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HARMONIC MAPS WITH POTENTIAL FROM $\mathbb{R}^2$ INTO S^2*

RUIQI JIANG†

Abstract. We study the existence problem of harmonic maps with potential from $\mathbb{R}^2$ into S^2. For a specific class of potential functions on S^2, we give the sufficient and necessary conditions for the existence of equivariant solutions of this problem. As an application, we generalize and improve the results on the Landau-Lifshitz equation from $\mathbb{R}^2$ into S^2 in [7] due to Gustafson and Shatah.

Key words. Harmonic maps with potential, Pohozaev identity, Landau-Lifshitz.

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1. Introduction. Let (M, g) and (N, h) be two Riemannian manifolds. A map $u_0 : M \to N$ is called a harmonic map iff it is critical with respect to the energy functional $E(u)$. See [5] for the precise definitions. The notion of harmonic maps with potential is first suggested by Ratto ([11]). Given a smooth "potential" function $H : N \to \mathbb{R}$, a map $u_0 : M \to N$ is called a *harmonic map with potential* H iff it is critical with respect to the functional

$$F(u) \equiv E(u) + \int_M H(u) dV_g.$$

In this paper, we consider the existence problem of harmonic maps with potential in the special case where (M, g) is the Euclidean 2-plane $\mathbb{R}^2$, and (N, h) is the unit 2-sphere S^2 in $\mathbb{R}^3$, i.e.

$$S^2 = \{x \in \mathbb{R}^3 : \ |x|^2 = 1\}.$$

Then, if we set $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, the energy is simply

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx,$$

where

$$|\nabla u|^2 = \sum_{i=1}^{3} |\nabla u_i|^2.$$

Also, we will assume $H(u) = G(d(u))$ for a function $G : [0, \pi] \to \mathbb{R}$, where $d(u)$ denotes the geodesic distance from $u \in S^2$ to the north pole $P = (0, 0, 1)$. This assumption on H enables us to seek for solutions which are equivariant with respect to the $S^1 = O(2)$ actions on both the domain $\mathbb{R}^2$ and the target S^2. If we identify the (x_1, x_2)-plane with the complex plane $\mathbf{C}$, and consider $\mathbb{R}^3 = \mathbf{C} \oplus \mathbb{R}^1$, then a ***m-equivariant map*** u takes the following form:

$$u(r, \theta) = \sin h(r) e^{im\theta} + \cos h(r) \cdot e_3 \tag{1.1}$$

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†College of Mathematics and Econometrics, Hunan University, Changsha, 410082, P. R. China (jiangruiqi@hnu.edu.cn); Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P. R. China (jiangruiqi@amss.ac.cn). Supported by CNSF , Grant No. 11471316.

where (r, θ) denotes the polar coordinates in $\mathbb{R}^2$, $h : [0, \infty) \to \mathbb{R}^1$ with $h(0) = 0$, m is a non-zero integer, and $e_3 = (0, 0, 1)$ is the unit vector. For such m-equivariant maps, the energy F reduces to a functional on the function h as follows. (We omit the factor 2π in the integrals.)

$$J(h) = \frac{1}{2} \int_0^{\infty} \left(h'^2 + \frac{m^2}{r^2} \sin^2 h \right) r dr + \int_0^{\infty} G(h)\, r dr. \tag{1.2}$$

Moreover, if h is a critical point of J, then the map u defined in (1.1) is a critical point of F, hence a harmonic map with potential for the chosen potential function H. Thus, the question of finding harmonic maps with potential is reduced to solving the following O.D.E., which is the Euler-Lagrange equation of $J(h)$, with suitable boundary conditions at $r = 0$ and $r = \infty$.

$$h'' + \frac{1}{r} h' - \frac{m^2}{r^2} \sin h \cos h = g(h) \tag{1.3}$$

where $g(\cdot) = G'(\cdot)$. The boundary conditions we assume will be

$$h(0) = 0, \qquad h(\infty) = \pi. \tag{1.4}$$

Such a problem has been considered by some authors in connection with Landau-Lifshitz type equations. Gustafson and Shatah [7], in order to look for the periodic solutions to certain Landau-Lifshitz type equation, studied the above problem with $g(h) = \lambda \sin h \cos h + \omega \sin h$. Their result shows that, if $\lambda > 0$, and $\omega > 0$ is small, then the problem admits a solution which has finite energy and increases monotonely from 0 to π. In [8], Hang and Lin also considered (1.3) with $g(h) = -\lambda \sin h \cos h$, but the condition (1.4) at infinity is replaced by $h(\infty) = \pi/2$. They proved that for each $\lambda > 0$ there exists a unique solution of infinite energy for the problem.

Another reason for studying such a problem is that the Q-solitons arised in the O(3) sigma model, some properties of which were discussed in [14] by qualitative analysis techniques and numerical computation method. Indeed, such Q-solitons are exactly the m-equivariant critical points of the functional $F(u)$ with $H(u) = V(u_3) - \frac{1}{2}(1 - u_3^2)$, where V is a smooth real function on $\mathbb{R}$. It's easy to verify that, to look for Q-solitons, one only needs to solve the equation (1.3) with boundary conditions: $h(0) = 0$, $h(\infty) = (2k + 1)\pi$, $k \in \mathbb{Z}$. For more details, readers can refer to [14].

In this paper, we will consider a class of functions $g(\cdot)$ in (1.3) for which the solvability question of problem (1.3)-(1.4) with $0 \le h(r) \le \pi$ in the interval $[0, \infty)$ can be completely answered. The functions $g(\cdot)$ in this class satisfy the following three conditions.

(**i**) There exists $\xi \in (0, \pi)$ such that

$$\begin{cases} g(0) = g(\xi) = g(\pi) = 0, \\ g(x) > 0, \quad x \in (0, \xi), \\ g(x) < 0, \quad x \in (\xi, \pi); \end{cases}$$

(**ii**) $\int_0^{\pi} g(x) dx > 0$;

(**iii**) $g'(\pi) > 0$.

We choose the potential function G in (1.2), which is a primitive of g, to be

$$G(x) = - \int_x^{\pi} g(t) dt, \tag{1.5}$$

so that $G(\pi) = 0$ and $G(0) < 0$.

Notice that, the function g in [7] falls into our class. Hence, our result is a generalization and improvement of [7].

REMARK 1. *It is worth to point out that the condition (**iii**) can guarantee the solutions converge to π exponentially as $r \to \infty$, the condition (**i**) and (**ii**) are quite natural since they are almost necessary conditions on solvability of the problem (1.3)-(1.4) with $0 \leq h(r) \leq \pi$ in the interval $[0, \infty)$.*

Indeed, under the assumption on the potential function H, it is easy to verify that $\nabla H(\cdot)$ vanish on the north pole and the south pole. It implies that $G'(0) = G'(\pi) = 0$, i.e.

$$g(0) = g(\pi) = 0.$$

The usual (Derrick) scaling argument shows that any harmonic map with potential H must satisfy the following identity

$$\int_{R^2} H(u)dx = 0,$$

which is equivalent to

$$\int_0^\infty G(h)rdr = 0. \tag{1.6}$$

Obviously, the finiteness of the integral $\int_0^\infty G(h)rdr$ implies that

$$G(\pi) = 0.$$

Hence, for $x \in [0, \pi]$, there holds that

$$G(x) = -\int_x^\pi g(t)dt.$$

For the purpose of finding solutions to the problem (1.3)-(1.4), by the identity (1.6), we know that $G(\cdot)$ must change sign in the interval $[0, \pi]$. It implies that the function $g(\cdot) = G'(\cdot)$ also has to change sign in the interval $[0, \pi]$, otherwise $G(\cdot) \geq 0$ or $G(\cdot) \leq 0$ in the interval $[0, \pi]$. For technical reasons, we only consider the case that the function $g(\cdot)$ has only one zero point ξ in the interval $(0, \pi)$, i.e.

$$g(\xi) = 0.$$

That's to say, the function $g(\cdot)$ only changes sign once in the interval $[0, \pi]$.

On the other hand, from the results in [9](see Lemma 2.1 and Theorem 2.3), we know that if there exists a solution $h(r)$ to the equation (1.3) with $g'(\pi) < 0$ and $\lim_{r\to\infty} h(r) = \pi$, then $h(r)$ will oscillate around π as $r \to \infty$ and

$$\int_{R_0}^\infty [h'^2 + \frac{m^2}{r^2}\sin^2 h]rdr = +\infty$$

for some $R_0 > 0$. So, in order to find the solutions of finite energy to the problem (1.3)-(1.4) with $0 \leq h(r) \leq \pi$, we assume in this paper

$$g'(\pi) > 0.$$

Since the function $g(\cdot)$ has only one zero point ξ in the interval $(0,\pi)$ and $g'(\pi)>0$, we can deduce that

$$g(x)>0, \quad x\in(0,\xi) \quad \textit{and} \quad g(x)<0, \quad x\in(\xi,\pi).$$

Therefore, the fact that the function $G(\cdot)$ changes sign in the interval $[0,\pi]$ implies $G(0)=-\int_0^\pi g(t)dt<0$, i.e.

$$\int_0^\pi g(t)dt>0.$$

Remark 2. *For the convenience of readers, we emphasize some properties of the potential $G(\cdot)$. Firstly, $G(\cdot)$ increases monotonically in the interval $(0,\xi)$ and decreases monotonically in the interval (ξ,π). Secondly, there exists an absolute positive constant C such that on the interval $[\xi,\pi]$ there holds*

$$C^{-1}(x-\pi)^2\le G(x)\le C(x-\pi)^2.$$

It is well known that, when $g\equiv 0$ in (1.3), there is a family of solutions φ_λ to (1.3)-(1.4) which corresponds to a family of harmonic maps from $\mathbb{R}^2$ onto S^2 of degree $m>0$. These solutions have the following explicit expression.

$$\varphi_\lambda(r)=2\arctan[(\lambda r)^m], \quad \lambda>0.$$

Now we can state our main result.

Theorem 1.1. *Assume that the function $g\in C^\infty([0,\pi])$ satisfies $(\boldsymbol{i})-(\boldsymbol{iii})$, $m\neq 0$ is an integer and that G is as in (1.5). The problem (1.3)-(1.4) admits a solution with $0\le h(r)\le\pi$ on $(0,\infty)$ if and only if there holds*

$$0<\int_0^\infty G(\varphi_1(r))\,rdr\le\infty.$$

Moreover, the solutions we obtain satisfy $h'(r)>0$ on $(0,\infty)$ and converge to π exponentially as $r\to\infty$.

Remark 3. *Since*

$$\int_0^\infty G(\varphi_\lambda(r))\,rdr=\frac{1}{\lambda^2}\int_0^\infty G(\varphi_1(r))\,rdr,$$

we can replace $\varphi_1(r)$ by $\varphi_\lambda(r)$ for $\lambda>0$.

Remark 4. *In fact, in Theorem 1.1 and throughout the paper, we only need to assume that $g\in C^{\alpha(m)}([0,\pi])$, where $\alpha(m)=max\{1,|m|-2\}$, if $m\neq 0$ is fixed.*

Our method for the proof is basically a combination of the shooting method for O.D.E.'s, the variational method for obtaining solutions to certain boundary value problems and the blow-up analysis for determining the behavior of solutions with large initial data. We will repeatedly use a Pohozaev type identity in our analysis. (The name "Pohozaev" usually means such an identity can be obtained by a domain variation along a conformal vector field, and in our case the vector field is $r\frac{\partial}{\partial r}$.)

In the next section we would consider an initial value problem of O.D.E. (1.3) with the singularity at $r = 0$ and prove its existence, uniqueness and continuous dependence on initial data. In Section 3, by qualitative analysis, we will establish a series of lemmas to characterize the behavior of solutions of O.D.E. (1.3) under suitable assumptions. In Section 4 we discuss the existence of the boundary value problems of O.D.E. (1.3) by variational methods. In Section 5, we give the proof of Theorem (1.1) by shooting method. Finally, we apply our result to certain Landau-Lifshitz type equations in section 6.

Convention: For convenience, we always assume that $m > 0$ without further comment.

2. The existence of solutions to the initial value problems. In this section, we consider the initial value problem of O.D.E. (1.3) with the singularity at $r = 0$ and prove its existence, uniqueness and the continuous dependence on the initial data.

For simplicity, we rewrite (1.3) as following form:

$$h'' + \frac{1}{r}h' - \frac{m^2}{r^2}\sin h \cos h - g(h) = 0.$$

Let us consider the following initial value problem:

$$h'' + \frac{1}{r}h' - \frac{m^2}{r^2}\sin h \cos h - g(h) = 0, \quad r \in (0, +\infty) \tag{2.1}$$

$$h(0) = 0, \quad h^{(m)}(0) = m!a, \tag{2.2}$$

where $g(x) \in C^\infty(\mathbb{R})$, $\|g\|_{C^1} \leq C < \infty$, $a \in \mathbb{R}$ and $h^{(m)}$ denotes the m-order derivative of h.

Definition 2.1. *If $h(r) \in C^m[0, +\infty) \cap C^\infty(0, +\infty)$ satisfies (2.1)-(2.2), then $h(r)$ is called a solution to (2.1)-(2.2).*

Remark 5. *If $h(r)$ is a solution of (2.1)-(2.2), by substituting the asymptotic expansion of $h(r)$ at $r = 0$ to the equation (2.1), we see that k-th derivative of $h(r)$ evaluated at the point 0 with $k \leqslant m - 1$ is zero, i.e.*

$$h^{(k)}(0) = 0, \quad 0 \leqslant k \leqslant m - 1.$$

So the initial value (2.2) is given reasonably for (2.1).

We shall employ the contraction map principle, which is different from the method of upper and lower solutions in [6] and the variational method in [8], to address the problem of existence and uniqueness of local solutions to (2.1)-(2.2). Then, by means of the standard existence and uniqueness theory on ordinary differential equation, we can extend the local solution to the whole interval $[0, +\infty)$.

It is easy to see that (2.1) can be rewritten in the following form:

$$(rh')' = \frac{m^2}{r}\sin h \cos h + g(h)r. \tag{2.3}$$

Hence, the problem (2.1)-(2.2) can be expressed in the following integral equation:

$$h(r) = \int_0^r \frac{1}{s}\int_0^s (\frac{m^2}{2t}\sin 2h + g(h)t)dtds. \tag{2.4}$$

Let

$$h(r) = ar^m + r^{m+2}\phi \tag{2.5}$$

and substitute it into (2.4), then we get

$$\phi = \frac{1}{r^{m+2}}\left\{\int_0^r \frac{1}{s}\int_0^s \big[\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t\big]dtds - ar^m\right\}.$$

Define a map

$$T : C[0,\, \delta] \to C[0,\, \delta]$$

by

$$\begin{aligned} &T(\phi) \\ &= \frac{1}{r^{m+2}}\left\{\int_0^r \frac{1}{s}\int_0^s \big[\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t\big]dtds - ar^m\right\}, \end{aligned} \tag{2.6}$$

where δ would be determined later.

First, we need to verify that T is well defined. Since, for any fixed continuous function ϕ, there holds true

$$\begin{aligned} &\left|\int_0^r \frac{1}{s}\int_0^s \big[\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t\big]dtds\right| \\ &\le \int_0^r \frac{1}{s}\int_0^s \big[\frac{m^2}{2t}2(|a|\, t^m + t^{m+2}\,|\phi|) + C(|a|\, t^m + t^{m+2}\,|\phi|)t\big]dtds \\ &\le C(|a| + \|\phi\|_{C[0,\delta]})\int_0^r \frac{1}{s}\int_0^s (t^{m-1} + t^{m+3})dtds \\ &\le C(|a| + \|\phi\|_{C[0,\delta]})(r^m + r^{m+4}) \\ &\le C(|a| + \|\phi\|_{C[0,\delta]})(\delta^m + \delta^{m+4}) < +\infty, \end{aligned} \tag{2.7}$$

where C is independent of ϕ, therefore we know that $T(\phi)(\,\cdot\,)$ is continuous on $(0, \delta]$. The remaining work is to verify that $T(\phi)(\,\cdot\,)$ is also continuous at $r = 0$. Indeed,

$$\begin{aligned} \lim_{r\to 0} T(\phi) &= \lim_{r\to 0}\frac{\int_0^r \frac{1}{s}\int_0^s [\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t]dtds - ar^m}{r^{m+2}} \\ &= \lim_{r\to 0}\frac{\frac{1}{r}\int_0^r [\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t]dt - mar^{m-1}}{(m+2)r^{m+1}} \\ &= \lim_{r\to 0}\frac{\int_0^r [\frac{m^2}{2t}\sin 2(at^m + t^{m+2}\phi) + g(at^m + t^{m+2}\phi)t]dt - mar^{m}}{(m+2)r^{m+2}} \\ &= \lim_{r\to 0}\frac{\frac{m^2}{2r}\sin 2(ar^m + r^{m+2}\phi) + g(ar^m + r^{m+2}\phi)r - m^2ar^{m-1}}{(m+2)^2r^{m+1}} \\ &= \lim_{r\to 0}\frac{\frac{m^2}{2}\sin 2(ar^m + r^{m+2}\phi) + g(ar^m + r^{m+2}\phi)r^2 - m^2ar^{m}}{(m+2)^2r^{m+2}} \\ &= \begin{cases} \frac{1}{9}(\phi(0) - \frac{2}{3}a^2 + g'(0)a), & m = 1, \\ \frac{1}{(m+2)^2}[m^2\phi(0) + g'(0)a], & m \ge 2. \end{cases} \end{aligned} \tag{2.8}$$

Next, we turn to discuss when T is a contraction mapping. We compute

$$\begin{aligned}|T(\phi_1)-T(\phi_2)| &\le \frac{1}{r^{m+2}}\int_0^r \frac{1}{s}\int_0^s \{\frac{m^2}{2t}\left|\sin 2(at^m+t^{m+2}\phi_1)-\sin 2(at^m+t^{m+2}\phi_2)\right| \\ &\quad + \left|g(at^m+t^{m+2}\phi_1)-g(at^m+t^{m+2}\phi_2)\right| t\}dtds \\ &\le \frac{1}{r^{m+2}}\int_0^r \frac{1}{s}\int_0^s \{\frac{m^2}{2t}2t^{m+2}\left|\phi_1-\phi_2\right| + Ct^{m+3}\left|\phi_1-\phi_2\right|\}dtds \\ &\le \frac{1}{r^{m+2}}\left\|\phi_1-\phi_2\right\|_{C[0,\delta]}\int_0^r \frac{1}{s}\int_0^s \{m^2t^{m+1}+Ct^{m+3}\}dtds \\ &\le \frac{1}{r^{m+2}}\left\|\phi_1-\phi_2\right\|_{C[0,\delta]}\left(\frac{m^2}{(m+2)^2}r^{m+2}+\frac{C}{(m+4)^2}r^{m+4}\right) \\ &\le \left(\frac{m^2}{(m+2)^2}+\frac{C}{(m+4)^2}\delta^2\right)\left\|\phi_1-\phi_2\right\|_{C[0,\delta]},\end{aligned}$$

where C is a positive constant independent of ϕ. It's easy to see that T is a contraction mapping if δ is chosen such that

$$\frac{m^2}{(m+2)^2}+\frac{C}{(m+4)^2}\delta^2<1. \tag{2.9}$$

Thus there exists a unique fixed point $\phi^* \in C[0,\delta]$ such that $T(\phi^*)=\phi^*$. So,

$$h(r)=ar^m+r^{m+2}\phi^*$$

satisfies (2.1). Moreover, by a simple calculation, we can verify that $h(r)$ also satisfies (2.2). This means that $h(r)$ is a local solution to (2.1)-(2.2). Then, by a standard argument we can extend the local solution to a global solution. Thus, the following theorem holds true.

THEOREM 2.1. *Assume that $g(x)\in C^\infty(\mathbb{R})$ satisfies $\|g\|_{C^1(\mathbb{R})}<\infty$. Then, (2.1)-(2.2) always admits a unique global solution.*

For simplicity, we denote the solution of (2.1)-(2.2) by $h_a(r)$ to emphasize the dependence on the initial value a. Now, let us consider the continuous dependence on the initial data of these obtained solutions. We need to establish the following theorem.

THEOREM 2.2. *Assume that $g(x)\in C^\infty(\mathbb{R})$ satisfies $\|g\|_{C^1(\mathbb{R})}<\infty$. If $h_{a_0}(r)$ is the solution of (2.1)-(2.2), then, $\forall R>0$, $\forall\varepsilon>0$, there exists $\eta=\eta(a_0,\varepsilon,R)>0$ such that, for $|a-a_0|<\eta$, there holds*

$$\|h_a(r)-h_{a_0}(r)\|_{C^1[0,R]}\le\varepsilon. \tag{2.10}$$

Proof. By the standard O.D.E. theory on continuous dependence on the initial data, we only need to prove the theorem in the case $R=\delta$, which has been determined in (2.9).

From the proof of Theorem 2.1, we know that h_a has the following form:

$$h_a(r)=ar^m+r^{m+2}\phi_a. \tag{2.11}$$

In order to get (2.10), we need to estimate the size of $\|\phi_a(r)-\phi_{a_0}(r)\|_{C[0,R]}$. Indeed,

$$
\begin{aligned}
&|\phi_a-\phi_{a_0}|\\
=&\,|T(\phi_a)-T(\phi_{a_0})|\\
\le&\,\frac{1}{r^{m+2}}\int_0^r\frac{1}{s}\int_0^s\{\frac{m^2}{2t}\left|(\sin 2(at^m+t^{m+2}\phi_a)-\sin 2(a_0t^m+t^{m+2}\phi_{a_0})-2(a-a_0)t^m\right|\\
&+\left|g(at^m+t^{m+2}\phi_a)-g(a_0t^m+t^{m+2}\phi_{a_0})\right|t\}dtds\\
\le&\,\frac{1}{r^{m+2}}\int_0^r\frac{1}{s}\int_0^s\{\frac{m^2}{2t}[2t^{m+2}|\phi_a-\phi_{a_0}|+(C+(a-a_0)^2)t^{2m}\\
&+(C+|\phi_a-\phi_{a_0}|^2)t^{2m+4}]+C(|a-a_0|t^{m+1}+t^{m+3}|\phi_a-\phi_{a_0}|)\}dtds\\
\le&\,\frac{1}{r^{m+2}}\left\{\|\phi_a-\phi_{a_0}\|_{C[0,\delta]}\left(\frac{m^2}{(m+2)^2}+C\delta^2\right)r^{m+2}\right.\\
&\left.+C|a-a_0|\frac{1}{(m+2)^2}r^{m+2}\right\}\\
\le&\,\|\phi_a-\phi_{a_0}\|_{C[0,\delta]}\left(\frac{m^2}{(m+2)^2}+C\delta^2\right)+C|a-a_0|\frac{1}{(m+2)^2},
\end{aligned}
$$

where C is a positive constant depending only on m and a_0. By choosing smaller δ, there holds

$$\frac{m^2}{(m+2)^2}+C\delta^2<1,$$

Hence, we obtain

$$\|\phi_a-\phi_{a_0}\|_{C[0,\delta]}\le C|a-a_0|, \tag{2.12}$$

where C is a positive constant depending on m and a_0. It follows that

$$
\begin{aligned}
|h_a(r)-h_{a_0}(r)|&\le|a-a_0|r^m+r^{m+2}|\phi_a-\phi_{a_0}|\\
&\le|a-a_0|\delta^m+C\delta^{m+2}|\phi_a-\phi_{a_0}|\\
&\le C|a-a_0|,
\end{aligned}\tag{2.13}
$$

and

$$
\begin{aligned}
\left|h'_a(r)-h'_{a_0}(r)\right|\le&\,\frac{1}{r}\int_0^r\{\frac{m^2}{2t}\left|(\sin 2(at^m+t^{m+2}\phi_a)-\sin 2(a_0t^m+t^{m+2}\phi_{a_0})\right|\\
&+\left|g(at^m+t^{m+2}\phi_a)-g(a_0t^m+t^{m+2}\phi_{a_0})\right|t\}dt\\
\le&\,\frac{1}{r}\int_0^r\{\frac{m^2}{2t}[2|a-a_0|t^m+2t^{m+2}|\phi_a-\phi_{a_0}|]\\
&+C(|a-a_0|t^{m+1}+t^{m+3}|\phi_a-\phi_{a_0}|)\}dt\\
\le&\,\frac{1}{r}\left\{|a-a_0|\left(mr^m+C\frac{1}{(m+2)^2}r^{m+2}\right)\right.\\
&\left.+\|\phi_a-\phi_{a_0}\|_{C[0,\delta]}(\frac{m^2}{m+2}r^{m+2}+\frac{C}{m+4}r^{m+4})\right\}\\
\le&\,C|a-a_0|,
\end{aligned}\tag{2.14}
$$

where C is a positive constant depending only on m and a_0. Thus, we obtain the desired estimate:

$$\|h_a(r) - h_{a_0}(r)\|_{C^1[0,\delta]} \leq C\,|a - a_0|\,.$$

□

REMARK 6. *By the standard elliptic regularity theory, it is not difficult to see that, the conclusions in Theorem 2.1 and Theorem 2.2 also hold true, when* $g \in C^{\alpha(m)}(\mathbb{R})$, *where* $\alpha(m) = max\{1, |m| - 2\}$.

3. Qualitative Analysis of the O.D.E. In this section, we will establish a series of lemmas to characterize the behavior of solutions to (2.1) under some suitable assumptions on the function g.

First, let us recall the Pohozaev identity of (2.1). By multiplying the both sides of equation (2.3) by $rh'(r)$ and integrating from s to r, we obtain the Pohozaev identity

$$(rh'(r))^2 - (sh'(s))^2 = m^2[\sin^2 h(r) - \sin^2 h(s)] + 2\int_s^r g(h(t))h'(t)t^2 dt, \quad (3.1)$$

or

$$\begin{aligned}(rh'(r))^2 - (sh'(s))^2 = m^2[\sin^2 h(r) - \sin^2 h(s)] + 2[G(h(r))r^2 - G(h(s))s^2]\\ -4\int_s^r G(h(t))tdt, \quad (3.2)\end{aligned}$$

where

$$G(x) = -\int_x^\pi g(t)dt.$$

LEMMA 3.1. *Assume that* $g(x) \in C^\infty([0, \pi])$ *satisfies condition* (***i***) *and* $h(r)$ *satisfies (2.1). If there exists* $r_0 \in (0, +\infty)$ *such that*

$$0 \leq h(r_0) < \min\{\pi - \xi, \xi\} \quad \textit{and} \quad h'(r_0) > 0, \quad (3.3)$$

then there exists $r_1 \in (r_0, \infty)$ *such that*

$$h(r_1) = \xi \quad \textit{and} \quad h'(r) > 0, \forall r \in [r_0, r_1].$$

Proof. By the Pohozaev identity, we have

$$(rh'(r))^2 = (r_0 h'(r_0))^2 + m^2[\sin^2 h(r) - \sin^2 h(r_0)] + 2\int_{r_0}^r g(h(t))h'(t)t^2 dt. \quad (3.4)$$

Set

$$r^* = \sup\{s \in [r_0, +\infty) \mid h'(r) > 0, \forall r \in [r_0, s)\}.$$

It is easy to see that $r_0 < r^* \leq +\infty$, since $h'(r_0) > 0$.

We claim that

$$h(r^*) > \xi. \quad (3.5)$$

If this was false, there holds that

$$0 \le h(r_0) \le h(r) < \xi, \quad \forall r \in [r_0, r^*). \tag{3.6}$$

Since $h(r)$ increases monotonically in the interval $[r_0, r^*)$ and $h(r_0) < \min\{\pi - \xi, \xi\}$, we obtain

$$\sin^2 h(r) \ge \sin^2 h(r_0), \forall r \in [r_0, r^*). \tag{3.7}$$

As $g(x) > 0$, $\forall x \in (0, \xi)$, there holds

$$g(h(r)) \ge 0, \quad \forall r \in [r_0, r^*). \tag{3.8}$$

Combining (3.4), (3.7) and (3.8) we obtain

$$rh'(r) \ge r_0 h'(r_0) > 0, \quad \forall r \in [r_0, r^*). \tag{3.9}$$

It follows that

$$h(r) = \int_{r_0}^{r} h'(t)dt + h(r_0) \ge r_0 h'(r_0) \int_{r_0}^{r} \frac{1}{t} dt + h(r_0).$$

This implies that $r^* < +\infty$. Otherwise, we would deduce that $h(r)$ is unbounded in the interval $[r_0, +\infty)$, which contradicts the fact (3.6).By the definition of r^*, we get $h'(r^*) = 0$ which contradicts (3.9). Hence, we prove our claim.

Finally, we can choose $r_1 \in (r_0, r^*)$ such that $h(r_1) = \xi$. By the definition of r^*, we get

$$h'(r) > 0, \quad \forall r \in [r_0, r_1].$$

□

COROLLARY 3.2. *Suppose that $g(x)$ satisfies condition (**i**). If $h_a(r)$ is the solution of (2.1)-(2.2) with $a > 0$, there exists $s_a \in (0, +\infty)$ such that $h(r)$ increases monotonically from 0 to ξ on the interval $[0, s_a]$.*

LEMMA 3.3. *Suppose that $g(x)$ satisfies conditions (**i**) and (**iii**). If $h(r)$, which is not a constant function, satisfies (2.1) on the interval $(r_0, +\infty)$, $\xi \le h(r) \le \pi$ for any $r \in [r_0, +\infty)$ and $\lim_{r\to\infty} h(r) = l > \xi$, then, there hold that for any $r \in [r_0, +\infty)$*

$$h'(r) > 0 \qquad \text{and} \qquad l = \pi.$$

Moreover, $h(r)$ converges to π exponentially as $r \to +\infty$.

Proof. As π is a trivial solution of equation (2.1) and $h(r)$ is not a constant function, we have

$$h(r) < \pi, \quad r \in (r_0, +\infty). \tag{3.10}$$

In fact, if there exists a $r_1 \in (r_0, +\infty)$ such that $h(r_1) = \pi$, by the condition of $\xi \le h(r) \le \pi$ for any $r \in [r_0, +\infty)$, we get $h'(r_1) = 0$. The uniqueness of solutions to initial value problem tells that $h(r) \equiv \pi$, which contradicts the fact that $h(r)$ is not a constant function.

Since $\lim_{r\to\infty} h(r) = l > \xi$ and $g'(\pi) > 0$, there exists $r_1 \in [r_0, +\infty)$ such that, for any $r \in [r_1, +\infty)$, we have

$$\frac{m^2}{r^2} \sin h(r) \cos h(r) + g(h(r)) < 0. \tag{3.11}$$

Indeed, if $l < \pi$, we have

$$\lim_{r\to\infty} \frac{m^2}{r^2} \sin h(r) \cos h(r) + g(h(r)) = g(l) < 0,$$

if $l = \pi$, there exist $R > 0$ and $\varepsilon > 0$ such that, for any $(r, x) \in [R, \infty) \times [\pi - \varepsilon, \pi)$, there holds

$$\frac{m^2}{r^2} \sin x \cos x + g(x) \le \frac{1}{2} g'(\pi)(x - \pi) < 0.$$

Hence (3.11) follows.

From (2.3) we have that, for $r_1 \le s \le r < +\infty$,

$$rh'(r) = sh'(s) + \int_s^r \left[\frac{m^2}{t^2} \sin h \cos h + g(h)\right] tdt \tag{3.12}$$

and

$$h(r) = h(s) + \int_s^r h'(t)dt. \tag{3.13}$$

Since $\xi \le h(r) \le \pi$ for any $r \in [r_0, +\infty)$, (3.13) implies that there exists $C > 0$ and $r_k \to +\infty$ such that

$$-h'(r_k) \le \frac{C}{r_k}.$$

Set

$$A(r) = \int_s^r \left[\frac{m^2}{t^2} \sin h \cos h + g(h)\right] tdt,$$

it follows from (3.12) that $A(r_k) \ge -C$. On the other hand, $A(r)$ is decreasing and negative since the integrand is negative on the interval $[r_1, +\infty)$, we see that $\lim_{r\to\infty} A(r)$ exists. It implies that

$$\lim_{r\to\infty} \frac{m^2}{r^2} \sin h \cos h + g(h) = g(l) = 0.$$

Then we get $l = \pi$.

By (3.12) we infer that $\lim_{r\to\infty} rh'(r) = l_0$ exists. Then, we can easily see that l_0 is zero. Otherwise, $h(r)$ would be unbounded by (3.13). Let $r \to +\infty$ in (3.12) and replace s by r, we get

$$rh'(r) = -\int_r^{+\infty} \left[\frac{m^2}{t} \sin h \cos h + g(h)t\right] dt. \tag{3.14}$$

From (3.14) and (3.11) we can deduce that for any $r \in [r_1, +\infty)$

$$h'(r) > 0.$$

By the Pohozaev identity (3.2), we have that, for $r_0 \leq s \leq r < +\infty$,

$$\begin{aligned}(rh'(r))^2 - (sh'(s))^2 = m^2[\sin^2 h(r) - \sin^2 h(s)] + 2[G(h(r))r^2 - G(h(s))s^2] \\ -4\int_s^r G(h(t))tdt. \end{aligned} \tag{3.15}$$

Since $h(r)$ converges to π exponentially as $r \to \infty$(the fact will be proved in the following) and Remark 2, we get

$$\lim_{r\to\infty} G(h(r))r^2 = 0 \quad \text{and} \quad 0 < \int_s^{+\infty} G(h(t))tdt < +\infty.$$

Letting $r \to +\infty$ in (3.15) and then replacing s by r, we obtain

$$(rh'(r))^2 = m^2 \sin^2 h(r) + 2G(h(r))r^2 + 4\int_r^{+\infty} G(h(t))tdt.$$

Since $\xi \leq h(r) < \pi$ for $r \in [r_0, +\infty)$, from the above identity, we get $h'(r) \neq 0$ for any $r \in [r_0, +\infty)$. As $h'(r)$ is continuous on the interval $[r_0, +\infty)$ and $h'(r) > 0$ for any $r \in [r_1, +\infty)$, we obtain $h'(r) > 0$ for any $r \in [r_0, +\infty)$.

Now, we are in the position to prove that $h(r)$ converges to π exponentially as $r \to +\infty$. Let

$$\widetilde{h}(r) = \pi - h(r).$$

Then, $\widetilde{h}(r) > 0$ on $(r_1, +\infty)$ satisfies the following equation

$$\widetilde{h}'' + \frac{1}{r}\widetilde{h}' = \frac{m^2 \sin 2\widetilde{h}}{2r^2} - g(\pi - \widetilde{h}). \tag{3.16}$$

Let $f(r) = be^{-\epsilon r}$. Then, it is easy to verify that $f(r)$ satisfies the following equation

$$f'' + \frac{1}{r}f' = (\epsilon^2 - \frac{\epsilon}{r})f. \tag{3.17}$$

Denote

$$\beta(r) = f(r) - \widetilde{h}(r).$$

Then, it follows from (3.16) and (3.17)

$$\beta'' + \frac{1}{r}\beta' = (\epsilon^2 - \frac{\epsilon}{r})f - \frac{m^2 \sin 2\widetilde{h}}{2r^2} + g(\pi - \widetilde{h}).$$

Since $\lim_{r\to+\infty} \widetilde{h}(r) = 0$, we choose $R_0 > 0$ such that

$$g(\pi - \widetilde{h}(r)) < -\frac{1}{2}g'(\pi)\widetilde{h}(r), \quad r \geq R_0.$$

By choosing $\epsilon = \sqrt{\frac{1}{2}g'(\pi)}$ and $b = b_0$ such that

$$\beta(R_0) = b_0 e^{-\epsilon R_0} - \widetilde{h}(R_0) > 0,$$

we have that, for $r \geq R_0$,

$$\beta'' + \frac{1}{r}\beta' = (\epsilon^2 - \frac{\epsilon}{r})f - \frac{m^2 \sin 2\widetilde{h}}{2r^2} + g(\pi - \widetilde{h}) < \frac{1}{2}g'(\pi)\beta$$

i.e.

$$\beta'' + \frac{1}{r}\beta' - \frac{1}{2}g'(\pi)\beta < 0, \quad r \in [R_0, +\infty).$$

Since $\beta(R_0) > 0$ and $\lim_{r\to+\infty} \beta(r) = 0$, by the maximum principle we have $\beta(r) > 0$ as $r \geq R_0$, i.e.

$$0 < \widetilde{h}(r) = \pi - h(r) < b_0 e^{-\sqrt{\frac{1}{2}g'(\pi)}r}, \quad r \in [R_0, +\infty).$$

Thus, we complete the proof. □

LEMMA 3.4. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies condition* (***i***). *If $h(r)$ satisfies (2.1) on the interval (s, μ) with $\xi = h(s) \leq h(r) \leq \pi = h(\mu)$, $r \in (s, \mu)$, then there holds true*

$$h'(r) > 0, \quad r \in [s, \mu].$$

Proof. Since π is a trivial solution to (2.1), we get

$$h(r) < \pi, \ r \in (s, \mu) \quad \text{and} \quad h'(\mu) > 0.$$

By the Pohozaev identity, we have that, for $r \in [s, \mu]$,

$$(rh'(r))^2 = (\mu h'(\mu))^2 + m^2 \sin^2 h(r) + 2G(h(r))r^2 + 4\int_r^\mu G(h(t))tdt.$$

Since $G(x) \geq 0$ for any $x \in [\xi, \pi]$, we have $h'(r) \neq 0$ for any $r \in [s, \mu]$. As $h'(r)$ is continuous on the interval $[s, \mu]$ and $h'(\mu) > 0$, we obtain $h'(r) > 0$ for any $r \in [s, \mu]$. □

LEMMA 3.5. *Suppose that $g(x)$ satisfies condition* (***i***). *Let $h_i(r)$, $i = 1, 2$, be two increasing functions satisfying (2.1). If they intersect at two different points in the domain $(0, +\infty) \times [\xi, \pi]$, then $h_1 \equiv h_2$ on the interval $(0, +\infty)$.*

Proof. If this lemma is false, without loss of generality, we assume there exist $0 < r_1 < r_2 < +\infty$ such that $h_1(r_i) = h_2(r_i)$, where $i = 1, 2$, and

$$h_1(r) > h_2(r) \quad \forall r \in (r_1, r_2). \tag{3.18}$$

Then, we get

$$h_1'(r_1) > h_2'(r_1) \geq 0 \quad \text{and} \quad 0 \leq h_1'(r_2) < h_2'(r_2). \tag{3.19}$$

By the Pohozaev identity (3.2), for $i = 1, 2$, we have

$$\begin{aligned}(r_2 h_i'(r_2))^2 - (r_1 h_i'(r_1))^2 = m^2[\sin^2 h_i(r_2) - \sin^2 h_i(r_1)] + 2[G(h_i(r_2))r_2^2 \\ -G(h_i(r_1)r_1^2] - 4\int_{r_1}^{r_2} G(h_i(t))tdt. \qquad (3.20)\end{aligned}$$

Substituting $h_1(r_i) = h_2(r_i), i = 1, 2$, into the above identities, we can obtain

$$r_2^2(h_1'^2(r_2) - h_2'^2(r_2)) + r_1^2(h_2'^2(r_1) - h_1'^2(r_1)) = 4\int_{r_1}^{r_2} [G(h_2(t)) - G(h_1(t))]tdt. \qquad (3.21)$$

(3.19) implies that the left hand side of (3.21) is negative. However, by (3.18) and the fact $G(x)$ is decreasing on the interval $[\xi, \pi]$, we infer that the right hand side of (3.21) is positive. There exists a contradiction. The lemma is proved. □

4. The Solvability of the boundary value problem of O.D.E.. In order to prove Theorem 1.1, we need to study the solvability of the boundary value problem of (2.1) to look for some comparison functions in this section. We will employ the variational method to approach the existence of such a boundary value problem. The following argument is the key ingredient of the proof of Theorem 1.1.

For $0 < s < \mu < +\infty$, we consider the following two problems:

$$P_s \begin{cases} (rh')' = \dfrac{m^2}{r}\sin h\cos h + g(h)r, & \xi \le h(r) \le \pi, \quad r \in (s, +\infty), \\ h(s) = \xi, \quad \lim\limits_{r\to+\infty} h(r) = \pi, \end{cases}$$

and

$$Q_{(s,\mu)} \begin{cases} (rh')' = \dfrac{m^2}{r}\sin h\cos h + g(h)r, & \xi \le h(r) \le \pi, \quad r \in (s, \mu), \\ h(s) = \xi, \quad h(\mu) = \pi. \end{cases}$$

THEOREM 4.1. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies conditions (**i**) and (**iii**). Then,there exists a unique solution $\widetilde{h}(r)$ to P_s. Moreover, $\widetilde{h}'(r) > 0$, $\forall r \in [s, +\infty)$.*

Proof. **Step 1**. First, we consider the functional J_s given by

$$J_s(h) = \frac{1}{2}\int_s^{+\infty} \left[(h'(r))^2 + \frac{m^2}{r^2}\sin^2 h(r)\right] rdr + \int_s^{+\infty} G(h(r))rdr,$$

which is defined on the space X_s given by

$$X_s = \{h(r) \mid \pi - h(r) \in H^1([s, +\infty), rdr),\ \xi \le h(r) \le \pi,\ h(s) = \xi\}.$$

From the Remark 2, we know that there exists a constant $C > 0$ such that

$$C^{-1}G(x) \le (\pi - x)^2 \le CG(x),\ x \in [\xi, \pi].$$

Hence, we know J_s is well defined in the space X_s.

It is easy to see that, for any $h(r) \in X_s$, $h(r)$ is continuous on the interval $[s, +\infty)$ and $\lim\limits_{r\to\infty} h(r) = \pi$. Moreover, the fact that $G(h(r)) \ge 0$ for any $r \in [s, +\infty)$ implies $J_s(h(r)) \ge 0$.

Choose a minimizing sequence $\{h_k\} \subseteq X_s$ such that

$$\inf_{X_s} J_s = \lim_{k\to\infty} J_s(h_k).$$

Since

$$\begin{aligned}\int_s^{+\infty} [((\pi - h_k)')^2 + (\pi - h_k)^2] r dr &= \int_s^{+\infty} [(h_k')^2 + (\pi - h_k)^2] r dr \\ &\le \int_s^{+\infty} (h_k')^2 r dr + C \int_s^{+\infty} G(h_k) r dr \\ &\le C J_s(h_k) \le C, \end{aligned} \tag{4.1}$$

where C is independent of k, up to a subsequence, there exists $\pi - \widetilde{h} \in H^1([s, +\infty), rdr)$ such that

$$\pi - h_k \to \pi - \widetilde{h} \quad \text{weakly in} \quad H^1([s, +\infty), rdr), \tag{4.2}$$

and

$$\forall R \in (s, +\infty),\ \pi - h_k \to \pi - \widetilde{h} \quad \text{in} \quad C[s, R]. \tag{4.3}$$

By (4.3), we get $\widetilde{h}(s) = \xi$ and $\lim_{k\to\infty} h_k(r) = \widetilde{h}(r)$ for any $r \in [s, +\infty)$. So $\widetilde{h} \in X_s$. By (4.2), we obtain

$$\int_s^{+\infty} (\widetilde{h}')^2 r dr \le \varliminf_{k\to\infty} \int_s^{+\infty} (h_k')^2 r dr.$$

By Fatou's lemma and $\lim_{k\to\infty} h_k(r) = \widetilde{h}(r)$ for any $r \in [s, +\infty)$, we have

$$0 \le \int_s^{+\infty} \frac{m^2}{r^2} \sin^2 \widetilde{h}(r) r dr \le \varliminf_{k\to\infty} \int_s^{+\infty} \frac{m^2}{r^2} \sin^2 h_k(r) r dr,$$

$$0 \le \int_s^{+\infty} G(\widetilde{h}) r dr \le \varliminf_{k\to\infty} \int_s^{+\infty} G(h_k) r dr.$$

Immediately, from the above three inequalities we obtain

$$J_s(\widetilde{h}) \le \varliminf_{k\to\infty} J_s(h_k) = \inf_{X_s} J_s.$$

Since $\widetilde{h} \in X_s$, we know that $\widetilde{h}$ is a minimal point of J_s in X_s, i.e.

$$J_s(\widetilde{h}) = \inf_{X_s} J_s.$$

Step 2. Next, we need to verify that $\widetilde{h}$ is just the solution to the boundary value problem P_s. Obviously, $\widetilde{h}$ satisfies the boundary conditions of P_s, now we turn to prove $\widetilde{h}$ satisfies (2.1) on the interval $(s, +\infty)$.

We say $\varphi \in C_0^\infty(s, +\infty)$ is an admissible variational function for $\widetilde{h}$, if there exists $\varepsilon > 0$ such that $\widetilde{h} + t\varphi \in X_s,\ t \in [0, \varepsilon)$.

Since $\widetilde{h}$ is the minimal point of $J_s(\,\cdot\,)$, then, for any admissible variational function φ, we have

$$\frac{d}{dt}J_s(\widetilde{h}+t\varphi)\Big|_{t=0} \geq 0. \tag{4.4}$$

More precisely,

$$\int_s^{+\infty}(\widetilde{h}'\varphi' + \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi)rdr + \int_s^{+\infty} g(\widetilde{h})\varphi rdr \geq 0. \tag{4.5}$$

(i). If $\xi < \widetilde{h} < \pi,\ r \in (r_1, r_2)$, where $s \leq r_1 < r_2 \leq +\infty$, by (4.5), it's easy to check $\widetilde{h}$ satisfies (2.1) on the interval (r_1, r_2).

(ii). We claim that $\widetilde{h}(r) < \pi$ for any $r \in (s, +\infty)$.
If the assertion were false, define

$$r^* = \inf\{r \in (s, +\infty)|\ \widetilde{h}(r) = \pi\}.$$

From the definition of X_s we can easily see that $s < r^* < +\infty$. Moreover, let's define the following continuous function

$$\widehat{h}(r) = \begin{cases} \widetilde{h}(r), & r < r^*, \\ \pi, & r \geq r^*. \end{cases}$$

Obviously, $\widehat{h} \in X_s$. It is easy to see that, if $\widehat{h} \neq \widetilde{h}$, from the definition of J_s and $\widehat{h}$ we infer

$$J_s(\widehat{h}) < J_s(\widetilde{h}).$$

This contradicts the fact $J_s(\widetilde{h}) = \inf\limits_{X_s} J_s$. This means that $\widehat{h} \equiv \widetilde{h}$.

By the definition of r^*, we choose small $\delta > 0$ such that

$$\widetilde{h}(r) < \pi, \quad \forall r \in [r^* - \delta,\ r^*).$$

So, by the conclusion of (i), we obtain that $\widetilde{h}(r)$ satisfies (2.1) in the interval $(r^*-\delta,\ r^*)$ and

$$\widetilde{h}'(r^*_-) = \lim_{r\to r^*_-}\widetilde{h}'(r) > 0.$$

Let's choose $\varphi \in C_0^\infty[r^* - \delta,\ r^* + \delta]$ such that $\varphi \leq 0,\ \varphi(r^*) = -1$. Then φ is an admissible variational function for $\widetilde{h}$. By (4.5), we have

$$\begin{aligned} 0 &\leq \int_s^{+\infty}(\widetilde{h}'\varphi' + \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi)rdr + \int_s^{+\infty} g(\widetilde{h})\varphi rdr \\ &= \int_{r^*-\delta}^{r^*}(\widetilde{h}'\varphi' + \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi)rdr + \int_{r^*-\delta}^{r^*} g(\widetilde{h})\varphi rdr \\ &= \widetilde{h}'(r)\varphi(r)r\Big|_{r^*-\delta}^{r^*} + \int_{r^*-\delta}^{r^*}[-(r\widetilde{h}')' + \frac{m^2}{r}\sin\widetilde{h}\cos\widetilde{h} + g(\widetilde{h})r]\varphi dr \\ &= -\widetilde{h}'(r^*_-)r^* < 0, \end{aligned} \tag{4.6}$$

there exists a contradiction. So, $\widetilde{h}(r) < \pi$ for any $r \in (s, +\infty)$.

(iii). We claim that $\widetilde{h}(r) > \xi,\ r \in (s, +\infty)$.
If there exists $r \in (s, +\infty)$ such that $\widetilde{h}(r) = \xi$, then we define

$$\widehat{r} = \sup\{r \in (s, +\infty) \mid \widetilde{h}(r) = \xi\},$$

and obtain $s < \widehat{r} < +\infty$.

By the definition of $\widehat{r}$ and the conclusion of (ii), we get

$$\xi < \widetilde{h}(r) < \pi,\ r \in (\widehat{r}, +\infty).$$

So, by the conclusion of (i), we know that $\widetilde{h}(r)$ satisfies (2.1) in the interval $(\widehat{r}, +\infty)$. Since $\lim\limits_{r\to\infty} \widetilde{h}(r) = \pi$, by Lemma 3.3, we obtain

$$\widetilde{h}'(\widehat{r}_+) = \lim_{r\to\widehat{r}_+} \widetilde{h}'(r) > 0.$$

In the following, we need to consider two cases:

Case I: There exists small $\delta > 0$ such that

$$\widetilde{h}(r) \equiv \xi, \quad \forall r \in [\widehat{r} - \delta,\, r^*).$$

For this case, let's choose $\varphi \in C_0^\infty[\widehat{r} - \delta,\, \widehat{r} + \delta]$ such that $0 \le \varphi \le 1,\ \varphi(\widehat{r}) = 1$. Then φ is an admissible variational function for $\widetilde{h}$. By (4.5),

$$\begin{aligned}
0 &\le \int_s^{+\infty} (\widetilde{h}'\varphi' + \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi) r dr + \int_s^{+\infty} g(\widetilde{h})\varphi r dr \\
&= \int_{\widehat{r}-\delta}^{\widehat{r}} \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi r dr + \widetilde{h}'(r)\varphi(r) r\Big|_{\widehat{r}}^{\widehat{r}+\delta} \\
&= m^2 \sin\xi\cos\xi \int_{\widehat{r}-\delta}^{\widehat{r}} \frac{\varphi}{r} dr - \widetilde{h}'(\widehat{r}_+)\widehat{r} \\
&\le m^2 \log\frac{\widehat{r}}{\widehat{r}-\delta} - \widetilde{h}'(\widehat{r}_+)\widehat{r}.
\end{aligned} \tag{4.7}$$

we can choose δ small enough such that

$$m^2 \log\frac{\widehat{r}}{\widehat{r}-\delta} - \widetilde{h}'(\widehat{r}_+)\widehat{r} < 0.$$

Obviously, there exists a contradiction.

Case II: There exists small $\delta > 0$ such that

$$\xi < \widetilde{h}(r) < \pi, \quad \forall r \in [\widehat{r} - \delta,\, \widehat{r}).$$

For this case, by the conclusion of (i), we obtain that $\widetilde{h}(r)$ satisfies (2.1) in the interval $(\widehat{r} - \delta,\, \widehat{r})$. Moreover, $\widetilde{h}'(\widehat{r}_-) = \lim\limits_{r\to\widehat{r}_-} \widetilde{h}'(r) \le 0$.

We can choose $\varphi \in C_0^\infty[\widehat{r} - \delta,\, \widehat{r} + \delta]$ such that $0 \le \varphi \le 1,\ \varphi(\widehat{r}) = 1$. Then φ is an admissible variational function for $\widetilde{h}$. By (4.5),

$$\begin{aligned}
0 &\le \int_s^{+\infty} (\widetilde{h}'\varphi' + \frac{m^2}{r^2}\sin\widetilde{h}\cos\widetilde{h}\varphi) r dr + \int_s^{+\infty} g(\widetilde{h})\varphi r dr \\
&= \widetilde{h}'(r)\varphi(r) r\Big|_{\widehat{r}-\delta}^{\widehat{r}} + \widetilde{h}'(r)\varphi(r) r\Big|_{\widehat{r}}^{\widehat{r}+\delta} \\
&= (\widetilde{h}'(\widehat{r}_-) - \widetilde{h}'(\widehat{r}_+))\widehat{r} < 0,
\end{aligned} \tag{4.8}$$

there exists a contradiction.

Since Case I and Case II cannot happen and $\widetilde{h}(r) \geq \xi$, we know that there always holds true $\widetilde{h}(r) > \xi$ for any $r \in (s, +\infty)$.

Combining (i), (ii) and (iii), we have that $\widetilde{h}(r)$ satisfies (2.1) in the interval $(s, +\infty)$. Moreover, there holds

$$\xi < \widetilde{h}(r) < \pi, \quad \forall r \in (s, +\infty).$$

Step 3. By Lemma 3.3, we immediately know that there holds true

$$\widetilde{h}'(r) > 0$$

for $r \in [s, +\infty)$.

The remaining work is to prove the uniqueness of the solution to P_s. Assume $\widetilde{h}_1(r)$ and $\widetilde{h}_2(r)$ are two different solutions of P_s. By the uniqueness of initial value problem, we obtain $\widetilde{h}'_1(s) \neq \widetilde{h}'_2(s)$. Without loss of generality, we assume $\widetilde{h}'_1(s) > \widetilde{h}'_2(s)$.

By Lemma 3.5, we get

$$\widetilde{h}_1(r) > \widetilde{h}_2(r), \quad r \in (s, +\infty). \tag{4.9}$$

By the Pohozaev identity (3.2), for $i = 1, 2$, we have

$$\left(s\widetilde{h}'_i(s)\right)^2 = m^2 \sin^2 \xi + 2G(\xi)s^2 + 4\int_s^{+\infty} G(\widetilde{h}_i(t))tdt. \tag{4.10}$$

Hence, from (4.9) and (4.10) we deduce that

$$0 < \left(s\widetilde{h}'_1(s)\right)^2 - \left(s\widetilde{h}'_2(s)\right)^2 = 4\int_s^{+\infty} [G(\widetilde{h}_1) - G(\widetilde{h}_2)]tdt. \tag{4.11}$$

However, since $G(x)$ is decreasing on the interval $[\xi, \pi]$, by (4.9) we know the right hand side of (4.11) is negative. This is a contradiction. So, the solution of P_s is unique. □

THEOREM 4.2. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies condtions (**i**) and (**iii**). Then, the problem $Q_{(s,\mu)}$ admits a unique solution $h(r)$. Moreover, $h'(r) > 0$ for any $r \in [s, \mu]$.*

Proof. By replacing $+\infty$ by μ, and X_s by Y_s, where

$$Y_s = \{h(r) \mid \pi - h(r) \in H^1([s, \mu], rdr),\ \xi \leq h(r) \leq \pi,\ h(s) = \xi \quad \text{and} \quad h(\mu) = \pi\},$$

in the proof of Theorem 4.1, we can address the existence of solution to the problem $Q_{(s,\mu)}$.

By Lemma 3.4, we get $h'(r) > 0,\ r \in [s, \mu]$. By Lemma 3.5, we get the uniqueness of solution to the problem $Q_{(s,\mu)}$. □

5. The proof of Theorem 1.1. In [1, 2], Ding has employed a mini-max argument to obtain the existence and uniqueness of equivariant harmonic maps from a sphere into another sphere. However, for our present case it seems that Ding's method is not valid. Here, we will employ the shooting target method to prove Theorem 1.1. To achieve this goal, we need to characterize the behavior $h_a(r)$, the solution of (2.1)-(2.2) with $a > 0$. Concretely, we need to establish some lemmas on when $h_a(r)$ increases monotonically from 0 to π on a finite interval. For simplicity, we would like to call such increasing $h_a(r)$ as *"solution of type* (I)*"* (see the following definition 5.1). We will employ the blow-up analysis to show that h_a is actually a solution of type (I) as $a > 0$ is small enough. As $a > 0$ is large enough, we combine the blow-up analysis and the Pohozaev identities to characterize the behaviors of h_a which is completely different from the case that $a > 0$ is small.

We know that there exists $\lambda_0 > 0$ such that $\varphi_{\lambda_0}(r) = 2\arctan[(\lambda_0 r)^m]$, corresponding to the equivariant harmonic map with degree m, satisfies the following initial value problem:

$$\begin{cases} h'' + \dfrac{1}{r}h' - \dfrac{m^2}{r^2}\sin h\cos h = 0, \quad r \in (0, +\infty), \\ h(0) = 0, \quad h^{(m)}(0) = m!. \end{cases} \tag{5.1}$$

For the sake of convenience, we denote φ_{λ_0} by ϕ. As ϕ is increasing monotonically from 0 to π, then there exists a unique $r_\xi \in (0, +\infty)$ such that $\phi(r_\xi) = \xi$.

Define

$$\phi_s(r) = \phi\left(\frac{r_\xi}{s} r\right).$$

Then $\phi_s(s) = \xi$ and $\phi_s(r)$ also satisfies the following equation

$$h'' + \frac{1}{r}h' - \frac{m^2}{r^2}\sin h\cos h = 0, \quad r \in (0, +\infty). \tag{5.2}$$

LEMMA 5.1. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$. If $\widetilde{h}_s$ is the solution of P_s, then, for any $r \in (s, +\infty)$ there holds true*

$$\phi_s(r) < \widetilde{h}_s(r).$$

Proof. By the Pohozaev identity (3.1), we have that, for $r \in [s, +\infty)$,

$$\left(r\widetilde{h}_s'(r)\right)^2 = m^2 \sin^2 \widetilde{h}_s(r) - 2\int_r^{+\infty} g(\widetilde{h}_s(t))\widetilde{h}_s'(t)t^2 dt \tag{5.3}$$

and

$$\left(r\phi_s'(r)\right)^2 = m^2 \sin^2 \phi_s(r). \tag{5.4}$$

By Theorem 4.1, we have $\widetilde{h}_s'(r) > 0$ for any $r \in [s, +\infty)$. Hence, we have, for $r \in [s, +\infty)$,

$$-2\int_r^{+\infty} g(\widetilde{h}_s(t))\widetilde{h}_s'(t)t^2 dt > 0,$$

here we also use the fact $g(x) \le 0$ for any $x \in [\xi, \pi]$. Since $\widetilde{h}_s(s) = \phi_s(s) = \xi$, by comparing (5.3) and (5.4) we obtain that

$$\phi_s'(s) < \widetilde{h}_s'(s).$$

In fact, there holds true that, for any $r \in (s, +\infty)$,

$$\phi_s(r) < \widetilde{h}_s(r).$$

If the above inequality fails, we define

$$r^* = \sup\{r \in (s, +\infty) \mid \widetilde{h}_s(t) > \phi_s(t),\, \forall t \in (s, r)\}$$

and obtain

$$s < r^* < +\infty.$$

By the definition of r^*, we have

$$\phi_s(r) < \widetilde{h}_s(r), \quad \forall r \in (s,\, r^*) \quad \text{and} \quad \phi_s(r^*) = \widetilde{h}_s(r^*).$$

This implies that

$$\phi_s'(r^*) \geq \widetilde{h}_s'(r^*).$$

However, from (5.3) and (5.4) we infer that

$$\phi_s'(r^*) < \widetilde{h}_s'(r^*),$$

as $\phi_s(r^*) = \widetilde{h}_s(r^*)$. A contradiction is derived. Thus, we complete the proof of the lemma. □

Let h_a be the solution of (2.1)-(2.2) with $a > 0$. By Corollary 3.2, we know there exists $s_a \in (0, +\infty)$ such that $h_a(r)$ increases monotonically from 0 to ξ in the interval $[0, s_a]$. Then,we have the following lemma.

Lemma 5.2. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$ and h_a is a solution of (2.1)-(2.2) with $a > 0$, which increases monotonically from 0 to ξ on the interval $[0, s_a]$. Then, there holds true $\phi_{s_a}(r) > h_a(r)$ for any $r \in (0,\, s_a)$.*

Proof. The proof is analogous to the proof of Lemma 5.1. We omit it. □

Lemma 5.3. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$. If $\widetilde{h}_s$ is the solution of P_s and h satisfies (2.1) on the interval $(s, +\infty)$ with $h(s) = \xi$, then, there exists $r_1 \in (s, +\infty)$ such that $h(r)$ increases monotonically from ξ to π on the interval $[s, r_1]$ if and only if $h'(s) > \widetilde{h}_s'(s)$.*

Proof. (1). If $h'(s) < \widetilde{h}_s'(s)$, there does not exist $r_1 \in (s, +\infty)$ such that $h(r)$ increases monotonically from ξ to π on the interval $[s, r_1]$. Otherwise, h will intersect $\widetilde{h}_s$ at two different points in the domain $[s, +\infty) \times [\xi, \pi]$ which contradicts Lemma 3.5.

(2). If $h'(s) = \widetilde{h}_s'(s)$, by the uniqueness of initial value problem, we get $h \equiv \widetilde{h}_s$. As $\widetilde{h}_s$ increases monotonically from ξ asymptotically to π on the interval $[s, +\infty)$, we can't pick $r_1 \in (s, +\infty)$ such that $h(r)$ increases monotonically from ξ to π in the interval $[s,\, r_1]$.

(3). For simplicity, let $h(r, a)$ be the solution of the following problem:

$$\begin{cases} (rh')' = \dfrac{m^2}{r} \sin h \cos h + g(h) r, & r \in (s, +\infty) \\ h(s) = \xi, \quad h'(s) = a. \end{cases}$$

Then, we have $h(r, \widetilde{h}'_s(s)) \equiv \widetilde{h}_s(r)$.

Define

$$A = \{a \in \mathbb{R} \,|\, h(r,a) \text{ increases monotonically from } \xi \text{ to } \pi \text{ in finite interval}\}.$$

By Theorem 4.2, we know that A is a non-empty set. Moreover, by the continuous dependence of the solutions on the initial data and Lemma 3.4, we know A is a non-empty open set. From the argument in (1) and (2), we get

$$\inf\{a \,|\, a \in A\} \geq \widetilde{h}'_s(s) > 0.$$

For any $\bar{a} \in \partial A$ (the boundary of A) and $\bar{a} < +\infty$, we can choose a sequence $\{a_k\} \subseteq A$ such that

$$\lim_{k\to\infty} a_k = \bar{a}.$$

Let r_k be the minimal number $r \geq s$ such that $h(r, a_k) = \pi$. Then, we have

$$\lim_{k\to\infty} r_k = +\infty.$$

Otherwise, by the continuous dependence of the solutions on the initial data, we have $\bar{a} \in A$ which contradicts the fact A is an open set. By the continuous dependence of the solutions on the initial data again, we have that, for any $r \in [s, +\infty)$,

$$h'(r, \bar{a}) \geq 0 \quad \text{and} \quad \xi \leq h(r, \bar{a}) \leq \pi.$$

It implies that

$$\lim_{r\to\infty} h(r, \bar{a}) = l > \xi.$$

By Lemma 3.3, we obtain

$$\xi < h(r, \bar{a}) < \pi, \quad r \in (s, +\infty) \quad \text{and} \quad l = \pi.$$

It means that $h(r, \bar{a})$ is also a solution of problem P_s. By the uniqueness of solution to the problem P_s, we get $\bar{a} = \widetilde{h}'_s(s)$. So, we get $A = (\widetilde{h}'_s(s), +\infty)$. The proof of the lemma is completed. □

Definition 5.1. *h_a is called a solution of type* (I) *to (2.1)-(2.2) if there exists $r_a \in (0, +\infty)$ such that h_a increases monotonically from* 0 *to* π *in the interval* $[0, r_a]$.

Remark 7. *By Corollary 3.2 and Lemma 3.4, if h_a is a solution to type* (I) *to (2.1)-(2.2) and increases monotonically from* 0 *to* π *in the interval* $[0, r_a]$, *then*

$$h'(r) > 0, \quad \forall r \in (0, r_a].$$

Let h_a be the solution of problem (2.1)-(2.2) with $a > 0$. By Corollary 3.2, let s_a be the minimal positive number such that $h_a(s) = \xi$. Then we have

(a). $s_a \to +\infty$ as $a \to 0$.

(b). By the Pohozaev identity

$$(rh_a'(r))^2 = m^2 \sin^2 h_a(r) + 2G(h_a(r))r^2 - 4\int_0^r G(h_a(t))tdt, \tag{5.5}$$

we get

$$\begin{aligned}(h_a'(r))^2 &\leq \frac{1}{r^2}\{m^2 \sin^2 h_a(r) + 2G(\xi)r^2 - 4\int_0^r G(0)tdt\} \\ &\leq m^2 \frac{\sin^2 h_a(r)}{r^2} + 2(G(\xi) - G(0)).\end{aligned}$$

On the other hand, by (5.5) and Lemma 5.2, we have

$$\begin{aligned}(h_a'(s_a))^2 &\geq \frac{1}{{s_a}^2}\{m^2 \sin^2 \xi + 2G(\xi){s_a}^2 - 4\int_0^{s_a} G(\phi_{s_a}(t))tdt\} \\ &\geq 2G(\xi) - 4\frac{1}{r_\xi^2}\int_0^{r_\xi} G(\phi(t))tdt > 0.\end{aligned}$$

Combining the above two inequalities with the properties of h_a near $r = 0$ (cf. Section 2), we have that there exists positive constant c_0 and c_1 such that, if $a \leq 1$, then

$$0 < h_a'(r) \leq c_1, \quad \forall r \in (0, s_a] \quad \text{and} \quad 0 < c_0 \leq h_a'(s_a) \leq c_1.$$

LEMMA 5.4. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$ and h_a is a solution to (2.1)-(2.2). Then, there exists a positive number $b > 0$ such that, if $a > 0$ is small enough, then, we have*

$$(h_a'(s_a))^2 \geq 2G(\xi) + b.$$

Proof. Let a_i be any sequence such that $a_i > 0$ and $a_i \to 0$. For simplicity, we denote $h_i = h_{a_i}$ and $s_i = s_{a_i} \to +\infty$.

Set

$$u_i(r) = h_i(r + s_i).$$

Then, we have that, on $(-s_i, 0]$,

$$u_i'' + \frac{1}{r + s_i}u_i' - \frac{m^2}{(r + s_i)^2}\sin u_i \cos u_i = g(u_i).$$

Note that we have $u_i(0) = \xi$, $c_0 \leq u_i'(0) \leq c_1$,

$$0 < u_i(r) \leq \xi \quad \text{and} \quad 0 < u_i'(r) \leq c_1, \quad \forall r \in (-s_i, 0].$$

By a diagonal subsequence argument, there exists a subsequence of $\{u_i\}$, still denoted by u_i, such that u_i converges uniformly in $C^2([-R, 0])$, for any given $R > 0$, to some $u \in C^2_{loc}(-\infty, 0])$. The limit u satisfies the following equation on $(-\infty, 0]$

$$u'' = g(u), \tag{5.6}$$

with $c_0 \leq u'(0) \leq c_1$. Moreover, for any $r \in (-\infty, 0]$ there holds true

$$0 \leq u(r) \leq \xi = u(0) \quad \text{and} \quad 0 \leq u'(r) \leq c_1.$$

Hence, we conclude that there exists $0 \le l \le \xi$ such that

$$\lim_{r \to -\infty} u(r) = l.$$

Since $u'' = g(u) \ge 0$, it is easy to see that there exists $\mu \in [0, c_1]$ such that

$$\lim_{r \to -\infty} u'(r) = \mu.$$

On the other hand, we also have

$$\int_{-\infty}^{0} u'(r)dr \le \xi.$$

Hence, we infer that $u'(r) \to 0$ as $r \to -\infty$, i.e. $\mu = 0$. It follows that

$$\int_{-\infty}^{0} u''(r)dr = u'(0).$$

As $u'' \ge 0$, from the integrability of u'' on $(-\infty, 0]$ we conclude that

$$\lim_{r \to -\infty} u''(r) = \lim_{r \to -\infty} g(u(r)) = g(l) = 0.$$

Hence, $l = 0$ or ξ. But, since u is increasing function on $(-\infty, 0]$ with $u(0) = \xi$ and $u'(0) \ge c_0 > 0$, we have $l < \xi$. Hence, $l = 0$.

By integrating the two sides of (5.6) we obtain

$$u'(r)^2 = 2G(u(r)) + C, \tag{5.7}$$

where $C = (u'(0))^2 - 2G(\xi)$. Let $r \to -\infty$ in (5.7), we deduce

$$u'(0)^2 = 2G(\xi) - 2G(0). \tag{5.8}$$

Notice that, the solution u of (5.6) with initial data $u(0) = \xi$ and $u'(0) = \sqrt{2G(\xi) - 2G(0)}$ is unique. The uniqueness implies that, if we denote $u_a(r) = h_a(r + s_a)$, there holds true

$$u_a(r) \to u(r) \text{ uniformly in } C^2([-R, 0]) \text{ for any } R > 0,$$

as $a \to 0$. Then, by (5.8), we obtain that, as $a \to 0$, there holds

$$h_a'(s_a)^2 \to 2G(\xi) - 2G(0).$$

If we first take $b = -G(0)$, then let $a > 0$ be small enough, then the desired conclusions follow. Thus we complete the proof. □

THEOREM 5.5. *Suppose that $g(x) \in C^\infty([0, \pi])$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$. If h_a is the solution of (2.1)-(2.2) with $a > 0$, then, there exists $\epsilon > 0$ such that for any $a \in (0, \epsilon)$, h_a is a solution of type* (I).

Proof. By Corollary 3.2, for any $a > 0$ there exists $s_a \in (0, +\infty)$ such that $h_a(r)$ increases monotonically from 0 to ξ on the interval $[0, s_a]$ with $h(s_a) = \xi$ and $h'(s_a) > 0$.

Let $\widetilde{h}_{s_a}$ be the solution of the problem P_s with $s = s_a$. Then, by Lemma 5.3, h_a is a solution of type (I) if and only if $h_a'(s_a) > \widetilde{h}_{s_a}'(s_a)$. So, to prove the theorem, it suffices to prove that there exists $\epsilon > 0$ such that, for all $a \in (0, \epsilon)$,

$$h_a'(s_a) > \widetilde{h}_{s_a}'(s_a).$$

By Theorem 4.1, we know $\widetilde{h}_{s_a}$ minimizes the functional J_{s_a} on space X_{s_a}. Let

$$\theta(r)=\begin{cases}(\pi-\xi)(r-s_a)+\xi, & r\in[s_a,s_a+1],\\ \pi, & r\in(s_a+1,+\infty)\end{cases}$$

then, $\theta\in X_{s_a}$. It follows that $J_{s_a}(\widetilde{h}_{s_a})\le J_{s_a}(\theta)$. Hence, we have

$$\begin{aligned}\int_{s_a}^{+\infty}G(\widetilde{h}_{s_a})tdt &\le J_{s_a}(\widetilde{h}_{s_a})\le J_{s_a}(\theta)\\ &\le \frac{1}{2}\int_{s_a}^{s_a+1}[(\pi-\xi)^2+\frac{m^2}{r^2}]rdr+\int_{s_a}^{s_a+1}G(\xi))rdr\\ &\le C\left\{(s_a+\frac{1}{2})+\log\frac{s_a+1}{s_a}\right\},\end{aligned}\tag{5.9}$$

where C is a positive constant independent of s_a.

On the other hand, by the Pohozaev identity we have

$$(s_a\widetilde{h}'_{s_a}(s_a))^2=m^2\sin^2\xi+2G(\xi){s_a}^2+4\int_{s_a}^{+\infty}G(\widetilde{h}_{s_a}(t))tdt.\tag{5.10}$$

Combining (5.9) and the above identity we obtain

$$(\widetilde{h}'_{s_a}(s_a))^2\le\frac{1}{s_a^2}\left\{m^2\sin^2\xi+2G(\xi){s_a}^2+4C((s_a+\frac{1}{2})+\log\frac{s_a+1}{s_a})\right\}.\tag{5.11}$$

Hence, it follows

$$\overline{\lim_{a\to 0}}(\widetilde{h}'_{s_a}(s_a))^2\le 2G(\xi).$$

Combining the last inequality with Lemma 5.4, there exists $\epsilon>0$ such that for all $a\in(0,\epsilon)$,

$$h'_a(s_a)>\widetilde{h}'_{s_a}(s_a).$$

Then the proof of the theorem is completed. □

THEOREM 5.6. *Suppose that $g(x)\in C^\infty([0,\pi])$ satisfies $(\boldsymbol{i})-(\boldsymbol{iii})$. Let ϕ be the solution of the problem (5.1).*
(1). *If*

$$-\infty<\int_0^{+\infty}G(\phi)rdr\le 0,$$

then h_a is a solution of type (I) *to (2.1)-(2.2) for all $a>0$.*
(2). *If*

$$0<\int_0^{+\infty}G(\phi)rdr\le+\infty,$$

then there exists $a_0>0$ such that h_a is not a solution of type (I) *to (2.1)-(2.2) for $a>a_0$.*

Proof. By Corollary 3.2, for $a > 0$, there exists $s_a \in (0, +\infty)$ such that $h_a(r)$ increases monotonically from 0 to ξ in the interval $[0,\, s_a]$ with $h(s_a) = \xi$ and $h'(s_a) > 0$. Let $\widetilde{h}_{s_a}$ be the solution of problem P_s with $s = s_a$. Then, by the Pohozaev identity, we have

$$(s_a h_a'(s_a))^2 = m^2 \sin^2 \xi + 2G(\xi){s_a}^2 - 4\int_0^{s_a} G(h_a(t))tdt, \tag{5.12}$$

$$(s_a \widetilde{h}_{s_a}'(s_a))^2 = m^2 \sin^2 \xi + 2G(\xi){s_a}^2 + 4\int_{s_a}^{+\infty} G(\widetilde{h}_{s_a}(t))tdt. \tag{5.13}$$

Now we discuss the case (1). By Lemma 5.3, h_a is a solution of type (I) if and only if

$$h_a'(s_a) > \widetilde{h}_{s_a}'(s_a).$$

Comparing (5.12) and (5.13), it suffices to prove that, for $a > 0$, the following inequality is true

$$\int_0^{s_a} G(h_a(t))tdt + \int_{s_a}^{+\infty} G(\widetilde{h}_{s_a}(t))tdt < 0. \tag{5.14}$$

Since $G(x)$ is increasing on the interval $[0,\, \xi]$ and decreasing on the interval $[\xi,\, \pi]$, by Lemma 5.2 and Lemma 5.1, we derive that, as $a > 0$,

$$\int_0^{s_a} G(h_a(t))tdt < \int_0^{s_a} G(\phi_{s_a}(t))tdt = \frac{s_a^2}{r_\xi^2}\int_0^{r_\xi} G(\phi(t))tdt, \tag{5.15}$$

$$\int_{s_a}^{+\infty} G(\widetilde{h}_{s_a}(t))tdt < \int_{s_a}^{+\infty} G(\phi_{s_a}(t))tdt = \frac{s_a^2}{r_\xi^2}\int_{r_\xi}^{+\infty} G(\phi(t))tdt. \tag{5.16}$$

Combining (5.15) and (5.16), we get that, for $a > 0$, there holds

$$\int_0^{s_a} G(h_a(t))tdt + \int_{s_a}^{+\infty} G(\widetilde{h}_{s_a}(t))tdt < \frac{s_a^2}{r_\xi^2}\int_0^{+\infty} G(\phi(t))tdt.$$

So, when

$$-\infty < \int_0^{+\infty} G(\phi)rdr \le 0,$$

(5.14) follows and the conclusion stated in (1) is true.

We turn to the discussion of the case (2). If the conclusion stated in (2) fails, then there exists a sequence $a_i \to +\infty$ such that h_{a_i} is a solution of type (I) to (2.1)-(2.2).

Set

$$\hbar_i({a_i}^{\frac{1}{m}} r) \equiv h_{a_i}(r).$$

Then $\hbar_i(r)$ is the solution of the following problem:

$$\begin{cases} \hbar_i''(r) + \dfrac{1}{r}\hbar_i'(r) - \dfrac{m^2}{r^2}\sin\hbar_i(r)\cos\hbar_i(r) - {a_i}^{-\frac{2}{m}} g(\hbar_i(r)) = 0. \\ \hbar_i(0) = 0, \quad \hbar_i^{(m)}(0) = m! \end{cases} \tag{5.17}$$

Let s_i be the minimal positive number $s > 0$ such that $\hbar_i(s) = \pi$. Comparing the problem (5.17) with the problem (5.1), we conclude that, for any $R > 0$, $\hbar_i$ converges to ϕ uniformly in $C^1[0, R]$ as $a_i \to +\infty$. Notice that, ϕ increases monotonically from 0 asymptotically to π, then, we have

$$\lim_{i\to+\infty} s_i = +\infty.$$

By the Pohozaev identity, we have

$$(s_i\hbar_i'(s_i))^2 = -4a_i{}^{-\frac{2}{m}} \int_0^{s_i} G(\hbar_i)rdr.$$

Since $\hbar_i'(s_i) > 0$, we get

$$\int_0^{s_i} G(\hbar_i)rdr < 0. \tag{5.18}$$

On the other hand, since

$$0 < \int_0^{+\infty} G(\phi)rdr \le +\infty,$$

we can pick $R_0 > 0$ such that $\phi(R_0) > \xi$ and

$$\int_0^{R_0} G(\phi)rdr > 0. \tag{5.19}$$

Hence,

$$\int_0^{R_0} G(\phi)rdr = \lim_{i\to+\infty} \int_0^{R_0} G(\hbar_i)rdr \le \varliminf_{i\to\infty} \int_0^{s_i} G(\hbar_i)rdr \le 0,$$

which contradicts (5.19). So, if

$$0 < \int_0^{+\infty} G(\phi)rdr \le +\infty,$$

there always exists $a_0 > 0$ such that, for $a > a_0$, h_a is not a solution of type (I) to (2.1)-(2.2). □

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let ϕ be the solution of problem (5.1). From the Remark 3, we can replace φ_1 by ϕ. Since ϕ increases from 0 asymptotically to π and $G(x) \ge 0$ for $x \in [\xi, \pi]$, we have

$$-\infty < \int_0^{+\infty} G(\phi)rdr \le +\infty.$$

Define

$$A = \{a > 0 \mid h_a \text{ is a solution of type (I) to (2.1)-(2.2)}\}.$$

By the continuous dependence of the solutions on the initial data (cf. Theorem 2.2), Corollary 3.2 and Lemma 3.4, we know A is an open set. By Theorem 5.5, we derive that A is a non-empty open set.

We need only to consider the following two cases:

Case One. If

$$-\infty < \int_0^{+\infty} G(\phi) r dr \leq 0,$$

Theorem 5.6 tells us that $A = (0, +\infty)$. It means that all solutions of (2.1)-(2.2) with $a > 0$ increase from 0 to π on finite interval. So the problem (1.3)-(1.4) with $0 \leq h(r) \leq \pi$ on $(0, \infty)$ does not admit any solution.

Case Two. If

$$0 < \int_0^{+\infty} G(\phi) r dr \leq +\infty,$$

set

$$a^* = \sup\{a \in A\},$$

by Theorem 5.6 we have $0 < a^* < +\infty$. We claim that h_{a^*} is a solution of (1.3)-(1.4) with $0 < h(r) < \pi$ on $(0, \infty)$.

In the following, we always assume $a \in A$. Let $r_a \in (0, +\infty)$ such that h_a increases monotonically from 0 to π on the interval $[0, r_a]$. Then, we have $\lim_{a \to a^*} r_a = +\infty$. Otherwise, there exists a sequence $a_k \to a^*$ such that

$$\lim_{a_k \to a^*} r_{a_k} = r^*$$

where $r^* \in (0, +\infty)$. By continuous dependence on initial data of the problem (2.1)-(2.2), we have that h_{a^*} is also a solution of type (I), which contradicts the definition of a^*.

As $\lim_{a \to a^*} r_a = +\infty$, we can infer that

$$h'_{a^*}(r) \geq 0 \quad \text{and} \quad 0 \leq h_{a^*}(r) \leq \pi, \quad r \in (0, +\infty).$$

Then, we have

$$\lim_{r \to +\infty} h_{a^*}(r) = l.$$

By Corollary 3.2 and the fact that π is a trivial solution of equation (2.1), we have that

$$0 < h_{a^*}(r) < \pi \quad \text{for any} \quad r \in (0, +\infty) \qquad \text{and} \qquad \xi < l \leq \pi.$$

By Lemma 3.3, we have that $l = \pi$ and $h_{a^*}(r)$ converges to π exponentially as $r \to +\infty$. Hence $h_{a^*}(r)$ is a solution of (1.3)-(1.4) with $0 < h(r) < \pi$ on $(0, \infty)$. Thus, we complete the proof of the theorem.

6. Applications to the equations of Landau-Lifshitz type. In this section, we will generalize and improve the results due to Gustafson and Shatah in [7] as an application of Theorem 1.1. First we recall the Landau-Lifshitz equation

$$\partial_t u = u \times \Delta u,$$

where $u : M \times \mathbb{R} \to S^2$ and " $\times$ " denotes the cross product in $\mathbb{R}^3$ (see [10]). We would like to consider the Schrödinger flows for maps $u : M \times \mathbb{R} \to S^2$ corresponding to the functional

$$\widetilde{F}(u) = \int_M |\nabla u|^2 dM + \int_M \widetilde{H}(u) dM$$

in the sense of [3], where $\widetilde{H} \in C^\infty(S^2, \mathbb{R})$. The equation of flows can be expressed as

$$u_t = u \times (\Delta u - \nabla \widetilde{H}(u)). \tag{6.1}$$

The stationary solutions of this equation satisfy

$$\Delta u + |\nabla u|^2 u = \nabla \widetilde{H}(u).$$

This is just the elliptic system of harmonic maps with potential $\widetilde{H}(u)$ from M into S^2. When M is a compact Riemann surface satisfying some symmetry and $\widetilde{H} \equiv 0$, Ding and Yin have studied the existence of special periodic solutions to the Landau-Lifshitz equation. For details readers can refer to [4]. Such a special class of periodic solutions is called "geometric solitons" in [12, 13]. For the case $M \equiv \mathbb{R}^2$ and $\widetilde{H}(u) \equiv \widetilde{G}(d(u))$, where $d(u)$ denotes the geodesic distance from $u \in S^2$ to the north pole $P = (0, 0, 1)$, we would like to consider the equivariant solutions to (6.1) written by

$$u(x, t) = (\sin h(r) \cos(m\theta + wt), \sin h(r) \sin(m\theta + wt), \cos h(r)), \tag{6.2}$$

where (r, θ) is the polar coordinates on $\mathbb{R}^2$ and $m \in \mathbb{Z}\backslash\{0\}$. Substituting (6.2) into (6.1), we obtain

$$h'' + \frac{1}{r}h' - \frac{m^2}{r^2} \sin h \cos h = \widetilde{g}(h) + \omega \sin h \tag{6.3}$$

where $\widetilde{g}(\cdot) = \widetilde{G}'(\cdot)$.

Set

$$g(h) = \widetilde{g}(h) + \omega \sin h,$$

then equation (6.3) is of the same form as (1.3).

In particular, the following equation of Landau-Lifshitz type has strong physical background

$$\partial_t u = u \times (\Delta u + \lambda u_3 \widehat{k}), \tag{6.4}$$

where

$$u(x, t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{S}^2, \quad \widehat{k} = (0, 0, 1), \quad \lambda > 0.$$

For more details readers can refer to [7] and the references therein. The equation (6.4) is of a Hamiltonian structure and its Hamiltonian energy functional is:

$$E = E_e + \lambda E_a.$$

Here, the exchange energy E_e and the anisotropy energy E_a are defined respectively by

$$E_e = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \quad \text{and} \quad E_a = \frac{1}{2}\int_{\mathbb{R}^2} 1 - u_3^2.$$

It is worth to point out that, when $\lambda \neq 0$, the static solutions with finite energy to (6.4) are ruled out by Pohozaev identity.

Gustafson and Shatah in [7] considered the existence of such equivariant solutions to (6.4) as (6.2). In the present case, the corresponding equation reads

$$h'' + \frac{1}{r}h' - \frac{m^2}{r^2}\sin h \cos h - g(h) = 0, \tag{6.5}$$

where

$$g(h) = (\omega + \lambda \cos h)\sin h.$$

Note (6.5) is a special case of the equation (1.3). By variational methods, Gustafson and Shatah in [7] obtained the following results.

Proposition 6.1. *For any $m \in \mathbb{Z}\backslash\{0\}$ there exists a positive number ω_0 with $0 < \omega_0 \leq \frac{1}{|m|}$ such that, if $0 < \omega < \omega_0\lambda$, the equation (6.4) with $\lambda > 0$ admits a solution of form (6.2) with $h(r)$ which increases monotonically from 0 asymptotically to π on $(0, \infty)$.*

A natural problem is whether or not ω_0 can be accurately determined. On the other hand, one also wants to know whether or not the equation (6.4) admits a solution of form (6.2) as $\omega \notin (0, \omega_0\lambda)$. In other words, when $g(h) = (\omega + \lambda\cos h)\sin h$, where $\omega \notin (0, \omega_0\lambda)$, is the problem (1.3)-(1.4) solvable? As a direct application of Theorem 1.1, we can completely answer the above questions. More precisely, we have the following more general results:

Theorem 6.2. *Let (M, g) be the Euclidean 2-plane $\mathbb{R}^2$. Assume that the potential function $\widetilde{H}(u) = \widetilde{G}(d(u)) : S^2 \to \mathbb{R}$ with*

$$\widetilde{G}'(\cdot) = g(\cdot) - \omega\sin(\cdot),$$

where $g(\cdot)$ satisfies $(\boldsymbol{i}) - (\boldsymbol{iii})$. Then the equation (6.1) admits a solution of form (6.2) with $h(r)$ satisfying $h(0) = 0 \leq h(r) \leq \pi = h(\infty)$ on $(0, \infty)$ if and only if G, which is defined by $G(x) = -\int_x^\pi g(t)dt$, satisfies

$$0 < \int_0^\infty G(\varphi_1(r))\, rdr \leq \infty.$$

Moreover, the solution we obtain satisfies $h'(r) > 0$ on $(0, \infty)$ and converges to π exponentially as $r \to \infty$.

Especially, for the equation (6.4) with $\lambda > 0$ we have the following theorem.

Theorem 6.3. *For any $m \in \mathbb{Z}\backslash\{0\}$ the equation (6.4) with $\lambda > 0$ admits a solution of form (6.2) with $h(r)$ satisfying $h(0) = 0 \leq h(r) \leq \pi = h(\infty)$ on $(0, \infty)$ if and only if*

$$0 < \omega < \frac{\lambda}{|m|}.$$

Moreover, the solution we obtain satisfies $h'(r) > 0$ on $(0, \infty)$ and converges to π exponentially as $r \to \infty$.

Proof. In order to prove the theorem, we only need to show the solvability of (1.3)-(1.4) with

$$g(h) = (\omega + \lambda \cos h) \sin h.$$

If the problem (1.3)-(1.4) with $g(h)$ as above admits a solution $h(r)$, by Pohozaev identity, we get

$$\int_0^{+\infty} G(h(r)) r dr = 0,$$

where

$$G(h) = \frac{\lambda}{2} \sin^2 h - \omega(1 + \cos h).$$

Obviously, when $\frac{\omega}{\lambda} \leq 0$, there holds true $G(x) \geq 0$ for any $x \in [0, \pi]$. On the other hand, when $\frac{\omega}{\lambda} \geq 1$ we have $G(x) \leq 0$ for any $x \in [0, \pi]$. No matter which case happens, for any function $h(r)$ with $0 \leq h(r) \leq \pi$ on $(0, \infty)$ there holds true

$$\int_0^{+\infty} G(h(r)) r dr = 0,$$

if and only if

$$G(h) \equiv 0.$$

This implies $h \equiv \pi$ or 0, since π is the only zero point of $G(x)$ on the interval $[0, \pi]$ as $\frac{\omega}{\lambda} < 0$ or $\frac{\omega}{\lambda} \geq 1$ while there exist two zero points of $G(x)$ on the interval $[0, \pi]$ in the case $\frac{\omega}{\lambda} = 0$, i.e. 0 and π. Hence the problem (1.3)-(1.4) doesn't admit a solution with $h(0) = 0 \leq h(r) \leq \pi = h(\infty)$ on $(0, \infty)$ with $\frac{\omega}{\lambda} \leq 0$ or $\frac{\omega}{\lambda} \geq 1$.

So, it suffices to consider the case $0 < \frac{\omega}{\lambda} < 1$. For this case, it's easy to check $g(x)$ satisfies the conditions (**i**) – (**iii**).

By a direct calculation, we have

$$\int_0^{+\infty} G(\varphi_1) r dr = \begin{cases} +\infty, & |m| = 1, \\ \left(\frac{\lambda}{|m|} - \omega\right) C_0, & |m| \geq 2. \end{cases}$$

Here

$$C_0 = \frac{1}{|m|} \mathrm{B}\left(\frac{1}{|m|}, 1 - \frac{1}{|m|}\right)$$

and $\mathrm{B}(x, y)$ is the Beta function and $\varphi_1 = 2\arctan(r^m)$. So, the conclusions of Theorem 6.3 follow directly from Theorem 1.1. □

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