ASIAN J. MATH.
Vol. 20, No. 3, pp. 449–474, July 2016

003

ACTION OF $\mathbb{R}$-FUCHSIAN GROUPS ON $\mathbb{CP}^2$*

ANGEL CANO†, JOHN R. PARKER‡, AND JOSÉ SEADE§

Abstract. We look at lattices in $\mathrm{Iso}_+(\mathbf{H}^2_{\mathbb{R}})$, the group of orientation preserving isometries of the real hyperbolic plane. We study their geometry and dynamics when they act on $\mathbb{CP}^2$ via the natural embedding of $\mathrm{SO}_+(2,1) \hookrightarrow \mathrm{SU}(2,1) \subset \mathrm{SL}(3,\mathbb{C})$. We use the Hermitian cross product in $\mathbb{C}^{2,1}$ introduced by Bill Goldman, to determine the topology of the Kulkarni limit set Λ_{Kul} of these lattices, and show that in all cases its complement Ω_{Kul} has three connected components, each being a disc bundle over $\mathbf{H}^2_{\mathbb{R}}$. We get that Ω_{Kul} coincides with the equicontinuity region for the action on $\mathbb{CP}^2$. Also, it is the largest set in $\mathbb{CP}^2$ where the action is properly discontinuous and it is a complete Kobayashi hyperbolic space. As a byproduct we get that these lattices provide the first known examples of discrete subgroups of $\mathrm{SL}(3,\mathbb{C})$ whose Kulkarni region of discontinuity in $\mathbb{CP}^2$ has exactly three connected components, a fact that does not appear in complex dimension 1 (where it is known that the region of discontinuity of a Kleinian group acting on $\mathbb{CP}^1$ has 0, 1, 2 or infinitely many connected components).

Key words. Group representation, Fuchsian group, limit set, complex projective space.

AMS subject classifications. 22E40, 37B05.

Introduction. The motivation for this work comes from the theory of lattices in $\mathrm{SO}(n,1)$, the group of linear automorphisms of $\mathbb{R}^{n+1}$ that preserve the quadratic form $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. The problem we study can be expressed as follows. Consider the natural inclusion $\rho : \mathrm{SO}(n,1) \to \mathrm{SU}(m,1)$ given by block diagonal matrices $\rho : A \mapsto (I_{m-n}, A)$, in the special linear group of automorphisms of the Hermitian space $\mathbb{C}^{m,1}$, which is $\mathbb{C}^{m+1}$ equipped with the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\overline{w}_1 + \cdots + z_m\overline{w}_m - z_{m+1}\overline{w}_{m+1}.$$

Let Ξ be the complex line spanned by the null vector $\xi = (0, \cdots, 0, 1, 1)^t$, let $\mathcal{L}$ be its Hermitian orthogonal complement, a hyperplane. Let $\widetilde{\mathbf{\Lambda}}$ be the orbit of $\mathcal{L}$ under this representation, that is the family of all hyperplanes $\rho(A)(\mathcal{L})$ for $A \in \mathrm{SO}(n,1)$. The problem is to study the algebraic, geometric and dynamical properties of this set, and of its image $\mathbf{\Lambda}$ under the projectivisation map $\mathbb{C}^{m+1} \setminus \{\mathbf{0}\} \xrightarrow{\mathbb{P}} \mathbb{CP}^m$.

Our interest in this question arose from the fact that if Γ is a lattice in $\mathrm{SO}(n,1)$, and if we consider the action of Γ on the projective space $\mathbb{CP}^m$ determined by the representation ρ, then we know from [11, 4] that Γ acts properly discontinuously on the complement $\mathbf{\Omega} := \mathbb{CP}^m \setminus \mathbf{\Lambda}$, which is the region of equicontinuity for the action of Γ. Furthermore, by [1, 9] (and Theorem 2.7.(iii) below), $\mathbf{\Omega}$ is a complete Kobayashi hyperbolic space where $\mathrm{SO}(n,1)$ acts by holomorphic isometries with respect to the Kobayashi metric. Moreover, if we restrict the discussion to the case $n = 2 = m$, as we do in this paper, then by [11] we know further that the set $\mathbf{\Lambda}$ is the Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ of every lattice Γ in $\mathrm{SO}_+(2,1) \subset \mathrm{SU}(2,1)$. This was recently proved in all dimensions [5].

*Received November 19, 2013; accepted for publication March 11, 2015.
†Instituto de Mathemáticas, UNAM, Cuernavaca, Morelos, México (angelcano@im.unam.mx).
‡Department of Mathematical Sciences, Durham University, Durham DH1 3LE, England (j.r.parker@durham.ac.uk).
§Instituto de Mathemáticas, UNAM, Cuernavaca, Morelos, México (jseade@im.unam.mx). Partially supported by grants from CONACYT, PAPIIT-UNAM, Mexico, and the Alan Richards Fellowship at Grey College, U. K.

Our approach relies on Bill Goldman's work [7] on linear algebra in the Hermitian space $\mathbb{C}^{2,1}$, and more specifically, on the Hermitian cross-product $\boxtimes$ in this space. This product $\boxtimes$ is an alternating "bilinear" map (in fact conjugate bilinear) that associates to each pair of vectors $\mathbf{z}$, $\mathbf{w}$ another vector $\mathbf{z}\boxtimes\mathbf{w}$, which is orthogonal to both $\mathbf{z}$ and $\mathbf{w}$ whenever these are linearly independent. See Section 2.1 for more details of the cross-product and its properties. We also consider the complex conjugation map $\mathbf{z}\longmapsto\overline{\mathbf{z}}$ in $\mathbb{C}^{2,1}$. We combine the complex conjugation map with the Hermitian cross-product to define a decomposition of $\mathbb{C}^{2,1}\setminus\{\mathbf{0}\}$ into three sets U_+, U_0, U_- which is closely related to, but different from, the classical decomposition of this Hermitian space into positive, null and negative vectors, respectively V_+, V_0, V_-. The sets U_+, U_0, U_- correspond to the points where the function defined by $f(\mathbf{z})=\langle i\mathbf{z}\boxtimes\overline{\mathbf{z}}, i\mathbf{z}\boxtimes\overline{\mathbf{z}}\rangle$ is positive, zero or negative, respectively. By definition this corresponds to the cases when the vector $i\mathbf{z}\boxtimes\overline{\mathbf{z}}$ is in V_+, $(V_0\cup\{\mathbf{0}\})$ and in V_-, respectively.

Let $\mathbb{R}^{2,1}\subset\mathbb{C}^{2,1}$ be the set of real points. It is clear that such points are fixed by the complex conjugation map. We let $\mathcal{P}_{\Re}$ denote the projectivisation of $\mathbb{R}^{2,1}\setminus\{\mathbf{0}\}$, which is a copy of $\mathbb{RP}^2$ embedded in $\mathbb{CP}^2$. We show (Lemma 2.4) that if $\mathbf{z}$ is a non-zero vector such that $f(\mathbf{z})=0$, then either the projectivisation $z=\mathbb{P}(\mathbf{z})$ is a point in the set $\mathbf{\Lambda}$ or else the projectivisation z is in the plane $\mathcal{P}_{\Re}$. The latter happens if and only if $i\mathbf{z}\boxtimes\overline{\mathbf{z}}=\mathbf{0}$. This is used to show that all vectors in $V_-\cup V_0$ whose projectivisation $\mathbb{P}$ is not contained in the Lagrangian plane $\mathcal{P}_{\Re}$ are contained in the set U_+, and the set $\mathbf{\Lambda}$ is $\mathbb{P}(U_0)\setminus\mathbb{P}(V_-)$. We then arrive to the following theorem:

Theorem 1. *The set $\mathbf{\Lambda}$ is a 3-dimensional semi-algebraic set that contains the Möbius strip $\mathcal{M}:=\mathcal{P}_{\Re}\setminus\mathbf{H}^2_{\mathbb{R}}$ as its singular set; every point in the interior of $\mathcal{M}$ is the meeting point of exactly two of the projective lines that form the set $\mathbf{\Lambda}$. Moreover, $\mathbf{\Lambda}\setminus\mathcal{M}$ is a fibre bundle over $\partial\mathbf{H}^2_{\mathbb{R}}$ with fibre at each $\xi\in\partial\mathbf{H}^2_{\mathbb{R}}$ the corresponding sphere $\mathcal{L}_\xi$ –tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at ξ– minus the circle $C_\xi:=\mathcal{L}_\xi\cap\mathcal{M}$. Thence $\mathbf{\Lambda}\setminus\mathcal{M}$ is diffeomorphic to a disjoint union of two solid tori $\mathbb{S}^1\times\mathbb{R}^2$.*

This is used to show:

Theorem 2. *The complement $\mathbf{\Omega}:=\mathbb{CP}^2\setminus\mathbf{\Lambda}$ has three connected components, $\Omega_+,\Omega_-^1,\Omega_-^2$, each being $\mathrm{SO}_+(2,1)$-invariant and each being diffeomorphic to an open 4-ball. In particular, the ball $\mathbf{H}^2_{\mathbb{C}}$ is contained in one of these components, namely Ω_+.*

Precise descriptions of Ω_+, Ω_-^1 and Ω_-^2 are given in (9), (10) and (11). Note that any matrix $A\in\mathrm{SO}(2,1)\setminus\mathrm{SO}_+(2,1)$ interchanges the components Ω_-^1 and Ω_-^2 and preserves Ω_+.

Our next theorem is:

Theorem 3. *Let Ω_+, Ω_-^1, Ω_-^2 be as in Theorem 2. Then one has a natural projection map $\Pi:\mathbf{\Omega}\to\mathbf{H}^2_{\mathbb{R}}$ which turns each of these three sets into an $\mathrm{SO}_+(2,1)$-equivariant fibre bundle over $\mathbf{H}^2_{\mathbb{R}}$ with fibre a 2-disc, and one has:*

1. *The fibre over $o:=[0:0:1]$ in Ω_+ is the Lagrangian 2-plane*

$$L_o=\Big\{[iy_1:iy_2:x_3]\ :\ y_1,y_2,x_3\in\mathbb{R},\ x_3\neq 0\Big\}.$$

2. *The fibres over o in Ω_-^1 and Ω_-^2 are the two open hemispheres D_o^1 and D_o^2 determined by the equator $\Im(z_1\overline{z}_2)=0$ in the line $S_o:=\big\{[z_1:z_2:0]:(z_1,z_2)\in\mathbb{C}^2\setminus\{(0,0)\}\big\}$, which is the projective dual of $[0:0:1]$. That is, the*

fibres are

$$D_o^1 = \Big\{[z_1 : z_2 : 0] \ : \ z_1,\, z_2 \in \mathbb{C},\ \Im(z_1\overline{z}_2) > 0\Big\},$$
$$D_o^2 = \Big\{[z_1 : z_2 : 0] \ : \ z_1, z_2 \in \mathbb{C},\ \Im(z_1\overline{z}_2) < 0\Big\}.$$

3. *These three fibres L_o, D_o^1, D_o^2 have as common boundary the circle C_o:*

$$\begin{aligned} C_o = \partial L_o^+ &= \Big\{[iy_1 : iy_2 : 0] \ : \ (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}\Big\} \\ &= \partial D_o^1 = \partial D_o^2 = \Big\{[z_1 : z_2 : 0] \ : \ (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\},\ \Im(z_1\overline{z}_2) = 0\Big\}. \end{aligned}$$

To determine the fibres of these bundles over a general point in $\mathbf{H}^2_{\mathbb{R}}$ we use the fact that $\mathrm{SO}_+(2,1)$ acts transitively on $\mathbf{H}^2_{\mathbb{R}}$ and the bundles in question are equivariant. So we can just translate the fibres over the special point $[0 : 0 : 1]$ to the fibres over any other point using the group action. We remark that $\mathbf{H}^2_{\mathbb{R}}$ is being regarded as the projectivisation of the set of negative vectors in the totally real 3-space $\mathbb{R}^3 \in \mathbb{C}^3$. Thence (see Section 3.3) a general point $x \in \mathbf{H}^2_{\mathbb{R}}$ can be described as $x = [\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1]$ for some $t \geq 0$ and $\theta \in [0, 2\pi)$. We get:

THEOREM 4. *Let $\Pi : \mathbf{\Omega} \to \mathbf{H}^2_{\mathbb{R}}$, L_o, D_o^1, D_o^2 and C_o be as in Theorem 3. Let x be any point of $\mathbf{H}^2_{\mathbb{R}}$ and let $A_x \in \mathrm{SO}_+(2,1)$ be any map sending o to x. Then*

1. *The fibre L_x over x in Ω_+ is the Lagrangian 2-plane $L_x = A_x(L_o)$.*
2. *The fibres over x in Ω_-^1 and Ω_-^2 are the two open hemispheres $D_x^1 = A_x(D_o^1)$ and $D_x^2 = A_x(D_o^2)$ in the sphere $S_x = A_x(S_o)$.*
3. *These three fibres L_x, D_x^1, D_x^2 have as common boundary the circle $C_x = A_x(C_o)$.*

We give explicit expressions for L_x, D_x^1, D_x^2 and C_x in Propositions 3.10 and 3.11. As a corollary to Theorem 4, we have that if P is an arbitrary fundamental domain for the action of a cofinite $\mathbb{R}$-Fuchsian group on $\mathbf{H}^2_{\mathbb{R}}$, then the inverse image of P by the projection $\Pi : \Omega \to \mathbf{H}^2_{\mathbb{R}}$ is a fundamental domain for the action of Γ on Ω. This is in the same vein as the construction of fundamental domains constructed by Parker and Platis in [13].

Finally, we discuss how the fibres behave as the base point in $\mathbf{H}^2_{\mathbb{R}}$ tends to the boundary $\partial\mathbf{H}^2_{\mathbb{R}}$.

THEOREM 5. *For $x \in \mathbf{H}^2_{\mathbb{R}}$ let L_x, D_x^1, D_x^2 and C_x be as in Theorem 4. For $\xi \in \partial\mathbf{H}^2_{\mathbb{R}}$ let $\mathcal{L}_\xi$ and C_ξ be as in Theorem 1. Then as the point $x \in \mathbf{H}^2_{\mathbb{R}}$ tends to $\xi \in \partial\mathbf{H}^2_{\mathbb{R}}$ we have:*

1. *The circle C_x tends to the circle C_ξ.*
2. *$L_x \cup C_x$, the closure of the fibre in Ω_+ over x, tends pointwise to $\mathcal{L}_\xi$.*
3. *$S_x = D_x^1 \cup D_x^2 \cup C_x$, the closure of the fibre in Ω_- over x, tends to $\mathcal{L}_\xi$.*

Perhaps the most surprising feature of this result is part (2), namely that $L_x \cup C_x$, which is a copy of $\mathbb{RP}^2$, tends to $\mathcal{L}_\xi$, which is a copy of $\mathbb{CP}^1$. The way this happens is the following. We can view $L_x \cup C_x$ as a copy of $\mathbb{R}^2$ together with a circle of directions at infinity. As x tends to ξ this circle of directions collapses to a single point. The limit is then a copy of $\mathbb{R}^2$ with a single point at infinity, which is a sphere.

Summarising, we have that duality in $\mathbb{RP}^2$ associates a real projective line (a circle) C_x in the interior of $\mathcal{M}$ to each point $x \in \mathbf{H}^2_{\mathbb{R}}$. Also, duality in $\mathbb{CP}^2$ associates to each such point x a complex projective line (a sphere) S_x in $V_+ \cup V_0$ which meets the totally real plane $\mathcal{P}_\Re$ in the circle C_x. The hemispheres $D_x^1 \cup D_x^2 = S_x \setminus C_x$ lie in $\mathbb{P}U_-$. The union of all these pairs of hemispheres $D_x^{1,2}$ fills the whole set $\mathbb{P}U_-$, which has two components and fibres over $\mathbf{H}^2_\mathbb{R}$ in the natural way. This gives an identification between $\mathbb{P}U_-$ and two components, Ω^1_- and Ω^2_- of $\mathbf{\Omega} := \mathbb{CP}^2 \setminus \mathbf{\Lambda}$. On the other hand, each point $x \in \mathbf{H}^2_\mathbb{R}$ determines a unique totally real Lagrangian plane orthogonal to $\mathbf{H}^2_\mathbb{R}$ in $\mathbf{H}^2_\mathbb{C}$ at x; such a plane extends naturally to a plane in $\mathbb{CP}^2$ that has the circle C_x as boundary. The union of all these 2-planes is the set $\mathbb{P}U_+ \cup \mathbf{H}^2_\mathbb{R}$. This is a third component Ω_+ of $\mathbf{\Omega}$. Thence for each $x \in \mathbf{H}^2_\mathbb{R}$ the three fibres L_x, D_x^1, D_x^2 are 2-discs, glued together along their boundary, which is the circle C_x. These 2-discs form a kind of "theta surface", *i.e.*, a Θ rotated around its vertical axis. Yet, the horizontal bar actually corresponds to a 2-disc, whose boundary is wrapping twice around the circle of singular points, together with which it forms a real projective plane.

An immediate consequence of Theorem 3 is:

COROLLARY 6. *The Kulkarni region of discontinuity $\Omega_{\rm Kul}$ of all $\mathbb{R}$-Fuchsian lattices has three connected components, each diffeomorphic to a 4-ball.*

This is interesting because if we look at Kleinian subgroups of $\mathrm{PSL}(2, \mathbb{C})$ acting on the projective line, we know that the region of discontinuity either has infinitely many connected components, or else it has at most two connected components. When we look at discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$, then the number of connected components in the Kulkarni region of discontinuity $\Omega_{\rm Kul}$ can be:

- Zero: For instance the suspension of every discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ whose limit set is the whole $\mathbb{CP}^1$. (We refer to [4] for the suspension construction.)
- One: For instance a lattice in $\mathrm{PU}(2, 1)$ where $\Omega_{\rm Kul}$ is exactly a copy of $\mathbf{H}^2_\mathbb{C}$, that is the unit complex ball (see [4, Corollary 7.2.11 (b)]).
- Two: For instance all the $\mathbb{C}$-Fuchsian lattices described in Section 1.3. More generally, all groups constructed by suspending a Fuchsian subgroup of $\mathrm{PSL}(2, \mathbb{C})$ of the first kind.
- Four: For instance the examples in [2] of complex Kleinian groups with exactly four lines in general position in the Kulkarni limit set.
- Infinite: For instance all groups constructed by suspending a Kleinian subgroup of $\mathrm{PSL}(2, \mathbb{C})$ with infinitely many connected components in its region of discontinuity.

The $\mathbb{R}$-Fuchsian lattices are the first known examples of discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ where the Kulkarni region of discontinuity has three connected components. We do not know whether or not the above list exhausts all possibilities, *i.e.*, whether there exist subgroups of $\mathrm{PSL}(3, \mathbb{C})$ where the number of connected components in $\Omega_{\rm Kul}$ is $\neq 0$, 1, 2, 3, 4 or ∞.

This paper is arranged as follows. In Section 1, for completeness we first give some background that we need on projective and complex hyperbolic geometry, limit sets and $\mathbb{R}$-Fuchsian groups in $\mathrm{PU}(2, 1)$. We define here the lambda and omega sets of $\mathrm{SO}_+(2, 1)$ in $\mathbb{CP}^2$, which we denote by $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ because they are reminiscent of the limit set and discontinuity region of discrete groups. We also discuss in this section the analogous problem in the much simpler case where $\mathrm{Iso}_+(\mathbf{H}^2_\mathbb{R})$, the group of orientation preserving isometries of the real hyperbolic plane, is represented in $\mathrm{SU}(2, 1)$ not via the representation $\mathrm{SO}_+(2, 1)$ considered above, but instead via the natural embedding $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1, 1)) \subset \mathrm{SU}(2, 1)$. This motivates the results that we describe below for

$\mathbb{R}$-Fuchsian groups, and it also highlights an interesting point. As we know already, the group $\mathrm{Iso}_+(\mathbf{H}^2_{\mathbb{R}})$ can be embedded in $\mathrm{SU}(2,1)$ in the two natural ways mentioned above: by thinking of it as being $\mathrm{SU}(1,1)$ or as being $\mathrm{SO}_+(2,1)$. In one case, it yields the subgroup of holomorphic isometries that preserve a complex geodesic, a 2-disc $\mathbf{H}^1_{\mathbb{C}}$, which inherits from $\mathbf{H}^2_{\mathbb{C}}$ a metric that turns it into the Poincaré disc model for the hyperbolic plane, with constant curvature -1. In the second case it yields a totally geodesic invariant 2-disc in $\mathbf{H}^2_{\mathbb{R}}$, which inherits from $\mathbf{H}^2_{\mathbb{C}}$ a metric that turns it into the Klein-Beltrami model of $\mathbf{H}^2_{\mathbb{R}}$, with constant curvature $-\frac{1}{4}$. So from the geometric viewpoint there are significant differences between these two cases. The results in this article show that there also significant topological differences between the two cases: In the first of them, the corresponding set $\mathbf{\Lambda}$ of lines tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at the points in $\partial\mathbf{H}^1_{\mathbb{C}}$ splits $\mathbb{CP}^2$ in two connected components, each diffeomorphic to a 4-ball; in the second case, the corresponding set $\mathbf{\Lambda}$ splits $\mathbb{CP}^2$ in three connected components, each diffeomorphic to a 4-ball.

In Section 2 we look at the set $\mathbf{\Lambda}$ and prove Theorem 1. Also, we define the projection $\mathbf{\Omega} \to \mathbf{H}^2_{\mathbb{R}}$ and show that this gives rise to the appropriate fibre bundles. In Section 3 we describe the fibre bundles in more detail. First, we give equations for the special fibre over the origin $[0:0:1]$ of these bundles, which together with the results of Section 2 proves Theorems 2 and 3. We go on to complete the proof of Theorem 4 by using the knowledge we gained about the special fibres over $[0:0:1]$ and then using the fact that the bundles in question are equivariant. Finally, we investigate the behaviour of the fibres as the base point tends to $\partial\mathbf{H}^2_{\mathbb{R}}$, which completes the proof of Theorem 5.

Acknowledgements. This research was partially supported by CONACYT and PAPIIT-UNAM, Mexico. In addition, part of the work was carried out while JRP was supported by Santander Bank to visit Mexico, and part of it was carried out while JS was visiting Durham as an Alan Richards Fellow at Grey College, and also supported by a Pascal Fellowship that allowed AC to visit Durham. We are grateful for all the support we received.

1. Preliminaries on $\mathbb{R}$-Fuchsian groups acting on $\mathbb{CP}^2$.

1.1. Real and complex hyperbolic space in $\mathbb{CP}^2$. The projective space $\mathbb{CP}^2$ is the quotient of the complex space $\mathbb{C}^3$ minus the origin, by the action of the non-zero complex numbers: $\mathbb{CP}^2 := (\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mathbb{C}^*$. We denote by $\mathbb{P}$ the projectivisation map $\mathbb{C}^3 \setminus \{\mathbf{0}\} \xrightarrow{\mathbb{P}} \mathbb{CP}^2$. Throughout this paper, points in $\mathbb{C}^3$ (or in $\mathbb{C}^{2,1}$, see below) will be denoted by $\mathbf{z}$, and z will denote the image in $\mathbb{CP}^2$ under projectivisation. We will think of $\mathbf{z}$ as a column vector in $\mathbb{C}^3$, as we want matrices to always act on the left. So, if $\mathbf{z} = (z_1, z_2, z_3)^t$ is a column vector in $\mathbb{C}^3$ then $z = \mathbb{P}(\mathbf{z}) = [z_1 : z_2 : z_3]$, using homogeneous coordinates to denote points in $\mathbb{CP}^2$.

Let $\mathbb{C}^{2,1}$ denote a copy of $\mathbb{C}^3$ equipped with the Hermitian form:

$$H(\mathbf{z},\mathbf{w}) := \langle \mathbf{z},\mathbf{w}\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3\,,$$

where $\mathbf{z} = (z_1, z_2, z_3)^t$ and $\mathbf{w} = (w_1, w_2, w_3)^t$ are (column) vectors in $\mathbb{C}^3$. Denote by V_-, V_0, V_+ the sets of negative, null and positive vectors in $\mathbb{C}^{2,1}\setminus\{\mathbf{0}\}$, respectively, *i.e.*, the non-zero vectors where the quadratic form $Q(\mathbf{z}) = |z_1|^2 + |z_2|^2 - |z_3|^2 = H(\mathbf{z},\mathbf{z})$ is negative, zero or positive.

In this article we often speak of orthogonality between vectors in $\mathbb{C}^{2,1}$. This means that the value of the Hermitian form H on these vectors is 0. Given $z = [z_1 : z_2 : z_3]$, by $z^\perp = [z_1 : z_2 : z_3]^\perp$ we mean the set of all points $w = [w_1 : w_2 : w_3]$ in $\mathbb{CP}^2$ such that $z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3 = 0$.

The set V_0 is often referred to as *the light cone*, and the set V_- is the interior of this light cone. The projectivisation $\mathbb{P}(V_-)$ of V_- plays a key role in what follows. We observe that each complex line in V_- meets the set

$$B := \left\{ \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \in \mathbb{C}^3 \,:\, |z_1|^2 + |z_2|^2 < 1 \right\},$$

in a unique point and therefore $\mathbb{P}(V_-)$ is a complex 2-dimensional open ball in $\mathbb{CP}^2$:

$$\mathbb{P}(B) = \Big\{ [z_1 : z_2 : 1] \in \mathbb{CP}^2 \ :\ z_1, z_2 \in \mathbb{R},\, |z_1|^2 + |z_2|^2 < 1 \Big\} = \mathbb{P}(V_-)\,.$$

The restriction of $(-Q)$ to V_- determines a positive definite quadratic form on this set, which defines a metric on $\mathbb{P}(V_-)$ and turns it into a model for the *complex hyperbolic space* $\mathbf{H}^2_{\mathbb{C}}$. The subgroup of $\mathrm{SL}(3, \mathbb{C})$ of maps that preserve the quadratic form Q is by definition $\mathrm{SU}(2,1)$ and its projectivisation $\mathrm{PU}(2,1)$ is the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. We set $\overline{\mathbf{H}}^2_{\mathbb{C}} = \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$; this is a closed real 4-ball with boundary the 3-sphere, which is the projectivisation of V_0, the set of null-vectors.

Notice that $\mathbf{H}^2_{\mathbb{C}}$ contains a copy of the 2-disc:

$$\mathbf{H}^2_{\mathbb{R}} = \Big\{ [x_1 : x_2 : 1] \in \mathbb{CP}^2 \ :\ x_1, x_2 \in \mathbb{R},\, x_1^2 + x_2^2 < 1 \Big\} = \mathbf{H}^2_{\mathbb{C}} \cap \mathcal{P}_{\Re},$$

and the induced metric turns this into the Klein-Beltrami model for the real hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ (see [7]). The orientation preserving isometries of $\mathbf{H}^2_{\mathbb{R}}$ in this model form the group $\mathrm{SO}_+(2,1)$, which is the connected component of $\mathrm{SO}(2,1)$ containing the identity. One has a natural embedding:

$$\iota_{\mathbb{R}} : \mathrm{SO}_+(2,1) \longrightarrow \mathrm{SU}(2,1)\,,$$

which allows us to think of $\mathrm{SO}_+(2,1)$ as a group of automorphisms of $\mathbb{CP}^2$, acting by isometries on $\mathbf{H}^2_{\mathbb{C}}$ as well as on the real hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$. In particular, every group of isometries of the real hyperbolic disc $\mathbf{H}^2_{\mathbb{R}}$ can be regarded as a group of isometries of $\mathbf{H}^2_{\mathbb{C}}$ via this embedding.

Recall that a (classical) Fuchsian group is by definition a discrete subgroup of $\mathrm{Iso}_+(\mathbf{H}^2_{\mathbb{R}})$, the group of orientation preserving isometries of the real hyperbolic plane. Given a Fuchsian group Γ, the identification of $\mathrm{Iso}_+(\mathbf{H}^2_{\mathbb{R}})$ with $\mathrm{SO}_+(2,1)$ provides a natural way of embedding Γ in $\mathrm{SU}(2,1)$:

Definition 1.1. *The image in* $\mathrm{SU}(2,1)$ *of a discrete subgroup* $\Gamma \subset \mathrm{SO}_+(2,1)$ *under the natural embedding* $\mathrm{SO}_+(2,1) \longrightarrow \mathrm{SU}(2,1)$, *is called an* $\mathbb{R}$*-Fuchsian subgroup.*

Of course there are other ways of embedding Fuchsian groups in $\mathrm{SU}(2,1)$. For instance one of these is as $\mathbb{C}$-Fuchsian groups (cf. [8]), and we look at these below.

It is clear that $\mathbb{R}$-Fuchsian groups act on $\mathbb{CP}^2$, leaving invariant the ball $\mathbf{H}^2_{\mathbb{C}}$ as well as the totally real Lagrangian plane $\mathcal{P}_{\Re}$.

1.2. The limit set and equicontinuity. A discrete subgroup Γ of $\mathrm{Iso}(\mathbf{H}^n_{\mathbb{R}})$ acts properly discontinuously on $\mathbf{H}^n_{\mathbb{R}}$. In contrast, the action of Γ on the boundary $\partial\mathbf{H}^n_{\mathbb{R}}$ divides this set into two subsets. First, the limit set of Γ is the set of accumulation points of Γ-orbits. This set is obviously closed and invariant. It also has many other remarkable properties. Its complement is called the region of discontinuity. When the limit set is all of $\partial\mathbf{H}^n_{\mathbb{R}}$ (and so the region of discontinuity is empty) then Γ is said to be of the *first kind*. Otherwise it is of the *second kind*. In this paper we will only consider Fuchsian groups (that is discrete groups Γ in $\mathrm{Iso}_+(\mathbf{H}^2_{\mathbb{R}})$) of the first kind.

Viewing the hyperbolic plane as the Poincaré disc (or the upper half plane) in $\mathbb{CP}^1$, we can naturally identify $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$ with the subgroup $\mathrm{PU}(1,1)$ (or $\mathrm{PSL}(2,\mathbb{R})$ respectively) of $\mathrm{PSL}(2,\mathbb{C})$. If Γ is a Fuchsian group of the first kind then its limit set is a circle in $\mathbb{CP}^1$. The region of discontinuity is a pair of discs, coincides with the region of equicontinuity and is the largest subset of $\mathbb{CP}^1$ where the action of Γ is properly discontinuous.

In higher dimensions, there are several possible notions of the concept of "limit set" for discrete groups of projective automorphisms of $\mathbb{CP}^n$, and we refer to [4] for a thorough discussion of this topic. One of these was introduced by Ravi Kulkarni in [10] and applies in a fairly general setting that includes the one we envisage here. This notion of limit set has the nice property of granting that its complement is an open invariant set where the action is properly discontinuous.

Let us recall the definition of the Kulkarni limit set. For simplicity we restrict the discussion to discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$, so we consider a discrete subgroup $\Gamma \subset \mathrm{PSL}(3,\mathbb{C})$. Its *Kulkarni limit set* $\Lambda_{\mathrm{Kul}}(\Gamma)$ is by definition the union of three Γ-invariant sets $\Lambda_0(\Gamma)$, $\Lambda_1(\Gamma)$ and $\Lambda_2(\Gamma)$:

- $\Lambda_0(\Gamma)$ is the closure of the set of points in $\mathbb{CP}^2$ with infinite isotropy.
- $\Lambda_1(\Gamma)$ is the closure in $\mathbb{CP}^2$ of the set of accumulation points of orbits of points in $\mathbb{CP}^2 \setminus \Lambda_0(\Gamma)$.
- $\Lambda_2(\Gamma)$ is the closure in $\mathbb{CP}^2$ of the set of accumulation points of orbits of compact sets in $\mathbb{CP}^2 \setminus (\Lambda_0(\Gamma) \cup \Lambda_1(\Gamma))$.

The complement $\Omega_{\mathrm{Kul}}(\Gamma) := \mathbb{CP}^2 \setminus \Lambda_{\mathrm{Kul}}(\Gamma)$ is *the Kulkarni region of discontinuity* of (Γ).

These ideas were introduced by Kulkarni in [10] where he also proved that the action on the set $\Omega_{\mathrm{Kul}}(\Gamma)$ is properly discontinuous. For discrete groups of $\mathrm{PSL}(2,\mathbb{C})$ this notion coincides with the usual region of discontinuity, and also with the region of equicontinuity. But in higher dimensions these notions are different.

We recall that a (possibly non-discrete) family of transformations on a manifold M is *equicontinuous* on an open invariant set $U \subset M$ if all the transformations have "equal variation". More precisely,

Definition 1.2. *A family $\mathcal{F}$ of continuous functions between complete metric spaces is equicontinuous at a point $x_0 \in U$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ (which depends only on ε) such that $d(g(x_0), g(x)) < \varepsilon$ for all $g \in \mathcal{F}$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous on U if it is equicontinuous at each point of U.*

The family $\mathcal{F}$ is called *normal* if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets to a continuous function. Moreover, by Arzelà-Ascoli's theorem these two notions –equicontinuity and normal family– are equivalent whenever the domain is a compact set.

Notice also that the union $\Lambda_0(\Gamma) \cup \Lambda_1(\Gamma)$ is the usual (Poincaré) limit set, *i.e.*, the set of accumulation points of all orbits of points in $\mathbb{CP}^2$.

Now suppose that Γ actually is a subgroup of $\mathrm{PU}(2,1)$, so it acts on $\mathbb{CP}^2$ leaving invariant the 4-ball of points in $\mathbb{CP}^2$ whose homogeneous coordinates $[z_1 : z_2 : z_3]$ satisfy $|z_1|^2 + |z_2|^2 < |z_3|^3$. In this case we also have another notion of limit set defined by Chen and Greenberg in [6], which we denote by $\Lambda_{\mathrm{CG}}(\Gamma)$. This is the subset of $\partial\mathbf{H}^2_{\mathbb{C}}$ where the orbits of points in $\mathbf{H}^2_{\mathbb{C}}$ accumulate. As in the classical setting of real hyperbolic groups, one has that if $\Lambda_{\mathrm{CG}}(\Gamma)$ has finite cardinality, then it consists of at most two points and such groups are called elementary.

So for a complex hyperbolic discrete group $\Gamma \subset \mathrm{PU}(2,1)$ we have two notions of a limit set: The Chen-Greenberg limit set, which takes into account only the action of the group on the ball $\mathbf{H}^2_{\mathbb{C}}$, and the Kulkarni limit set, which looks at the action globally on all of $\mathbb{CP}^2$. We also have the complement of the equicontinuity region. For non-elementary discrete subgroups of $\mathrm{PU}(2,1)$, the relation between these three sets was established by J.-P. Navarrete in [11] (see also [4]).

To explain Navarrete's results in [11] we recall that the boundary of $\mathbf{H}^2_{\mathbb{C}}$ is a 3-sphere and at each point z in $\partial\mathbf{H}^2_{\mathbb{C}}$ there is a unique complex projective line in $\mathbb{CP}^2$, denoted $\mathcal{L}_z$, which is tangent to the 3-sphere $\partial\mathbf{H}^2_{\mathbb{C}} := \mathbb{P}(V_0)$ at z. This line is the projectivisation of the set of vectors in $\mathbb{C}^{2,1}$ which are H-orthogonal to $\mathbf{z}$. The collection of these lines will play an important role in our construction.

The main result in [11] says:

THEOREM 1.3. *If* $\Gamma \subset \mathrm{PU}(2,1)$ *is non-elementary then:*

(i) The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ *is the union of all projective lines in* $\mathbb{CP}^2$ *which are tangent to* $\partial\mathbf{H}^2_{\mathbb{C}} \cong \mathbb{S}^3$ *at points in the Chen-Greenberg limit set* $\Lambda_{\mathrm{CG}}(\Gamma)$*.*

(ii) The Kulkarni region of discontinuity $\Omega_{\mathrm{Kul}}(\Gamma)$ *is the largest open invariant set in* $\mathbb{CP}^2$ *where the action is properly discontinuous.*

(iii) $\Omega_{\mathrm{Kul}}(\Gamma)$ *coincides with the region of equicontinuity.*

REMARK 1.4. We may naturally consider the generalisation of this theorem to higher dimensions: Given a discrete subgroup $\Gamma \subset \mathrm{PU}(n,1)$, we have its Chen-Greenberg limit set $\Lambda_{\mathrm{CG}}(\Gamma)$ defined in the same way. This is contained in the $(2n-1)$-sphere $\partial\mathbf{H}^n_{\mathbb{C}} \subset \mathbb{CP}^n$. At each point z of this sphere there is a unique complex projective hyperplane $\mathcal{L}_z$ tangent to $\partial\mathbf{H}^n_{\mathbb{C}}$ at z. The union $\Lambda(\Gamma)$ of all these hyperplanes at points in $\Lambda_{\mathrm{CG}}(\Gamma)$ is a closed Γ-invariant set. It is proved in [3] that the action of Γ on the complement $\mathbb{CP}^n \setminus \Lambda(\Gamma)$ is properly discontinuous and this actually is also the region of equicontinuity. Furthermore, we know that $\Lambda(\Gamma)$ is the Kulkarni limit set of the action [5].

Consider a Fuchsian subgroup $\Gamma \subset \mathrm{SO}_+(2,1)$. The (Poincaré) limit set Λ of Γ is contained in the boundary of $\mathbf{H}^2_{\mathbb{R}}$ which is a circle $\partial\mathbf{H}^2_{\mathbb{R}} = \mathbb{S}^1$. If $\Gamma \subset \mathrm{SO}_+(2,1)$ is thought of as a subgroup of $\mathrm{SU}(2,1)$, then its Chen-Greenberg limit set $\Lambda_{\mathrm{CG}}(\Gamma)$ coincides with the usual limit set Λ contained in $\partial\mathbf{H}^2_{\mathbb{R}}$. The group is cofinite if and only if its limit set Λ, and hence also its Chen-Greenberg limit set Λ_{CG}, is the whole circle $\partial\mathbf{H}^2_{\mathbb{R}}$; such groups are also called lattices in $\mathrm{SO}_+(2,1)$. We also know, from Theorem 1.3, that in this case the Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ is the union of all complex projective lines in $\mathbb{CP}^2$ which are tangent to the 3-sphere $\partial\mathbf{H}^2_{\mathbb{R}}$ at points in $\Lambda_{\mathrm{CG}}(\Gamma)$.

Inspired by these constructions we observe that if we regard $\mathrm{SO}_+(2,1)$ as a subgroup of $\mathrm{SU}(2,1)$, then $\mathrm{SO}_+(2,1)$ itself leaves invariant the circle $\partial\mathbf{H}^2_{\mathbb{R}} = \mathcal{P}_{\Re} \cap \partial\mathbf{H}^2_{\mathbb{C}}$. Furthermore, since the (projective) action of $\mathrm{SU}(2,1)$ on $\mathbb{CP}^2$ is by holomorphic transformations, every complex projective line which is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at a given point ξ, is carried by each $A \in \mathrm{SU}(2,1)$ into the unique complex projective line which is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at the point $A(\xi)$.

Definition 1.5. *We let $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{SO}_+(2,1))$ be the set defined by:*

$$\mathbf{\Lambda} := \Big\{\mathcal{L}_\xi : \xi \in \partial\mathbf{H}^2_{\mathbb{R}} \textit{ and } \mathcal{L}_\xi \textit{ is the unique complex projective line tangent to } \partial\mathbf{H}^2_{\mathbb{C}} \textit{ at } \xi\Big\}.$$

We denote by $\mathbf{\Omega} = \mathbf{\Omega}(\mathrm{SO}_+(2,1))$ its complement $\mathbf{\Omega} = \mathbb{CP}^2 \setminus \mathbf{\Lambda}$.

By definition $\mathbf{\Lambda}$ is a closed $\mathrm{SO}_+(2,1)$-invariant subset of $\mathbb{CP}^2$. The notation is chosen in analogy with the traditional concepts of limit set and discontinuity region, since we know from Navarrete [11] that these sets coincide with the Kulkarni limit set and the Kulkarni region of discontinuity of every cofinite $\mathbb{R}$-Fuchsian subgroup of $\mathrm{PU}(2,1)$.

Proposition 1.6. *The set $\mathbf{\Omega} = \mathbf{\Omega}\big(\mathrm{SO}_+(2,1)\big)$ is the equicontinuity set $Eq\big(\mathrm{SO}_+(2,1);\mathbb{CP}^2\big)$ for the action of $\mathrm{SO}_+(2,1)$ on $\mathbb{CP}^2$.*

Proof. Observe first that there are infinitely many lines in general position contained in $\mathbf{\Lambda}$. Hence the theorem of Cartan-Montel for normal families (see [12, Chapter VIII]) implies $\mathbf{\Omega} \subset Eq\big(\mathrm{SO}_+(2,1);\mathbb{CP}^2\big)$. Conversely, let $\mathcal{L}$ be a line in $\mathbf{\Lambda}$, tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at a point $\xi \in \partial\mathbf{H}^2_{\mathbb{R}}$, and let $A \in \mathrm{SO}_+(2,1)$ be a parabolic element (see [7]) that leaves ξ invariant. Then

$$Eq\big(\mathrm{SO}_+(2,1);\mathbb{CP}^2\big) \subset Eq(\langle A\rangle) = \mathbb{CP}^2 \setminus \mathcal{L}\,,$$

which obviously implies

$$Eq\big(\mathrm{SO}_+(2,1);\mathbb{CP}^2\big) \subset \mathbb{CP}^2 \setminus \overline{\mathrm{SO}_+(2,1)\,\mathcal{L}}\,,$$

and the result follows because $\overline{\mathrm{SO}_+(2,1)\,\mathcal{L}} = \mathbf{\Lambda}\big(\mathrm{SO}_+(2,1)\big)$. □

We remark that the same statement and the same proof extend to the more general setting of lattices in $\mathrm{SO}_+(n,1) \subset \mathrm{SU}(m,1)$ for $m \geq n$ considered at the beginning of the introduction.

Remark 1.7. It is worth saying that the subset $\widetilde{\mathbf{\Lambda}} \subset \mathbb{C}^{2,1}$ considered in the introduction is the inverse image of $\mathbf{\Lambda}$ under the projectivisation map $\mathbb{P} : \mathbb{C}^3 \setminus \{\mathbf{0}\} \to \mathbb{CP}^2$. Hence $\widetilde{\mathbf{\Lambda}}$ is a holomorphic line bundle over $\mathbf{\Lambda}$, namely the restriction of the tautological bundle over $\mathbb{CP}^2$.

We now observe that the circle $\partial\mathbf{H}^2_{\mathbb{R}}$ can be parametrised as follows:

$$\partial\mathbf{H}^2_{\mathbb{R}} = \Big\{[\cos(\theta) : \sin(\theta) : 1] \in \mathbb{CP}^2 :\ \theta \in [0,2\pi)\Big\}. \tag{1}$$

To determine the corresponding lines in $\mathbf{\Lambda}$ we remark that for ξ in $\partial\mathbf{H}^2_{\mathbb{C}}$, if the line $\mathcal{L}_\xi$ passes through the point $\xi = [\cos(\theta) : \sin(\theta) : 1]$, then it passes also through the orthogonal point $[-\sin(\theta) : \cos(\theta) : 0]$. Hence the corresponding line is:

$$\begin{aligned}\mathcal{L}_\xi &= \Big\{[\lambda\cos(\theta) - \mu\sin(\theta) : \lambda\sin(\theta) + \mu\cos(\theta) : \lambda] \in \mathbb{CP}^2\ :\ \theta \in [0,2\pi),\ [\lambda : \mu] \in \mathbb{CP}^1\Big\}\\ &= \Big\{[\cos(\theta) : \sin(\theta) : 1]^{\perp}\Big\}\,.\end{aligned}$$

We arrive to the following proposition:

Proposition 1.8. *The set $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{SO}_+(2,1))$ in $\mathbb{CP}^2$ is:*

$$\mathbf{\Lambda} = \Big\{[\lambda\cos\theta - \mu\sin\theta : \lambda\sin\theta + \mu\cos\theta : \lambda] \in \mathbb{CP}^2\ :\ \theta \in [0,2\pi] \textit{ and } [\lambda : \mu] \in \mathbb{CP}^1\Big\},$$

the set of projective lines tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at points in the circle $\partial\mathbf{H}^2_{\mathbb{R}}$ given by (1).

1.3. $\mathbb{C}$-Fuchsian groups. Recall now that the projective line $\mathbb{CP}^1$ can be embedded in $\mathbb{CP}^2$ in many ways, as for instance as the set of points with homogeneous coordinates $[0 : z_2 : z_3]$. Its group of automorphisms is $\mathrm{PSL}(2,\mathbb{C})$ and one has a group isomorphism:

$$\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Iso}_+(\mathbf{H}^3_{\mathbb{R}}),$$

where $\mathrm{Iso}_+(\mathbf{H}^3_{\mathbb{R}})$ is the group of orientation preserving isometries of real hyperbolic 3-space. Its subgroup $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SO}_+(2,1)$. The upper-half plane is biholomorphic to the unit disc $\big\{[0 : z_2 : 1] : |z_2|^2 < 1\big\}$, and we can identify $\mathrm{PSL}(2,\mathbb{R})$ with the subgroup $\mathrm{PU}(1,1)$ of $\mathrm{PSL}(2,\mathbb{C})$ consisting of maps that preserve this disc.

The group $\mathrm{PSL}(2,\mathbb{C})$ has a natural lifting to its double cover $\mathrm{SL}(2,\mathbb{C})$ and this latter group has a canonical embedding in $\mathrm{SL}(3,\mathbb{C})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

The projective space $\mathbb{CP}^2$ can be regarded as being a compactification of $\mathbb{C}^2$ by attaching to it a line $\mathcal{L}_\infty \cong \mathbb{CP}^1$ at infinity, the "line of directions". The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ naturally extends to an action on $\mathbb{CP}^2$ that leaves $\mathcal{L}_\infty$ invariant and the action on this line is the usual action of $\mathrm{PSL}(2,\mathbb{C})$. This yields a natural embedding,

$$\iota : \mathrm{PSL}(2,\mathbb{C}) \longrightarrow \mathrm{PSL}(3,\mathbb{C})\,.$$

This method of embedding subgroups of $\mathrm{PSL}(2,\mathbb{C})$ in $\mathrm{PSL}(3,\mathbb{C})$ is a special type of the *suspension groups* studied in [4].

It is easy to see that we can actually choose the line $\mathcal{L}_\infty$ to be

$$\mathcal{L}_\infty = \big\{[0 : z_2 : z_3] : (z_2, z_3) \in \mathbb{C}^2 \setminus \{(0,0)\}\big\}.$$

This projective line intersects the complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ in a complex slice, a copy of $\mathbf{H}^1_{\mathbb{C}}$, which is a 2-disc isometric to $\mathbf{H}^2_{\mathbb{R}}$ (with the Poincaré disc-model, see [7]). The restriction of ι to $\mathrm{PSL}(2,\mathbb{C})$ preserves the complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ and determines an embedding:

$$\iota_{\mathbb{C}} : \mathrm{PU}(1,1) \longrightarrow \mathrm{PU}(2,1)\,.$$

Hence every group of isometries of the hyperbolic plane, viewed as $\mathbf{H}^1_{\mathbb{C}}$, can be regarded as a group of isometries of $\mathbf{H}^2_{\mathbb{C}}$ via this embedding. In fact, passing to the double cover $\mathrm{SU}(1,1)$ we may consider, more generally, the natural embedding $\mathrm{S}\big(\mathrm{U}(1)\times\mathrm{U}(1,1)\big) \subset \mathrm{SU}(2,1)$:

$$\iota^\theta : \left(e^{i\theta}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \longmapsto \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{-i\theta}a & e^{-i\theta}b \\ 0 & e^{-i\theta}c & e^{-i\theta}d \end{pmatrix}.$$

Projectivising the latter group we get an embedding $\iota^\theta_{\mathbb{C}}$ of $\mathrm{PU}(1,1)$ into $\mathrm{PU}(2,1)$.

As in [8], we call the image in $\mathrm{PU}(2,1)$ of a discrete subgroup $\Gamma \subset \mathrm{PU}(1,1)$ under this map a $\mathbb{C}$-*Fuchsian subgroup*. Such a group leaves invariant the sphere $\partial\mathbf{H}^2_{\mathbb{C}}$ and also leaves invariant the projective line $\mathcal{L}_\infty = \big\{[0 : z_2 : z_3] : (z_2, z_3) \in \mathbb{C}^2 \setminus \{(0,0)\}\big\}$. Hence it leaves invariant the circle

$$\partial\mathbf{H}^1_{\mathbb{C}} = \partial\mathbf{H}^2_{\mathbb{C}} \cap \mathcal{L}_\infty = \big\{[0 : e^{i\phi} : 1] : \phi \in [0, 2\pi)\big\}.$$

Notice that one also has a set $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{PU}(1,1))$ defined similarly to the $\mathbb{R}$-Fuchsian case (Definition 1.5): It consists of all complex projective lines $\mathcal{L}_\xi$ in $\mathbb{CP}^2$ tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at points ξ in $\partial\mathbf{H}^1_{\mathbb{C}}$. Its complement is the omega set $\mathbf{\Omega} = \mathbf{\Omega}(\mathrm{PU}(1,1))$.

Again, if Γ is a cofinite Fuchsian group in $\mathrm{PU}(1,1)$ and we embed it in $\mathrm{PU}(2,1)$ as via $\iota^\theta_{\mathbb{C}}$, we get a $\mathbb{C}$-Fuchsian group and by Theorem 1.3 we have that the set $\mathbf{\Lambda}(\mathrm{U}(1,1))$ coincides with the Kulkarni limit set of Γ; its complement $\mathbf{\Omega}(\mathrm{U}(1,1))$ is the Kulkarni region of discontinuity of Γ and coincides with the region of equicontinuity.

At each point $\xi = [0 : e^{i\phi} : 1] \in \partial\mathbf{H}^1_{\mathbb{C}}$ the corresponding line $\mathcal{L}_\xi$ is the unique projective line passing through ξ and the orthogonal point $[1 : 0 : 0]$. Thus $\mathcal{L}_\xi$ is:

$$\begin{aligned}\mathcal{L}_\xi &= \Big\{[\mu : \lambda e^{i\phi} : \lambda] \in \mathbb{CP}^2 \,:\, \phi \in [0, 2\pi) \text{ and } [\lambda : \mu] \in \mathbb{CP}^1\Big\}\\ &= \Big\{[0 : e^{i\phi} : 1]^\perp \,;\, \phi \in [0, 2\pi)\Big\},\end{aligned}$$

where the latter term denotes the set of all points in $\mathbb{CP}^2$ orthogonal to $\xi = [0 : e^{i\phi} : 1]$ for the given $(2,1)$-Hermitian form.

If z is a point in $\mathbf{\Omega}\big(\mathrm{PU}(1,1)\big) := \mathbb{CP}^2 \setminus \mathbf{\Lambda}\big(\mathrm{PU}(1,1)\big)$, then the line passing through ξ and the point $[1 : 0 : 0]$ meets the projective line $\mathcal{L}_\infty$ in a unique point, which necessarily is away from the circle $\partial\mathbf{H}^1_{\mathbb{C}}$. This determines a projection $\mathbf{\Omega} \to \mathcal{L}_\infty \setminus \partial\mathbf{H}^1_{\mathbb{C}}$, which is easily seen to be a fibre bundle with fibre $\mathbb{C}$. Since $\mathcal{L}_\infty \setminus \partial\mathbf{H}^1_{\mathbb{C}}$ consists of two open hemispheres, this bundle is trivial, and we arrive to the following:

PROPOSITION 1.9. *The set* $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{PU}(1,1))$ *is:*

$$\mathbf{\Lambda} = \Big\{[\mu : \lambda e^{i\phi} : \lambda] \in \mathbb{CP}^2 \,:\, \phi \in [0, 2\pi) \textit{ and } [\lambda : \mu] \in \mathbb{CP}^1\Big\},$$

tangent to $\partial\mathbf{H}^2_{\mathbb{C}} \cong S^3$ *at points in the circle* $\partial\mathbf{H}^1_{\mathbb{C}} = \partial H^2_{\mathbb{C}} \cap \mathcal{L}_\infty$ *and meeting at the point* $[1 : 0 : 0]$. *Thence:*

1. *The set* $\mathbf{\Lambda}$ *is homeomorphic to the complex cone over the circle* $\partial\mathbf{H}^1_{\mathbb{C}}$ *with vertex at* $[1 : 0 : 0]$, *and therefore* $\mathbf{\Lambda} \setminus \big\{[1 : 0 : 0]\big\}$ *is diffeomorphic to a solid torus* $S^1 \times \mathbb{C}$.
2. *Its complement* $\mathbf{\Omega} = \mathbf{\Omega}(\mathrm{PU}(1,1))$ *is a trivial fibre bundle over the projective line* $\mathcal{L}_\infty$ *minus the equator* $\partial\mathbf{H}^1_{\mathbb{C}}$, *with fibre* $\mathbb{C}$. *Hence this set has two connected components, each diffeomorphic to a 4-ball* $D^2 \times \mathbb{C}$ *where* D^2 *is an open 2-disc.*

This result motivates what we do below for $\mathrm{SO}_+(2,1)$.

2. The Hermitian cross-product.

2.1. The linear algebra of the Hermitian cross-product. We recall that one has on $\mathbb{C}^{2,1}$ the Hermitian cross-product $\boxtimes$ introduced by Bill Goldman in [7, p. 43], which is an alternating (essentially) bilinear map $\mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ that can be defined by (see [7, p. 45]):

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \boxtimes \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \overline{z}_3\overline{w}_2 - \overline{z}_2\overline{w}_3 \\ \overline{z}_1\overline{w}_3 - \overline{z}_3\overline{w}_1 \\ \overline{z}_1\overline{w}_2 - \overline{z}_2\overline{w}_1 \end{pmatrix}.$$

In fact for every $\lambda,\ \mu \in \mathbb{C}^*$ and for every $\mathbf{z},\ \mathbf{w} \in \mathbb{C}^{2,1}$ one has:

$$(\lambda\mathbf{z}) \boxtimes (\mu\mathbf{w}) = \overline{\lambda}\,\overline{\mu}\,(\mathbf{z} \boxtimes \mathbf{w})\,. \tag{2}$$

Thus $\boxtimes$ is bilinear, except that scalars act via their complex conjugate. It is also clear that if the vectors $\mathbf{z}$ and $\mathbf{w}$ are linearly independent, then the Hermitian cross-product is a vector orthogonal to both $\mathbf{z}$ and $\mathbf{w}$, with respect to the Hermitian form $H = \langle\, ,\, \rangle$. This construction will play a key role in what follows, so we develop here some theory about it, which can be of interest on its own.

Let $\mathbf{z}$ and $\mathbf{w}$ be linearly independent vectors in $\mathbb{C}^{2,1}$, so they span a 2-plane $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ and determine a projective line $\overleftrightarrow{z,w}$ in $\mathbb{CP}^2$. We are interested in characterising the 2-planes in $\mathbb{C}^{2,1}$ which give rise to projective lines which are tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$, particularly –but not only– at points in $\partial\mathbf{H}^2_{\mathbb{R}}$. Of course these are the 2-planes which are tangent to the light cone V_0 in $\mathbb{C}^{2,1}$.

LEMMA 2.1. *Suppose that* $\mathbf{z}$ *and* $\mathbf{w}$ *are linearly independent vectors in* $\mathbb{C}^{2,1}$. *Let* $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ *denote the complex 2-plane spanned by* $\mathbf{z}$ *and* $\mathbf{w}$.

(a) $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ *is contained in* V_+ *if and only if* $\mathbf{z}\boxtimes\mathbf{w}$ *is in* V_-.

(b) $\overleftrightarrow{\mathbf{z},\mathbf{w}}\cap V_-\neq\emptyset$ *if and only if* $\mathbf{z}\boxtimes\mathbf{w}$ *is in* V_+.

(c) $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ *is tangent to* V_0 *if and only if* $\mathbf{z}\boxtimes\mathbf{w}$ *is in* V_0. *Moreover,* $\overleftrightarrow{\mathbf{z},\mathbf{w}}\cap V_0$ *is spanned by* $\mathbf{z}\boxtimes\mathbf{w}$.

Proof. We know from [7] that the cross-product $\mathbf{z}\boxtimes\mathbf{w}$ spans the 1-dimensional space $\overleftrightarrow{\mathbf{z},\mathbf{w}}^{\perp}$ orthogonal to the 2-plane $\overleftrightarrow{\mathbf{z},\mathbf{w}}$.

We begin by showing that $\mathbf{z}\boxtimes\mathbf{w}\in\overleftrightarrow{\mathbf{z},\mathbf{w}}$ if and only if $\mathbf{z}\boxtimes\mathbf{w}\in V_0$. Suppose $\mathbf{z}\boxtimes\mathbf{w}\in\overleftrightarrow{\mathbf{z},\mathbf{w}}$. Since $\mathbf{z}\boxtimes\mathbf{w}$ is orthogonal to all points in $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ we see that $\langle\mathbf{z}\boxtimes\mathbf{w},\mathbf{z}\boxtimes\mathbf{w}\rangle=0$ and so $\mathbf{z}\boxtimes\mathbf{w}\in V_0$. If $\mathbf{z}\boxtimes\mathbf{w}$ is not contained in $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ then $\{\mathbf{z},\,\mathbf{w},\,\mathbf{z}\boxtimes\mathbf{w}\}$ spans $\mathbb{C}^{2,1}$ and so is a basis. Thus, if $\mathbf{z}\boxtimes\mathbf{w}$ were in V_0 then it would be orthogonal to all three basis vectors, and so to all vectors in $\mathbb{C}^{2,1}$. This is a contradiction, since the Hermitian form is non-degenerate.

Suppose $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ is tangent to V_0. Without loss of generality suppose that $\mathbf{z}\in V_+$ and $\mathbf{w}\in V_0$. We claim that $\langle\mathbf{z},\mathbf{w}\rangle=0$. If not, then consider $\mathbf{v}_\tau=\mathbf{z}-\tau\langle\mathbf{z},\mathbf{w}\rangle\mathbf{w}$ where $\tau\in\mathbb{R}_+$. Clearly $\mathbf{v}_\tau$ is in $\overleftrightarrow{\mathbf{z},\mathbf{w}}$. However,

$$\langle\mathbf{v}_\tau,\mathbf{v}_\tau\rangle=\langle\mathbf{z}-\tau\langle\mathbf{z},\mathbf{w}\rangle\mathbf{w},\mathbf{z}-\tau\langle\mathbf{z},\mathbf{w}\rangle\mathbf{w}\rangle=\langle\mathbf{z},\mathbf{z}\rangle-2\tau|\langle\mathbf{z},\mathbf{w}\rangle|^2.$$

By taking τ sufficiently large, we can force this point to be in V_-, a contradiction. Hence $\mathbf{w}$ is orthogonal to $\mathbf{z}$. By construction, $\mathbf{w}$ is in V_0 and so $\mathbf{w}$ is orthogonal to all points in $\overleftrightarrow{\mathbf{z},\mathbf{w}}$, the complex span of $\mathbf{z}$ and $\mathbf{w}$. Thus, $\mathbf{w}$ is a multiple of $\mathbf{z}\boxtimes\mathbf{w}$.

Conversely, suppose $\mathbf{z}\boxtimes\mathbf{w}$ lies in $\overleftrightarrow{\mathbf{z},\mathbf{w}}$, in particular $\mathbf{z}\boxtimes\mathbf{w}\in V_0$ as above. Any two (complex) dimensional space must contain a vector in V_+, so write $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ as the span of $\mathbf{z}\boxtimes\mathbf{w}$ and some $\mathbf{z}\in V_+$. Then it is clear that any linear combination of $\mathbf{z}\boxtimes\mathbf{w}$ and $\mathbf{z}$ lies in $V_+\cup V_0$, and lies in V_0 only when it is a multiple of $\mathbf{z}\boxtimes\mathbf{w}$. Hence $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ is tangent to V_0 and $\overleftrightarrow{\mathbf{z},\mathbf{w}}\cap V_0$ is spanned by $\mathbf{z}\boxtimes\mathbf{w}$.

If $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ is contained in V_+, then, $\{\mathbf{z},\,\mathbf{w},\,\mathbf{z}\boxtimes\mathbf{w}\}$ is a basis of $\mathbb{C}^{2,1}$. Since V_- is non-empty, we must have $\mathbf{z}\boxtimes\mathbf{w}\in V_-$. Conversely, if $\mathbf{z}\boxtimes\mathbf{w}\in V_-$ then, as the form has signature $(2,1)$, any vector orthogonal to $\mathbf{z}\boxtimes\mathbf{w}$ must be in V_+.

If $\overleftrightarrow{\mathbf{z},\mathbf{w}}\cap V_-\neq\emptyset$ then the restriction of the Hermitian form to $\overleftrightarrow{\mathbf{z},\mathbf{w}}$ has signature $(1,1)$. Since $\{\mathbf{z},\,\mathbf{w},\,\mathbf{z}\boxtimes\mathbf{w}\}$ is a basis of $\mathbb{C}^{2,1}$ and the form has signature $(2,1)$, then $\mathbf{z}\boxtimes\mathbf{w}$ must be in V_+. Conversely, if $\mathbf{z}\boxtimes\mathbf{w}$ is in V_+ then, since $\{\mathbf{z},\,\mathbf{w},\,\mathbf{z}\boxtimes\mathbf{w}\}$ is a basis for $\mathbb{C}^{2,1}$ and the form has signature $(2,1)$ we can find a vector orthogonal to $\mathbf{z}\boxtimes\mathbf{w}$ that lies in V_-. $\square$

As a consequence of this lemma, given a complex 2-plane $\mathfrak{P}\subset\mathbb{C}^{2,1}$ passing through the origin, the corresponding projective line in $\mathbb{CP}^2$ is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ if and only if for each pair of linearly independent vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathfrak{P}$ one has that the vector $\mathbf{z} \boxtimes \mathbf{w}$ is in V_0; and in that case $z \boxtimes w \in \mathbb{CP}^2$ is the point of tangency of $\mathbb{P}(\mathfrak{P})$ with $\partial \mathbf{H}^2_{\mathbb{C}}$.

Now we observe that given vectors $\mathbf{z}$, $\mathbf{w}$ in $\mathbb{C}^{2,1}$ one has: $\mathbf{z} \boxtimes \mathbf{w} = -\mathbf{w} \boxtimes \mathbf{z}$ so their projectivisation coincides:

$$z \boxtimes w := \mathbb{P}(\mathbf{z} \boxtimes \mathbf{w}) = \mathbb{P}(\mathbf{w} \boxtimes \mathbf{z}) = w \boxtimes z\,.$$

For all $\mathbf{z}$, $\mathbf{w} \in \mathbb{C}^{2,1}$ we have:

$$\overline{\mathbf{z}} \boxtimes \overline{\mathbf{w}} = \overline{\mathbf{z} \boxtimes \mathbf{w}}.$$

In particular, for all $\mathbf{z}$ we have:

$$\mathbf{z} \boxtimes \overline{\mathbf{z}} = -\overline{\mathbf{z}} \boxtimes \mathbf{z} = -\overline{\mathbf{z} \boxtimes \overline{\mathbf{z}}}\,, \tag{3}$$

and therefore, $i\mathbf{z} \boxtimes \overline{\mathbf{z}} = \overline{i\mathbf{z} \boxtimes \overline{\mathbf{z}}}$ and so $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in \mathbb{R}^{2,1} \subset \mathbb{C}^{2,1}$. Moreover, provided $\mathbf{z} \boxtimes \overline{\mathbf{z}} \neq \mathbf{0}$, we have:

$$z \boxtimes \overline{z} = \mathbb{P}(i\mathbf{z} \boxtimes \overline{\mathbf{z}}) = \mathbb{P}(\overline{i\mathbf{z} \boxtimes \overline{\mathbf{z}}}) = \overline{z \boxtimes \overline{z}}\,.$$

This implies that the point $z \boxtimes \overline{z} \in \mathbb{CP}^2$ is invariant under complex conjugation and therefore it is in $\mathcal{P}_{\mathfrak{R}}$. We state these facts as a lemma:

Lemma 2.2. *Suppose that* $\mathbf{z} \in \mathbb{C}^{2,1}$ *is a non-zero vector for which* $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \neq \mathbf{0}$. *Then we have* $z \boxtimes \overline{z} = \mathbb{P}(i\mathbf{z} \boxtimes \overline{\mathbf{z}})$, *the image of this cross-product under the projectivisation map, is a well defined point in the real Lagrangian plane* $\mathcal{P}_{\mathfrak{R}}$ *of points in* $\mathbb{CP}^2$ *that can be represented by homogeneous coordinates in* $\mathbb{R}$.

Consider the lambda set $\mathbf{\Lambda}$ of $\mathrm{SO}_+(2,1)$ given in Definition 1.5. Recall that $\mathbf{\Lambda}$ is, by definition, the set of all complex projective lines in $\mathbb{CP}^2$ which are tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ at points in $\partial \mathbf{H}^2_{\mathbb{R}} = \partial \mathbf{H}^2_{\mathbb{C}} \cap \mathcal{P}_{\mathbb{R}}$. A complex projective line in $\mathbb{CP}^2$ is tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ if and only if the corresponding plane in $\mathbb{C}^{2,1}$ is tangent to V_0. The point of tangency in $\partial \mathbf{H}^2_{\mathbb{C}}$ is in $\mathcal{P}_{\mathbb{R}}$ if the line of tangency to V_0 is preserved by the complex conjugation map. Using Lemma 2.1 (c) and Lemma 2.2 we can use this to characterise complex projective lines in $\mathbf{\Lambda}$.

Corollary 2.3. *If* $\mathbf{z} \in \mathbb{C}^{2,1}$ *is such that* $\mathbf{z}$ *and* $\overline{\mathbf{z}}$ *are linearly independent and their product* $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ *is in* V_0, *then the projective line* $\overleftrightarrow{z, \overline{z}}$ *is in* $\mathbf{\Lambda}$ *and it is tangent to* $\partial \mathbf{H}^2_{\mathbb{C}}$ *at the point* $z \boxtimes \overline{z} \in \partial \mathbf{H}^2_{\mathbb{R}}$.

Recall that V_0 is by definition the set of null vectors for the Hermitian form and so if $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_0$ then we have $\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = 0$. Conversely, suppose that $\mathbf{z}$ is a vector in $\mathbb{C}^{2,1}$ for which $\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = 0$. Then it may be that either $i\mathbf{z} \boxtimes \overline{\mathbf{z}} = \mathbf{0}$ or $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \neq \mathbf{0}$. In the latter case we must have that $\mathbf{z}$ and $\overline{\mathbf{z}}$ span a complex 2-plane orthogonal to $\mathbf{z} \boxtimes \overline{\mathbf{z}}$, so these vectors are linearly independent and we are in the setting of Corollary 2.3. On the other hand, if $i\mathbf{z} \boxtimes \overline{\mathbf{z}} = \mathbf{0}$ then the two vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly dependent, which implies they represent the same point in $\mathbb{CP}^2$, so this point is in $\mathcal{P}_{\mathfrak{R}}$. Thus we get:

Lemma 2.4. *Suppose that* $\mathbf{z}$ *is a vector in* $\mathbb{C}^{2,1}$ *for which* $\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = 0$. *Then either:*

(a) The projectivisation z *of* $\mathbf{z}$ *is a point in the set* $\mathbf{\Lambda} := \mathbf{\Lambda}(\mathrm{SO}_+(2,1)) \setminus \mathcal{P}_{\mathfrak{R}}$, *and this happens if and only if* $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \neq \mathbf{0}$; *or else*

(b) *the projectivisation z of $\mathbf{z}$ is in the plane $\mathcal{P}_{\Re}$, and this happens if and only if $i\mathbf{z} \boxtimes \overline{\mathbf{z}} = \mathbf{0}$.*

One clearly has:

$$i\mathbf{z} \boxtimes \overline{\mathbf{z}} = i \begin{pmatrix} \overline{z}_3 z_2 - \overline{z}_2 z_3 \\ \overline{z}_1 z_3 - \overline{z}_3 z_1 \\ \overline{z}_1 z_2 - \overline{z}_2 z_1 \end{pmatrix} = 2 \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_2 y_1 - x_1 y_2 \end{pmatrix}.$$

Therefore $f(\mathbf{z}) = \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle$ can be expressed as the real-valued polynomial

$$f(\mathbf{z}) = 4\big((x_3y_2 - x_2y_3)^2 + (x_1y_3 - x_3y_1)^2 - (x_1y_2 - x_2y_1)^2\big). \tag{4}$$

Corollary 2.5. *The set $\mathbf{\Lambda}$ is the semi-algebraic subset of $\mathbb{CP}^2$ consisting of points whose homogeneous coordinates $[x_1 + iy_1 : x_2 + iy_2 : x_3 + iy_3]$ satisfy:*

$$\begin{array}{lllll} 0 & \leq & \langle \mathbf{z}, \mathbf{z} \rangle & = & x_1^2 + y_1^2 + x_2^2 + y_2^2 - x_3^2 - y_3^2 \quad \textit{and} \\ 0 & = & \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle & = & 4(x_3y_2 - x_2y_3)^2 + 4(x_1y_3 - x_3y_1)^2 - 4(x_1y_2 - x_2y_1)^2. \end{array}$$

Proof. It is clear that the points z in $\mathbf{\Lambda}$ are outside the ball $\mathbf{H}^2_{\mathbb{C}}$ and therefore the correspond to vectors $\mathbf{z}$ with $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$. Furthermore, consider a line $\mathcal{L} \subset \mathbf{\Lambda}$ and a point $z \in (\mathcal{L} \setminus \mathcal{P}_{\Re})$. The point z is then the projectivisation of a point $\mathbf{z} \in \mathbb{C}^{2,1}$ such that $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent and $\mathcal{L}$ is the projectivisation of the plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$. Then Lemma 2.1 implies $f(\mathbf{z}) = 0$.

Conversely, Lemma 2.4 ensures that if $\mathbf{z}$ is such that $\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = 0$ then either z is in $\mathbf{\Lambda}$ or else it is in the plane $\mathcal{P}_{\Re}$. So we must show that the points in $\mathcal{P}_{\Re}$ which are in $\mathbf{\Lambda}$ are exactly those in the Möbius strip $\mathcal{M} := \mathcal{P}_{\Re} \setminus \mathbf{H}^2_{\mathbb{R}}$. One side is obvious: If $z \in \mathbf{H}^2_{\mathbb{R}}$ then $z \notin \mathbf{\Lambda}$. On the other hand, by definition of the set $\mathbf{\Lambda}$, this contains all points in $\partial\mathbf{H}^2_{\mathbb{R}}$. Now consider a point x in the interior of $\mathcal{M}$. This point determines exactly two real projective lines passing through x and tangent to $\partial\mathbf{H}^2_{\mathbb{R}}$, and each of these lines determines a complex projective line which is in $\mathbf{\Lambda}$ and contains the point x. □

To finish this section we have:

Proposition 2.6. *The function $f(\mathbf{z}) = \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle$ is invariant under the standard action of $\mathrm{SO}_+(2,1)$ as a subgroup of $\mathrm{SU}(2,1)$.*

Proof. Recall that a basic property of the elements in $\mathrm{SO}_+(2,1)$ is that these matrices satisfy $\overline{A\mathbf{z}} = A\overline{\mathbf{z}}$. Then one has

$$\begin{aligned} f(A\mathbf{z}) &:= \big\langle i(A\mathbf{z}) \boxtimes \overline{(A\mathbf{z})}, i(A\mathbf{z}) \boxtimes \overline{(A\mathbf{z})} \big\rangle \\ &= \big\langle i(A\mathbf{z}) \boxtimes (A\overline{\mathbf{z}}), i(A\mathbf{z}) \boxtimes (A\overline{\mathbf{z}}) \big\rangle \\ &= \big\langle A(i\mathbf{z} \boxtimes \overline{\mathbf{z}}), A(i\mathbf{z} \boxtimes \overline{\mathbf{z}}) \big\rangle \\ &= \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle \\ &= f(\mathbf{z}). \end{aligned}$$

□

A consequence of this result is that $\mathbf{\Lambda}$, as defined algebraically in Corollary 2.5 is invariant under the action of $\mathrm{SO}_+(2,1)$. This proves the first part of Theorem 1 stated in the introduction.

2.2. A partition of $\mathbb{CP}^2$ determined by the cross-product. Given the function $f(\mathbf{z}) = \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle$, we consider the sets:

$$U_+ = \{\mathbf{z} \in \mathbb{C}^3 \ : \ f(\mathbf{z}) > 0\} = \{\mathbf{z} \in \mathbb{C}^3 \ : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_+\}, \tag{5}$$

$$\begin{aligned} U_0 &= \{\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathbf{0}\} \ : \ f(\mathbf{z}) = 0\} \\ &= \{\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathbf{0}\} : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_0\} \cup \{\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathbf{0}\} : i\mathbf{z} \boxtimes \overline{\mathbf{z}} = \mathbf{0}\}, \end{aligned} \tag{6}$$

$$U_- = \{\mathbf{z} \in \mathbb{C}^3 \ : \ f(\mathbf{z}) < 0\} = \{\mathbf{z} \in \mathbb{C}^3 \ : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_-\}. \tag{7}$$

By definition of the Hermitian cross-product, given $\lambda \in \mathbb{C}^*$ we have:

$$f(\lambda\mathbf{z}) = \left\langle i(\lambda\mathbf{z}) \boxtimes \overline{(\lambda\mathbf{z})}, i(\lambda\mathbf{z}) \boxtimes \overline{(\lambda\mathbf{z})} \right\rangle = \left\langle |\lambda|^2 (i\mathbf{z} \boxtimes \overline{\mathbf{z}}), |\lambda|^2 (i\mathbf{z} \boxtimes \overline{\mathbf{z}}) \right\rangle = |\lambda|^4 f(\mathbf{z}).$$

Hence the partition $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\} = U_+ \cup U_0 \cup U_-$ descends to a partition of $\mathbb{CP}^2$, and these sets are $\mathrm{SO}_+(2,1)$-invariant by Proposition 2.6.

In this section we prove the following theorem, which completes the proof of Theorem 1.

THEOREM 2.7. *Let U_+, U_0, U_- be as in* (5), (6) *and* (7). *The induced partition of $\mathbb{CP}^2$ into the three $\mathrm{SO}_+(2,1)$-invariant sets $\mathbb{P}U_+$, $\mathbb{P}U_0$, $\mathbb{P}U_-$ has the following properties:*

(i) The set $\mathbb{P}U_0 \setminus \mathbf{H}^2_{\mathbb{R}} = \mathbb{P}U_0 \setminus \mathbb{P}V_-$ is the lambda set $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathrm{SO}_+(2,1))$.

(ii) The omega set is $\mathbf{\Omega} = \mathbf{\Omega}(\mathrm{SO}_+(2,1)) = (\mathbb{P}U_+ \cup \mathbf{H}^2_{\mathbb{R}}) \cup \mathbb{P}U_-$.

(iii) Each point in the interior of the Möbius strip $\mathcal{M} := \mathcal{P}_{\Re} \setminus \mathbf{H}^2_{\mathbb{R}}$ is the meeting place of exactly two of the projective lines in $\mathbf{\Lambda}$, and the set $\mathbf{\Lambda} \setminus \mathcal{M}$ is a smooth 3-manifold which fibres over $\partial \mathbf{H}^2_{\mathbb{R}}$ with fibre two open 2-discs. Hence this set is diffeomorphic to the disjoint union of two solid tori $S^1 \times \mathbb{R}^2$.

(iv) The omega set $\mathbf{\Omega}$ is a complete Kobayashi hyperbolic space.

We set $\Omega_+ = \mathbb{P}U_+ \cup \mathbf{H}^2_{\mathbb{R}}$ and $\Omega_- = \mathbb{P}U_-$, so the omega set $\mathbf{\Omega} := \mathbb{CP}^2 \setminus \mathbf{\Lambda}$ is the union $\Omega_+ \cup \Omega_-$. We remark that statements (i) and (ii) follow immediately from Corollary 2.5 and the lemma below.

LEMMA 2.8. *If $\mathbf{z}$ is in U_- or in $U_0 \setminus \mathbb{R}^{2,1}$, then $\mathbf{z}$ is in V_+. Or equivalently, if $\mathbf{z}$ is in $V_- \cup V_0$ and $\mathbf{z} \notin \mathbb{R}^{2,1}$, then it is in U_+*

Proof. This lemma is basically a restatement of Lemma 2.1 in the case where $\mathbf{w} = \overline{\mathbf{z}}$. Suppose that $\mathbf{z}$ is not in $\mathbb{R}^{2,1}$. Then $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent. Hence $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is non-zero and orthogonal to the plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ spanned by these two vectors.

Observe that if $\mathbf{z}$ (and hence $\overline{\mathbf{z}}$) is in V_- or in V_0, then $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ intersects V_- and therefore $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_+ by Lemma 2.1 (b). By definition, this means that $\mathbf{z}$ is in U_+.

Next, if $\mathbf{z}$ (and hence $\overline{\mathbf{z}}$) is in V_+ then one of three things can happen:

(a) The plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ intersects V_-. Then, as above, its orthogonal vector $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_+ and so $\mathbf{z}$ is in U_+.

(b) The plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ is tangent to V_0. Then $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_0 and $\mathbf{z}$ is in U_0.

(c) The plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ is entirely in V_+. Then $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_- and $\mathbf{z}$ is in U_-.

Conversely (still assuming $\mathbf{z} \notin \mathbb{R}^{2,1}$) we have three possibilities, again corresponding to the different parts of Lemma 2.1:

(a) If $\mathbf{z}$ is in U_- then $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_- and the plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ lies entirely outside the light cone. In particular, $\mathbf{z}$ is in V_+.

(b) If $\mathbf{z}$ is in U_0 then $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in V_0, the plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ is tangent to the light cone and the tangency lies in $\mathcal{P}_{\Re}$. So $\mathbf{z}$ is not the point of tangency, hence $\mathbf{z}$ is in V_+.

(c) If $\mathbf{z}$ is in U_+ then $i\mathbf{z}\boxtimes\overline{\mathbf{z}}$ is in V_+ and the plane $\overleftrightarrow{\mathbf{z},\overline{\mathbf{z}}}$ intersects V_-, V_0 and V_+. It is not clear which of these contain $\mathbf{z}$.

Thus, the only way that $\mathbf{z}\in\mathbb{C}^{2,1}\setminus\mathbb{R}^{2,1}$ can be in $V_-\cup V_0$ is for it to be in U_+. □

Let us prove statement (iii) in Theorem 2.7. Every complex projective line $\mathcal{L}$ in $\mathbf{\Lambda}$ meets the Lagrangian plane $\mathcal{P}_{\mathbb{R}}\cong\mathbb{RP}^2$ in a real projective line ℓ which necessarily is contained in $\mathcal{M}$ and is tangent to $\partial\mathbf{H}^2_{\mathbb{R}}$. Recall that every two complex projective lines in $\mathbb{CP}^2$ meet in exactly one point, and every two real projective lines in $\mathbb{RP}^2$ meet in exactly one point. Therefore if we are given two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbf{\Lambda}$, then their intersection is the meeting point of the corresponding real projective lines ℓ_1 and ℓ_2, which a point in $\mathcal{M}$. Then the claim that each interior point in $\mathcal{M}$ is the meeting place of exactly two lines in $\mathbf{\Lambda}$ follows from the fact that given the circle $\partial\mathbf{H}^2_{\mathbb{R}}$ in $\mathbb{RP}^2$ and a point x in the interior of $\mathcal{M}$, there are exactly two projective lines passing through x and which are tangent to $\partial\mathbf{H}^2_{\mathbb{R}}$.

As remarked above, every complex projective line $\mathcal{L}$ in $\mathbf{\Lambda}$ meets the Lagrangian plane $\mathcal{P}_{\mathbb{R}}\cong\mathbb{RP}^2$ in a real projective line ℓ which necessarily is contained in $\mathcal{M}$, and the meeting point of any two of such lines is a point in $\mathcal{M}$. Hence every point in $\mathbf{\Lambda}\setminus\mathcal{M}$ is contained in a unique line in $\mathbf{\Lambda}$. This determines in the obvious way a projection

$$\pi:\mathbf{\Lambda}\setminus\mathcal{M}\longrightarrow\partial\mathbf{H}^2_{\mathbb{R}}\,,$$

and this is a fibre bundle with fibre a complex projective line –that is a 2-sphere– minus its intersection with $\mathcal{M}$, which is a real projective line, –that is a circle. This proves the statement in Theorem 2.7 (iii).

It remains to prove statement (iv) to complete the proof of Theorem 2.7. Recall that Kobayashi hyperbolic means that the Kobayashi pseudo metric is an actual metric. Using [9] and [1, Theorem 1.3], a subset of $\mathbb{CP}^2$ that misses more than 4 lines in general position is necessarily a complete Kobayashi hyperbolic space. In our case, the omega set is the complement of the lambda set, $\mathbf{\Omega}:=\mathbb{CP}^2\setminus\mathbf{\Lambda}$. The previous arguments show that $\mathbf{\Lambda}$ has infinitely many lines: namely, one for each point in the circle $\partial\mathbf{H}^2_{\mathbb{R}}$. Furthermore, through each point in the Möbius strip $\mathcal{M}$ one has exactly two such lines. Thence $\mathbf{\Lambda}$ has infinitely many lines in general position and the result follows.

Notice that this contrasts with the $\mathrm{PU}(1,1)$ case where, by Proposition 1.9, all the lines in the corresponding set $\mathbf{\Lambda}(\mathrm{PU}(1,1))$ pass through the focal point of the complex geodesic which is $\mathrm{PU}(1,1)$-invariant.

3. The fibre bundle $\Omega\to\mathbf{H}^2_{\mathbb{R}}$.

3.1. The cross-product and the fibre bundle. Consider the sets U_+, U_-, U_0 and their projectivisations $\mathbb{P}U_+$, $\mathbb{P}U_-$, $\mathbb{P}U_0$ as in Theorem 2.7. We know, Lemma 2.8, that U_+ contains the space of negative vectors V_- in $\mathbb{C}^{2,1}\setminus\mathbb{R}^{2,1}$. Their projectivisation is contained in $\mathbf{H}^2_{\mathbb{C}}\setminus\mathbf{H}^2_{\mathbb{R}}$. Goldman in Section 3.3.6 (page 107) of [7] studies the orthogonal projection $\Pi_{\mathbb{R}}:\mathbf{H}^n_{\mathbb{C}}\longrightarrow\mathbf{H}^n_{\mathbb{R}}$. He shows that $\Pi_{\mathbb{R}}$ assigns to each $z\in\mathbf{H}^n_{\mathbb{C}}$ the midpoint $m(z)$ of the real geodesic segment joining z and $\overline{z}$. For this (restricting the discussion to the case $n=2$) he introduces a function η by choosing, for each $\mathbf{z}\in\mathbb{C}^{2,1}$, a number $\eta(\mathbf{z})$ so that

$$\eta^2(\mathbf{z})=-\langle\mathbf{z},\overline{\mathbf{z}}\rangle=z_3^2-z_1^2-z_2^2\,.$$

Then, for negative vectors $\mathbf{z}$ the orthogonal projection $\Pi_{\mathbb{R}}$ carries the point $z=\mathbb{P}(\mathbf{z})$ into the projectivisation $m(z)$ of the vector defined by:

$$\mathbf{m}(\mathbf{z}):=\mathbf{z}\,\overline{\eta}(\mathbf{z})+\overline{\mathbf{z}}\,\eta(\mathbf{z})\,.$$

Note that taking a different square root of $\eta^2(\mathbf{z})$ changes $\mathbf{m}(\mathbf{z})$ by a sign. Hence $m(\mathbf{z}) = \mathbb{P}(\mathbf{m}(\mathbf{z}))$ is not affected by this choice.

Following this construction we may now define a function $\widetilde{\Pi} : U_+ \cup U_- \to \mathbb{C}^{2,1}$ by:

$$\widetilde{\Pi}(\mathbf{z}) = \begin{cases} \mathbf{z}\,\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\,\eta(\mathbf{z}) & \text{if } \mathbf{z} \in U_+, \text{ that is } f(\mathbf{z}) > 0, \text{ where } \eta^2(\mathbf{z}) = -\langle \mathbf{z}, \overline{\mathbf{z}} \rangle\,, \\ i\mathbf{z} \boxtimes \overline{\mathbf{z}} & \text{if } \mathbf{z} \in U_-, \text{ that is } f(\mathbf{z}) < 0. \end{cases}$$

By construction, including use of (3), the image of $\widetilde{\Pi}$ is contained in $\mathbb{R}^{2,1} \subset \mathbb{C}^{2,1}$, the Lagrangian subspace comprising points with real coordinates.

Lemma 3.1. *The projection* $\widetilde{\Pi} : U_+ \cup U_- \longrightarrow \mathbb{R}^{2,1}$ *is* $\mathrm{SO}_+(2,1)$*-equivariant: If* $A \in \mathrm{SO}_+(2,1)$ *then* $\widetilde{\Pi}(A\mathbf{z}) = A\,\widetilde{\Pi}(\mathbf{z})$.

Proof. We consider separately the two cases $U_\pm$. Given $\mathbf{z} \in U_-$ we have:

$$\widetilde{\Pi}(A\mathbf{z}) = i(A\mathbf{z}) \boxtimes \overline{(A\mathbf{z})} = i(A\mathbf{z}) \boxtimes (A\overline{\mathbf{z}}) = A(i\mathbf{z} \boxtimes \overline{\mathbf{z}}) = A\,\widetilde{\Pi}(\mathbf{z})\,.$$

So $\widetilde{\Pi}$ is equivariant in this case.

Now suppose that $\mathbf{z} \in U_+$. First, note that every $A \in \mathrm{SO}_+(2,1)$ preserves the $(2,1)$-Hermitian form and satisfies $\overline{A\mathbf{z}} = A\overline{\mathbf{z}}$ for all $\mathbf{z}$, so we have:

$$\eta^2(A\mathbf{z}) = -\langle A\mathbf{z}, \overline{A\mathbf{z}} \rangle = -\langle A\mathbf{z}, A\overline{\mathbf{z}} \rangle = -\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \eta^2(\mathbf{z})\,.$$

Hence η is $\mathrm{SO}_+(2,1)$-invariant. In particular, $\eta(A\mathbf{z}) = \eta(\mathbf{z})$. Therefore, we have:

$$\widetilde{\Pi}(A\mathbf{z}) = A\mathbf{z}\,\overline{\eta}(A\mathbf{z}) + \overline{A\mathbf{z}}\,\eta(A\mathbf{z}) = A\mathbf{z}\,\overline{\eta}(\mathbf{z}) + A\overline{\mathbf{z}}\,\eta(\mathbf{z}) = A(\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\,\eta(\mathbf{z})) = A\,\widetilde{\Pi}(\mathbf{z}).$$

Hence $\widetilde{\Pi}$ is equivariant in this case as well. □

Lemma 3.2. *For each* $\mu \in \mathbb{C} \setminus \{0\}$ *one has* $\widetilde{\Pi}(\mu\mathbf{z}) = |\mu|^2\widetilde{\Pi}(\mathbf{z})$.

Proof. If $\mathbf{z} \in U_-$ then by definition we have:

$$\widetilde{\Pi}(\mu\,\mathbf{z}) = i(\mu\,\mathbf{z}) \boxtimes (\overline{\mu}\,\overline{\mathbf{z}}) = \overline{\mu}\,\mu(i\mathbf{z} \boxtimes \overline{\mathbf{z}}) = |\mu|^2(i\mathbf{z} \boxtimes \overline{\mathbf{z}}) = |\mu|^2\widetilde{\Pi}(\mathbf{z}),$$

as stated. Now we consider points in U_+. Observe first that we have:

$$\eta^2(\mu\mathbf{z}) = -\langle \mu\mathbf{z}, \overline{\mu\overline{\mathbf{z}}} \rangle = -\langle \mu\mathbf{z}, \overline{\mu}\,\overline{\mathbf{z}} \rangle = -\mu^2\langle \mathbf{z}, \bar{\mathbf{z}} \rangle = \mu^2\eta^2(\mathbf{z}).$$

Hence for all $\mathbf{z} \in U_+$ we have:

$$\widetilde{\Pi}(\mu\,\mathbf{z}) = \mu\,\mathbf{z}\,\overline{\eta}(\mu\mathbf{z}) + \overline{(\mu\,\mathbf{z})}\,\eta(\mu\mathbf{z}) = \mu\,\mathbf{z}\,\overline{\mu}\,\overline{\eta}(\mathbf{z}) + \overline{\mu}\,\overline{\mathbf{z}}\,\mu\,\eta(\mathbf{z}) = |\mu|^2\widetilde{\Pi}(\mathbf{z}).$$

□

An immediate consequence of Lemma 3.2 is that the map $\widetilde{\Pi}$ determines a well-defined projection map $\Pi : \mathbb{P}U_+ \cup \mathbb{P}U_- \longrightarrow \mathcal{P}_{\Re} \subset \mathbb{CP}^2$ by

$$\Pi(z) = \mathbb{P}(\widetilde{\Pi}(\mathbf{z})).$$

We can extend this map continuously across $\mathbf{H}^2_{\mathbb{R}}$ by requiring that

$$\Pi(z) = z, \qquad \text{when } z \in \mathbf{H}^2_{\mathbb{R}}.$$

This is consistent with the definition of $\widetilde{\Pi}$ on U_+: If $\mathbf{z}$ has real entries and lies in V_- then, using $\overline{\mathbf{z}} = \mathbf{z}$, we have

$$\eta^2(\mathbf{z}) = \big|\langle \mathbf{z}, \mathbf{z}\rangle\big| ,$$

and so $\overline{\eta}(\mathbf{z}) = \eta(\mathbf{z})$. Hence $\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z}) = 2\eta(\mathbf{z})\,\mathbf{z}$, and so $\mathbb{P}\big((\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})\big) = z$. Moreover, using Lemma 3.1, the map $\Pi : (\mathbb{P}U_+ \cup \mathbf{H}^2_{\mathbb{R}}) \cup \mathbb{P}U_- \longrightarrow \mathcal{P}_{\Re}$ is $\mathrm{SO}_+(2,1)$-equivariant. The next lemma shows that the image of Π is contained in the real hyperbolic disc $\mathbf{H}^2_{\mathbb{R}} \subset \mathcal{P}_{\Re}$.

LEMMA 3.3. *The image of $\widetilde{\Pi}$ is in $V_- \cap \mathbb{R}^{2,1}$. Hence the image of Π is in $\mathbf{H}^2_{\mathbb{R}}$.*

Proof. We saw by construction that $\widetilde{\Pi}(\mathbf{z})$ is invariant under complex conjugation and so the image of $\widetilde{\Pi}$ is in $\mathbb{R}^{2,1} \subset \mathbb{C}^{2,1}$. So it suffices to show $\widetilde{\Pi}(\mathbf{z}) \in V_-$.

The proof that the image of $\widetilde{\Pi}$ is in V_- is again is by cases. If $\mathbf{z} \in U_-$ then, $\widetilde{\Pi}(\mathbf{z}) = i\mathbf{z}\boxtimes\overline{\mathbf{z}}$, by definition of $\widetilde{\Pi}$, and $i\mathbf{z}\boxtimes\overline{\mathbf{z}} \in V_-$ by the definition of U_-. This proves the result in the first case.

Now suppose $\mathbf{z} \in U_+$. By definition, this means that $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_+$. In this case identity (2.16) in [7] implies:

$$\begin{aligned} 0 &< \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}}\rangle \\ &= \langle \overline{\mathbf{z}}, \mathbf{z}\rangle\langle \mathbf{z}, \overline{\mathbf{z}}\rangle - \langle \mathbf{z}, \mathbf{z}\rangle\langle \overline{\mathbf{z}}, \overline{\mathbf{z}}\rangle \\ &= (-\overline{\eta}^2(\mathbf{z}))(-\eta^2(\mathbf{z})) - \langle \mathbf{z}, \mathbf{z}\rangle^2 \\ &= |\eta(\mathbf{z})|^4 - \langle \mathbf{z}, \mathbf{z}\rangle^2 . \end{aligned}$$

So, if $\mathbf{z} \in U_+$ then $|\eta(\mathbf{z})|^2 > \langle \mathbf{z}, \mathbf{z}\rangle$. Note that this includes the case where $\langle \mathbf{z}, \mathbf{z}\rangle < 0$. Since $\mathbf{z} \in U_+$, we have $\widetilde{\Pi}(\mathbf{z}) = \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})$. We now show this is in V_-.

$$\begin{aligned} \big\langle \widetilde{\Pi}(\mathbf{z}), \widetilde{\Pi}(\mathbf{z})\big\rangle &= \langle \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z}), \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})\rangle \\ &= \langle \mathbf{z}\overline{\eta}(\mathbf{z}), \mathbf{z}\overline{\eta}(\mathbf{z})\rangle + \langle \mathbf{z}\overline{\eta}(\mathbf{z}), \overline{\mathbf{z}}\eta(\mathbf{z})\rangle + \langle \overline{\mathbf{z}}\eta(\mathbf{z}), \mathbf{z}\overline{\eta}(\mathbf{z})\rangle + \langle \overline{\mathbf{z}}\eta(\mathbf{z}), \overline{\mathbf{z}}\eta(\mathbf{z})\rangle \\ &= |\eta(\mathbf{z})|^2\langle \mathbf{z}, \mathbf{z}\rangle + \overline{\eta}^2(\mathbf{z})\langle \mathbf{z}, \overline{\mathbf{z}}\rangle + \eta^2(\mathbf{z})\langle \overline{\mathbf{z}}, \mathbf{z}\rangle + |\eta(\mathbf{z})|^2\langle \overline{\mathbf{z}}, \overline{\mathbf{z}}\rangle \\ &= |\eta(\mathbf{z})|^2\langle \mathbf{z}, \mathbf{z}\rangle - \overline{\eta}^2(\mathbf{z})\eta^2(\mathbf{z}) - \eta^2(\mathbf{z})\overline{\eta}^2(\mathbf{z}) + |\eta(\mathbf{z})|^2\langle \overline{\mathbf{z}}, \overline{\mathbf{z}}\rangle \\ &= 2|\eta(\mathbf{z})|^2\big(\langle \mathbf{z}, \mathbf{z}\rangle - |\eta(\mathbf{z})|^2\big) \\ &< 0 . \end{aligned}$$

Hence $\widetilde{\Pi}(\mathbf{z}) \in V_-$ both when $\mathbf{z} \in U_-$ and when $\mathbf{z} \in U_+$. Putting this together we also see that $\Pi(z) = \mathbb{P}\big(\widetilde{\Pi}(\mathbf{z})\big) \in \mathbb{P}(V_- \cap \mathbb{R}^{2,1}) = \mathbf{H}^2_{\mathbb{R}}$. ◻

The proof of this result also yields the following corollary, which means that we can extend the definition of $\Pi(z)$ continuously to all $z \in \mathbb{P}(U_0 \setminus \mathbb{R}^{2,1})$. In fact, we can extend it continuously to $\mathbb{CP}^2 \setminus \mathcal{M}^\circ$, *i.e.*, all points of $\mathbb{CP}^2$ not in the interior of the Möbius strip $\mathcal{M}$.

COROLLARY 3.4. *Let $\mathbf{z}$ be a vector in V_+ for which $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent and $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_0$. Then $\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})$ is in the subspace spanned by $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$.*

Proof. Arguing as in the proof of Lemma 3.3, the fact that $i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_0$ implies

$$0 = \langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}}\rangle = |\eta(\mathbf{z})|^4 - \langle \mathbf{z}, \mathbf{z}\rangle^2 .$$

Since $\mathbf{z} \in V_+$, this implies that $|\eta(\mathbf{z})|^2 = \langle \mathbf{z}, \mathbf{z}\rangle > 0$.

By construction, the point $\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})$ is in the complex 2-plane $\overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}}$ spanned by $\mathbf{z}$ and $\overline{\mathbf{z}}$. Again arguing as in Lemma 3.3 we see that

$$\langle \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z}), \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z})\rangle = 2|\eta(\mathbf{z})|^2 \big(\langle \mathbf{z}, \mathbf{z}\rangle - |\eta(\mathbf{z})|^2\big) = 0.$$

Hence we see that $\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z}) \in \overleftrightarrow{\mathbf{z}, \overline{\mathbf{z}}} \cap V_0$. From Lemma 2.1 (c) we know this subspace is spanned by $i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ as required. □

Furthermore, the above results enable us to show that U_- has two components. If $\mathbf{z} \in U_-$ then $\widetilde{\Pi}(\mathbf{z}) = i\mathbf{z} \boxtimes \overline{\mathbf{z}}$ is in $V_- \cap \mathbb{R}^{2,1}$, the interior of the real light cone, that is, it is a timelike vector in Minkowski space. The interior of the real light cone has two components, corresponding to future pointing vectors and past pointing vectors. These two components are distinguished by the sign of the third entry (which is necessarily non-zero). Therefore, we make the following definition:

For $\mathbf{z} \in U_-$ we define $\alpha(\mathbf{z})$ to be the bottom entry in the vector $\widetilde{\Pi}(\mathbf{z}) = i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_- \cap \mathbb{R}^{2,1}$. That is

$$\alpha(\mathbf{z}) = \big(i\mathbf{z} \boxtimes \overline{\mathbf{z}}\big)_3 \in \mathbb{R} \setminus \{0\}. \tag{8}$$

Hence $\alpha(\mathbf{z}) > 0$ (respectively < 0) if and only if $i\mathbf{z}\boxtimes\overline{\mathbf{z}}$ is future pointing (respectively past pointing).

Lemma 3.5. *The set U_- has two components characterised by the sign of α:*

$$U_-^1 = \Big\{\mathbf{z} \in U_- : \alpha(\mathbf{z}) > 0\Big\}, \qquad U_-^2 = \Big\{\mathbf{z} \in U_- : \alpha(\mathbf{z}) < 0\Big\}.$$

Moreover, any $A \in \mathrm{SO}_+(2,1)$ preserves these components, and for any $\mu \in \mathbb{C} \setminus \{0\}$ and $\mathbf{z} \in U_-$, the vector $\mu\mathbf{z}$ is in the same component as $\mathbf{z}$.

Proof. Using (8) we see that α is a continuous function of $\mathbf{z}$. Since it never takes the value 0, it distinguishes two components.

Using Lemma 3.1, $\alpha(A\mathbf{z})$ is the bottom entry in the vector $Ai\mathbf{z} \boxtimes \overline{\mathbf{z}}$. Then the well known fact that the component $\mathrm{SO}_+(2,1)$ of $\mathrm{SO}(2,1)$ is characterised by sending future pointing vectors to future pointing vectors means that $\alpha(A\mathbf{z}) > 0$ (respectively < 0) if and only if $\alpha(\mathbf{z}) >$ (respectively < 0). Therefore A maps U_-^1 to itself and maps U_-^2 to itself.

From Lemma 3.2, for any $\mu \in \mathbb{C}\setminus\{0\}$, we have $\alpha(\mu\mathbf{z}) = |\mu|^2\, \alpha(\mathbf{z})$. Thus $\alpha(\mu\mathbf{z}) > 0$ (respectively < 0) if and only if $\alpha(\mathbf{z}) > 0$ (respectively < 0). □

Recall that $\Omega_+ = \mathbb{P}U_+ \cup \mathbf{H}^2_{\mathbb{R}}$ and $\Omega_- = \mathbb{P}U_-$, so the omega set $\boldsymbol{\Omega} := \mathbb{CP}^2 \setminus \boldsymbol{\Lambda}$ is the union $\Omega_+ \cup \Omega_-$. Moreover, from Lemma 3.5 the components U_-^1 and U_-^2 of U_- are projectively invariant and so we define $\Omega_-^1 = \mathbb{P}U_-^1$ and $\Omega_-^2 = \mathbb{P}U_-^2$. That is

$$\Omega_+ = \Big\{z \in \mathbb{CP}^2 : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_+ \text{ or } \mathbf{z} \in V_- \text{ for any } \mathbf{z} \text{ with } \mathbb{P}(\mathbf{z}) = z\Big\}, \tag{9}$$

$$\Omega_-^1 = \Big\{z \in \mathbb{CP}^2 : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_- \text{ and } \alpha(\mathbf{z}) > 0 \text{ for any } \mathbf{z} \text{ with } \mathbb{P}(\mathbf{z}) = z\Big\}, \tag{10}$$

$$\Omega_-^2 = \Big\{z \in \mathbb{CP}^2 : i\mathbf{z} \boxtimes \overline{\mathbf{z}} \in V_- \text{ and } \alpha(\mathbf{z}) < 0 \text{ for any } \mathbf{z} \text{ with } \mathbb{P}(\mathbf{z}) = z\Big\}. \tag{11}$$

In the next two sections we will give a precise description of Ω_+, Ω_-^1 and Ω_-^2. We will show they are all connected, and hence $\boldsymbol{\Omega}$ does have exactly three components.

Summarising the previous discussion we have:

THEOREM 3.6. *The projection* $\Pi : \mathbf{\Omega} \to \mathbf{H}^2_{\mathbb{R}}$ *is an* $\mathrm{SO}_+(2,1)$*-equivariant smooth fibre bundle.*

The fact that this actually is a fibre bundle follows immediately from the fact the projection is equivariant, which implies that each fibre has a product neighbourhood. This is the fibre bundle in Theorem 3.

3.2. The fibre over the origin. In this section we consider the pre-image under Π of the point $o = [0 : 0 : 1] \in \mathbb{CP}^2$, which corresponds to the origin in the Klein-Beltrami disc embedded in the ball $\mathbf{H}^2_{\mathbb{C}}$, where Π is the projection map in Theorem 3.6.

Since $\mathrm{SO}_+(2,1)$ acts transitively on $\mathbf{H}^2_{\mathbb{R}}$ and the map Π is equivariant, to determine the fibre of Π over an arbitrary point $x = [x_1 : x_2 : x_3] \in \mathbf{H}^2_{\mathbb{R}}$, we need only to determine the fibre over the special point o, and then see how this fibre moves under the action of $\mathrm{SO}_+(2,1)$ on points in $\mathbf{H}^2_{\mathbb{R}}$. That is what we do in the next section; here we focus on the fibres over o. We consider the components of the fibre in $\Omega_+ = \mathbb{P}U_+ \cup \mathbf{H}^2_{\mathbb{R}}$ and $\Omega_- = \mathbb{P}U_-$ separately.

LEMMA 3.7. *Suppose that* $z = [z_1 : z_2 : z_3] \in \Omega_+$ *is a point for which* $\Pi(z) = o = [0 : 0 : 1]$*. Then* $z_3 \neq 0$ *and* z_1/z_3*,* z_2/z_3 *are both purely imaginary. Hence* $L_o = (\Pi|_{\Omega_+})^{-1}(o)$*, the fibre over* o *of the projection* Π *restricted to* Ω_+ *is the Lagrangian plane*

$$L_o = \Big\{ [iy_1 : iy_2 : x_3] : y_1,\, y_2,\, x_3 \in \mathbb{R},\ x_3 \neq 0 \Big\}.$$

Its boundary consists of the circle C_o *comprising all points in* $\mathbb{CP}^2$ *that can be represented by homogeneous coordinates of the same form but with* $x_3 = 0$*. That is*

$$C_o = \Big\{ [iy_1 : iy_2 : 0] : (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0\} \Big\}.$$

Proof. Since $z \in \Omega_+$ we have $\mathbf{z} \in U_+$ or $z \in \mathbf{H}^2_{\mathbb{R}}$. If $\Pi(z) = o$ then

$$\widetilde{\Pi}(\mathbf{z}) = \mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{\mathbf{z}}\eta(\mathbf{z}) = \begin{pmatrix} z_1\overline{\eta} + \overline{z}_1\eta \\ z_2\overline{\eta} + \overline{z}_2\eta \\ z_3\overline{\eta} + \overline{z}_3\eta \end{pmatrix} = \mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\eta^2 = \eta^2(\mathbf{z}) = z_3^2 - z_1^2 - z_2^2$. Then

$$0 = z_1\overline{\eta} + \overline{z}_1\eta = z_2\overline{\eta} + \overline{z}_2\eta \quad \text{and} \quad 1 = z_3\overline{\eta} + \overline{z}_3\eta\,.$$

These inequalities imply

$$0 = z_1\eta(z_1\overline{\eta} + \overline{z}_1\eta) = z_1^2|\eta|^2 + |z_1|^2\eta^2, \qquad 0 = z_2\eta(z_2\overline{\eta} + \overline{z}_2\eta) = z_2^2|\eta|^2 + |z_2|^2\eta^2.$$

Hence, if we set $\eta^2 = |\eta|^2 e^{i\theta}$ then $z_1^2 = -|z_1|^2 e^{i\theta}$ and $z_2^2 = -|z_2|^2 e^{i\theta}$. Therefore

$$|\eta|^2 e^{i\theta} = \eta^2 = z_3^2 - z_1^2 - z_2^2 = z_3^2 + |z_1|^2 e^{i\theta} + |z_2|^2 e^{i\theta}.$$

Thus, $z_3^3 = \varepsilon |z_3|^2 e^{i\theta}$ where $\varepsilon = \pm 1$ and $|\eta|^2 = \varepsilon|z_3|^2 + |z_1|^2 + |z_2|^2$. However,

$$1 = (z_3\overline{\eta} + \overline{z}_3\eta)^2 = z_3^2\overline{\eta}^2 + 2|z_3|^2|\eta|^2 + \overline{z}_3^2\eta^2 = 2\varepsilon|z_3|^2|\eta|^2 + 2|z_3|^2|\eta|^2 = 2(\varepsilon+1)|z_3|^2|\eta|^2.$$

As the left hand side is non-zero we must have $z_3 \neq 0$ and $\varepsilon = +1$. This means that $z_3^2 = |z_3|^2 e^{i\theta}$ and so, $z_1^2/z_3^2 = -|z_1|^2/|z_3|^2 \leq 0$ and $z_2^2/z_3^2 = -|z_2|^2/|z_3|^2 \leq 0$. Hence z_1/z_3 and z_2/z_3 are both purely imaginary. □

This generalises the result of Goldman in [7], used by Parker and Platis in [13], that the pre-image of the origin in $\mathbf{H}^2_{\mathbb{C}}$ under orthogonal projection onto the real Lagrangian plane $\mathbf{H}^2_{\mathbb{R}}$ is the purely imaginary Lagrangian plane

$$\Big\{[iy_1 : iy_2 : 1] \ : \ y_1, y_2 \in \mathbb{R},\ y_1^2 + y_2^2 < 1\Big\}.$$

In [7, 13] the extra condition $y_1^2 + y_2^2 < 1$ was imposed to ensure that L_o was contained in $\mathbf{H}^2_{\mathbb{C}}$. We drop this condition and use L_o as given by Lemma 3.7.

Note that the closure $\overline{L}_o = L_o \cup C_o$ of the Lagrangian plane L_o is the fixed point set of the antiholomorphic involution in $\mathbb{C}^{2,1}$ given by

$$R_o : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \longmapsto \begin{pmatrix} -\overline{z}_1 \\ -\overline{z}_2 \\ \overline{z}_3 \end{pmatrix}. \tag{12}$$

Now we consider the fibre over o of the bundle $\Omega_- \xrightarrow{\Pi} \mathbf{H}^2_{\mathbb{R}}$.

Lemma 3.8. *Suppose that $z = [z_1 : z_2 : z_3] \in \Omega_-$ is a point for which $\Pi(z) = o = [0 : 0 : 1]$. Then $z_3 = 0$ and $\Im(z_1\overline{z}_2) \neq 0$. Hence $L_o = (\Pi|_{\Omega_-})^{-1}(o)$, the fibre over o of the projection Π restricted to Ω_- consists of two open hemispheres D_o^1, D_o^2 contained in the projective line $S_o = \big\{[z_1 : z_2 : 0] \in \mathbb{CP}^2\big\}$ and defined by $\Im(z_1\overline{z}_2) \neq 0$. In other words, the components of the fibre are:*

$$D_o^1 = \Big\{[z_1 : z_2 : 0] : z_1,\, z_2 \in \mathbb{C},\ \Im(z_1\overline{z}_2) > 0\Big\},$$
$$D_o^2 = \Big\{[z_1 : z_2 : 0] : z_1,\, z_2 \in \mathbb{C},\ \Im(z_1\overline{z}_2) < 0\Big\}.$$

The common boundary of D_o^1 and D_o^2 is the circle C_o from Lemma 3.7 , namely

$$C_o = \Big\{[z_1 : z_2 : 0] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0\},\ \Im(z_1\overline{z}_2) = 0\Big\}.$$

Proof. Since $z \in \Omega_-$ we have $\mathbf{z} \in U_-$. If $\Pi(z) = o$ then

$$\widetilde{\Pi}(\mathbf{z}) = i\mathbf{z} \boxtimes \overline{\mathbf{z}} = i \begin{pmatrix} \overline{z}_3 z_2 - \overline{z}_2 z_3 \\ \overline{z}_1 z_3 - \overline{z}_3 z_1 \\ \overline{z}_1 z_2 - \overline{z}_2 z_1 \end{pmatrix} = \begin{pmatrix} 2\Im(z_3\overline{z}_2) \\ 2\Im(z_1\overline{z}_3) \\ 2\Im(z_1\overline{z}_2) \end{pmatrix} = \mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{13}$$

Moreover $\alpha(\mathbf{z}) = \big(i\mathbf{z} \boxtimes \overline{\mathbf{z}}\big)_3 = 2\Im(z_1\overline{z}_2)$ so our description of D_o^1 and D_o^2 given above is consistent with Lemma 3.5. In particular, using (10) and (11), we see that D_o^1 is contained in Ω_-^1 and D_o^2 is contained in Ω_-^2.

We claim that any solution to (13) must have $z_3 = 0$. If we were to have $z_3 \neq 0$, then $z_1\overline{z}_2 = (z_1\overline{z}_3)(z_3\overline{z}_2)/|z_3|^2$. Since $\Im(z_1\overline{z}_3) = \Im(z_3\overline{z}_2) = 0$, then we also have $\Im(z_1\overline{z}_2) = 0$, which is a contradiction. Putting $z_3 = 0$ gives

$$\langle i\mathbf{z} \boxtimes \overline{\mathbf{z}}, i\mathbf{z} \boxtimes \overline{\mathbf{z}} \rangle = -4\big(\Im(z_1\overline{z}_2)\big)^2.$$

Since this should be negative, it is clear that $\Im(z_1\overline{z}_2) \neq 0$.

The projectivisation of vectors in $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\}$ with $z_3 = 0$ form the complex projective line (that is sphere):

$$S_o = \Big\{[z_1 : z_2 : 0] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\Big\}.$$

The condition $\Im(z_1\overline{z}_2) \neq 0$ divides this sphere into the two hemispheres D_o^1 and D_o^2 given above. These hemispheres are interchanged by the involution R_0 given in (12). □

We remark that Theorem 3 in the introduction is now an immediate consequence of Theorem 3.6, together with Lemmas 3.7 and 3.8.

3.3. The general fibre. Now we use Lemmas 3.7 and 3.8 to determine the general fibres of the bundle $\mathbf{\Omega} \to \mathbf{H}^2_{\mathbb{R}}$. This yields Theorem 4 in the introduction. For this it is convenient to think of $\mathbf{H}^2_{\mathbb{R}}$ as being the unit ball in $[x_1 : x_2 : 1]$ in $\mathbb{RP}^2$. Hence the coordinates $(x_1 : x_2)$ can be described in polar coordinates by $\big(\tanh(t)\cos(\theta), \tanh(t)\sin(\theta)\big)$. Thus in homogeneous coordinates we have:

$$x = [x_1 : x_2 : 1] = [\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1],$$

As t and θ vary, we obtain all points in the real hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ embedded in $\mathbf{H}^2_{\mathbb{C}} \subset \mathbb{CP}^2$.

The proof of the following lemma is left as an exercise to the reader:

LEMMA 3.9. *Let $x := [\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1]$ be an arbitrary point in $\mathbf{H}^2_{\mathbb{R}}$. Then the matrix A defined by:*

$$A := \begin{pmatrix} \cosh(t)\cos(\theta) & -\sin(\theta) & \sinh(t)\cos(\theta) \\ \cosh(t)\sin(\theta) & \cos(\theta) & \sinh(t)\sin(\theta) \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}$$

is in $\mathrm{SO}_+(2,1)$ and projectively carries $o = [0 : 0 : 1]$ into x.

We may now use this matrix A to translate the fibres over the special point o given by lemmas (3.7) and (3.8), to the fibres over an arbitrary point in $\mathbf{H}^2_{\mathbb{R}}$. In doing so, the matrix A will allow us to use a new basis $\mathcal{B}_x$ adapted to x. The new basis is

$$\mathcal{B}_x = \left\{ \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \tanh(t) \end{pmatrix}, \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} \tanh(t)\cos(\theta) \\ \tanh(t)\sin(\theta) \\ 1 \end{pmatrix} \right\}. \tag{14}$$

We note that the first and last vectors are projective images of the first and last basis vector under A. In fact we have scaled by $1/\cosh(t)$ in each case. Since $1/\cosh(t)$ is a positive real number, this does not have a significant effect on the fibres.

First consider the fibre in Ω_+ over x. Applying the matrix A immediately gives the following description of the fibre.

PROPOSITION 3.10. *Let $z \in \Omega_+$ be a point for which $\Pi(z) = x$, where $x \in \mathbf{H}^2_{\mathbb{R}}$ is the point*

$$x := [\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1].$$

Then z is the image under the map A given in Lemma 3.9 of a point in L_o. Hence $L_x = (\Pi|_{\Omega_+})^{-1}(x)$ is the Lagrangian plane

$$L_x = \mathbb{P}\left\{ iy_1 \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \tanh(t) \end{pmatrix} + iy_2 \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \tanh(t)\cos(\theta) \\ \tanh(t)\sin(\theta) \\ 1 \end{pmatrix} : \begin{matrix} y_1, y_2, x_3 \in \mathbb{R}, \\ x_3 \neq 0 \end{matrix} \right\}.$$

The boundary of L_o is the circle

$$C_x = \Big\{[iy_1\cos(\theta) - iy_2\sin(\theta) : iy_1\sin(\theta) + iy_2\cos(\theta) : iy_1\tanh(t)] : y_1, y_2 \in \mathbb{R}^2 \setminus \{(0,0)\}\Big\}.$$

We remark that each such plane L_x intersects $\mathbf{H}^2_{\mathbb{C}}$ in the set of all points in $\mathbf{H}^2_{\mathbb{C}} \setminus \mathbf{H}^2_{\mathbb{R}}$ which are contained in the totally real Lagrangian plane of points Hermitian orthogonal to $\mathbf{H}^2_{\mathbb{R}}$ at x.

We now consider the fibre in Ω_- over the point x. Once again, simply applying A gives the fibre.

Proposition 3.11. *Let $z \in \Omega_-$ be a point for which $\Pi(z) = x$, where $x \in \mathbf{H}^2_{\mathbb{R}}$ is the point*

$$x = [\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1].$$

Then z is the image under the map A given in Lemma 3.9 of a point in D^1_o or D^2_o. Hence $(\Pi|_{\Omega_-})^{-1}(x)$ comprises the two discs

$$D^1_x = \Big\{[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\tanh(t)] : \Im(z_1\overline{z}_2) > 0\Big\},$$
$$D^2_x = \Big\{[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\tanh(t)] : \Im(z_1\overline{z}_2) < 0\Big\}.$$

The common boundary of D^1_x and D^2_x is the circle C_x from Proposition 3.10, namely

$$C_x = \Big\{[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\tanh(t)] : \Im(z_1\overline{z}_2) = 0\Big\}.$$

Consider a vector $\mathbf{z}$ that projects to one of D^1_x, D^2_x, C_x as in Proposition 3.11, namely

$$\mathbf{z} = \begin{pmatrix} z_1\cos(\theta) - z_2\sin(\theta) \\ z_1\sin(\theta) + z_2\cos(\theta) \\ z_1\tanh(t) \end{pmatrix}.$$

Then using (8), we see that $\alpha(\mathbf{z}) = \big(i\mathbf{z} \boxtimes \overline{\mathbf{z}}\big)_3 = 2\Im(z_1\overline{z}_2)$. Therefore, using (10) and (11) we see that D^1_x is contained in Ω^1_- and D^2_x is contained in Ω^2_-.

Observe that the construction above shows that Ω_+, Ω^1_- and Ω^2_- are all connected. Thus $\mathbf{\Omega}$ indeed does have exactly three components. This completes the proof of Theorem 4 and hence also of Theorem 2 stated in the introduction.

3.4. Limiting behaviour of the fibres as x tends to $\partial\mathbf{H}^2_{\mathbb{R}}$. In this section we investigate the limiting behaviour of the fibres as the base point tends to the boundary of $\mathbf{H}^2_{\mathbb{R}}$. Our goal will be to prove Theorem 5.

As above, we parametrise points x in $\mathbf{H}^2_{\mathbb{R}}$ via $t \in \mathbb{R}_+$ and $\theta \in [0, 2, \pi)$ as

$$x = \big[\tanh(t)\cos(\theta) : \tanh(t)\sin(\theta) : 1\big].$$

Note that as t tends to infinity then x tends to $\partial\mathbf{H}^2_{\mathbb{R}}$. Therefore to describe the behaviour as points of $\mathbf{H}^2_{\mathbb{R}}$ tend to $\partial\mathbf{H}^2_{\mathbb{R}}$ then we should consider a sequence x_j

parametrised by t_j and θ_j with the property that there exists θ so that $\lim_{j\to\infty} t_j = \infty$ and $\lim_{j\to\infty} \theta_j = \theta$. The limiting point will be

$$\xi = \big[\cos(\theta) : \sin(\theta) : 1\big] \in \partial\mathbf{H}^2_{\mathbb{R}}.$$

For simplicity of exposition, it is sufficient to fix θ and simply to let t tend to ∞ (so x tends radially towards ξ). It is straightforward to adapt our arguments to more general ways that x can tend towards ξ. We use the basis $\mathcal{B}_x$ of $\mathbb{C}^{2,1}$ given in (14). Note that as t tends to infinity then $\tanh(t)$ tends to 1. Thus the first and third basis vectors tend to the same limit.

Let $\mathbf{z}$ be a vector in $U_0 - \mathbb{R}^{2,1}$ that projects to the fibre $\mathcal{L}_\xi$, namely take

$$\mathbf{z} = \begin{pmatrix} z_1\cos(\theta) - z_2\sin(\theta) \\ z_1\sin(\theta) + z_2\cos(\theta) \\ z_1 \end{pmatrix}.$$

Then it is easy to see that $\eta^2(\mathbf{z}) = -z_2^2$ and so we define $\eta(\mathbf{z}) = iz_2$. A short calculation yields

$$\mathbf{z}\overline{\eta}(\mathbf{z}) + \overline{z}\eta(\mathbf{z}) = i\mathbf{z} \boxtimes \overline{\mathbf{z}} = 2\Im(z_1\overline{z}_2)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 1 \end{pmatrix}.$$

Hence the definition of $\widetilde{\Pi}(\mathbf{z})$ extends continuously to the same limit, whether we approach U_0 from U_+ or U_-. Compare this to Corollary 3.4.

First consider $S_x = D_x^1 \cup D_x^2 \cup C_x$, which is given by

$$S_x = \Big\{\big[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\tanh(t)\big] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\Big\}.$$

It is clear that as t tends to infinity, then S_x tends to

$$\mathcal{L}_\xi = \Big\{\big[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\big] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\Big\}$$

as given in Proposition 1.8. Moreover,

$$C_x = \Big\{\big[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\tanh(t)\big] \in S_x : \Im(z_1\overline{z}_2) = 0\Big\},$$

tends to

$$C_\xi = \Big\{\big[z_1\cos(\theta) - z_2\sin(\theta) : z_1\sin(\theta) + z_2\cos(\theta) : z_1\big] \in \mathcal{L}_\xi : \Im(z_1\overline{z}_2) = 0\Big\} = \mathcal{L}_\xi \cap \mathcal{M}.$$

This proves parts (1) and (3) of Theorem 5.

We now consider the limit as t tends to infinity of $\overline{L}_x = L_x \cup C_x$:

$$\overline{L}_x = \mathbb{P}\left\{ y_1\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \tanh(t) \end{pmatrix} + y_2\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} + -ix_3\begin{pmatrix} \tanh(t)\cos(\theta) \\ \tanh(t)\sin(\theta) \\ 1 \end{pmatrix} : y_1,\, y_2,\, x_3 \in \mathbb{R} \right\}.$$

Note we have multiplied our homogeneous coordinates by $-i$. First consider the chart P_x where $y_2 \neq 0$ on which we select the inhomogeneous coordinates given by $y_2 = 1$. This chart is a copy of $\mathbb{R}^2$:

$$P_x = \left\{ y_1\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ \tanh(t) \end{pmatrix} + \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} + -ix_3\begin{pmatrix} \tanh(t)\cos(\theta) \\ \tanh(t)\sin(\theta) \\ 1 \end{pmatrix} : y_1,\, x_3 \in \mathbb{R} \right\}.$$

It is clear that as t tends to ∞ this tends to the following copy of $\mathbb{R}^2$ in $\mathcal{L}_\xi$, which we denote by P_ξ:

$$P_\xi = \left\{ (y_1 - ix_3)\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\\ 1\end{pmatrix} + \begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{pmatrix} : y_1, x_3 \in \mathbb{R} \right\}.$$

In order to get the whole of $\overline{L}_x$ we must add to P_x the collection of points where $y_2 = 0$. This is a circle of directions

$$Q_x = \left\{ \cos(\phi)\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\\ \tanh(t)\end{pmatrix} + i\sin(\phi)\begin{pmatrix}\tanh(t)\cos(\theta)\\ \tanh(t)\sin(\theta)\\ 1\end{pmatrix} : \phi \in [0,\pi) \right\},$$

and $P_x \cup Q_x$ is a copy of $\mathbb{RP}^2$. As t tends to ∞ we see that Q_x tends to

$$Q_\xi = \left\{ e^{i\phi}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\\ 1\end{pmatrix} : \phi \in [0,\pi) \right\}.$$

After projectivising, we see that $\mathbb{P}(Q_\xi) = \xi$, so the whole circle of directions collapses to a single point, namely ξ itself. Hence $P_\xi \cup Q_\xi = P_\xi \cup \{\xi\}$ is a sphere. In fact it is $\mathcal{L}_\xi$. Putting this together, we see that $\overline{L}_x$ tends pointwise to $\mathcal{L}_\xi$ as claimed. This completes the proof of Theorem 5.

REFERENCES

[1] W. Barrera, A. Cano, and J.-P. Navarrete, *The limit set of discrete subgroups of* $\mathrm{PSL}(3,\mathbb{C})$, Math. Proc. Cambridge Philos. Soc., 150 (2011), pp. 129–146.

[2] W. Barrera, A. Cano, and J.-P. Navarrete, *Subgroups of* $\mathrm{PSL}(3,\mathbb{C})$ *with four lines in general position in its limit set*, Conform. Geom. Dyn., 15 (2011), pp. 160–176.

[3] A. Cano and J. Seade, *On the equicontinuity region of discrete subgroups of* $\mathrm{PU}(1,n)$, J. Geom. Anal., 20 (2010), pp. 291–305.

[4] A. Cano, J.-P. Navarrete, and J. Seade, *Complex Kleinian Groups*, Progress in Mathematics 303. Birkhaüser, 2013.

[5] A. Cano, B. Liu, and M. López, *The limit set for discrete complex hyperbolic groups*, to appear in Indiana University Journal of Mathematics.

[6] S. S. Chen and L. Greenberg, *Hyperbolic spaces*, in “Contributions to Analysis”, Academic Press, New York, 1974, pp. 49–87.

[7] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford University Press, Oxford, 1999.

[8] N. Gusevskii and J. R. Parker, *Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant*, Geom. Dedicata, 97 (2003), pp. 151–185.

[9] Sh. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der Mathematischen Wissenschaften 318. Springer-Verlag, Berlin, 1998.

[10] R. S. Kulkarni, *Groups with domains of discontinuity*, Math. Ann., 237 (1978), pp. 253–272.

[11] J.-P. Navarrete, *On the limit set of discrete subgroups of* $\mathrm{PU}(2,1)$, Geom. Dedicata, 122 (2006), pp. 1–13.

[12] S. Lang, *Introduction to Complex Hyperbolic Spaces*, Springer-Verlag, New York, Berlin, Heidelberg, 1987.

[13] J. R. Parker and I. D. Platis, *Open sets of maximal dimension in complex hyperbolic quasi-Fuchsian space*, J. Differential Geometry, 73 (2006), pp. 319–350.