$ and real-analytic in their variables, the matrix $A = (A_{ij})$ is coercive on $\overline{\mathbb{D}}$, and B is zero when $\eta = 0$.

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