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DEFORMATIONS OF COISOTROPIC SUBMANIFOLDS IN LOCALLY CONFORMAL SYMPLECTIC MANIFOLDS*

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Abstract. In this paper, we study deformations of coisotropic submanifolds in a locally conformal symplectic manifold. Firstly, we derive two equivalent equations that govern C^∞ deformations of coisotropic submanifolds. Using the first equation we define the corresponding C^∞-moduli space of coisotropic submanifolds modulo the Hamiltonian isotopies. Secondly, we prove that the formal deformation problem is governed by an L_∞-structure which is a $\mathfrak{b}$-deformation of strong homotopy Lie algebroids introduced in [OP] in the symplectic context. Then we study deformations of locally conformal symplectic structures and their moduli space. Using the second equation we study the corresponding bulk (extended) deformations of coisotropic submanifolds. Finally we revisit Zambon's obstructed infinitesimal deformation [Za] in this enlarged context and prove that it is still obstructed.

Key words. Locally conformal symplectic manifold, coisotropic submanifold, $\mathfrak{b}$-twisted differential, bulk deformation, Zambon's example.

AMS subject classifications. Primary 53D35.

1. Introduction.

Symplectic manifolds (M, ω) have been of much interest in *global* study of Hamiltonian dynamics, and symplectic topology via analysis of pseudoholomorphic curves. In this regard closedness of the two-form ω plays an important role in relation to the dynamics of Hamiltonian diffeomorphisms and the global analysis of pseudoholomorphic curves. On the other hand when one takes the coordinate chart definition of symplectic manifolds and implements the covariance property of Hamilton's equation, there is no compulsory reason why one should require the two-form to be closed. Indeed in the point of view of canonical formalism in Hamiltonian mechanics and construction of the corresponding *bulk physical space*, it is more natural to require the locally defined canonical symplectic forms

$$\omega_\alpha = \sum_{i=1}^n dq_i^\alpha \wedge dp_i^\alpha$$

to satisfy the cocycle condition

$$\omega_\alpha = \lambda_{\beta\alpha}\omega_\beta, \quad \lambda_{\beta\alpha} \equiv \text{const.} \tag{1.1}$$

with $\lambda_{\gamma\beta}\lambda_{\beta\alpha} = \lambda_{\gamma\alpha}$ as the gluing condition. (See introduction [V2] for a nice explanation on this point of view) The corresponding bulk constructed in this way naturally becomes *locally conformal symplectic manifolds* (abbreviated as l.c.s manifolds) whose definition we first recall. For the consistency of the definition, we will mostly assume $\dim M > 2$ in this paper.

DEFINITION 1.1. An l.c.s. manifold is a triple $(X, \omega, \mathfrak{b})$ where $\mathfrak{b}$ is a closed one-form and ω is a nondegenerate 2-form satisfying the relation

$$d\omega + \mathfrak{b} \wedge \omega = 0. \tag{1.2}$$

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We refer to [V2], [HR], [Ba1], [Ba2] for more detailed discussion of general properties of l.c.s. manifolds and non-trivial examples.

Locally by choosing $\mathfrak{b} = dg$ for a local function $g : U \to \mathbb{R}$ on an open neighborhood U, (1.2) is equivalent to

$$d(e^{-g}\omega) = 0 \tag{1.3}$$

and so the local geometry of l.c.s manifold is exactly the same as that of symplectic manifolds. In particular one can define the notion of Lagrangian submanifolds, isotropic submanifolds, and coisotropic submanifolds in the same way as in the symplectic case since the definitions require only nondegeneracy of the two-form ω.

The main questions of our interest in this paper are following. Firstly, we like to know whether the global geometry of coisotropic submanifolds is any different from that of symplectic case. Secondly, we like to treat the problem of extended deformation of coisotropic submanifolds posed by Oh-Park [OP, p.355], i.e. both the symplectic form ω and the coisotropic submanifold are allowed to vary, in even a larger setting of l.c.s. manifolds. Note that the problem of extended deformations of coisotropic submanifolds of certain Poisson manifolds have been recently considered by Schaetz and Zambon [SZ], see also Remark 12.7.

We recall that by the results from [Za], [OP], deformation theory of coisotropic submanifolds in symplectic manifolds is generally obstructed. In particular, the set of coisotropic submanifolds with a given rank does not form a smooth Frechet manifold [Za], and the relevant (formal) deformation theory thereof is described by an L_∞-structure called *strong homotopy Lie algebroids* [OP]. In the present paper, we show that Oh-Park's deformation theory naturally extends to that of l.c.s. manifolds, once appropriate normal form theorem of canonical neighborhoods of coisotropic submanifolds (Theorem 4.2) and the theory of bulk-deformed strong homotopy Lie algebroids (sections 9, 11) are developed. For this purpose, we need to prove the l.c.s analog of Darboux-type theorem [We] and develop the l.c.s. analog to Moser's trick, for which usage of Novikov-type cohomology instead of the ordinary de-Rham cohomology is essential. (See [HR] for relevant exposition of this cohomology theory.) We derive two equivalent equations that govern C^∞-deformations of coisotropic submanifolds (Theorems 6.2, 8.1) and develop a theory of bulk deformations of l.c.s. forms and of coisotropic submanifolds in this larger context of l.c.s. manifolds (sections 11, 12).

Some more motivations of the present study are in order. First of all, we would like to see if the obstructed example of Zambon [Za] in the symplectic context is still obstructed in this enlarged deformations of coisotropic submanifolds together with bulk deformations of l.c.s. structures with replacement of closedness of ω by the Novikov-closedness of $\mathfrak{b}$-twisted differential. We then prove that Zambon's example still remains obstructed even under this enlarged setting of bulk deformations (Theorem 12.6).

Another source of motivation comes from the study of J-holomorphic curves in this enlarged bulk of l.c.s. manifolds. Again all the local theory of J-holomorphic curves go through without change. The only difference lies in the global geometry of J-holomorphic curves and it is not completely clear at this moment whether Novikov-closedness of l.c.s. structure $(X, \omega, \mathfrak{b})$ would give reasonably meaningful implication to the study of moduli problem of J-holomorphic curves in the context of closed strings or open strings attached to suitably physical D-branes. We refer to [KaOr] for some physical motivation of coisotropic D-branes and to [CF] for a generalization of study of deformations of coisotropic submanifolds in the Poisson context.

We would like to thank Yoshihiro Ohnita for inviting us to the Pacific Rim Geometry Conference in 2011 where the first named author gave a talk on l.c.s. manifolds, which triggered our collaboration. We thank Marco Zambon for alerting their preprint [SZ] to us. We also thank Luca Vitagliano for a useful communication on the normalization theorem.

2. Locally conformal pre-symplectic manifolds. Suppose $Y \subset (X, \omega_X, \mathfrak{b})$ is a coisotropic submanifold. Then the restriction (Y, ω, b) satisfies the same equation

$$d^b\omega := d\omega + b \wedge \omega = 0 \tag{2.1}$$

except that ω is no longer nondegenerate but has constant rank.

This gives rise to the notion of *locally conformal pre-symplectic manifolds*, abbreviated as l.c.p-s. manifold.

Definition 2.1. A triple (Y, ω, b) is called an l.c.p-s. manifold if b is a closed one-form and ω is a two-form with constant rank that satisfy

$$d^b\omega := d\omega + b \wedge \omega = 0. \tag{2.2}$$

Remark 2.2. If the rank of ω is at least 4, then the wedge product with ω defines a linear injective map from $\Omega^1(Y)$ to $\Omega^3(Y)$. Hence b is defined uniquely by the equation (2.2). If the rank of ω is 2, then the restriction of b to the null space TY^ω of ω is defined by (2.2). The kernel of the wedge product $\Omega^1(Y) \to \Omega^3(Y)$, $\gamma \mapsto \omega \wedge \gamma$, is the two-dimensional annihilator $T\mathcal{F}^\circ$ of TY^ω. In particular, if rank ω is 2 and (Y, ω, b) is an l.c.p-s. manifold, then $(Y, \omega, b + b')$ is also an l.c.p-s. manifold for any $b' \in T\mathcal{F}^\circ$ such that $db' = 0$.

From now on, we consider a general l.c.p-s. manifold (Y, ω, b).

We next introduce morphisms between l.c.p-s. manifolds and automorphisms of (Y, ω, b) generalizing those of l.c.s. manifolds (see [HR] for the corresponding definitions for the l.c.s. case.)

Definition 2.3. Let (Y, ω, b) and (Y', ω', b') be two l.c.p-s. manifolds. A diffeomorphism $\phi : Y \to Y'$ is called *l.c.p-s.* if there exists $a \in C^\infty(Y, \mathbb{R} \setminus \{0\})$ such that

$$\phi^*\omega' = (1/a)\omega, \quad \phi^*b' = b + d(\ln|a|).$$

By setting $a = e^{tu}$, it is easy to check that the following is the infinitesimal version of Definition 2.3.

Definition 2.4. Let (Y, ω, b) be a l.c.p-s. manifold. A vector field ξ on Y is called *l.c.p-s.* if there exists a function $u \in C^\infty(Y)$ such that

$$\mathcal{L}_\xi\omega = -u\omega, \quad \mathcal{L}_\xi b = du$$

We denote by $Diff(Y, \omega, b)$ the set of l.c.p-s. diffeomorphisms.

Definition 2.5. We call any such function $u \in C^\infty(Y)$ that appears in Definition 2.4 is called an l.c.p-s. function. We denote by $C^\infty(Y; \omega, b)$ the set of l.c.p-s. functions.

It is easy to see that $C^\infty(Y; \omega, b)$ is a vector subspace of $C^\infty(X)$. We say an l.c.p-s. diffeomorphism (resp. vector field) an l.c.s. diffeomorphism (resp. vector field), if (Y, ω, b) is an l.c.s. manifold.

3. Canonical neighborhoods of coisotropic submanifolds. In this section, we develop the l.c.s. analog to Gotay's coisotropic neighborhood theorem [Go] in the symplectic case.

As in the symplectic case, we denote

$$E = (TY)^\omega$$

the *characteristic distribution* on Y. The following lemma is one of the important ingredients that enables us to develop deformation theory of coisotropic submanifolds in l.c.s. manifolds in a way similar to the symplectic case as done in [OP].

LEMMA 3.1. *The distribution E on Y is integrable.*

Proof. This is an immediate consequence (2.2) which shows that the ideal generated by ω is a differential ideal. □

We call the corresponding foliation the *null foliation* on Y and denote it by $\mathcal{F}$.

We now consider the dual bundle $\pi : E^* \to Y$ of E. The bundle $TE^*|_Y$ where $Y \subset E^*$ is the zero section of E^* carries the canonical decomposition

$$TE^*|_Y = TY \oplus E^*.$$

In the standard notation in the foliation theory, E and E^* are denoted by $T\mathcal{F}$ and $T^*\mathcal{F}$ and called the tangent bundle (respectively cotangent bundle) of the foliation $\mathcal{F}$.

REMARK 3.2. When Y is a coisotropic submanifold of an l.c.s. manifold $(X, \omega_X, \mathfrak{b})$ and (Y, ω, b) is the associated l.c.p-s. structure, then it is easy to check that the canonical isomorphism

$$\widetilde{\omega}_0 : TX \to T^*X$$

maps $E = TY^\omega$ to the conormal $N^*Y \subset T^*X$ and so its adjoint $(\widetilde{\omega}_0)^\dagger : TX \to T^*X$ induces an isomorphism between $NY = TX/TY$ and E^* where $E = (TY)^\omega$.

We choose a splitting

$$TY = G \oplus E, \quad E = (TY)^\omega, \tag{3.1}$$

and denote by

$$p_G : TY \to E$$

the projection to E along G in the splitting (3.1). Using this splitting, we can write a conformal symplectic form on a neighborhood of the zero section $Y \hookrightarrow E^*$ in the following way similarly as in the symplectic case. We have the bundle map

$$TE^* \xrightarrow{T\pi} TY \xrightarrow{p_G} E. \tag{3.2}$$

Let $\widehat{\alpha} \in E^*$ and $\xi \in T_{\widehat{\alpha}}E^*$. We define the one-form θ_G on E^* by its value

$$\theta_{G,\widehat{\alpha}}(\xi) := \widehat{\alpha}(p_G \circ T\pi(\xi)) \tag{3.3}$$

at each $\widehat{\alpha} \in E^*$. Then the two form

$$\omega_G := \pi^*\omega - d\theta_G - \pi^*b \wedge \theta_G \tag{3.4}$$

is nondegenerate in a neighborhood $U \subset E^*$ of the zero section (See the coordinate expression (7.6) of $d\theta_G$ and ω_G). Later, we use ω_G and ω_U interchangeably depending on context.

Then a straightforward computation proves

PROPOSITION 3.3. *Then the pair $(U, \omega_U, \mathfrak{b}_U)$ with $\mathfrak{b}_U := \pi^* b|_U$ defines an l.c.s. structure.*

REMARK 3.4. If Y is Lagrangian, then $E = TY$, $E^*_{|Y} = T^*Y$, and hence $p_G = Id$. For this special case, Proposition 3.3 is known as Example 3.1 in [HR], where θ_G is the Liouville 1-form on T^*Y.

In the next section, we will prove that this provides a general normal form of the l.c.s. neighborhood of the triple (Y, ω, b) which depends only on (Y, ω, b) and the splitting (3.1), and that this normal form is unique up to diffeomorphism. We call the pair $(U, \omega_U, \mathfrak{b}_U)$ a *(canonical) l.c.s. thickening* of the l.c.p-s. manifold (Y, ω, b).

4. Normal form theorem of coisotropic submanifolds in l.c.s. manifold. Let Y be a compact coisotropic submanifold in a l.c.s. manifold $(X, \omega_X, \mathfrak{b})$. Denote by (Y, ω, b) the induced l.c.p-s. structure given by

$$\omega = i^*\omega_X,\ b = i^*\mathfrak{b},$$

where $i : Y \to X$ is the canonical embedding. We denote by $(Ti)^*$ the associated bundle map $T^*X \to T^*Y$. Consider the bundle $\pi : E^* \to Y$ where $E = (TY)^\omega = T\mathcal{F}$ as in the previous section.

By Remark 3.2, the adjoint isomorphism

$$(\widetilde{\omega}_0^\dagger) : TX \to T^*X$$

induces an isomorphism

$$\widetilde{\omega}_X : NY = TX/TY \to E^*. \tag{4.1}$$

More precisely, we have the following lemma

LEMMA 4.1. *The nondegenerate two-form ω_X induces a canonical bundle isomorphism $\widetilde{\omega}_X : TX|_Y/TY \to E^*$ given by* (4.2).

Proof. We define the bundle map

$$\widetilde{\omega}_{XY} : TX|_Y \to T^*Y,\ v \mapsto (Ti)^*(v \rfloor \omega_X).$$

Denote by $j^* : T^*Y \to E^*$ the adjoint of $j : E \to TY$. Then $E = \ker \omega$ implies that

$$TY \subset \ker(j^* \circ \widetilde{\omega}_{XY}).$$

Hence $j^* \circ \widetilde{\omega}_{XY}$ descends to a bundle map $\widetilde{\omega}_X : NY = TX|_Y/TY \to E^*$ by setting

$$[v] \in NY \mapsto j^* \circ \widetilde{\omega}_{XY}(v). \tag{4.2}$$

The nondegeneracy of ω_X implies that $\widetilde{\omega}_X$ induces a canonical bundle isomorphism. □

Using this isomorphism, we identify the pair (NY, Y) with (E^*, o_{E^*}).

Now let g be a Riemannian metric on X. Then g gives rise to a splitting

$$TX|_Y \cong TY \oplus NY.$$

We also have a canonical isomorphism

$$TE^*|_Y \cong TY \oplus E^*. \tag{4.3}$$

Combining (4.3) with Lemma 4.1 we get

$$TE^*|_{o_{E^*}} \cong To^*_E \oplus E^* \cong TY \oplus NY \cong \quad TX|_Y.$$

Through this identification, we regard a neighborhood $U_1 \subset X$ of Y as a neighborhood of the zero section $o_{E^*} = Y \subset E^*$.

For any open set $U \subset X$ we denote the restriction of ω_X (resp. $\mathfrak{b}$) to U also by ω_X (resp. by $\mathfrak{b}$). In this section, we prove the following normal form theorem.

THEOREM 4.2 (Normal form). *Assume Y is compact. There exist an open neighborhood $U \subset X$ of Y, a neighborhood $V \subset E^*$ of the zero section Y, and a l.c.s. diffeomorphism*

$$\phi : (U, \omega_X, \mathfrak{b}) \to (V, (\omega_G)|_V, \pi^* b)$$

such that $\phi|_Y = Id$ and $d\phi|_{NY} = \widetilde{\omega}_X$ under the identification (4.3). *More specifically, ϕ satisfies*

$$\phi^*(\pi^*\omega - d\theta_G - \pi^* b \wedge \theta_G) = e^{-f}\omega_X$$

for some $f \in C^\infty(U)$.

Proof. Assume that U_1 be a neighborhood of Y in X which can be identified with a neighborhood W_1 of Y in NY via the exponential map $Exp_Y : NY \to U_1$. Set

$$\omega_1 := Exp^*_Y(\omega_X), \quad \mathfrak{b}_1 := Exp^*_Y(\mathfrak{b}). \tag{4.4}$$

Then $(W_1, \omega_1, \mathfrak{b}_1)$ is a l.c.s. manifold. Since the restriction of $dExp_Y$ to Y is equal to identity, Y is also a coisotropic submanifold in $(W_1, \omega_1, \mathfrak{b}_1)$.

Let $i_X : Y \hookrightarrow X$ be the inclusion. Set

$$\begin{aligned} V &:= \widetilde{\omega}_X(W_1), \quad b := i_X^* \mathfrak{b} \in \Omega^1(Y), \\ \widetilde{\omega}_1 &:= (\widetilde{\omega}_X^{-1})^*(\omega_1) \in \Omega^2(V), \quad \widetilde{\mathfrak{b}}_1 := (\widetilde{\omega}_X^{-1})^*(\mathfrak{b}_1) \in \Omega^1(V). \end{aligned} \tag{4.5}$$

Denote by $i_{E^*} : Y \hookrightarrow E^*$ the inclusion as the zero section and by $H^*_b(Y)$ the cohomology group $\ker d^b / \operatorname{im} d^b$.

LEMMA 4.3. *The embedding $i_{E^*} : Y \to E^*$ induces an isomorphism between $H^*_b(Y)$ and $H^*_{\pi^* b}(E^*)$. In particular, there exists a one-form $\eta \in \Omega^1(E^*)$ such that $\widetilde{\omega}_1 - \pi^*(\omega_1|_Y) = d^{\pi^* b}(\eta)$.*

Proof. Denote by $\mathcal{S}$ the following locally constant sheaf on Y

$$U \mapsto \mathcal{S}(U) := \{ f \in C^\infty(U, \mathbb{R}) |\, d^{b|_U} f = 0 \}.$$

It is known that $H_b(Y) = H(Y, \mathcal{S})$, see e.g. [HR, Remark 1.10]. The first assertion of Lemma 4.3 follows from the homotopy invariant property of cohomology with values in locally constant sheaf. The second assertion of Lemma 4.3 is a consequence of the first assertion. □

Since $H^1(E^*, \mathbb{R}) = H^1(Y, \mathbb{R})$ there exists a function $f \in C^\infty(E^*)$ such that $\eta = \pi^*(i^*\eta) + df$. Then $e^f\widetilde{\omega}$ is an l.c.s. form on E^* with the Lee form $\widetilde{\mathfrak{b}}_1 - df = \pi^*(\mathfrak{b}|_Y)$ [L].

Now we apply Moser's deformation to the normal form. Set

$$\widetilde{\omega}_0 := \omega_G|_V.$$

By (3.4) and (4.4) we have

$$\widetilde{\omega}_0(y) = \widetilde{\omega}_1(y) \text{ for all } y \in Y. \tag{4.6}$$

Since $H^*_{\pi^*b}(V) \cong H^*_{\widetilde{\mathfrak{b}}_1}(V)$, there exists a one form σ on V such that

$$\widetilde{\omega}_1 - \widetilde{\omega}_0 = d^{\pi^*b}\sigma.$$

Set

$$\widetilde{\omega}_t := \widetilde{\omega}_0 + td^{\pi^*b}\sigma = \pi^*(\omega_1|_Y) - d^{\pi^*b}(\theta_G - t\sigma).$$

By making V smaller if necessary, taking into account (4.6) and the compactness of Y, we assume that $\widetilde{\omega}_t$ are nondegenerate for all $t \in [0, 1]$. To prove Theorem 4.2, it suffices to solve the equation

$$\psi_t^*(\widetilde{\omega}_t) = e^{f_t(x)}\widetilde{\omega}_0 \tag{4.7}$$

for a family of diffeomorphism ψ_t of V and a function f_t with $f_1 = f$. Let ξ_t be the generating vector field of ψ_t i.e.

$$\frac{d}{dt}\psi_t = \xi_t(\psi_t),\ \psi_0 = Id.$$

Differentiating (4.7), we obtain

$$\psi_t^*\left(\frac{d}{dt}\widetilde{\omega}_t + \mathcal{L}_{\xi_t}\widetilde{\omega}_t\right) = \frac{\partial f}{\partial t}\widetilde{\omega}_0$$

which is equivalent to

$$\frac{d}{dt}\widetilde{\omega}_t + \mathcal{L}_{\xi_t}\widetilde{\omega}_t = \frac{\partial f}{\partial t} \circ \psi_t^{-1}.$$

But by definition of ω_t and Cartan's formula, this becomes

$$d^{\pi^*b}\sigma + \xi_t\rfloor d\widetilde{\omega}_t + d(\xi_t\rfloor\widetilde{\omega}_t) = g_t, \quad g_t(x) = \frac{\partial f}{\partial t}(\psi_t^{-1}(x)),$$

which in turn becomes

$$g_t = d^{\pi^*b}\sigma - \xi_t\rfloor(\pi^*b \wedge \widetilde{\omega}_t) + d^{\pi^*b}(\xi_t\rfloor\widetilde{\omega}_t) - \pi^*b \wedge (\xi_t\rfloor\widetilde{\omega}_t).$$

In other words, we obtain

$$(g_t - \pi^*b(\xi_t))\widetilde{\omega}_t = d^{\pi^*b}(\sigma + \xi_t\rfloor\widetilde{\omega}_t).$$

Using non-degeneracy of $\widetilde{\omega}_t$, we first solve

$$\sigma + \xi_t \rfloor \widetilde{\omega}_t = 0$$

for ξ_t on V and then define g_t by

$$g_t = \pi^* b(\xi_t)$$

for all $(t, x) \in [0,1] \times V$ again shrinking V, if necessary. We denote by ψ_t the flow of ξ_t which then determines f_t by $f_t = g_t \circ \psi_t$.

This proves Theorem 4.2. □

Remark 4.4. A careful examination of the above proof in fact shows that the compactness hypothesis of Y is not necessary as in the case of Weinstein's normal form theorem in [We]. We thank L. Vitagliano for pointing this out.

5. Geometry of the null foliation of l.c.p-s. manifold. Let (Y, ω, b) be an l.c.p-s. manifold of dimension $n + k$ and denote by $\mathcal{F}$ the associated null foliation. Set $2n := \dim X$, $n - k := \dim \mathcal{F}$, $l := 2k$. We now formulate a uniqueness statement in the symplectic thickening of (Y, ω) (Proposition 5.1), extending an analogous result in [OP]. We also prove the existence of a transverse l.c.s. form (Proposition 5.2), which is important for later sections.

Recalling that the l.c.s. form ω_G of (3.4) depends on the choice of the splitting Π, in this section we re-denote by ω_Π the l.c.s. form ω_G associated to the splitting Π.

Proposition 5.1. *(cf.* [OP, Proposition 5.1]*) For given two splittings* Π, Π', *there exist neighborhoods* U, U' *of the zero section* $Y \subset E^*$ *and a diffeomorphism* $\phi : U \to U'$ *and a function* $f : U \to \mathbb{R}$ *such that*

1. $\phi^* \omega_{\Pi'} = e^f \omega_\Pi$,
2. $\phi|_Y \equiv id$, *and* $T\phi|_{T_Y E^*} \equiv id$ *where* $T_Y E^*$ *is the restriction of* TE^* *to* Y.

Proof. Since $\mathcal{A}_E(TY)$ is contractible, we can choose a smooth family

$$\{\Pi_t\}_{0 \le t \le 1}, \quad \Pi_0 = \Pi, \, \Pi_1 = \Pi'.$$

Denoting $\omega_t := \omega_{\Pi_t}$, applying the isomorphism $H^1_b(Y) \cong H^1_{\pi^* b}(E^*)$, we have

$$\omega_t - \omega_0 = d^{\pi^* b} \sigma_t.$$

From the definition, we have

$$\sigma_t|_{T_Y E^*} \equiv 0.$$

for all $0 \le t \le 1$. With these, we imitate the proof of Theorem 4.2 to finish off the proof. □

For the study of the deformation problem of l.c.p-s. structures it is crucial to understand the transverse geometry of the null foliation. First we note that the l.c.p-s. form ω carries a natural *transverse l.c.p-s. form.* This defines the l.c.s. analog to the transverse symplectic form to the null foliation of pre-symplectic manifold. (See [Go], [OP], for example).

Proposition 5.2. *(cf.* [OP, Proposition 5.2]*) Let* $\mathcal{F}$ *be the null foliation of the l.c.p-s. manifold* (Y, ω, b). *Then it defines a transverse l.c.s. form on* $\mathcal{F}$ *in the following sense:*

1. $\ker(\omega_x) = T_x\mathcal{F}$ *for any* $x \in Y$, *and*
2. $\mathcal{L}_\xi\omega = -b(\xi)\omega$ *for any vector field* ξ *on* Y *tangent to* $\mathcal{F}$.

Proof. The first statement is trivial by definition of the null foliation and the second is an immediate consequence of the Cartan identity

$$\mathcal{L}_\xi\omega = d(\xi\rfloor\omega) + \xi\rfloor d\omega.$$

The first term vanishes since X is tangent to the null foliation $\mathcal{F}$. On the other hand, the second term becomes

$$\xi\rfloor d\omega = -\xi\rfloor(b\wedge\omega) = -b(\xi)\omega + b\wedge(\xi\rfloor\omega) = -b(\xi)\omega$$

which finishes the proof. □

One immediate consequence of the presence of the transverse l.c.p-s. form above is that any *transverse section* T of the foliation $\mathcal{F}$ carries a natural l.c.s. form: in any foliation coordinates, it follows from $E = \ker\omega = \mathrm{span}\{\frac{\partial}{\partial q^\alpha}\}_{1\le\alpha\le n-k}$ that we have

$$\pi^*\omega = \frac{1}{2}\sum_{2k\ge i>j\ge 1}\omega_{ij}dy^i\wedge dy^j, \tag{5.1}$$

where $\omega_{ij} = \omega(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ is skew-symmetric and invertible.

The condition (2) above implies

$$\omega_{ij;\beta} = b_\beta\omega_{ij} \tag{5.2}$$

where $b = \sum_j b_j dy^j + \sum_\beta b_\beta dq^\beta$ in the foliation coordinates $(y^1,\cdots,y^{2k},q^1,\cdots q^{n-k})$. Note that this expression is *independent* of the choice of splitting as long as $y^1,\cdots,y^{2k}$ are those coordinates that characterize the leaves of E by

$$y^1 = c^1,\cdots,y^\ell = c^\ell,\quad c^i\text{'s constant.} \tag{5.3}$$

By the closedness of the one-form b, (5.2) gives rise to the following proposition.

Proposition 5.3. *Let* L *be a leaf of the null foliation* $\mathcal{F}$ *on* (Y,ω), λ *a path in* L, *and let* T *and* S *be transverse sections of* $\mathcal{F}$ *with* $\lambda(0)\in T$ *and* $\lambda(1)\in S$. *Then the holonomy map*

$$hol^{S,T}(\lambda) : (T,\lambda(0)) \to (S,\lambda(1))$$

defines the germ of a l.c.s. diffeomorphism. In particular, each transversal T *to the null foliation carries a natural holonomy-invariant l.c.s. structure.*

6. Master equation and equivalence relations; classical part. Let us recall the proof of the fact that a graph of a 1-form $s\in\Omega^1(L)$ is Lagrangian with respect to the canonical symplectic form on TL if and only if $ds = 0$. This fact is a direct consequence of the following formula

$$s^*(\theta) = s,$$

which is obtained by

$$\langle s^*(\theta),\delta q\rangle = \langle\theta, s_*(\delta q)\rangle = s(\pi_* s_*\delta q) = s(\delta q)$$

where δq stands for the infinitesimal variation of q. Similarly we will derive the second equation for the graph Γ_s of a section $s : Y \to E^* \cong NY$ to be coisotropic with respect to ω_G (Theorem 6.2). We also call the corresponding equation the *classical part* of the master equation (cf. Theorem 8.1). We will study the full (local) moduli (with respect to different equivalence relations) problem of coisotropic submanifolds by analyzing the condition that the graph of a section $s : Y \to U$ in the symplectic thickening U is to be coisotropic with respect to ω_G (Lemmas 6.9, 6.10).

6.1. Description of coisotropic Granssmannian. In this subsection, we recall some basic algebraic facts on the coisotropic subspace C (with real dimensions $n + k$ where $0 \leq k \leq n$) in $\mathbb{C}^n$. We denote by C^ω the ω-orthogonal complement of C in $\mathbb{R}^{2n}$ and by Γ_k the set of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega)$. In other words,

$$\Gamma_k = \Gamma_k(\mathbb{R}^{2n}, \omega) =: \{C \in Gr_{n+k}(\mathbb{R}^{2n}) \mid C^\omega \subset C\}. \tag{6.1}$$

From the definition, we have the canonical flag,

$$0 \subset C^\omega \subset C \subset \mathbb{R}^{2n}$$

for any coisotropic subspace. We call (C, C^ω) a *coisotropic pair.* Combining this with the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, we have the splitting

$$C = H_C \oplus C^\omega \tag{6.2}$$

where H_C is the complex subspace of C.

Next we give a parametrization of all the coisotropic subspaces near given $C \in \Gamma_k$. Up to the unitary change of coordinates we may assume that C is the canonical model

$$C = \mathbb{C}^k \oplus \mathbb{R}^{n-k}.$$

We denote the (Euclidean) orthogonal complement of C by $C^\perp = i\mathbb{R}^{n-k}$ which is canonically isomorphic to $(C^\omega)^*$ via the isomorphism $\widetilde{\omega} : \mathbb{C}^n \to (\mathbb{C}^n)^*$. Then any nearby subspace of dimension $\dim C$ that is transverse to $C^\perp$ can be written as the graph of the linear map

$$A : C \to C^\perp \cong (C^\omega)^*$$

i.e., has the form

$$C_A := \{(x, Ax) \in C \oplus C^\perp = \mathbb{R}^{2n} \mid x \in C\}. \tag{6.3}$$

Denote $A = A_H \oplus A_I$ where

$$A_H : H = \mathbb{C}^k \to C^\perp \cong (C^\omega)^*,$$
$$A_I : C^\omega = \mathbb{R}^{n-k} \to C^\perp \cong (C^\omega)^*.$$

Note that the symplectic form ω induce the canonical isomorphism

$$\widetilde{\omega}^H : \mathbb{C}^k \to (\mathbb{C}^k)^*,$$
$$\widetilde{\omega}^I : \mathbb{R}^{n-k} = C^\omega \to (C^\omega)^* \cong C^\perp = i\mathbb{R}^{n-k}.$$

With this identification, the symplectic form ω has the form

$$\omega = \pi^*\omega_{0,k} + \sum_{i=1}^{n-k} dx_i \wedge dy^i, \tag{6.4}$$

where $\omega_{0,k}$ is the standard symplectic form in $\mathbb{C}^k$, $\pi : \mathbb{C}^n \to \mathbb{C}^k$ the projection, and $(x_1, \cdots, x_{n-k})$ the standard coordinates of $\mathbb{R}^{n-k}$ and $(y^1, \cdots, y^{n-k})$ its dual coordinates of $(\mathbb{R}^{n-k})^*$. We also denote by $\pi_H : (\mathbb{C}^k)^* \to \mathbb{C}^k$ the inverse of the above mentioned canonical isomorphism $\widetilde{\omega}^H$.

The following statements are fundamental in our work.

Proposition 6.1. *Let C_A be as in* (6.3).

1. *The subspace C_A is coisotropic if and only if A_H and A_I satisfies*

$$A_I - (A_I)^* + A_H\pi_H(A_H)^* = 0. \tag{6.5}$$

2. *The subspace C_A is coisotropic, if and only if $\omega^{k+1}|_{C_A} = 0$.*

Proof. The first assertion of Proposition 6.1 is Proposition 2.2 in [OP]. The second assertion of Proposition 6.1 follows from the observation that C_A is coisotropic, if and only the restriction $\omega|_{C_A}$ is maximally degenerate, i.e. $\mathrm{rank}(\omega|_{C_A}) = \mathrm{rank}\, \pi^*\omega_{0,k} = k$. □

6.2. The equation for coisotropic sections. Note that the projection $p_G : TY \to E$ induces a bundle map $p_G^* : E^* \to T^*Y$ by

$$\langle p_G^*(s), \delta q\rangle := \langle s, p_G(\delta q)\rangle$$

for any $s \in E^*$ and $\delta q \in T_{\pi(s)}Y$.

As before, assume that $m = \dim Y = \dim E^* - (n-k) = n + k$.

Theorem 6.2. *The graph Γ_s of a section $s : Y \to (U \subset E^*, \omega_G, \pi^*b)$ is coisotropic if and only if the 2-form $\omega_G(s) := \omega|_Y - d^b(p_G^*(s)) \in \Omega^2(Y)$ is maximally degenerate, i.e. $(\omega_G(s))^{k+1} = 0$.*

Proof. By Proposition 6.1, Γ_s is coisotropic if and only if the restriction of ω_G to Γ_s is maximally degenerate, or equivalently $(\omega_G)^{k+1}_{|\Gamma_s} = 0$. Since $\omega_G \circ s_* = s^*(\omega_G)$ we get

$$(\omega_G)^{k+1}|_{\Gamma_s} = 0 \Longleftrightarrow (s^*(\omega_G)|_Y)^{k+1} = 0. \tag{6.6}$$

By (3.4) we have

$$s^*(\omega_G)|_Y = s^*(\pi^*(\omega|_Y) - d^{\pi^*b}\theta_G) = \omega|_Y - s^*(d^{\pi^*b}\theta_G) = \omega|_Y - d^b(s^*\theta_G). \tag{6.7}$$

Using (3.3) we obtain for any $y \in Y$ and any $\partial q \in T_yY$

$$\langle s^*(\theta_G), \delta q\rangle_y = \langle \theta_G, s_*(\partial q)\rangle = s(p_G \circ \pi_* \circ s_*\delta q) = s(p_G(\delta q)). \tag{6.8}$$

It follows from (6.7) and (6.8)

$$s^*(\omega_G)|_Y = (\omega|_Y) - d^b(p_G^*(s)). \tag{6.9}$$

Theorem 6.2 follows immediately from (6.9). □

6.3. (Pre-)Hamiltonian equivalence and infinitesimal equivalence. In this section, we clarify the relation between the *intrinsic* pre-Hamiltonian equivalence (resp. intrinsic l.c.p-s. equivalence) between the l.c.p-s. structures (ω, b) and the *extrinsic* Hamiltonian equivalence (resp. extrinsic l.c.s. equivalence) between coisotropic embeddings in (U, ω_G, π^*b). The intrinsic pre-Hamiltonian equivalence is provided by the pre-Hamiltonian diffeomorphisms (Definition 6.5) on the l.c.p-s. manifold (Y, ω, b) and the extrinsic ones by Hamiltonian deformations of its coisotropic embedding into $(U, \omega_U, \pi^* b)$ (Definition 6.8). The intrinsic l.c.p-s. equivalence is provided by l.c.p-s. diffeomorphisms and the extrinsic ones by l.c.s. deformations of its coisotropic embedding into $(U, \omega_U, \pi^* b)$. The infinitesimal ((pre-)Hamiltonian) equivalence is a natural infinitesimal version of the intrinsic/extrinsic ((pre-)Hamiltonian) equivalence.

First we shall prove

LEMMA 6.3. *A vector field ξ on an l.c.p-s. manifold (Y, ω, b) is l.c.p-s. if and only if on each connected component of Y*

$$d^b(\xi \rfloor \omega) = c\,\omega \text{ for some } c \in \mathbb{R}. \tag{6.10}$$

Proof. Assume that ξ is l.c.p-s. vector field on Y. To prove (6.10) we note that a l.c.p-s. vector field ξ on a l.c.p-s. manifold (Y, ω, b) satisfies the following equation for some smooth function $u \in C^\infty(Y)$ (see Definition 2.4)

$$\begin{aligned} & \mathcal{L}_\xi \omega = -u\omega \\ \Longleftrightarrow\ & \xi \rfloor d\omega + d(\xi \rfloor \omega) = -u\omega \\ \Longleftrightarrow\ & -b(\xi)\,\omega + d^b(\xi \rfloor \omega) = -u\omega \\ \Longleftrightarrow\ & d^b(\xi \rfloor \omega) = (-u + b(\xi))\,\omega. \end{aligned} \tag{6.11}$$

By Definition (2.4) $\mathcal{L}_\xi b = du$, or equivalently

$$d(b(\xi) - u) = 0.$$

Comparing this with (6.11) we obtain (6.10) immediately.

Now assume that a vector field ξ on Y satisfies (6.10). Set

$$u := b(\xi) - c.$$

Then (6.11) holds. The above computations yield

$$\mathcal{L}_\xi \omega = -u\omega.$$

Since $\mathcal{L}_\xi b = d(b(\xi)) = du$, we conclude that ξ is l.c.p-s. This completes the proof of Lemma 6.3. □

We define a b-deformed Lie derivative as follows

$$\mathcal{L}^b_\xi(\phi) := d^b \circ i_\xi + i_\xi \circ d^b. \tag{6.12}$$

The following statements are direct consequences of Lemma 6.3, hence we omit their proof.

COROLLARY 6.4. *Let (Y, ω, b) be an l.c.p-s. manifold.*

1. *Assume that $[\omega] \neq 0 \in H^2_b(Y, \mathbb{R})$. Then any l.c.p-s. vector field ξ on (Y, ω, b) satisfies $d^b(\xi \rfloor \omega) = 0$ (the constant c in (6.10) is zero), equivalently, $\mathcal{L}^b_\xi(\omega) = 0$.*
2. *Assume that $\omega = d^b\theta$ for some $\theta \in \Omega^1(Y)$. Then $\xi \rfloor \omega = c\theta + \alpha$ for some $\alpha \in \ker d^b \cap \Omega^1(Y)$. In this case $\mathcal{L}^b_\xi \omega = c\,\omega$.*

Lemma 6.3 motivates the following definition

DEFINITION 6.5. A vector field ξ on an l.c.p-s. manifold (Y, ω, b) is called *pre-Hamiltonian*, if $\xi\rfloor\omega = d^b f$ for some $f \in C^\infty(Y)$. A diffeomorphism ϕ is called *pre-Hamiltonian*, if it is generated by a time-dependent pre-Hamiltonian vector field.

REMARK 6.6.
1. If Y is an l.c.s. manifold, our definition of a pre-Hamiltonian vector field coincides with the definition of a Hamiltonian vector field given by Vaisman [V2, (2.3)]. For $b = 0$, our definition of a pre-Hamiltonian vector field agrees with the definition in [OP, Definition 3.3].
2. Clearly, any vector field ξ on Y tangent to $\mathcal{F}$ is pre-Hamiltonian with the Hamiltonian $f = 0$. Using Lemma 6.3 we obtain again the second assertion of Proposition 5.2, noting that the corresponding constant c is zero.

The following Theorem is an extension of Theorem 8.1 in [OP].

THEOREM 6.7. *Any l.c.p-s. (resp. pre-Hamiltonian) vector field ξ on an l.c.p-s. manifold (Y, ω, b) can be extended to an l.c.s. (resp. Hamiltonian) vector field on (U, ω_G, π^*b).*

Proof. Let ξ be a l.c.p-s. (resp. pre-Hamiltonian) vector field on (Y, ω, b). We decompose

$$\xi = \xi_G + \xi_E$$

where $\xi_G \in G$ and ξ_E is tangent to $\mathcal{F}$. By Remark 6.6 (2), ξ_E is pre-Hamiltonian, hence it suffices to show that
1. ξ_E extends to a Hamiltonian vector field on (U, ω_G, π^*b);
2. ξ_G extends to a l.c.s. (resp. Hamiltonian) vector field on (U, ω_G, π^*b).

To prove (1), we define a Hamiltonian function $\widehat{f}$ on $(U \subset E^*, \omega_G, \pi^*b)$ as follows

$$\widehat{f}(\widehat{\alpha}) := \langle \widehat{\alpha}, \xi_E \rangle.$$

It is straightforward to check that

$$\widehat{f}|_Y = 0 = f, \text{ and } (d^b\widehat{f})|_Y = (d\widehat{f})|_Y. \tag{6.13}$$

Denote by $(d^{\pi^*b}\widehat{f})_{\#\omega_G}$ the associated Hamiltonian vector field on U. Using (6.13), (7.5) and the coordinate expression of ω_G in (7.8), we obtain easily that

$$\xi_E(y) = (d^{\pi^*b}\widehat{f})_{\#\omega_G}(y)$$

for all $y \in Y$. This proves (1).

Now we shall show (2). Since $\omega_G|_{\mathcal{F}} = 0$, we have

$$(\xi_G\rfloor\omega_G)(y) = (\xi_G\rfloor\omega)(y) \tag{6.14}$$

for all $y \in Y$. Suppose $[\omega] = 0 \in H^2_b(Y) = H^2_{\pi^*b}(U)$. Then $\omega_G = d^{\pi^*b}\theta_U$ for some 1-form θ_U on Y. Since

$$(d^{\pi^*b}\theta_U)|_Y = d^b(\theta_U)|_Y,$$

the one-form

$$\theta := (\theta_U)|_Y \tag{6.15}$$

satisfies the condition in Corollary 6.4 (2). Using Corollary 6.4 (2), formulas (6.14), (6.15) and the non-degeneracy of ω_U, we observe that the extendability of ξ_G is equivalent to the extendability of the one-form α associated to ξ_G as in Corollary 6.4 (2) to a one-form α_U on U satisfying the following condition:

$$d^{\pi^* b}\alpha_U = 0 \text{ and } (\alpha_U)|_Y = \alpha. \tag{6.16}$$

Set $\alpha_U := \pi^*(\alpha)$. Then α_U satisfies (6.16). This proves Theorem 6.7 for the case $[\omega] = 0 \in H^2_b(Y)$.

Now assume that $[\omega] \neq 0 \in H^2_b(Y) = H^2_{\pi^* b}(U)$. By Corollary 6.4(1) $d^b(\xi \rfloor \omega) = 0$, or equivalently, $\xi_G \rfloor \omega = \gamma$, where $d^b\gamma = 0$. Since ω_U is nondegenerate, using (6.14), we note that the required extendability of ξ_G is equivalent to the extendability γ to a one-form γ_U on U such that $d^{\pi^* b}\gamma_U = 0$. Clearly $\gamma_U := \pi^*(\gamma)$ satisfies the required condition. This proves (2) and completes the proof of Theorem 6.7. □

Now we study the geometry of the master equation for coisotropic sections $s \in E^*$. By Proposition 6.1, the coisotropic condition for s is given by

$$(\omega - d^b(p^*_G s))^{k+1} = 0 \in \Omega^{2k+2}(Y). \tag{6.17}$$

Abbreviate $p^*_G s$ as s_G and $p^*_G(E^*)$ as G°. Note that $G^\circ \subset T^*Y$ is the annihilator of G. We rewrite the master equation (6.17) in the following form

$$\begin{aligned} s_G &\in G^\circ, \\ (\omega - d^b s_G)^{k+1} &= 0. \end{aligned} \tag{6.18}$$

Since $p^*_G|_{E^*} : E^* \to G^\circ$ is a bundle isomorphism, Y is also a coisotropic submanifold in $(p^*_G(U) \subset G^\circ, (p^*_G|^{-1}_{E^*})^*(\omega_G))$. Abusing the notation, we abbreviate $(p^*_G|^{-1}_{E^*})^*(\omega_G)$ as ω_G and $p^*_G(U)$ as U. Clearly, the linearized equation of (6.18) at the zero section is

$$\omega^k \wedge d^b\alpha = 0. \tag{6.19}$$

Since $\omega^k|_G \neq 0$ and $\omega^{k+1} = 0$, the linearized equation of (6.17) is equivalent to the following equation

$$d^{\bar b}_{\mathcal{F}}\alpha = 0 \tag{6.20}$$

for a section $\alpha \in E^*$. Here $\bar b$ denote the restriction of b to $\mathcal{F}$.

Definition 6.8. Two sections $s_0, s_1 : Y \to U \subset G^\circ$ are called *Hamiltonian equivalent* (resp. l.c.s. *equivalent*), if there exists a family of Hamiltonian diffeomorphisms (resp. l.c.s. diffeomorphisms) ψ_t of (U, ω_G) and a family of diffeomorphisms $g_t \in Diff(Y)$, $t \in [0,1]$, such that $g_0 = Id|_Y$, $\psi_0 = Id|_U$ and $s_1 = \psi_1 \circ s_0 \circ g_1$.

Two sections $\xi_0, \xi_1 : Y \to U$ are called *infinitesimally Hamiltonian equivalent* (resp. *infinitesimally l.c.s. equivalent*), if $\xi_0 - \xi_1$ is the vertical (fiber) component of a Hamiltonian (resp. l.c.s.) vector field on (U, ω_G).

Clearly, if s_0 and s_1 are (Hamiltonian) equivalent sections, and s_0 is a coisotropic section, then s_1 is also a coisotropic section.

LEMMA 6.9. *Two solutions of the linearized equation (6.20) are infinitesimally Hamiltonian equivalent if and only if they are cohomologous as elements in $\Omega^1_b(Y,\omega)$. Consequently, the set of equivalence classes of the infinitesimally Hamiltonian equivalent solutions of the linearized equations is $H^1_b(Y,\omega)$.*

Proof. It suffices to prove Lemma 6.9 for the case where one of the two solutions is the zero section. For $f \in C^\infty(U)$ denote by $(d^{\pi^*b} f)_{\#\omega_G}$ the associated Hamiltonian vector field on U. We identify Y with the zero section of $G^\circ \supset U$ and for $y \in Y$ we denote by $T^{ver}_y G^\circ$ the vertical (fiber) component of $T_y G^\circ = T_y Y \oplus G^\circ_y$. For any $V \in T_y G^\circ$ denote by V^{ver} the vertical component of V. Now assume that a section $\xi : Y \to U \subset G^\circ$ is infinitesimally Hamiltonian equivalent to the zero section, i.e. there exists $f \in C^\infty(U)$ such that $\xi = (d^{\pi^*b} f)^{ver}_{\#\omega_G}$. Abusing notation we denote by π the projection $G^\circ \to Y$. Using (7.5) and (7.8), we obtain for all $y \in Y$

$$(d^{\pi^*b} f)^{ver}_{\#\omega_G}(y) = (d^{\pi^*b} f|_{\pi^{-1}(\mathcal{F})})^{ver}_{f^*_\beta \wedge dp_\beta}(y) = d^{\bar{b}}_{\mathcal{F}} f(y), \tag{6.21}$$

where $(d^{\pi^*b} f|_{\pi^{-1}(\mathcal{F})})^{ver}_{f^*_\beta \wedge dp_\beta}(y)$ denotes the vertical (fiber) component of the vector in $T_y(\pi^{-1}(\mathcal{F})) = E(y) \oplus E^*(y)$ that is dual to the one-form $d^{\pi^*b} f|_{\pi^{-1}(\mathcal{F})}(y)$ with respect to the nondegenerate two-form $\sum_\beta f^*_\beta \wedge dp_\beta(y) \in \Lambda^2 T^*_y(\pi^{-1}(\mathcal{F}))$. Hence $[\xi] = 0 \in H^1_b(Y,\omega)$.

Now assume that $\xi = d^{\bar{b}}_{\mathcal{F}} f$ where $f \in \Omega^0(U)$. By (6.21) ξ is infinitesimally Hamiltonian equivalent to the zero section. This proves the first assertion of Lemma 6.9. The second assertion is an immediate consequence of the first one and (6.20). This completes the proof of Lemma 6.9. □

Next, let us consider the case where ξ is infinitesimally l.c.s. equivalent to the zero section, i.e. there is a l.c.s. vector field $\widehat{\xi}$ on U such that $\xi(y)$ is the vertical (fiber) component of $\widehat{\xi}(y)$ for all $y \in Y$.

1. The case with $[\omega_G] \neq 0 \in H^2_{\pi^*b}(U,\mathbb{R}) = H^2_b(Y,\mathbb{R})$: Corollary 6.4 (1) implies that $d^{\bar{b}}(\widehat{\xi}\rfloor\omega_G) = 0$. The same argument as in the proof of Lemma 6.9 yields that for all $y \in Y$
$$\xi(y) = (\widehat{\xi}\rfloor\omega_G)|_{\mathcal{F}}(y) \in E^*(y).$$
This leads to specify a subgroup $H^1_{\bar{b},ext}(\mathcal{F})$ of the group $H^1_{\bar{b}}(\mathcal{F})$ whose elements are the restriction of d^{π^*b}-closed one-forms on U. It is easy to see that
$$H^1_{\bar{b},ext}(\mathcal{F}) = i^*(H^1_b(Y)),$$
where $i : \mathcal{F} \to Y$ is the natural inclusion.
2. The case with $\omega_G = d^b\theta_U$ for some $\theta_U \in \Omega^1(U)$: Clearly $d^b_{\mathcal{F}}(\theta_U|_{\mathcal{F}}) = 0$. Corollary 6.4 (2) implies that $(\widehat{\xi}\rfloor\omega_G)|_{\mathcal{F}} = c\theta_U|_{\mathcal{F}} + \alpha|_{\mathcal{F}}$, where $\alpha \in \Omega^1(U)$ and $d^{\pi^*b}\alpha = 0$. Using the argument in the proof of Lemma 6.9 we get
$$\xi(y) = c\theta_U|_{\mathcal{F}}(y) + \alpha|_{\mathcal{F}}(y) \in E^*(y).$$

The discussion above immediately yields

LEMMA 6.10. *Denote $H^1_b(Y,\omega) = H^1_{\bar{b}}(\mathcal{F})$. The set of the infinitesimal l.c.s. equivalence classes of the solutions ξ of the linearized equation (6.20) has one-one correspondence with*

1. $H^1_b(Y,\omega)/i^*(H^1_b(Y))$ *if* $[\omega] \neq 0$ *in* $H^2_b(Y)$ *and*
2. $H^1_b(Y,\omega)/(i^*(H^1_b(Y)) + \langle\theta|_{\mathcal{F}}\rangle_{\otimes\mathbb{R}})$ *if* $\omega = d^b\theta$.

7. Geometry of the l.c.s. thickening of a l.c.p-s. manifold. In this section, imitating the scheme performed for the pre-symplectic manifolds in [OP], we introduce special coordinates in the l.c.s. thickening (U, ω_U, π^*b) of a l.c.p-s. manifold and we compute important geometric structures in these coordinates ((7.4), (7.8), (7.11)), preparing for our study of deformations of compact coisotropic submanifolds in l.c.s. manifolds in the next two sections.

Again we start with a splitting

$$TY = G \oplus E,$$

the associated bundle projection $\Pi : TY \to TY$, the associated canonical one form θ_G, and the l.c.s. form

$$\omega_U = \pi^*\omega - d^{\pi^*b}\theta_G \tag{7.1}$$

on $U \subset E^*$. Let

$$(y^1, \cdots, y^{2k}, q^1, \cdots, q^{n-k})$$

be coordinates on Y adapted to the null foliation on an open subset $V \subset Y$. By choosing the frame

$$\{f_1^*, \cdots, f_{n-k}^*\}$$

of E^* that is dual to the frame $\{\frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^{n-k}}\}$ of E, we introduce the *canonical coordinates* on E^* by writing an element $\widehat{\alpha} \in E^*$ as a linear combination of $\{f_1^*, \cdots, f_{n-k}^*\}$

$$\widehat{\alpha} = p_\beta f_\beta^*,$$

and taking

$$(y^1, \cdots, y^{2k}, q^1, \cdots, q^{n-k}, p_1, \cdots, p_{n-k})$$

as the associated coordinates.

For a given splitting $\Pi : TY = G \oplus T\mathcal{F}$, there exists the unique splitting of TU

$$TU = G^\sharp \oplus T\pi^{-1}(\mathcal{F}) \tag{7.2}$$

that satisfies

$$G^\sharp = (T_{\widehat{\alpha}}\pi^{-1}(\mathcal{F}))^{\omega_U} \tag{7.3}$$

for any $\widehat{\alpha} \in U$, which is invariant under the action of l.c.s. diffeomorphisms on (U, ω_U, π^*b) that preserve the leaves of $\pi^{-1}(\mathcal{F})$.

Definition 7.1. We call the above unique splitting the *leafwise l.c.s. connection of* $U \to Y$ *compatible to the splitting* $\Pi : TY = G \oplus T\mathcal{F}$ or simply a *leafwise l.c.s.* Π-*connection* of $U \to Y$.

We would like to emphasize that this connection is not a vector bundle connection of E^* although U is a subset of E^*, which reflects *nonlinearity* of this connection. We refer the reader to subsection 13.1 for more detailed explanation.

Note that the splitting Π naturally induces the splitting

$$\Pi_* : T^*Y = (T\mathcal{F})^\circ \oplus G^\circ.$$

For the given splitting $TY = G \oplus E$ we can write, as in [OP, (4.5)],

$$G_x = \text{span}\left\{ \frac{\partial}{\partial y^i} + \sum_{\alpha=1}^{m-l} R_i^\alpha \frac{\partial}{\partial q^\alpha} \right\}_{1 \le i \le l}$$

for some R_i^αs, which are uniquely determined by the splitting and the given coordinates.

To derive the coordinate expression of θ_G, we compute

$$\begin{aligned}
\theta_G\Big(\frac{\partial}{\partial y^i}\Big) &= \widehat{\alpha}\Big(p_G \circ T\pi(\frac{\partial}{\partial y^i})\Big) = \widehat{\alpha}\Big(p_G(\frac{\partial}{\partial y^i})\Big) \\
&= p_\beta f_\beta^*\Big(- R_i^\alpha \frac{\partial}{\partial q^\alpha}\Big) = -p_\alpha R_i^\alpha, \\
\theta_G\Big(\frac{\partial}{\partial q^\beta}\Big) &= p_\beta, \qquad \theta_G\Big(\frac{\partial}{\partial p_\beta}\Big) = 0.
\end{aligned}$$

Hence we derive

$$\theta_G = p_\beta (dq^\beta - R_i^\beta dy^i). \tag{7.4}$$

Here we note that

$$(dq^\beta - R_i^\beta dy^i)|_{G_x} \equiv 0.$$

This shows that if we identify $E^* = T^*\mathcal{F}$ with G°, then we may write the dual frame on $T^*\mathcal{F}$ as

$$f_\beta^* = dq^\beta - R_i^\beta dy^i. \tag{7.5}$$

Motivated by this, we write

$$\begin{aligned}
d\theta_G &= dp_\beta \wedge (dq^\beta - R_i^\beta dy^i) - p_\beta dR_i^\beta \wedge dy^i \\
&= dp_\beta \wedge (dq^\beta - R_i^\beta dy^i) - (dq^\gamma - R_j^\gamma dy^j) \wedge p_\beta \frac{\partial R_i^\beta}{\partial q^\gamma} dy^i \\
&\quad -p_\beta \left(\frac{\partial R_i^\beta}{\partial y^j} - R_j^\gamma \frac{\partial R_i^\beta}{\partial q^\gamma} \right) dy^j \wedge dy^i
\end{aligned} \tag{7.6}$$

and

$$\pi^* b \wedge \theta_G = (b_\gamma dq^\gamma + b_i dy^i) \wedge p_\delta (dq^\delta - R_j^\delta dy^j). \tag{7.7}$$

Combining (7.6), (7.7) and (5.1), we derive

$$\begin{aligned}
\omega_U &= \frac{1}{2}\Big(\omega_{ij} - p_\beta F_{ij}^\beta\Big) dy^i \wedge dy^j \\
&\quad - \left(dp_\nu + p_\nu (b_\gamma dq^\gamma + b_i dy^i) + p_\beta \left(\frac{\partial R_i^\beta}{\partial q^\nu} \right) dy^i \right) \wedge (dq^\nu - R_j^\nu dy^j) \\
&= \frac{1}{2}\Big(\omega_{ij} - p_\beta F_{ij}^\beta\Big) dy^i \wedge dy^j \\
&\quad - \left(dp_\nu + p_\nu b_\gamma \left(dq^\gamma - R_i^\gamma dy^i\right) + \left(p_\nu (b_i + b_\gamma R_i^\gamma) + p_\beta \frac{\partial R_i^\beta}{\partial q^\nu} \right) dy^i \right) \\
&\qquad \wedge (dq^\nu - R_j^\nu dy^j)
\end{aligned} \tag{7.8}$$

similarly as in the derivation of [OP, (6.8)], where F_{ij}^{β} are the components of the transverse Π-curvature of the null-foliation given by (13.5) in the Appendix.

Note that we have

$$T\pi^{-1}(\mathcal{F}) = \operatorname{span}\Big\{\frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^{n-k}}, \frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_{n-k}}\Big\}$$

which is independent of the choice of the above induced foliation coordinates of TU.

Now we compute $G^\sharp = (T\pi^{-1}(\mathcal{F}))^{\omega_U}$ in TU in terms of these induced foliation coordinates. We will determine when the expression

$$a^j\Big(\frac{\partial}{\partial y^j} + R_j^\alpha \frac{\partial}{\partial q^\alpha}\Big) + d^\beta \frac{\partial}{\partial q^\beta} + c_\gamma \frac{\partial}{\partial p_\gamma}$$

satisfies

$$\omega_U\Big(a^j\Big(\frac{\partial}{\partial y^j} + R_j^\alpha \frac{\partial}{\partial q^\alpha}\Big) + d^\beta \frac{\partial}{\partial q^\beta} + c_\gamma \frac{\partial}{\partial p_\gamma}, T\pi^{-1}(\mathcal{F})\Big) = 0.$$

It is immediate to see by pairing with $\frac{\partial}{\partial p_\mu}$

$$d^\beta = 0, \quad \beta = 1, \cdots, n-k. \tag{7.9}$$

Next we study the equation

$$0 = \omega_U\Big(a^j\Big(\frac{\partial}{\partial y^j} + R_j^\alpha \frac{\partial}{\partial q^\alpha}\Big) + c_\gamma \frac{\partial}{\partial p_\gamma}, \frac{\partial}{\partial q^\nu}\Big)$$

for all $\nu = 1, \cdots, n-k$. A straightforward check provides

$$a^j\left(p_\nu(b_i + b_\gamma R_i^\gamma) + p_\beta \frac{\partial R_i^\beta}{\partial q^\nu}\right) + c_\nu = 0 \tag{7.10}$$

for all ν and j. Combining (7.9) and (7.10), we have obtained

$$\begin{aligned}&(T\pi^{-1}(\mathcal{F}))^{\omega_U}\\ &= \operatorname{span}\left\{\frac{\partial}{\partial y^j} + R_j^\alpha \frac{\partial}{\partial q^\alpha} - \left(p_\nu(b_i + b_\gamma R_i^\gamma) + p_\beta \frac{\partial R_i^\beta}{\partial q^\nu}\right)\frac{\partial}{\partial p_\nu}\right\}_{1\le j\le 2k}.\end{aligned} \tag{7.11}$$

Remark 7.2. Just as we have been considering $\Pi : TY = G \oplus T\mathcal{F}$ as a "connection" over the leaf space, we may consider the splitting $\Pi^\sharp : TU = G^\sharp \oplus T(\pi^{-1}\mathcal{F})$ as the leaf space connection *canonically induced from* Π *under the fiber-preserving map*

$$\pi : U \to Y$$

over the same leaf space $Y/\sim$: Note that the space of leaves of $\mathcal{F}$ and $\pi^{-1}\mathcal{F}$ are canonically homeomorphic.

8. Master equation in coordinates. We will derive the first equation for the graph of a section $s : Y \to E^* \cong NY$ to be coisotropic with respect to ω_U (Theorem 8.1), which is a natural extension of a similar equation in the symplectic setting obtained in [OP]. We call the corresponding equation the *classical part* of the master equation for the deformation theory of coisotropic submanifolds.

Recall that an Ehresmann connection of $U \to Y$ with a structure group H is a splitting of the exact sequence

$$0 \to VTU \longrightarrow TU \xrightarrow{T\pi} TY \to 0$$

that is invariant under the action of the group H. Here H is not necessarily a finite dimensional Lie group. In other words, an Ehresmann connection is a choice of decomposition

$$TU = HTU \oplus VTU$$

that is invariant under the fiberwise action of H. Recalling that there is a canonical identification $V_{\widehat{\alpha}}TU \cong V_{\widehat{\alpha}}TE^* \cong E^*_{\pi(\widehat{\alpha})}$, a connection can be described as a *horizontal* lifting $HT_{\widehat{\alpha}}U$ of TY to TU at each point $y \in Y$ and $\widehat{\alpha} \in U \subset E^*$ with $\pi(\widehat{\alpha}) = y$. We denote by $F^\# \subset HTU$ the horizontal lifting of a subbundle $F \subset TY$ in general.

Let $(y^1, \cdots, y^{2k}, q^1, \cdots, q^{n-k})$ be foliation coordinates of $\mathcal{F}$ on Y and

$$(y^1, \cdots, y^{2k}, q^1, \cdots, q^{n-k}, p_1, \cdots, p_{n-k})$$

be the induced foliation coordinates of $\pi^{-1}(\mathcal{F})$ on U. Then $G^\# = (T\pi^{-1}(\mathcal{F}))^{\omega_U}$ has the natural basis given by

$$e_j = \frac{\partial}{\partial y^j} + R_j^\alpha \frac{\partial}{\partial q^\alpha} - \left(p_\nu (b_i + b_\gamma R_i^\gamma) + p_\beta \frac{\partial R_i^\beta}{\partial q^\nu} \right) \frac{\partial}{\partial p_\nu} \tag{8.1}$$

which are basic vector fields of $T(\pi^{-1}\mathcal{F})$. We also denote

$$f_\alpha = \frac{\partial}{\partial q^\alpha}.$$

We define a local lifting of E

$$E^\sharp = \operatorname{span}\Big\{ f_1, \cdots, f_{n-k} \Big\}. \tag{8.2}$$

The lifting (8.2) of E provides a local splitting

$$TU = (G^\sharp \oplus E^\sharp) \oplus VTU \to TY$$

and defines a locally defined Ehresmann connection on the bundle $U \to Y$. From the expression (7.8) of ω_U, it follows that $G^\sharp \oplus E^\sharp$ is a coisotropic lifting of TY to TU. We denote by $\Pi^v : TU \to VTU$ the vertical projection with respect to this splitting.

With this preparation, we are finally ready to derive the master equation. Let $s : Y \to U \subset E^*$ be a section and denote

$$\nabla s := \Pi^v \circ ds \tag{8.3}$$

its locally defined "covariant derivative". In coordinates $(y^1, \cdots, y^{2k}, q^1, \cdots, q^{n-k})$, we have

$$\begin{aligned} ds\Big(\frac{\partial}{\partial y^j}\Big) &= \frac{\partial}{\partial y^j} + \frac{\partial s_\alpha}{\partial y^j}\frac{\partial}{\partial p_\alpha} \\ &= e_j - R_j^\alpha \frac{\partial}{\partial q^\alpha} + \left(\frac{\partial s_\nu}{\partial y^j} + s_\nu(b_i + b_\gamma R_i^\gamma) + s_\beta \frac{\partial R_i^\beta}{\partial q^\nu}\right)\frac{\partial}{\partial p_\nu}. \end{aligned}$$

Therefore we have derived

$$\nabla s\Big(\frac{\partial}{\partial y^j}\Big) = \left(\frac{\partial s_\nu}{\partial y^j} + s_\nu(b_i + b_\gamma R_i^\gamma) + s_\beta \frac{\partial R_i^\beta}{\partial q^\nu}\right)\frac{\partial}{\partial p_\nu}. \tag{8.4}$$

Similarly we compute

$$\begin{aligned} ds\Big(\frac{\partial}{\partial q^\nu}\Big) &= \frac{\partial}{\partial q^\nu} + \frac{\partial s_\alpha}{\partial q^\nu}\frac{\partial}{\partial p_\alpha} \\ &= \frac{\partial}{\partial q^\nu} + \frac{\partial s_\alpha}{\partial q^\nu}\frac{\partial}{\partial p_\alpha}, \end{aligned}$$

and so

$$\nabla s\Big(\frac{\partial}{\partial q^\nu}\Big) = \frac{\partial s_\alpha}{\partial q^\nu}\frac{\partial}{\partial p_\alpha}. \tag{8.5}$$

Recalling that $T_{\widehat{\alpha}}U = (E^\#_{\widehat{\alpha}} \oplus VT_{\widehat{\alpha}}U)^{\omega_U} \oplus E^\#_{\widehat{\alpha}} \oplus VT_{\widehat{\alpha}}U$, we conclude that the graph of ds with respect to the frame

$$\Big\{e_1, \cdots, e_{2k}, f_1, \cdots, f_{n-k}, \frac{\partial}{\partial p_1}, \cdots, \frac{\partial}{\partial p_{n-k}}\Big\}$$

can be expressed by the linear map

$$\begin{aligned} A_H : (E^\# \oplus VTU)^{\omega_U} \to VTU \cong E^*; \quad (A_H)^i_\alpha = \nabla_i s_\alpha, \\ A_I : E^\# \to VTU \cong E^*; \quad (A_I)^\beta_\alpha = \nabla_\beta s_\alpha, \end{aligned}$$

where

$$\begin{aligned} \nabla s\Big(\frac{\partial}{\partial y^i}\Big) &= (\nabla_i s_\alpha)\frac{\partial}{\partial q^\alpha}, \quad \nabla_i s_\alpha := \frac{\partial s_\alpha}{\partial y^j} + s_\alpha(b_i + b_\gamma R_i^\gamma) + s_\beta \frac{\partial R_i^\beta}{\partial q^\alpha}, \\ \nabla s\Big(\frac{\partial}{\partial q^\beta}\Big) &= (\nabla_\beta s_\alpha)\frac{\partial}{\partial q^\alpha}, \quad \nabla_\beta s_\alpha := \frac{\partial s_\alpha}{\partial q^\beta}. \end{aligned} \tag{8.6}$$

Finally we note that

$$\omega_U(s)(e_i, e_j) = w_{ij} - s_\beta F^\beta_{ij} := \widetilde{\omega}_{ij}$$

and denote its inverse by $(\widetilde{\omega}^{ij})$. Note that $(\widetilde{\omega}_{ij})$ is invertible if s_β is sufficiently small, i.e., if the section s is C^0-close to the zero section, or its image stays inside of U. Now Proposition 2.2 immediately implies

THEOREM 8.1. *Let ∇s be the vertical projection of ds as in (8.4). Then the graph of the section $s : Y \to U$ is coisotropic with respect to ω_U if and only if s satisfies*

$$\nabla_i s_\alpha \widetilde{\omega}^{ij} \nabla_j s_\beta = \nabla_\beta s_\alpha - \nabla_\alpha s_\beta \tag{8.7}$$

for all $\alpha > \beta$ *or*

$$\frac{1}{2}(\nabla_i s_\alpha \widetilde{\omega}^{ij} \nabla_j s_\beta) f_\alpha^* \wedge f_\beta^* = (\nabla_\beta s_\alpha) f_\alpha^* \wedge f_\beta^* \tag{8.8}$$

where f_α^* *is the dual frame of* $\{\frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^{n-k}}\}$ *defined by (7.5).*

Note that (8.8) involves terms of all order of s_β because the matrix $(\widetilde{\omega}^{ij})$ is the inverse of the matrix

$$\widetilde{\omega}_{ij} = \omega_{ij} - s_\beta F^\beta_{ij}.$$

There is a special case where the curvature vanishes i.e., satisfies

$$F_G = F^\beta_{ij} \frac{\partial}{\partial q^\beta} \otimes dy^i \wedge dy^j = 0 \tag{8.9}$$

in addition to (5.3). In this case, $\widetilde{\omega}_{ij} = \omega_{ij}$ which depends only on y^i's and so does ω^{ij}. Therefore (8.8) is reduced to the quadratic equation

$$\frac{1}{2}(\nabla_i s_\alpha \omega^{ij} \nabla_j s_\beta) f_\alpha^* \wedge f_\beta^* = (\nabla_\beta s_\alpha) f_\alpha^* \wedge f_\beta^*. \tag{8.10}$$

9. Deformation of strong homotopy Lie algebroids. In this section, we provide an invariant description of the master equation we have derived in the previous section. This can be regarded as the equation for the coisotropic submanifolds in the formal power series version of the equation or in the formal manifold in the sense of Kontsevich [K], [AKSZ].

9.1. b-deformed Oh-Park's strong homotopy Lie algebroids [OP]. We start with the normal form (7.1) of the symplectic thickening. We also note that the discussion of leaf space connection and the curvature, in particular the one-form θ_G does not depend on the closed one-form b but only depends on the conformal presymplectic form ω and the splitting $TY = G \oplus E$ only. In this regard, we can view the normal form in (7.1) as a deformation of the nondegenerate two form

$$\pi^*\omega - d\theta_G$$

to a conformal symplectic form relative to $\pi^* b$. So from now on, we denote $\omega_U = \pi^*\omega - d\theta_G$ and

$$\omega_U^b = \omega_U - d^{\pi^* b}\theta_G. \tag{9.1}$$

This deformation is responsible for the appearance of b-terms in (7.8) and then (7.11), (8.1) and eventually for the covariant derivative (8.4).

Again we regard (8.6) as the deformation of the old covariant derivative formula appearing in [OP, (7.3)] and denote the full covariant derivative

$$\nabla^b s = \nabla s + b|_{\mathcal{F}} s + \langle b|R| s \tag{9.2}$$

where $b|_{\mathcal{F}}$ is the restriction to the null-foliation of the one-form b and $\langle b|R|$ is the pairing of b and R which produces a one-form on G with values in $T\mathcal{F}$.

Now we give a deformed version of the notion of *strong homotopy Lie algebroid* introduced in [OP].

DEFINITION 9.1. Let $E \to Y$ be a Lie algebroid. A b-deformed L_∞*-structure* over the Lie algebroid is a structure of strong homotopy Lie algebra $(\mathfrak{l}[1], \mathfrak{m})$ on the associated b-deformed E-de Rham complex $\mathfrak{l}^\bullet = \Omega^\bullet(E) = \Gamma(\Lambda^\bullet(E^*))$ such that $\mathfrak{m}_1$ is the E-differential ${}^E d^{\bar b}$ induced by the (deformed) Lie algebroid structure on E as described in subsection 13.2. We call the pair $(E \to Y, \mathfrak{m})$ a b-deformed strong homotopy Lie algebroid.

Here we refer to [NT] or Appendix 13.2 for the definition of E-differential used in this definition.

With this definition of a b-deformed strong homotopy Lie algebroid, we will show that for given l.c.p-s. manifold (Y, ω, b) each splitting $\Pi : TY = G \oplus T\mathcal{F}$ induces a canonical L_∞-structure over the Lie algebroid $T\mathcal{F} \to Y$.

The following linear map and quadratic map are introduced in [OP] which play crucial roles in the construction of L_∞-structure on the foliation de Rham complex: a linear map

$$\widetilde{\omega} : \Omega^1(Y; \Lambda^\bullet E^*) \to \Gamma(\Lambda^{\bullet+1} E^*) = \Omega^{\bullet+1}(\mathcal{F}), \tag{9.3}$$

a quadratic map

$$\langle \cdot, \cdot \rangle_\omega : \Omega^1(Y; \Lambda^{\ell_1} E^*) \otimes \Omega^1(Y; \Lambda^{\ell_2} E^*) \to \Omega^{\ell_1+\ell_2}(\mathcal{F}), \tag{9.4}$$

and the third map that is induced by the transverse Π-curvature, whose definition is now in order.

We recall the definitions of those maps. The linear map $\widetilde{\omega}$ is defined by

$$\widetilde{\omega}(A) := (A|_E)_{skew}. \tag{9.5}$$

Here note that an element $A \in \Omega^1(Y; \Lambda^k E^*)$ is a section of $T^*Y \otimes \Lambda^k E^*$. Restricting A to E for the first factor we get $A|_E \in E^* \otimes \Lambda^k E^*$. Then $(A|_E)_{skew}$ is the skew-symmetrization of $A|_E$. The quadratic map is defined by

$$\langle A, B \rangle_\omega := \langle A|\pi|B \rangle - \langle B|\pi|A \rangle$$

where π is the transverse Poisson bi-vector on $N^*\mathcal{F}$ associated to the transverse symplectic form ω on $N\mathcal{F}$.

We will denote

$$F^\# := F\omega^{-1} = F_i^{\alpha j} dy^i \otimes \Big(\frac{\partial}{\partial y^j} + R_j^\beta \frac{\partial}{\partial q^\beta} \Big) \otimes \frac{\partial}{\partial q^\alpha} \in \Gamma(G^* \otimes G \otimes E), \tag{9.6}$$

where $F_i^{\alpha j} = F_{ik}^\alpha \omega^{kj}$. Note that we can identify $\Gamma(G^* \otimes G \otimes E)$ with $\Gamma(N^*\mathcal{F} \otimes N\mathcal{F} \otimes E)$ via the isomorphism $\pi_G : G \to N\mathcal{F}$.

For given $\xi \in \Omega^\ell(\mathcal{F})$, we define

$$d^b_{\mathcal{F}}(\xi) := (\nabla^b \xi|_E)_{skew}, \tag{9.7}$$

and deformed bracket

$$\{\xi_1, \xi_2\}^b_\Pi := \langle \nabla^b \xi_1, \nabla^b \xi_2 \rangle_\omega = \sum_{i<j} \omega^{ij} (\nabla^b_i \xi_1) \wedge (\nabla^b_j \xi_2). \tag{9.8}$$

Here the map in (9.7) is nothing but the $\bar b$-deformed leafwise differential of the null foliation which is indeed independent of the choice of splitting $\Pi : TY = G \oplus T\mathcal{F}$

but depends only on the foliation and the projection of the one-form b to $\mathcal{F}$, see also subsection 13.2. By Remark 2.2 the obtained leafwise differential depends only on ω. We use $d^b_{\mathcal{F}}$ and $d^{\bar{b}}_{\mathcal{F}}$ interchangeably.

The second is a bracket in the transverse direction which is a b-deformation of the one given in [OP, (9.13)].

Now we promote the maps $d^b_{\mathcal{F}}$ and $\{\cdot,\cdot\}^b$ to an infinite family of graded multilinear maps

$$\mathfrak{m}^b_\ell = (\Omega[1]^\bullet(\mathcal{F}))^{\otimes \ell} \to \Omega[1]^\bullet(\mathcal{F}) \tag{9.9}$$

so that the structure

$$\left(\bigoplus_{j=0}^{n-k} \mathfrak{l}[1]^j ; \{\mathfrak{m}^b_\ell\}_{1\le \ell<\infty} \right)$$

defines a *strong homotopy Lie algebroid* on $E = T\mathcal{F} \to Y$ in the above sense. Here $\Omega[1]^\bullet(\mathcal{F})$ is the shifted complex of $\Omega^\bullet(\mathcal{F})$, i.e., $\Omega[1]^k(\mathcal{F}) = \Omega^{k+1}(\mathcal{F})$ and $\mathfrak{m}_1$ is defined by

$$\mathfrak{m}^b_1(\xi) = (-1)^{|\xi|} d^b_{\mathcal{F}}(\xi)$$

and $\mathfrak{m}_2$ is given by

$$\mathfrak{m}^b_2(\xi_1, \xi_2) = (-1)^{|\xi_1|(|\xi_2|+1)} \{\xi_1, \xi_2\}^b_\Pi .$$

On the un-shifted group $\mathfrak{l}$, $d^b_{\mathcal{F}}$ defines a differential of degree 1 and $\{\cdot,\cdot\}_\omega$ is a graded bracket of degree 0 and $\mathfrak{m}^b_\ell$ is a map of degree $2-\ell$.

We now define $\mathfrak{m}^b_\ell$ for $\ell \ge 3$. Here enters the transverse Π-curvature $F = F_\Pi$ of the splitting Π of the null foliation $\mathcal{F}$. We define

$$\mathfrak{m}^b_\ell(\xi_1, \cdots, \xi_\ell) := \sum_{\sigma \in S_\ell} (-1)^{|\sigma|} \langle \nabla^b \xi_{\sigma(1)}, (F^\# \rfloor \xi_{\sigma(2)}) \cdots (F^\# \rfloor \xi_{\sigma(\ell-1)}) \nabla^b \xi_{\sigma(\ell)} \rangle_\omega \tag{9.10}$$

where $|\sigma|$ is the standard Koszul sign in the suspended complex. We have now arrived at our definition of strong homotopy Lie algebroid associated to the coisotropic submanifolds, which is a b-deformation of the one introduced in [OP, section 9], but which is applied *after enlarging our category* to that of locally conformal pre-symplectic two forms instead of pre-symplectic two forms.

THEOREM 9.2. *Let (Y, ω, b) be a l.c.p.s. manifold and $\Pi : TY = G \oplus T\mathcal{F}$ be a splitting. Then Π canonically induces a structure of strong homotopy Lie algebroid on $T\mathcal{F}$ in that the graded complex*

$$\left(\bigoplus_\bullet \Omega[1]^\bullet(\mathcal{F}), \{\mathfrak{m}^b_\ell\}_{1\le \ell<\infty} \right)$$

defines the structure of strong homotopy Lie algebra. We denote by $\mathfrak{l}_{(Y,\omega,b;\Pi)}$ the corresponding strong homotopy Lie algebra.

Proof. The proof of this theorem follows the strategy used in the proof of Theorem 9.4 [OP], which uses the formalism of super-manifolds and odd symplectic structure on the super tangent bundle $T[1]U$ [AKSZ] of the l.c.p.s. thickening U of (Y, ω, b).

We change the parity of TU along the fiber and denote by $T[1]U$ the corresponding super tangent bundle of U. One considers a multi-vector field on U as a (fiberwise) polynomial function on $T^*[1]U$. For example, the bi-vector field P, inverse to the *nondegenerate* form ω_U (cf. (9.1)), defines a quadratic function, which we denote by H. This also coincides with the push-forward of the even function $H^* : T[1]U \to \mathbb{R}$ induced by ω_U. We denote by $\{\cdot,\cdot\}_\Omega$ the (super-)Poisson bracket associated to the odd symplectic form Ω on $T[1]U$. Then the bracket operation

$$Q := \{H^*, \cdot\}_\Omega$$

defines a derivation on the set $\mathcal{O}_{T[1]U}$ of "functions" on $T[1]U$: Here $\mathcal{O}_{T[1]U}$ is the set of differential forms on U considered as fiberwise polynomial functions on $T[1]U$. We refer to [Gz] or [OP, Appendix] for the precise mathematical meaning for this correspondence. Therefore it defines an odd vector field.

Restricting ourselves to a Darboux neighborhood of $\mathbb{L} = T\mathcal{F}[1] \subset T[1]U$, we identify the neighborhood with a neighborhood of the zero section $T^*[1]\mathbb{L}$. Using the fact that (9.14) depends only on ξ, not on the extension, we will make a convenient choice of coordinates to write H in the Darboux neighborhood and describe how the derivation $Q = \{H,\cdot\}_\Omega$ acts on $\Omega^\bullet(\mathcal{F})$ in the canonical coordinates of $T^*[1]\mathbb{L}$. In this way, we can apply the canonical quantization which provides a canonical correspondence between functions on "the phase space" $T^*[1]\mathbb{L}$ and the corresponding operators acting on the functions on the "configuration space" $\mathbb{L}$, later when we find out how the deformed differential δ^b (9.15) acts on $\Omega^\bullet(\mathcal{F})$.

We denote by $(y^i, q^\alpha, p_\alpha, y_i^*, q_\alpha^*, p_*^\alpha)$ the canonical coordinates $T^*\mathbb{L}$ associated with the coordinates $(y^i, q^\alpha, p_\alpha)$ of $N^*\mathcal{F}$. Note that these coordinates are nothing but the canonical coordinates of $N^*Y \subset T^*U$ pulled-back to $T\mathcal{F} \subset TU$ and its Darboux neighborhood, with the corresponding parity change: We denote the (super) canonical coordinates of $T^*[1]\mathbb{L}$ associated with $(y^i, q^\alpha \mid p_\alpha)$ by

$$\begin{pmatrix} y^i, & q^\alpha & \mid p_*^\alpha \\ y_i^*, & q_\alpha^* & \mid p_\alpha \end{pmatrix}.$$

Here we note that the degree of y^i, q^α and p_α are 0 while their anti-fields, i.e., those with $*$ in them have degree 1. And we want to emphasize that $\mathbb{L}$ is given by the equation

$$y_i^* = p_\alpha = p_*^\alpha = 0 \tag{9.11}$$

and (y^i, y_i^*), (p_α, q_α^*) and (p_*^α, q^α) are conjugate variables. In terms of these coordinates, [OP, (9.23)] provides the formula

$$H^* = \frac{1}{2}\widetilde{\omega}^{ij} y_i^\# y_j^\# + p_*^\delta q_*^\delta. \tag{9.12}$$

Here, we define $y_i^\#$ to be

$$y_i^\# := y_i^* + R_i^\delta p_*^\delta - \left(p_\nu(b_i + b_\gamma R_i^\gamma) + p_\beta \frac{\partial R_i^\beta}{\partial q^\delta} q_\delta^* \right)$$

arising from (8.1) similarly as in [OP, (9.23)]. When ω is a closed symplectic form as in [OP], we have $\{H^*, H^*\} = 0$. However in the current l.c.s. case, this is no longer the case.

LEMMA 9.3. *Consider the l.c.p.s. manifold* (Y, ω, b) *and let* $(U, \omega_U, \pi^* b)$ *be its c.p.s. thickening constructed before. Regard the closed one-form* $\pi^* b$ *as the odd function* $b = b_{q^\gamma} p_*^\gamma + b_{p_\delta} q_\delta^* + b_i \psi_i^\#$ *where* $p_*^\gamma = dq^\gamma$, $\psi_i^\# = dy_i^\#$, $q_\delta^* = dp_\delta$. *Then* $\{H^*, H^*\} = bH^*$.

Proof. In this proof, we use Einstein's summation convention, whenever we feel convenient. Using the canonical bracket relations, we compute

$$\{H^*, H^*\} = \frac{1}{2} \left(\frac{\partial \widetilde{\omega}^{ij}}{\partial q_\delta} q_\delta^* + \frac{\partial \widetilde{\omega}^{ij}}{\partial p^\delta} p_*^\delta \right) y_i^\# y_j^\# + \frac{\partial \widetilde{\omega}^{ij}}{\partial y_k^\#} y_i^\# y_j^\# \psi_k^\#.$$

However the d^b-closedness of ω_U means that this sum is precisely equivalent to

$$(b_{q^\gamma} q_*^\gamma + b_{p_\delta} p_\delta^* \omega^{ij} + b_i \psi_i^\#) \frac{1}{2} \omega^{ij} y_i^\# y_j^\# = bH^*.$$

This finishes the proof. ∎

We will be interested in whether one can canonically restrict the vector field Q to $\mathbb{L} = T\mathcal{F}[1]$ or equivalently whether the function H has constant value on $\mathbb{L}$. Here comes the coisotropic condition naturally. The following was proved in [OP], which still holds for the current l.c.s. case.

LEMMA 9.4. [OP, Lemma 9.5] *Let* H *be the even function on* $T[1]X$ *induced by the symplectic form* ω_X, *and* $H^* : T^*[1]X \to \mathbb{R}$ *be its push-forward by the isomorphism* $\widetilde{\omega}_X : T[1]X \to T^*[1]X$. *When* $Y \subset (X, \omega)$ *is a coisotropic submanifold we have* $H^*|_{N^*[1]Y} = 0$. *Conversely, any (conic) Lagrangian subspace* $\mathbb{L}^* \subset T^*[1]X$ *satisfying* $H^*|_{\mathbb{L}^*} = 0$ *is equivalent to* $N^*[1]Y$, *for some coisotropic submanifold* Y *of* (X, ω).

Now we consider the b-deformed Hamiltonian $H_b^* : T[1]U \to \mathbb{R}$ the even function induced by the l.c.s. form ω_U^b. From the expression (7.8), a straightforward calculation leads to

$$H_b^* = H^* + p_\nu q_\nu^* b \tag{9.13}$$

where $q_\nu^* = \chi^\nu - R_j^\nu \psi_j$ is the conjugate variable of p_ν and $b = b_\gamma \chi^\gamma + b_i \psi^i$.

PROPOSITION 9.5. $\{H_b^*, H_b^*\} = 0$ *when restricted to* $\mathbb{L}^*$.

Proof. This is an immediate translation of the equality $d^b d^b = 0$. For readers' convenience, we prove this by direct calculation. By (9.11), we have

$$\{p_\nu q_\nu^* b, p_\nu q_\nu^* b\} = 0$$

when restricted to $\mathcal{O}_{\mathbb{L}^*}$. Finally we note that the vector field $Q = \{H^*, \cdot\}$ restricts to

$$\psi_*^i \frac{\partial}{\partial y^i} + \eta_\alpha^* \frac{\partial}{\partial q_\alpha^*} + \eta_\alpha \frac{\partial}{\partial q^\alpha}.$$

Therefore having (9.11) in our mind, we compute

$$\begin{aligned} \{H^*, p_\nu q_\nu^* b\} &= \{H^*, p_\nu q_\nu^* (b_\gamma p_*^\gamma + b^i \psi^i)\} \\ &= \{H^*, H^*\} + \{H^*, p_\nu q_\nu^* b^i y_i^*\} + \{p_\nu q_\nu^* b^i y_i^*, H^*\}. \end{aligned}$$

But we have $\{H^*, H^*\} = bH^*$ and compute

$$\{p_\nu q_\nu^* b, H^*\} = -p_\nu \left(\frac{\partial H^*}{\partial p_\nu} + q_\nu^* \{b, H^*\} \right).$$

Both restricts to zero on $\mathbb{L}^*$ by (9.11) and Lemma 9.4. This completes the proof of Proposition 9.5. □

Noting that $\mathbb{L}^* = N^*[1]Y$ is mapped to $\mathbb{L} = T\mathcal{F}[1]$ under the isomorphism $\widetilde{\omega}_X$, this lemma enables us to restrict the odd vector field Q to $T\mathcal{F}[1]$. We need to describe the Lagrangian embedding $T\mathcal{F}[1] \subset T[1]X$ more explicitly, and describe the induced directional derivative acting on $\Omega^\bullet(\mathcal{F})$ regarded as a subset of "functions" on $T\mathcal{F}[1]$. (Again we refer to [OP, Appendix] or [Gz] for the precise explanations of this).

Now we define $\delta'_b : \Omega^\bullet(\mathcal{F}) \to \Omega^\bullet(\mathcal{F})$ by the formula

$$\delta'_b(\xi) := \{H_b^*, \widetilde{\xi}\}_\Omega\Big|_{\mathbb{L}} \tag{9.14}$$

where $\widetilde{\xi}$ is the extension of ξ in a neighborhood of $\mathbb{L} \subset T[1]X$: the extension that we use is the lifting of $\xi \in \Omega^\bullet(\mathcal{F})$ to an element of $\Omega^\bullet(U)$ obtained by the (local) Ehresman connection constructed in section 8. The condition $Q|_{\mathbb{L}} \equiv 0$ implies that this formula is independent of the choice of (local) Ehresman connection. We will just denote $\delta'_b(\xi) = \{H_b^*, \xi\}_\Omega$ instead of (9.14) as long as there is no danger of confusion.

Obviously, δ'_b satisfies $\delta'_b\delta'_b = 0$ because of $\{H_b^*, H_b^*\} = 0$ by Proposition 9.5. Now it remains to verify that this is translated into the L_∞-relation $\delta^b\delta^b = 0$ in the tensorial language which is exactly what we wanted to prove. For this purpose, we need to describe the map $\delta'_b : \Omega^\bullet(\mathcal{F}) \to \Omega^\bullet(\mathcal{F})$ more explicitly.

The rest of the argument is precisely the same as the end of the proof of [OP, Theorem 9.4]. By expanding the even function H_b^* above into the power series

$$H_b^* = \sum_{\ell=1} H_\ell, \quad H_\ell \in \mathfrak{l}^\ell,$$

in terms of the degree (i.e., the number of factors of odd variables $(y_i^*, p_*^\alpha, p_\alpha)$ or the 'ghost number' in the physics language) our definition of $\mathfrak{m}_\ell^b$ exactly corresponds to the ℓ-linear operator

$$(\xi_1, \xi_2, \cdots, \xi_\ell) \mapsto \{\cdots\{H_\ell, \xi_1\}_\Omega, \cdots\}_\Omega, \xi_\ell\}_\Omega.$$

Note that the above power series acting on $(\xi_1, \cdots, \xi_\ell)$ always reduces to a finite sum and so is well-defined as an operator. Then by definition, the coderivation

$$\delta^b = \sum_{\ell=1}^\infty \widehat{\mathfrak{m}}_\ell^b \tag{9.15}$$

precisely corresponds to $\delta'_b = \{H_b^*, \cdot\}_\Omega$. Here $\mathfrak{m}_\ell^b$ is the b-deformed $\mathfrak{m}_\ell^b$-map defined by

$$\mathfrak{m}_\ell^b(\xi_1, \xi_2, \cdots, \xi_\ell) = \sum_{k=0}^\infty \frac{1}{k!}\mathfrak{m}_{\ell+k}(\underbrace{b, \cdots, b}_{k\,\text{times}}, \xi_1, \cdots, \xi_\ell).$$

(See [FOOO, section 3.6] for a general discussion on the deformation of A_∞ structures. This definition is the symmetrized version thereof.) The L_∞ relation $\delta^b\delta^b = 0$ then immediately follows from $\delta'_b\delta'_b = 0$. This finishes the proof. □

For example, under the above translation, the action of the odd vector field $Q\,|_{\mathbb{L}}$ on

$$\mathfrak{l} = \bigoplus_{\ell=0}^{n-k} \mathfrak{l}^\ell \cong \bigoplus_{\ell=0}^{n-k} \Omega^\ell(\mathcal{F}),$$

translates into to the leafwise differential $d_{\mathcal{F}}^{\bar{b}}$. This finishes the proof of Theorem 9.2. □

9.2. Gauge equivalence. In this section we prove that two strong homotopy Lie algebroids we have associated to two different splittings are *gauge equivalent* or L^∞-isomorphic. This is the formal analog to the (C^∞)-Hamiltonian equivalence of the coisotropic submanifolds.

DEFINITION 9.6. Let $(C[1], \mathfrak{m})$, $(C'[1], \mathfrak{m}')$ be L_∞-algebras and δ, δ' be the associated coderivation. A sequence $\varphi = \{\varphi_k\}_{k=1}^\infty$ with $\varphi_k : E_k C[1] \to C'[1]$ is said to be an *L_∞-homomorphism* if the corresponding coalgebra homomorphism $\widehat{\varphi} : EC[1] \to EC'[1]$ satisfies

$$\widehat{\varphi} \circ \delta = \delta' \circ \widehat{\varphi}.$$

We say that φ is an *L_∞-isomorphism,* if there exists a sequence of homomorphisms $\psi = \{\psi_k\}_{k=1}^\infty$, $\psi : E_k C'[1] \to C'[1]$ such that its associated coalgebra homomorphism $\widehat{\psi} : EC'[1] \to EC[1]$ satisfies

$$\widehat{\psi} \circ \widehat{\varphi} = id_{EC[1]}, \quad \widehat{\varphi} \circ \widehat{\psi} = id_{EC'[1]}.$$

In this case, we say that two L_∞ algebras, $(C[1], \mathfrak{m})$ and $(C'[1], \mathfrak{m}')$ are L_∞ isomorphic.

The following theorem is the b-deformed version of [OP, Theorem 10. 1]. (See also Definition 8.3.6 [Fu].)

THEOREM 9.7. *The two structures of strong homotopy Lie algebroid on the null distribution $E = T\mathcal{F}$ of (Y, ω, b) induced by two choices of splitting Π, Π' are canonically L_∞-isomorphic.*

Proof. We start with the expression of the l.c.s. form ω_U

$$\omega_U = \pi_Y^* \omega - d\theta_G - \pi_Y^* b \wedge \theta_G$$

given in (3.4) that is canonically constructed on a neighborhood U of the zero section $E^* = T^*\mathcal{F}$ when a splitting $\Pi : TY = G \oplus T\mathcal{F}$ is provided. To highlight dependence on the splitting, we denote by θ_Π and ω_Π^b the one-form θ_G and the l.c.s. form ω_U. We will also denote by δ_Π^b the $\delta : EC[1] \to EC[1]$ corresponding to the splitting Π. Here we would like to emphasize that the one-form θ_G depends only on the splitting Π but not depend on the one-form b.

Then for a given splitting Π_0, we have

$$\omega_\Pi^b - \omega_{\Pi_0}^b = d^{\pi_Y^* b}(\theta_{\Pi_0} - \theta_\Pi). \tag{9.16}$$

In the super language, this is translated into

$$H_\Pi^b - H_{\Pi_0}^b = \{H_{\Pi_0}^b, \Gamma\}_\Omega = -\{\Gamma, H_{\Pi_0}^b\}_\Omega \tag{9.17}$$

where Γ is the function associated to the one-form $\theta_{\Pi_0} - \theta_\Pi$ which has $deg'(\Gamma) = 0$ (or equivalently has $deg(\Gamma) = 1$). This function does *not* depend on b. The last identity comes from the super-commutativity of the bracket and the fact that $deg(H_{\Pi_0}^b) = 2$ and $deg(\Gamma) = 1$. Once we have established these, the rest of the proof is the same as that of Theorem 10.1 [OP] and so omitted, referring the readers thereto. □

This theorem then associates a canonical (L_∞-)isomorphism class of strong homotopy Lie algebras to each l.c.p-s manifold (Y, ω, b) and so to each coisotropic submanifold of l.c.s manifold $(X, \omega_X, \mathfrak{b}_X)$. As in the symplectic case, it is obvious from the construction that pre-Hamiltonian diffeomorphisms induce canonical isomorphism by pull-backs in our strong homotopy Lie algebroids.

In the point of view of coisotropic embeddings in l.c.s manifolds, this theorem implies that our strong homotopy Lie algebroids for two Hamiltonian isotopic coisotropic submanifolds are canonically isomorphic and so the isomorphism class of the strong homotopy Lie algebroids is an invariant of coisotropic submanifolds modulo the Hamiltonian isotopy as in the symplectic case [OP]. (See the relevant discussion in section 11 on the general bulk deformations of coisotropic submanifolds. According to the definitions therein, Hamiltonian deformations of coisotropic submanifolds correspond to equivalent bulk deformations.) This enables us to study the moduli problem of deformations of l.c.p-s. structures on Y in the similar way as done in [OP]. Up until now, most of our discussions correspond to the l.c.s. analogues of the deformation theory developed in [OP] in the symplectic context. This effort will finally pay off when we study the moduli problems of coisotropic submanifolds and its obstruction-deformation theory. We will be particulary interested in the deformation problems of Zambon's example in this enlarged categorical setting of conformally symplectic manifolds.

10. Moduli problem and the Kuranishi map. In this section, we write down the defining equation (8.8) for the graph $\operatorname{Graph} s \subset TU \subset TE^*$ to be coisotropic in a formal neighborhood, i.e., in terms of the power series of the section s with respect to the fiber coordinates in U and study them using (6.17). Using the concept of gauge equivalence of the solutions of a Maurer-Cartan equation in [FOOO, 4.3] we will study the moduli problem of the Maurer-Cartan equation (10.1) of $\mathfrak{l}^\infty_{(Y,\omega,b)}$.

First we state the following b-deformed analogue of Theorem 11.1 [OP] whose proof is the same as that of the latter and so omitted.

THEOREM 10.1. *The equation of the formal power series solutions* $\Gamma \in \mathfrak{l}^1$ *of* (8.8) *is given by*

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell!}\mathfrak{m}^b_\ell(\Gamma, \cdots, \Gamma) = 0 \quad on\ \Omega^2(\mathcal{F}) \tag{10.1}$$

where

$$\Gamma = \sum_{k=1}^{\infty} \varepsilon^k \Gamma_k \tag{10.2}$$

where Γ_k*'s are sections of* $T^*\mathcal{F}$ *and* ε *is a formal parameter.*

As in [OP, Remark 11.1] it is possible to interpret (10.1) as the condition for the gauge changed weak (or curved) L_∞-structure to define a (strong) L_∞-structure, i.e., $\mathfrak{m}^{b,\Gamma}_0 = 0$ or the Maurer-Cartan equation $\widehat{\mathfrak{m}}^{b,\Gamma}_1 \circ \widehat{\mathfrak{m}}^{b,\Gamma}_1 = 0$ for the deformation problem of the corresponding l.c.p-s. structure (Y, ω, b). In what follows, we will use (6.17) to study a formal solution of (10.1). This description seems to be more suitable for the study of C^∞ Maurer-Cartan equation which we hope to pursue in a sequel to the present paper.

By Theorem 6.2, using (6.9), a solution Γ of (8.8) is also a solution of (6.17), and therefore a formal solution of (8.8) is also a formal solution of (6.17). Let us plug a formal solution Γ, identified with $p_G^*\Gamma$, into (6.17), denoting $a_i := -d^b(\Gamma_i)$ and expanding $(\omega - d^b(\Gamma))^{k+1}$. Noting that a_i and ω^p are differential forms of even degree, we abbreviate wedge products of them as usual products. As a result we obtain

$$\sum_{N=0}^{\infty} \epsilon^N \sum_{\substack{1 \le i_1 < \cdots\cdots < i_p,\\ 1 \le s_i,\\ s_1 + \cdots + s_p \le (k+1)\\ i_1 s_1 + \cdots + i_p s_p = N}} a_{i_1}^{s_1} \cdots a_{i_p}^{s_p} \binom{k+1}{s_1} \cdots \binom{k+1}{s_p} \omega^{k+1-s_1 \cdots - s_p}(y) = 0.$$

Consequently for each $N \ge 0$ we have

$$0 = a_N (k+1)\omega^k +$$

$$+ \sum_{\substack{1 \le i_1 < \cdots\cdots < i_p \le N-1,\\ 1 \le s_i,\\ s_1 + \cdots + s_p \le (k+1)\\ i_1 s_1 + \cdots + i_p s_p = N}} a_{i_1}^{s_1} \cdots a_{i_p}^{s_p} \binom{k+1}{s_1} \cdots \binom{k+1}{s_p} \omega^{k+1-s_1 \cdots - s_p}(y). \tag{10.3}$$

Note that the number p entering in (10.3) is bounded by $(k+1)$.

Let us examine (10.3) for small numbers N. For $N = 0$ the corresponding term in (10.3) is $\omega^{k+1}(y) = 0$.

For $N = 1$ the corresponding values in (10.3) are $p = 1 = i_1 = s_1$ and the corresponding term is

$$\omega^k a_1(y) = 0$$

which is equivalent to

$$d_{\mathcal{F}}^{\bar{b}} \Gamma_1(y) = 0.$$

(See (6.19), (6.20).) So Γ_1 is a solution of the linearized equation, which is assumed to be given.

For $N = 2$ the corresponding values in (10.3) are $p = 1$, $i_1 = 1, s_1 = 2$ or $i_1 = 2, s_1 = 1$. The equation (10.3) in this case has the following form

$$(k+1)\omega^k a_2 + \binom{k+1}{2} \omega^{k-1} a_1^2 = 0,$$

which, since $d_{\mathcal{F}}^{\bar{b}} \Gamma_1(y) = 0$, is equivalent to

$$\begin{aligned} -\omega^k (d_{\mathcal{F}}^{\bar{b}} \Gamma_2) &= \frac{k}{2} \omega^{k-1} a_1^2, \\ \Leftrightarrow -d_{\mathcal{F}}^{\bar{b}} \Gamma_2 &= \frac{k}{2} P_\omega \rfloor (\omega^{k-1} a_1^2), \\ \Leftrightarrow -\omega \wedge d_{\mathcal{F}}^{\bar{b}} \Gamma_2 &= \frac{k}{2} a_1^2, \\ \Leftrightarrow -d_{\mathcal{F}}^{\bar{b}} \Gamma_2 &= \frac{1}{2} P_\omega \rfloor (a_1^2), \end{aligned} \tag{10.4}$$

where P_ω is the bi-vector in $\Lambda^2 G$ dual to the restriction of ω to G, so $\omega(P_\omega) = k$.

Thus, taking into account $d^{\bar b}_{\mathcal F}\Gamma_1(y) = 0$, the RHS of (10.4) is $\frac{1}{2}\mathfrak{m}^b_2(\Gamma_1, \Gamma_1)$ (cf. (9.8)). Note that $\mathfrak{m}^b_1$ is a derivation of $\mathfrak{m}^b_2$ the map, hence

$$\Omega^1(\mathcal F) \times \Omega^1(\mathcal F) \to \Omega^2(\mathcal F),\ (\Gamma_1, \Gamma_2) \mapsto \frac{1}{2} P_\omega \rfloor (d^b\Gamma_1 \wedge d^b\Gamma_2)$$

induces the Kuranishi-Gerstenhaber bracket

$$KG : H^1_b(Y, \omega) \times H^1_b(Y, \omega) \to H^2_b(Y, \omega),\ ([\Gamma_1], [\Gamma_2]) \mapsto \frac{1}{2}[\mathfrak{m}^b_2(\Gamma_1, \Gamma_2)].$$

Since $\mathfrak{m}^b_2$ is symmetric, the Kuranishi-Gerstenhaber bracket is defined by the Kuranishi map [OP]

$$Kr : H^1_b(Y, \omega) \to H^2_b(Y, \omega),\ [\Gamma_1] \mapsto [\mathfrak{m}^b_2(\Gamma_1, \Gamma_1)].$$

COROLLARY 10.2. *(cf.* [OP, Corollary11.5]*). The moduli problem is formally unobstructed only if Kr vanishes.*

The following Theorem is a b-deformed analog of [OP, Theorem 11.2] derived in the symplectic case, so we omit its proof.

THEOREM 10.3. *Let $\mathcal F$ be the null foliation of (Y, ω, b) and $\mathfrak l = \oplus_{\ell=1}^{n-k}\mathfrak l^\ell$ be the associated complex. Suppose that $H^2_b(Y, \omega) = \{0\}$. Then for any given class $\alpha \in H^1_b(Y, \omega)$, (10.1) has a solution $\Gamma = \sum_{k=1}^\infty \varepsilon^k \Gamma_k$ such that $d^b_{\mathcal F}(\Gamma_1) = 0$ and $[\Gamma_1] = \alpha \in H^1_b(Y, \omega)$. In other words, the formal moduli problem is unobstructed.*

In general, we say that an element $\Gamma_1 \in \ker d^b_{\mathcal F} \cap \Omega^1(\mathcal F)$ is *formally unobstructed*, if there exists a formal solution to (10.1) whose first summand Γ_1 is the given one. Similarly, $\Gamma_1 \in \ker d^b_{\mathcal F} \cap \Omega^1(\mathcal F)$ is called *smoothly unobstructed*, if it is tangent to a curve of smooth coisotropic deformations.

Note that Hamiltonian diffeomorphisms on U that are close to the identity act on the space of formal deformations by acting on each summand Γ_l in (10.2). (If a diffeomorphism $\phi : U \to U$ is close to the identity and $\Gamma : Y \to U \subset G^\circ$ is a section, then the composition $\pi \circ \phi \circ \Gamma : Y \to Y$ is a diffeomorphism, hence there exists a diffeomorphism $g \in Diff(Y)$ such that $\phi \circ \Gamma \circ g : Y \to U \subset G^\circ$ is a section.) They also act on the space of smooth coisotropic deformations by an obvious way. The following Lemma is straightforward, so we omit its proof.

LEMMA 10.4. *Given a function $f \in C^\infty(Y)$ and an element $a \in \Omega^1(\mathcal F) \cap \ker d^b_{\mathcal F}$ the following assertions hold:*

1. *a is formally unobstructed, if and only if $a + d^{\bar b}_{\mathcal F} f$ is formally unobstructed.*
2. *a is tangent to a curve $\gamma(t)$ of coisotropic deformations of Y in (U, ω_G, π^b) then $a + d^{\bar b}_{\mathcal F} f$ is tangent to the curve $\phi_t \circ \gamma_t$, where ϕ_t's are Hamiltonian diffeomorphisms close to the identity on $(U, \omega_G, \pi^* b)$ and generated by an extension of Hamiltonian f to U.*

Now we study the moduli of the solution of the Maurer-Cartan equation under the action induced by Hamiltonian diffeomorphisms on a neighborhood $(U, \omega_U, \pi^* b)$ of Y, taking into account Theorem 6.7. Here we follow the ideology in [FOOO, 4.3]. First we need introduce the notion of a model of the product of $[0, 1]$ with an L_∞-algebra

$(C[1], \mathfrak{m})$, which is also a L_∞-algebra, imitating the analogous notion for A_∞-algebras, introduced in [FOOO, 4.2].

Definition 10.5. (cf. [FOOO, Definition 4.2.1]) An L_∞-algebra $(\bar{C}[1], \bar{\mathfrak{m}})$ together with L_∞-homomorphisms

$$Incl : EC[1] \to \bar{C}[1],\ Eval_{s=0} : E\bar{C}[1] \to C,\ Eval_{s=1} : E\bar{C} \to C$$

is said to be *a model of* $[0,1] \times (C[1], \mathfrak{m})$, if the following holds:

1. $Incl : E_k C[1] \to \bar{C}[1]$ is zero unless $k = 1$ The same holds for $Eval_{s=0}$ and $Eval_{s=1}$.
2. $Eval_{s=0} \circ Incl = Eval_{s=1} \circ Incl =$ identity.
3. $Incl_1 : (C[1], \mathfrak{m}_1) \to (\bar{C}[1], \bar{\mathfrak{m}})$ is a cochain homotopy equivalence and $(Eval_{s=0})_1$, $(Eval_{s=1})_1 : (\bar{C}[1], \bar{\mathfrak{m}}_1) \to (C[1], \mathfrak{m}_1)$ are cochain homotopy equivalences.
4. The (cochain) homomorphism $(Eval_{s=0})_1 \oplus (Eval_{s=1})_1 : \bar{C}[1] \to C[1] \oplus C[1]$ is surjective.

Definition 10.6 (Maurer-Cartan moduli space). (cf. [FOOO, Definition 4.3.1]) We say that two solutions Γ_1, Γ_2 of (10.1) are gauge-equivalent if there exist a model $(\bar{C}[1], \bar{\mathfrak{m}})$ of $[0,1] \times (C[1], \mathfrak{m})$ and a solution $\tilde{\Gamma}$ of the Maurer-Cartan equation of $(\bar{C}[1], \bar{\mathfrak{m}})$ such that $Eval_{s=0*}(\tilde{\Gamma}) = \Gamma_1$, $Eval_{s=1*}(\tilde{\Gamma}) = \Gamma_2$. We say such a $\tilde{\Gamma}$ *a homotopy* from Γ_1 to Γ_2. We denote by $\mathfrak{M}_{formal}(Y, \omega, b)$ the set of gauge equivalence classes of Maurer-Cartan solutions.

Note that any Hamiltonian diffeomorphisms ϕ_t, $\phi_0 = Id$, on $(U, \omega_U, d^{\pi^* b})$ which are close to the identity, provides a homotopy between a formal solution Γ to (10.1). Denote by $\mathfrak{M}_{Ham}(Y, \omega, b)$ the set of Hamiltonian equivalent classes of formal coisotropic deformations of Y in U. Using the argument of the proof Theorem 9.7 we obtain easily a natural map $\mathfrak{M}_{Ham}(Y, \omega, b) \to \mathfrak{M}_{formal}(Y, \omega, b)$.

Remark 10.7. Theorem 6.2 implies that there is a one-one correspondence between of coisotropic deformations of a coisotropic submanifold $Y \subset (U, \omega_U, \pi^* b)$ and the set of deformations (ω', b) of the l.c.p-s. form (ω, b) that are of the same rank as ω and $\omega' - \omega = d^b s$ for some $s \in \Omega^1(\mathcal{F})$. The Hamiltonian equivalence of the coisotropic deformations induces a "Hamiltonian" equivalence on this set of l.c.p-s. forms on (Y, ω, b).

A deeper analysis on the relationship between the equations (10.1) and (10.3), using some ideas in [LV], will be given in a sequel to the present paper. In particular, it was raised as a question in [OP] whether the C^∞-analog to Theorem 10.3 holds or not. We hope to study and answer to this question in the sequel.

Remark 10.8. Among coisotropic deformations of Y there are special deformations respecting the leaf $\mathcal{F}$, i.e. those deformations Γ whose associated null foliation $\mathcal{F}$ stay unchanged, or equivalently, $\mathcal{F} \subset \ker d^b \Gamma$. For instance, if Y is Lagrangian all coisotropic deformations respect $\mathcal{F} = Y$. These deformations form a linear space, therefore, they are smoothly unobstructed. Clearly, they are invariant under infinitesimally Hamiltonian actions. A particular case has been considered by Ruan [R].

11. Deformations of l.c.s. structures on X. In this section we derive formulas (11.5), (11.7) describing the Zariski tangent space of the set $\mathfrak{M}_{lcs}(X)$ of equivalent classes of l.c.s. structures on a manifold X.

Definition 11.1. We call a smooth one-parameter family $(X, \omega_t, \mathfrak{b}_t)$ of l.c.s structures for $-\varepsilon \le t \le \varepsilon$ a *bulk-deformation.*

Since nondegeneracy is an open condition, we can represent a deformation ω_t with

$$\frac{\partial \omega_t}{\partial t}\Big|_{t=0} = \kappa.$$

The l.c.s. condition can be written as

$$\begin{cases} d\omega_t + \mathfrak{b}_t \wedge \omega_t = 0, \\ d\mathfrak{b}_t = 0. \end{cases} \tag{11.1}$$

In fact, since we assume $\dim X \ge 4$, ω_t uniquely determines $\mathfrak{b}_t$. So we will focus on the deformation of ω_t. By differentiating (11.1) with respect to t at 0, we obtain

$$d\kappa + \mathfrak{b}_0 \wedge \kappa + \frac{\partial \mathfrak{b}_t}{\partial t}\Big|_{t=0} \wedge \omega_X = 0. \tag{11.2}$$

Therefore we immediately derive the following description of Zariski tangent space of the set of l.c.s. structures.

Lemma 11.2. *Let $(X, \omega_t, \mathfrak{b}_t)$ be a bulk-deformation of l.c.s. structure on X with $(\omega_0, \mathfrak{b}_0) = (\omega_X, \mathfrak{b})$. Denote*

$$\frac{\partial \omega_t}{\partial t}\Big|_{t=0} = \kappa, \quad \frac{\partial \mathfrak{b}_t}{\partial t}\Big|_{t=0} = \mathfrak{c}.$$

Then $(\kappa, \mathfrak{c})$ satisfies

$$d^{\mathfrak{b}}\kappa = -\mathfrak{c} \wedge \omega_X, \quad d\mathfrak{c} = 0. \tag{11.3}$$

Since two l.c.s. forms $e^{f_t}\omega_t$ and ω_t are equivalent for $f_t \in C^\infty(X)$, two infinitesimal deformations $(\kappa, \mathfrak{c})$ and $(\kappa', \mathfrak{c}')$ of $(X, \omega_X, \mathfrak{b})$ are equivalent if there is a function $f \in C^\infty(X)$ such that

$$\kappa = -f\omega_X + \kappa' \text{ and } \mathfrak{c} = \mathfrak{c}' + df. \tag{11.4}$$

Definition 11.3. We call a pair $(\kappa, \mathfrak{c})$ an infinitesimal deformation of $(X, \omega_X, \mathfrak{b})$ when it satisfies (11.3) or equivalently

$$d^{\mathfrak{b}}\kappa = -\mathfrak{c} \wedge \omega_X, \quad d\mathfrak{c} = 0.$$

Now we recall the following from Definition 2.3

Definition 11.4. We say $(X, \omega, \mathfrak{b})$ is diffeomorphic to $(X', \omega', \mathfrak{b}')$ if there exists a l.c.s. diffeomorphism $\phi : X \to X'$. We denote by $\mathfrak{LCS}(X)$ the set of l.c.s. structures on X and $\mathfrak{M}_{lcs}(X)$ the set of equivalence classes of l.c.s. structures on X.

The following is the infinitesimal analog to this definition, taking into account (11.4).

DEFINITION 11.5. We say two infinitesimal deformations $(\kappa', \mathfrak{c}')$, $(\kappa, \mathfrak{c})$ of $(X, \omega_X, \mathfrak{b})$ are *equivalent*, if there exist a vector field ξ of X and a function $f \in C^\infty(X)$ such that

$$\kappa' = -f \cdot \omega_X + \kappa + \mathcal{L}_\xi \omega_X, \quad \mathfrak{c}' = \mathfrak{c} + \mathcal{L}_\xi \mathfrak{b} + df.$$

We denote by $\mathrm{Def}(X, \omega_X, \mathfrak{b})$ the set of equivalence classes of infinitesimal deformations of $(X, \omega_X, \mathfrak{b})$.

By definition, $\mathrm{Def}(X, \omega_X, \mathfrak{b})$ is the Zariski (or formal) tangent space of $\mathfrak{M}_{lcs}(X)$ at $(\omega_X, \mathfrak{b})$.

Next, we provide an explicit description of the Zariski tangent space $\mathrm{Def}(X, \omega_X, \mathfrak{b})$.

DEFINITION 11.6. Define a map $S(\omega_X, \mathfrak{b}) : Vect(X) \oplus \mathbb{R} \to (\ker d^{\mathfrak{b}} \cap \Omega^2(X))$ by

$$S(\omega_X, \mathfrak{b})(\xi, c) := d^{\mathfrak{b}}(\xi \rfloor \omega_X) - c\,\omega_X.$$

We divide our description of $\mathrm{Def}(X, \omega_X, \mathfrak{b})$ into two different cases depending on the cohomological property of $[\omega_X] \in H^2_{\mathfrak{b}}(X)$.

We start with the case where the linear map $L : H^1(X, \mathbb{R}) \to H^3_{\mathfrak{b}}(X, \mathbb{R})$, $[\alpha] \mapsto [\alpha] \wedge [\omega_X]$, is injective. In this case, any solution of (11.3) is of the form

$$(\kappa = -f \cdot \omega_X + \beta,\ \mathfrak{c} = df),$$

where

$$f \in C^\infty(X) \text{ and } \beta \in \ker d^{\mathfrak{b}} \cap \Omega^2(X).$$

By Definition 11.5 $(-f \cdot \omega_X + \beta, df)$ is equivalent to $(\beta, 0)$. The Cartan formula yields

$$\begin{aligned} \mathcal{L}_\xi \mathfrak{b} &= d(\mathfrak{b}(\xi)) \\ \mathcal{L}_\xi(\omega_X) &= \xi \rfloor d\omega_X + d(\xi \rfloor \omega_X) = -\mathfrak{b}(\xi)\omega_X + \mathfrak{b} \wedge (\xi \rfloor \omega_X) + d(\xi \rfloor \omega_X). \end{aligned}$$

Hence $(\beta, 0)$ is infinitesimally equivalent to zero if and only if there exist a function $g \in C^\infty(X)$ and a vector field ξ on X such that

$$\begin{aligned} &\beta = d^{\mathfrak{b}}(\xi \rfloor \omega_X) - (g + \mathfrak{b}(\xi))\omega_X \text{ and } 0 = d(g + (\mathfrak{b}(\xi)). \\ &\quad \Leftrightarrow \beta = d^{\mathfrak{b}}(\xi \rfloor \omega_X) - c\omega_X \text{ and } g + \mathfrak{b}(\xi) = c. \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathrm{Def}(X, \omega_X, \mathfrak{b}) &= ((\ker d^b \cap \Omega^2(X))/S(\omega, \mathfrak{b})(Vect(X) \oplus \mathbb{R}) \\ &= H^2_{\mathfrak{b}}(X)/\langle \omega_X \rangle_{\otimes \mathbb{R}}. \end{aligned} \tag{11.5}$$

In particular, if $\mathfrak{b} = 0$, i.e. $(X, \omega_X, \mathfrak{b})$ is actually a symplectic manifold, then

$$\mathrm{Def}(X, \omega_X) = H^2(X, \mathbb{R})/\langle \omega_X \rangle_{\otimes \mathbb{R}}. \tag{11.6}$$

Next, we consider the case where the linear map $L : H^1(X, \mathbb{R}) \to H^3_{\mathfrak{b}}(X, \mathbb{R})$, $[\alpha] \mapsto [\alpha] \wedge [\omega_X]$, is not injective. In this case any solution of (11.3) is of form

$$(\kappa = -f \cdot \omega_X + \beta + \theta,\ \mathfrak{c} = df + \gamma),$$

where

$$f \in C^\infty(X), [\gamma] \neq 0 \in H^1(X, \mathbb{R}), \gamma \wedge \omega_X = d^{\mathfrak{b}}\theta, \text{ and } \beta \in \ker d^{\mathfrak{b}} \cap \Omega^2(X).$$

Again the argument above implies that ($\kappa = -f \cdot \omega_X + \beta + \theta$, $\mathfrak{c} = df + \gamma$) is infinitesimally equivalent to zero, if and only if there exist a function $g \in C^\infty(X)$ and a vector field ξ on X such that

$$\gamma = -d(\mathfrak{b}(\xi) + g) \text{ and } \beta + \theta = d^{\mathfrak{b}}(\xi \rfloor \omega_X) - (g + \mathfrak{b}(\xi))\omega_X.$$

It follows that $[\gamma] = 0 \in H^1(X)$. Hence in this case we have

$$\mathrm{Def}(X, \omega_X, \mathfrak{b}) = \ker L \oplus (\ker d^{\mathfrak{b}} \cap \Omega^2(X)/S(\omega_X, \mathfrak{b})(Vect(X) \oplus \mathbb{R}) =$$

$$= \ker L \oplus H^2_{\mathfrak{b}}(X)/\langle \omega_X \rangle_{\otimes \mathbb{R}}. \tag{11.7}$$

In particular, if $\mathfrak{b} = 0$, i.e. $(X, \omega_X, \mathfrak{b})$ is actually a symplectic manifold, then

$$\mathrm{Def}(X, \omega_X) = \ker L \oplus H^2(X, \mathbb{R})/\langle \omega_X \rangle_{\otimes \mathbb{R}}. \tag{11.8}$$

Remark 11.7. In [Ba1, Theorem 2] Banyaga considered deformations of l.c.s. forms with a given Lee one-form.

12. Bulk deformations of coisotropic submanifolds; Zambon's example re-visited. In this section we consider bulk deformations of l.c.s. forms on a l.c.s. manifold X under which a given compact coisotropic submanifold Y stays coisotropic. Then we study bulk coisotropic deformations of Y under such bulk deformations of l.c.s. forms (Definition 12.1, Lemma 12.3, Lemma 12.2, Theorem 12.4). Finally we re-examine the Zambon example under bulk coisotropic deformations and show that it is still obstructed (Theorem 12.6.)

Given a coisotropic submanifold $i : Y \to (X, \omega_X, \mathfrak{b})$ we say that a bulk deformation $(\omega_t, \mathfrak{b}_t)$ respects Y, if Y remains coisotropic in $(X, \omega_t, \mathfrak{b}_t)$. By the normal form theorem 4.2, if Y is compact, there exist a neighborhood U of Y in X, a family of diffeomorphisms $\phi_t : U \to U$ and a family of smooth function $f_t \in C^\infty(U)$ such that for all $t \in [-\varepsilon, \varepsilon]$ we have

$$\phi_t(Y) = Id,$$

$$\phi_t^*(\pi^* i^* \omega_t - d^{\pi^* i^* b_t}\theta_G) = e^{f_t}\omega_t.$$

Here we identify U with a neighborhood of the zero section of $E^* = E_t^*$ as in section 4.

Definition 12.1. Assume that Y is a coisotropic submanifold of $(U, \omega_U, d^{\pi^* b}\theta_G)$. A deformation $\Gamma_t : Y \to U$ is called *a bulk coisotropic deformation*, if there exists a family of l.c.p-s. form $(\bar\omega_t, b_t)$ of constant rank on Y with $\bar\omega_0 = i^*\omega_U$, $b_0 = b$ and for each $t \in [-\varepsilon, \varepsilon]$ (the graph of) Γ_t is coisotropic in $(U, \pi^*\bar\omega_t, d^{\pi^* b_t}\theta_G)$. An infinitesimal coisotropic deformation $\Gamma_1 \in \ker d^{\bar b}_{\mathcal{F}} \cap \Omega^1(\mathcal{F})$ is called *formal bulk unobstructed*, if there exist a formal bulk deformation of $(\bar\omega_0, b_0)$ and a formal bulk coisotropic deformation Γ whose first term is the given Γ_1. An infinitesimal coisotropic deformation $\Gamma_1 \in$

$\ker d^{\bar{b}}_{\mathcal{F}} \cap \Omega^1(\mathcal{F})$ is called *smoothly bulk unobstructed*, if there exists a smooth bulk coisotropic deformation Γ_t such that $(d/dt)_{t=0}\Gamma_t = \Gamma_1$.

The following Lemma is a direct consequence of Theorem 6.2.

LEMMA 12.2. *A deformation Γ_t is a bulk coisotropic deformation, if and only if there is a bulk deformation $(\bar{\omega}_t, b_t)$ of the l.c.p-s. form $(\bar{\omega}_0, b_0)$ on Y such that*

$$(\bar{\omega}_t)^{k+1} = 0, \tag{12.1}$$

$$(\bar{\omega}_t - d^{b_t}\Gamma_t)^{k+1} = 0. \tag{12.2}$$

The following Lemma is obtained straightforward.

LEMMA 12.3. *Let (ω_t, b_t) be a smooth family of l.c.p-s. structures of constant rank $2k$ on (Y, ω_0) and denote*

$$\frac{\partial \omega_t}{\partial t}\Big|_{t=0} = \kappa, \quad \frac{\partial b_t}{\partial t}\Big|_{t=0} = c.$$

Then (κ, c) satisfies

$$d^b\kappa = -c \wedge \omega_Y, \quad dc = 0 \tag{12.3}$$

$$\omega_0^k \wedge \kappa = 0 \Leftrightarrow \kappa|_{\mathcal{F}} = 0. \tag{12.4}$$

Here $\mathcal{F}$ is the null foliation of (Y, ω_0). Furthermore two equivalent deformations generates equivalent $(\kappa, \mathfrak{b})$.

The discussion in the previous section can be repeated word-for-word for bulk-deformations of an l.c.p-s. form $(\bar{\omega}, b)$ on Y, except that we need to take care of κ, β (resp. $\beta + \theta$) so that their restriction to $\mathcal{F}$ vanishes. Equivalently, they are in the differential ideal $\mathcal{I}(\mathcal{F})$ generated by $T\mathcal{F}^\circ$. We define a subset

$$\Omega^i(Y, \omega, \mathcal{F}) := \Omega^i(Y) \cap \mathcal{I}(\mathcal{F})$$

which defines a differential submodule of $\Omega^i(Y)$ with respect to d^b. Denote its cohomology by

$$H^i_b(Y, \omega, \mathcal{F}) := \frac{\ker d^b \cap \Omega^i(Y, \omega, \mathcal{F})}{d^b(\Omega^{i-1}(Y, \omega, \mathcal{F}))}.$$

Note that the wedge product with ω restricts to a map $\mathcal{I}(\mathcal{F}) \to \mathcal{I}(\mathcal{F})$. The map descends to a map $L : H^1_0(Y, \omega, \mathcal{F}) \to H^3_b(Y, \omega, \mathcal{F})$.

The following theorem is obtained using the same arguments in the previous section, so we omit its proof.

THEOREM 12.4. *The space $Def(Y, \omega, b)$ of infinitesimal equivalent bulk-deformations of l.c.p-s. form (ω, b) on Y is isomorphic to the space $H^2_b(Y, \omega, \mathcal{F})/\langle\omega\rangle_{\otimes\mathbb{R}} \oplus \ker L$.*

Now we are ready to analyze Zambon's example.

EXAMPLE 12.5. We recall Zambon's example from [Za], [OP]. Let (Y, ω) be the standard 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with coordinates (y^1, y^2, q^1, q^2) with the pre symplectic form

$$\omega_Y = \bar{\omega}_0 = dy^1 \wedge dy^2,\ b_0 = 0.$$

Note that the null foliation is provided by the 2-tori

$$\{y^1 = const,\ y^2 = const\},$$

and it also carries the transverse foliation given by

$$\{q^1 = const,\ q^2 = const\}.$$

The canonical symplectic thickening is given by

$$\begin{aligned} E^* &= T^4 \times \mathbb{R}^2 = T^2 \times T^*(T^2), \\ \omega &= dy^1 \wedge dy^2 + (dq^1 \wedge dp^1 + dq^2 \wedge dp^2), \end{aligned}$$

where p^1, p^2 are the canonical conjugate coordinates of q^1, q^2.

It follows that the transverse curvature $F \equiv 0$ and so all $\mathfrak{m}_\ell = 0$ for $\ell \geq 3$ and the Maurer-Cartan equation (10.1) becomes the quadratic equation (cf. (10.4))

$$-d_{\mathcal{F}}(\Gamma_2) = \frac{1}{2} P_{\omega_Y} \rfloor (d\Gamma_1)^2. \tag{12.5}$$

In [Za], [OP], the one-form

$$\Gamma_1 = \sin(2\pi y^1) dq^1 + \sin(2\pi y^2) dq^2,$$

was shown to be obstructed by showing that $Kr([\Gamma_1]) \neq 0$. This can be also shown by computing the RHS of (12.5)

$$\begin{aligned} \frac{1}{2} P_{\omega_Y} \rfloor (d\Gamma_1)^2 &= (\partial y_1 \wedge \partial y_2) \rfloor 4\pi^2 \cos(2\pi y^1) \cos(2\pi y^2) dy^1 dq^1 dy^2 dq^2 \\ &= -4\pi^2 \cos(2\pi y^1) \cos(2\pi y^2) dq^1 dq^2, \end{aligned}$$

which cannot be a differential $d_{\mathcal{F}}(-\Gamma_2)$, since the integration of it over a generic leaf $T^2(y^1, y^2)$ of $\mathcal{F}$ is not zero.

THEOREM 12.6. Γ_1 *is formally bulk obstructed.*

Proof. Assume the opposite, i.e. there is a formal bulk deformation $\omega_t = \sum_{i=0}^\infty t^i \bar{\omega}_i$ and bulk deformation $\Gamma_t = \sum_{i=1}^\infty t^i \Gamma_i$. The equation (12.1) implies

$$\sum_{i=0}^{l} \bar{\omega}_i \bar{\omega}_{l-i} = 0 \text{ for all } 0 \leq l \leq \infty. \tag{12.6}$$

Furthermore $d^{b_t} \omega_t = 0$ is equivalent to the following:

$$db_i = 0 \text{ for all } 0 \leq i \leq \infty \tag{12.7}$$

$$\text{and } 0 = d^{b_0} \bar{\omega}_i + \sum_{1 \leq k \leq i} b_k \wedge \bar{\omega}_{i-k} \text{ for all } 1 \leq i \leq \infty. \tag{12.8}$$

Next, from (12.6), for $l = 1$, we obtain

$$\bar{\omega}_0 \wedge \bar{\omega}_1 = 0$$

$$\Leftrightarrow \bar{\omega}_1 = dy^1 \wedge \alpha^1 + dy^2 \wedge \alpha^2, \tag{12.9}$$

where $\alpha^i \in \Omega^1(Y)$. Further, we obtain from (12.8) for $i = 1$

$$d\bar{\omega}_1 = -b_1 \wedge dy^1 \wedge dy^2. \tag{12.10}$$

From (12.10) and (12.7) it follows that $[b_1] \in \ker L : H^1(Y) \to H^3(Y), \gamma \mapsto \gamma \wedge [\omega_Y]$. Hence we obtain

$$b_1 = b_1^1 dy^1 + b_1^2 dy^2 + df \text{ where } b_1^i \in \mathbb{R} \text{ and } f \in C^\infty(Y). \tag{12.11}$$

Now from (12.6) for $l = 2$ we obtain

$$\bar{\omega}_0 \wedge \bar{\omega}_2 + \bar{\omega}_1^2 = 0. \tag{12.12}$$

Next, using (12.12) and (12.11), we derive from (12.2), looking at the coefficient of t^2,

$$\omega_0 \wedge (d\Gamma_2 + b_1 \wedge \Gamma_1) + (d\Gamma_1)^2 = 0$$
$$\Leftrightarrow d_{\mathcal{F}}\Gamma_2 + d_{\mathcal{F}}(f \cdot \Gamma_1) = 4\pi^2 \cos 2\pi y^1 \cos 2\pi y^2 dq^1 dq^2. \tag{12.13}$$

As we have computed, the RHS of (12.13) is not equal to zero in $H_b^2(Y, \omega_Y)$. So the equation (12.13) for Γ_2 and f does not have a solution. This completes the proof of Theorem 12.6. □

Remark 12.7. In [SZ, Proposition 3.5, Corollary 3.6] Schaetz and Zambon also consider extended deformation of coisotropic submanifolds in certain Poisson manifolds and revisited the Zamnon example, using different methods. Note that the class of extended deformations considered by them for the Zambon example is smaller than the class of extended deformations considered in our paper. In particular, their Corollary 3.6 is a consequence of our Theorem 12.6.

13. Appendix.

13.1. Leaf space connection and curvature. In this subsection, we recall some basic definitions and properties of the leaf space connection borrowing the exposition of [OP, section 3]. We refer readers thereto for the proofs of all the results stated without proof in the present subsection.

Let $\mathcal{F}$ be an arbitrary foliation on a smooth manifold Y. Following the standard notations in the foliation theory, we define the normal bundle $N\mathcal{F}$ and conormal bundle $N^*\mathcal{F}$ of the foliation $\mathcal{F}$ by

$$N_y\mathcal{F} := T_yY/E_y, \quad N_y^*\mathcal{F} := (T_y/E_y)^* \cong E_y^\circ \subset T_y^*Y.$$

In this vein, we will denote $E = T\mathcal{F}$ and $E^* = T^*\mathcal{F}$ respectively, whenever it makes our discussion more transparent. We have the natural exact sequences

$$0 \to T\mathcal{F} \to TY \to N\mathcal{F} \to 0, \tag{13.1}$$
$$0 \leftarrow T^*\mathcal{F} \leftarrow T^*Y \leftarrow N^*\mathcal{F} \leftarrow 0. \tag{13.2}$$

The choice of splitting $TY = G \oplus T\mathcal{F}$ may be regarded as a "connection" of the "E-bundle" $TY \to Y/\sim$ where $Y/\sim$ is the space of leaves of the foliation on Y. Note that $Y/\sim$ is *not* Hausdorff in general. We will indeed call a choice of splitting *a leaf space connection of* $\mathcal{F}$ in general.

We can also describe the splitting in a more invariant way as follows: Consider bundle maps $\Pi : TY \to TY$ that satisfy

$$\Pi_x^2 = \Pi_x, \ \operatorname{im}\Pi_x = T_x\mathcal{F}$$

at every point of Y, and denote the set of such projections by

$$\mathcal{A}_E(TY) \subset \Gamma(Hom(TY, TY)) = \Omega_1^1(Y).$$

There is a one-one correspondence between the choice of splittings (3.1) and the set $\mathcal{A}_E(TY)$ provided by the correspondence

$$\Pi \leftrightarrow G := \ker \Pi.$$

If necessary, we will denote by Π_G the element with $\ker \Pi = G$ and by G_Π the complement to E determined by Π. We will use either of the two descriptions, whichever is more convenient depending on the circumstances.

Next we recall the notion of curvature of the Π-connection.

Definition 13.1. Let $\Pi \in \mathcal{A}_E(TY)$ and denote by $\Pi : TY = G_\Pi \oplus T\mathcal{F}$ the corresponding splitting. The *transverse* Π-*curvature* of the foliation $\mathcal{F}$ is a $T\mathcal{F}$-valued two form defined on $N\mathcal{F}$ as follows: Let $\pi : TY \to N\mathcal{F}$ be the canonical projection and

$$\pi_\Pi : G_\Pi \to N\mathcal{F}$$

be the induced isomorphism. Then we define

$$F_\Pi : \Gamma(N\mathcal{F}) \otimes \Gamma(N\mathcal{F}) \to \Gamma(T\mathcal{F})$$

by

$$F_\Pi(\eta_1, \eta_2) := \Pi([X, Y]) \tag{13.3}$$

where $X = \pi_\Pi^{-1}(\eta_1)$ and $Y = \pi_\Pi^{-1}(\eta_2)$ and $[X, Y]$ is the Lie bracket on Y.

The following proposition justifies the name *transverse* Π-*curvature* which plays a crucial role in our description of the strong homotopy Lie algebroid associated to the pre-symplectic manifold (Y, ω_Y) (and so of coisotropic submanifolds) and its Maurer-Cartan equation. We refer to [OP] for its proof.

Proposition 13.2. *Let F_Π be as above. For any smooth functions f, g on Y and sections η_1, η_2 of $N\mathcal{F}$, we have the identity*

$$F_\Pi(f\eta_1, g\eta_2) = fgF_\Pi(\eta_1, \eta_2)$$

i.e., the map F_Π defines a well-defined section as an element in $\Gamma(\Lambda^2(N^*\mathcal{F}) \otimes T\mathcal{F})$.

In the foliation coordinates $(y^1, \cdots, y^\ell, q^1, \cdots, q^{m-\ell})$, F_Π has the expression

$$F_\Pi = F_{ij}^\beta \frac{\partial}{\partial q^\beta} \otimes dy^i \wedge dy^j \in \Gamma(\Lambda^2(N^*\mathcal{F}) \otimes T\mathcal{F}), \tag{13.4}$$

where

$$F^\beta_{ij} = \frac{\partial R^\beta_j}{\partial y^i} - \frac{\partial R^\beta_i}{\partial y^j} + R^\gamma_i \frac{\partial R^\beta_j}{\partial q^\gamma} - R^\gamma_j \frac{\partial R^\beta_i}{\partial q^\gamma}. \tag{13.5}$$

We next recall the relationship between F_{Π_0} and F_Π. Note that with respect to the given splitting

$$\Pi_0 : TY = G_0 \oplus T\mathcal{F} \cong N\mathcal{F} \oplus T\mathcal{F}$$

any other projection $\Pi : TY \to TY$ can be written as the following block matrix

$$\Pi = \begin{pmatrix} 0 & 0 \\ B & Id \end{pmatrix}$$

where $B = B_{\Pi_0 \Pi} \circ \pi_{G_0} : G_0 \to T\mathcal{F}$ is the bundle map which is uniquely determined by Π_0 and Π and vice versa. The following lemma shows their relationship in coordinates.

Lemma 13.3. *Let F_Π and F_{Π_0} be the transverse Π-curvatures with respect to Π and Π_0 respectively, and let $B = B_{\Pi_0\Pi}$ be the bundle map mentioned above. In terms of the foliation coordinates, we have*

$$\begin{aligned} F^\beta_{ij} = F^\beta_{0,ij} &+ \Big(\frac{\partial B^\beta_j}{\partial y^i} - \frac{\partial B^\beta_i}{\partial y^j} + R^\alpha_i \frac{\partial B^\beta_j}{\partial q^\alpha} - R^\alpha_j \frac{\partial B^\beta_i}{\partial q^\alpha} + B^\alpha_i \frac{\partial R^\beta_j}{\partial q^\alpha} - B^\alpha_j \frac{\partial R^\beta_i}{\partial q^\alpha} \Big) \\ &+ \Big(B^\alpha_i \frac{\partial B^\beta_j}{\partial q^\alpha} - B^\alpha_j \frac{\partial B^\beta_i}{\partial q^\alpha} \Big). \end{aligned} \tag{13.6}$$

Now we provide an invariant description of the above formula (13.6). Consider the sheaf $\Lambda^\bullet(N^*\mathcal{F}) \otimes T\mathcal{F}$ and denote by

$$\Omega^\bullet(N^*\mathcal{F}; T\mathcal{F}) := \Gamma(\Lambda^\bullet(N^*\mathcal{F}) \otimes T\mathcal{F})$$

the group of (local) sections thereof. For an invariant interpretation of the above basis of G_x and the transformation law (13.6), we need to use the notion of *basic vector fields* (or *projectable vector fields*) which is standard in the foliation theory (see e.g., [MM]) : Consider the Lie subalgebra

$$L(Y, \mathcal{F}) = \{\xi \in \Gamma(TY) \mid ad_\xi(\Gamma(T\mathcal{F})) \subset \Gamma(T\mathcal{F})\}$$

and its quotient Lie algebra

$$\ell(Y, \mathcal{F}) = L(Y, \mathcal{F})/\Gamma(T\mathcal{F}).$$

An element from $\ell(Y, \mathcal{F})$ is called a *transverse vector field* of $\mathcal{F}$. In general, there may not be a global basic lifting Y of a given transverse vector field. But the following lemma shows that this is always possible locally.

Lemma 13.4. *Let $x_0 \in Y$ and $v \in N_{x_0}\mathcal{F}$. Then there exists a local basic vector field ξ in a neighborhood of x_0 such that it is tangent to G*

$$\pi(\xi(x_0)) = v$$

where $\pi : TY \to N\mathcal{F}$ is the canonical projection.

DEFINITION 13.5. Let $\mathcal{F}$ be a foliation on Y. Let $\Pi \in \mathcal{A}_E(TY)$ and $\Pi : TY = G_\Pi \oplus T\mathcal{F}$ be the Π-splitting. We call a basic vector field ξ tangent to G_Π a Π-*basic vector field* or a G-*basic vector field.*

In this point of view, the vector field

$$Y_i := \frac{\partial}{\partial y^i} + \sum_{\alpha=1}^{n-k} R_i^\alpha \frac{\partial}{\partial q^\alpha}$$

is the unique G-basic vector field that satisfies

$$Y_j \equiv \frac{\partial}{\partial y^i} \mod T\mathcal{F},$$

i.e., defines the same transverse vector field as $\frac{\partial}{\partial y^i}$.

DEFINITION 13.6. Let X be any (local) basic vector field of $\mathcal{F}$ tangent to G_Π. We define the Π-*Lie derivative* of B with respect to X by the formula

$$L_X^\Pi B = \sum_{i_1 < \cdots < i_\ell} L_X(B_{i_1 i_2 \cdots i_\ell}) dy^{i_1} \wedge \cdots \wedge dy^{i_\ell} \tag{13.7}$$

where $B_{i_1 i_2 \cdots i_\ell}$ is a local section of $T\mathcal{F}$ given by the local representation of B

$$B = \sum_{i_1 < \cdots < i_\ell} B_{i_1 \cdots i_\ell} dy^{i_1} \wedge \cdots \wedge dy^{i_\ell}$$

in any given foliation coordinates. Here $B_{i_1 \cdots i_\ell}$ is the (locally defined) leafwise tangent vector field given by

$$B_{i_1 \cdots i_\ell} = B^\beta_{i_1 \cdots i_\ell} \frac{\partial}{\partial q^\beta}.$$

From now on without mentioning further, we will always assume that B is locally defined, unless otherwise stated.

DEFINITION 13.7. For any element $B \in \Gamma(\Lambda^\ell(N^*\mathcal{F}); TF)$, we define

$$d^\Pi B \in \Gamma(\Lambda^{\ell+1}(N^*\mathcal{F}); TF)$$

by the formula

$$d^\Pi B = \sum_{j=1}^{2k} dy^j \wedge L^\Pi_{Y_j} B \tag{13.8}$$

where we call the operator d^Π the Π-*differential.*

For given splitting Π and a vector field ξ, we denote by ξ^Π the projection of ξ to $G = G_\Pi$, i.e.,

$$\xi^\Pi = \xi - \Pi(\xi).$$

Then the definition of d^Π can be also given by the same kind of formula as that of the usual exterior derivative d: For given $B \in \Omega^k(N^*\mathcal{F}; T\mathcal{F})$ and local sections $\eta_1, \cdots, \eta_{k+1} \in N_x\mathcal{F}$, we define

$$\begin{aligned} & d^\Pi B(v_1, \cdots, v_k, v_{k+1}) \\ = & \sum_i (-1)^{i-1} X_i(B(\eta_1, \cdots, \widehat{\eta_i}, \cdots, \eta_{k+1})) \\ & + \sum_{i<j} (-1)^{i+j-1} B(\pi([X_i, X_j]), \eta_1, \cdots, \widehat{\eta_i}, \cdots, \widehat{\eta_j}, \cdots, \eta_{k+1}). \end{aligned} \tag{13.9}$$

Here X_i is a Π-basic vector field with $\pi(X_i(x)) = \eta_i(x)$ for each given point $x \in Y$.

It is straightforward to check that this definition coincides with (13.8).

Next we introduce the analog of the "bracket"

$$[\cdot,\cdot]_\Pi : \Omega^{\ell_1}(N^*\mathcal{F}; T\mathcal{F}) \otimes \Omega^{\ell_2}(N^*\mathcal{F}; T\mathcal{F}) \to \Omega^{\ell_1+\ell_2}(N^*\mathcal{F}; T\mathcal{F}).$$

Definition 13.8. Let $B \in \Omega^{\ell_1}(N^*\mathcal{F}; T\mathcal{F})$, $C \in \Omega^{\ell_2}(N^*\mathcal{F}; T\mathcal{F})$. We define their bracket

$$[B, C]_\Pi \in \Omega^{\ell_1+\ell_2}(N\mathcal{F}; T\mathcal{F})$$

by the formula

$$\begin{aligned} & [B, C]_\Pi(v_1, \cdots, v_{\ell_1}, v_{\ell_1+1}, \cdots, v_{\ell_1+\ell_2}) \\ = & \sum_{\sigma \in S_n} \frac{\mathrm{sign}(\sigma)}{(\ell_1+\ell_2)!} [B(X_{\sigma(1)}, \cdots, X_{\sigma(\ell_1)}), C(X_{\sigma(\ell_1+1)}, \cdots, X_{\sigma(\ell_1+\ell_2)})] \qquad (13.10) \\ = & \sum_{\tau \in Shuff(n)} \frac{\mathrm{sign}(\tau)}{\ell_1! \ell_2!} [B(X_{\tau(1)}, \cdots, X_{\tau(\ell_1)}), \\ & \qquad C(X_{\tau(\ell_1+1)}, \cdots, X_{\tau(\ell_1+\ell_2)})] \qquad (13.11) \end{aligned}$$

for each $x \in Y$ and $v_i \in N_x\mathcal{F}$, and X_i's are (local) Π-basic vector fields such that $\pi(X_i(x)) = v_i$ as before. Here S_n is the symmetric group with size n and $Shuff(n) \subset S_n$ is the subgroup of all "shuffles". $[\cdot,\cdot]$ is the usual Lie bracket of leafwise vector fields.

For the case $\ell_1 = \ell_2 = 1$, we derive the coordinate formula

$$[B, C]_\Pi = \Big(B_i^\alpha \frac{\partial C_j^\beta}{\partial q^\alpha} - C_j^\alpha \frac{\partial B_i^\beta}{\partial q^\alpha} \Big) \frac{\partial}{\partial q^\beta} \otimes dy^i \wedge dy^j. \tag{13.12}$$

With these definitions, we have the following "Bianchi identity" in our context.

Proposition 13.9. *Let $\Pi : TY = G \oplus T\mathcal{F}$ and d^Π be the associated Π-differential. Then we have*

$$\begin{aligned} d^\Pi F_\Pi &= 0 \\ (d^\Pi)^2 B &= [F_\Pi, B]_\Pi. \end{aligned}$$

Combining the above discussion, the transformation law (13.6) in coordinates is translated into the following invariant form.

PROPOSITION 13.10. *Let* Π, Π_0 *be two splittings as in Lemma 13.3 and* $B_{\Pi_0\Pi} \in \Gamma(N^*\mathcal{F} \otimes T\mathcal{F})$ *be the associated section. Then we have*

$$F_\Pi = F_{\Pi_0} + d^{\Pi_0} B_{\Pi_0\Pi} + [B_{\Pi_0\Pi}, B_{\Pi_0\Pi}]_{\Pi_0}. \tag{13.13}$$

13.2. Lie algebroid and its $\bar{b}$-deformed E-cohomology. We start with recalling the definition of Lie algebroid and its associated E-de Rham complex and E-cohomology. The leafwise de Rham complex $\Omega^\bullet(\mathcal{F})$ is a special case of the *E-de Rham complex* associated to the general Lie algebroid E.

We quote the following definitions from [NT].

DEFINITION 13.11. Let M be a smooth manifold. A *Lie algebroid* on M is a triple $(E, \rho, [\,,\,])$, where E is a vector bundle on M, $[\,,\,]$ is a Lie algebra structure on the sheaf of sections of E, and ρ is a bundle map, called the *anchor map*,

$$\rho : E \to TM$$

such that the induced map

$$\Gamma(\rho) : \Gamma(M; E) \to \Gamma(TM)$$

is a Lie algebra homomorphism and, for any sections σ and τ of E and a smooth function f on M, the identity

$$[\sigma, f\tau] = \rho(\sigma)[f] \cdot \tau + f \cdot [\sigma, \tau].$$

DEFINITION 13.12. Let $(E, \rho, [\,,\,])$ be a Lie algebroid on M. The *E-de Rham complex* $({}^E\Omega^\bullet(M), {}^E d)$ is defined by

$$\begin{aligned} {}^E\Omega(\Lambda^\bullet(E^*)) &= \Gamma(\Lambda^\bullet(E^*)) \\ {}^E d\omega(\sigma_1, \cdots, \sigma_{k+1}) &= \sum_i (-1)^i \rho(\sigma_i)\omega(\sigma_1, \cdots, \widehat{\sigma_i}, \cdots, \sigma_{k+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \omega([\sigma_i, \sigma_j], \sigma_1, \cdots, \widehat{\sigma}_i, \cdots, \widehat{\sigma}_j, \cdots, \sigma_{k+1}). \end{aligned}$$

The cohomology of this complex will be denoted by ${}^E H^*(M)$ and called the E-de Rham cohomology of M.

Now assume that $\bar{b} \in {}^E\Omega^1(E^*)$ is a cocycle: ${}^E d\bar{b} = 0$. Then ${}^E d^{\bar{b}} := {}^E d + \bar{b}\wedge$ satisfies: $({}^E d^{\bar{b}})^2 = 0$. The cohomology of $({}^E\Omega^\bullet(M), {}^E d^{\bar{b}})$ will be denoted by ${}^E H^*_{\bar{b}}(M)$ and called *the $\bar{b}$-deformed E-de Rham cohomology.*

In [OP] the authors have noticed that for a coisotropic submanifold Y in a symplectic manifold X the triple

$$(E = TY^\omega, \rho = i, [\,,\,])$$

defines the structure of Lie algebroid and the E-differential is the exterior derivative $d_\mathcal{F}$ along the null foliation $\mathcal{F}$. Now assume that (Y, ω, b) is a coisotropic submanifold in (X, ω_X, α). Then the restriction $\bar{b}$ of b to $\mathcal{F}$ is a closed 1-form in the complex $(\Omega(\Lambda^\bullet E), d_{\mathcal{F}})$ and ${}^E d^{\bar{b}}$ coincides with $d^{\bar{b}}_{\mathcal{F}}$, which we also denote by $d^b_{\mathcal{F}}$.

The $\bar{b}$-deformed E-de Rham differential is related to the infinitesimal deformation space of coisotropic submanifolds in a l.c.s. manifold. For this, we introduce the space

$$\mathcal{C}oiso_k = \mathcal{C}oiso_k(X, \omega_X)$$

the set of coisotropic submanifolds with nullity $n-k$ for $0 \leq k \leq n$ and characterize its infinitesimal deformation space at $Y \subset E^*$, the zero section of E^*. By the coisotropic neighborhood theorem, the infinitesimal deformation space, denoted as $T_Y\mathcal{C}oiso_k(X, \omega_X) = T_Y\mathcal{C}oiso_k(U, \omega_U)$ with some abuse of notion, depends only on (Y, ω) where $\omega = i^*\omega_X$, but not on (X, ω_X). An element in $T_Y\mathcal{C}oiso_k(U, \omega_U)$ is a section of the bundle $E^* = T^*\mathcal{F} \to Y$.

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