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A LOOP GROUP METHOD FOR MINIMAL SURFACES IN THE THREE-DIMENSIONAL HEISENBERG GROUP*

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Abstract. We characterize constant mean curvature surfaces in the three-dimensional Heisenberg group by a family of flat connections on the trivial bundle $\mathbb{D} \times \mathrm{GL}_2\mathbb{C}$ over a simply connected domain $\mathbb{D}$ in the complex plane. In particular for minimal surfaces, we give an immersion formula, the so-called Sym-formula, and a generalized Weierstrass type representation via the loop group method. Our generalized Weierstrass type representation produces all simply-connected non-vertical minimal surfaces in the Heisenberg group.

Key words. Constant mean curvature, Heisenberg group, spinors, generalized Weierstrass type representation.

AMS subject classifications. Primary 53A10, 58D10; Secondary 53C42.

Introduction. Surfaces of constant curvature or of constant mean curvature in space forms (of both definite and indefinite type) have been investigated since the beginning of differential geometry. For more than fifteen years now a loop group technique has been used to investigate these surfaces, see [21, 36].

During the last few years, surfaces of constant mean curvature in more general three-dimensional manifolds have been investigated. A natural target were the model spaces of Thurston geometries, see [19].

According to Thurston [43], there are eight model spaces of three-dimensional geometries, Euclidean 3-space $\mathbb{R}^3$, 3-sphere $\mathbb{S}^3$, hyperbolic 3-space $\mathbb{H}^3$, Riemannian products $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, the three-dimensional Heisenberg group Nil_3, the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of the special linear group and the space Sol_3. The geometrization conjecture posed by Thurston (and solved by Perelman) states that these eight model spaces are the *building blocks* to construct any three-dimensional manifolds. The dimension of the isometry group of the model spaces is greater than 3, except in the case Sol_3. In particular, the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ have 6-dimensional isometry groups. The model spaces with the exception of Sol_3 and $\mathbb{H}^3$ belong to the following 2-parameter family $\{E(\kappa,\tau) \mid \kappa,\tau \in \mathbb{R}\}$ of homogeneous Riemannian 3-manifolds: Let

$$E(\kappa,\tau) = (\mathcal{D}_{\kappa,\tau}, ds^2_{\kappa,\tau}),$$

where the domain $\mathcal{D}_{\kappa,\tau}$ is the whole 3-space $\mathbb{R}^3$ for $\kappa \geq 0$ and

$$\mathcal{D}_{\kappa,\tau} := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < -4/\kappa\}$$

for $\kappa < 0$. The Riemannian metric $ds^2_{\kappa,\tau}$ is given by

$$ds^2_{\kappa,\tau} = \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2} + \left(dx_3 + \frac{\tau(x_2 dx_1 - x_1 dx_2)}{1 + \frac{\kappa}{4}(x_1^2+x_2^2)}\right)^2.$$

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This 2-parameter family can be seen in the local classification of all homogeneous Riemannian metrics on $\mathbb{R}^3$ due to Bianchi, [10], see also Vranceanu [46, p. 354]. Cartan classified transitive isometric actions of 4-dimensional Lie groups on Riemannian 3-manifolds [17, pp. 293–306]. Thus the family $E(\kappa, \tau)$ with $\kappa \in \mathbb{R}$ and $\tau \geq 0$ is referred to as the *Bianchi-Cartan-Vranceanu family*, [5]. By what was said above the Bianchi-Cartan-Vranceanu family includes all local three-dimensional homogeneous Riemannian metrics whose isometry groups have dimension greater than 3 except constant negative curvature metrics. The parameters κ and τ are called the *base curvature* and *bundle curvature* of $E(\kappa, \tau)$, respectively.

The Heisenberg group Nil_3 together with a standard left-invariant metric is isometric to the homogeneous Riemannian manifold $E(\kappa, \tau)$ with $\kappa = 0$ and $\tau \neq 0$. Without loss of generality we can normalize $\tau = 1/2$ for the Heisenberg group: $\mathrm{Nil}_3 = E(0, 1/2)$. Note that $E(0, 1)$ is the *Sasakian space form*, $\mathbb{R}^3(-3)$, see [12, 13].

An important piece of progress of surface geometry in $E(\kappa, \tau)$ was a result of Abresch and Rosenberg [1]: A certain quadratic differential turned out to be holomorphic for all surfaces of constant mean curvature in the above model spaces $E(\kappa, \tau)$.

Since in the classical case of surfaces in space forms the holomorphicity of the (unperturbed) Hopf differential was crucial for the existence of a loop group approach to the construction of those surfaces, the question arose, to what extent a loop group approach would also exist for the more general class of constant mean curvature surfaces in Thurston geometries.

All the model spaces are Riemannian homogeneous spaces. Minimal surfaces in Riemannian homogeneous spaces are regarded as conformally harmonic maps from Riemann surfaces. Conformally harmonic maps of Riemann surfaces into Riemannian symmetric spaces admit a zero curvature representation and hence loop group methods can be applied.

More precisely, the loop group method has two key ingredients. One is the zero curvature representation of harmonic maps. The zero curvature representation is equivalent with the existence of a loop of flat connections and this representation enables us to use loop groups. The other one is an appropriate loop group decomposition. A loop group decomposition recovers the harmonic map (minimal surfaces) from holomorphic potentials. The construction of harmonic maps from prescribed potentials is now referred to as the *generalized Weierstrass type representation* for harmonic maps, see [24].

Every (compact) semi-simple Lie group G equipped with a bi-invariant (semi-)Riemannian metric is represented by $G \times G/G$ as a (semi-)Riemannian symmetric space. Thus we can apply the loop group method to harmonic maps into G. Harmonic maps from the two-sphere $\mathbb{S}^2$ or the two-torus $\mathbb{T}^2$ into compact semi-simple Lie groups have been studied extensively, see [14, 15, 41, 45]. The three-sphere $\mathbb{S}^3$ is identified with the special unitary group SU_2 equipped with a bi-invariant Riemannian metric of constant curvature 1. Thus we can study minimal surfaces in $\mathbb{S}^3$ by a loop group method. Note that harmonic tori in $\mathbb{S}^3$ have been classified by Hitchin [30] via the spectral curve method.

It is known that model spaces, except $\mathbb{S}^2 \times \mathbb{R}$, can be realized as Lie groups equipped with left-invariant Riemannian metrics. Thus it has been expected to generalize the loop group method for harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric to those for maps into more general Lie groups. However the bi-invariant property is essential for the application of the loop group method.

Thus to establish a generalized Weierstrass type representation for minimal surfaces (or more generally CMC surfaces) in model spaces of Thurston geometries, another key ingredient is required. Since Nil_3 seems to be a particularly simple example of a Thurston geometry and more work has been done for this target space than for other ones, we would like to attempt to introduce a loop group approach to constant mean curvature surfaces in Nil_3.

The procedure is as follows: Consider a conformal immersion $f : \mathbb{D} \to \mathrm{Nil}_3 = E(0, 1/2)$ of a simply connected domain of the complex plane $\mathbb{C}$. Then define a matrix valued function $\varPhi$ by $\varPhi = f^{-1}\partial_z f$ and expand it as $\varPhi = \sum_{k=1}^{3} \phi_k e_k$ relative to the natural basis $\{e_1, e_2, e_3\}$ of the Lie algebra of Nil_3. The new key ingredient is the *spin structure* of Riemann surfaces. Represent (ϕ_1, ϕ_2, ϕ_3) by $(\phi_1, \phi_2, \phi_3) = ((\overline{\psi_2})^2 - \psi_1^2,\ i((\overline{\psi_2})^2 + \psi_1^2),\ 2\psi_1\overline{\psi_2})$ in terms of *spinors* $\{\psi_1, \psi_2\}$ that are unique up to a sign.

It has been shown by Berdinsky [7], that the spinor field $\psi = (\psi_1, \psi_2)$ satisfies the matrix system of equations $\partial_z \psi = \psi U$, $\partial_{\bar z}\psi = \psi V$, where the coefficients of U and V have a simple form in terms of the *mean curvature* H, the *conformal factor* e^u of the metric, the spin geometric *support function* h of the normal vector field of the immersion and the *Abresch-Rosenberg quadratic differential* Qdz^2. In [8], another quadratic differential, $\tilde{A}dz^2$ where $\tilde{A} = Q/(2H + i)$ was introduced. For $H = \text{const}$ there is, obviously, not much of a difference. However for Nil_3 it turns out that $\tilde{A}$ is holomorphic if and only if f has constant mean curvature, [8], but Q is holomorphic for all constant mean curvature surfaces and, in addition, also for one non constant mean curvature surface, the so-called *Hopf cylinder* (Theorem A.1). A similar situation occurs for other Thurston geometries, see [26].

One can show that for every conformal constant mean curvature immersion f into Nil_3 the Berdinsky system describes a harmonic map into a symmetric space $\mathrm{GL}_2\mathbb{C}/\operatorname{diag}$ (Theorem 4.1). Thus the corresponding system can be constructed by the loop group method, that is, there exists an *associated family* of surfaces parametrized by a *spectral parameter*. However, it is not clear so far, how one can make sure that a solution spinor $\psi = (\psi_1, \psi_2)$ to the Berdinsky system induces, via $f^{-1}\partial_z f = \sum_{k=1}^{3} \phi_k e_k$, with ϕ_k the "Weierstrass type representations" formed with ψ_1 and ψ_2 as above, a (real !) immersion into Nil_3.

Unfortunately, a result of Berdinsky [6] shows that a naturally associated family of surfaces cannot stay in Nil_3 for all values of the spectral parameter (Corollary 4.6). However, for the case of minimal surfaces in Nil_3 this problem does not arise. Therefore, as a first attempt to introduce a loop group method for the discussion of constant mean curvature surfaces in Thurston geometries, we present in this paper a loop group approach to minimal surfaces in Nil_3.

Moreover, for the case of minimal surfaces, the normal Gauss maps, which are maps into hyperbolic 2-space $\mathbb{H}^2$, are harmonic (Theorem 5.3) and an immersion formula is obtained from the frame of the normal Gauss map, the so-called *Sym-formula* (Theorem 6.1), see also [18]. Thus the loop group method can be applied without restrictions to the case of minimal surfaces, that is, a pair of meromorphic 1-forms, through the loop group decomposition, determines a minimal surface, the so-called *generalized Weierstrass type representation.* All simply-connected non-vertical minimal surfaces are obtained by our generalized Weierstrass type representation. It is worthwhile to note that the associated family of a minimal surface in Nil_3 preserves the support but not the metric. This gives a geometric characterization of the associated family of a minimal surface in Nil_3 which is different from the case of constant mean curvature surfaces in $\mathbb{R}^3$, where the associated family preserves the metric (Corollary 6.3).

This paper is organized as follows: In Sections 1–3, we give basic results for harmonic maps and surfaces in Nil_3. In Section 4, constant mean curvature surfaces in Nil_3 are characterized by a family of flat connections on $\mathbb{D} \times \mathrm{GL}_2\mathbb{C}$. In Sections 5 and 6, we will concentrate on minimal surfaces. In particular, minimal surfaces in Nil_3 are characterized by a family of flat connections on $\mathbb{D} \times \mathrm{SU}_{1,1}$ and an immersion formula in terms of an extended frame is given. In Sections 7 and 8, a generalized Weierstrass type representation for minimal surfaces in Nil_3 is given via the loop group method, that is, a minimal surface is recovered by a pair of holomorphic functions through the loop group decomposition. In Section 9, several examples are given by the generalized Weierstrass type representation established in this paper.

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1. Minimal surfaces in Lie groups.

1.1. Let $G \subset \mathrm{GL}_n\mathbb{R}$ be a closed subgroup of the real general linear group of degree n. Denote by $\mathfrak{g}$ the Lie algebra of G, that is, the tangent space of G at the identity. We equip $\mathfrak{g}$ with an inner product $\langle \cdot, \cdot \rangle$ and extend it to a left-invariant Riemannian metric $ds^2 = \langle \cdot, \cdot \rangle$ on G.

Now let $f : M \to G$ be a smooth map of a Riemann surface M into G. Then $\alpha := f^{-1}df$ satisfies the *Maurer-Cartan equation*:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Take a local complex coordinate $z = x + iy$ defined on a simply connected domain $\mathbb{D} \subset M$ and express α as

$$\alpha = \Phi\, dz + \bar{\Phi}\, d\bar{z}.$$

Here the coefficient matrices Φ and $\bar{\Phi}$ are computed as

$$\Phi = f^{-1} f_z, \quad \bar{\Phi} = f^{-1} f_{\bar{z}}.$$

The subscripts z and $\bar{z}$ denote the partial differentiations $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, respectively. We note that $\bar{\Phi}$ is the complex conjugate of Φ, since f takes values in $G \subset \mathrm{GL}_n\mathbb{R}$.

Denote the complex bilinear extension of $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}^{\mathbb{C}}$ by the same letter. Then f is a conformal immersion if and only if

$$\langle \Phi, \Phi \rangle = 0, \quad \langle \Phi, \bar{\Phi} \rangle > 0. \tag{1.1}$$

For a conformal immersion $f : M \to G$, the induced metric (also called the first fundamental form) $\langle df, df \rangle$, is represented as $e^u dz d\bar{z}$. The function $e^u := 2\langle f_z, f_{\bar{z}} \rangle$ is called the *conformal factor* of the metric with respect to z.

Next take an orthonormal basis $\{e_1, e_2, \cdots, e_\ell\}$ of the Lie algebra $\mathfrak{g}$ ($\ell = \dim \mathfrak{g}$). Expand Φ as $\Phi = \phi_1 e_1 + \phi_2 e_2 + \cdots + \phi_\ell e_\ell$. Then we have the following fundamental fact.

PROPOSITION 1.1. *Let $f : M \to G \subset \mathrm{GL}_n\mathbb{R}$ be a conformal immersion with the conformal factor e^u. Moreover, set $\Phi = f^{-1}f_z = \sum_{k=1}^{\ell} \phi_k e_k$. Then the following statements hold*:

$$f_z = f\Phi, \qquad f_{\bar{z}} = f\bar{\Phi}, \tag{1.2}$$

$$\sum_{k=1}^{\ell} \phi_k^2 = 0, \tag{1.3}$$

$$\sum_{k=1}^{\ell} |\phi_k|^2 = \frac{1}{2}e^u. \tag{1.4}$$

In particular, Φ and $\bar{\Phi}$ satisfy the integrability condition

$$\Phi_{\bar{z}} - \bar{\Phi}_z + [\bar{\Phi}, \Phi] = 0. \tag{1.5}$$

Conversely, let $\mathbb{D}$ be a simply-connected domain and $\Phi = \sum_{k=1}^{\ell} \phi_k e_k$ a non-zero 1-*form on $\mathbb{D}$ which takes values in the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ satisfying the conditions* (1.3) *and* (1.5). *Then for any initial condition in G given at some base point in $\mathbb{D}$ there exists a unique conformal immersion f into G.*

Proof. By (1.5) the integrability condition for the equations (1.2) is satisfied. Therefore, there exists a map f into the complexification $G^{\mathbb{C}}$ of G satisfying (1.2). Since the metric of G is left-invariant, the conformality and the non-degeneracy of a metric of f follows from (1.3) and (1.4). It is straightforward to verify that the partial derivatives of $f\bar{f}^{-1}$ vanish. Hence $f\bar{f}^{-1}$ is constant. If we have chosen an initial condition in G for f, then this constant matrix is I, the identity element in G. □

1.2. Let $f : M \to G$ be a smooth map of a 2-manifold M. Then f induces a vector bundle f^*TG over M by

$$f^*TG = \bigcup_{p \in M} T_{f(p)}G,$$

where TG is the tangent bundle of G. The space of all smooth sections of f^*TG is denoted by $\Gamma(f^*TG)$. A section of f^*TG is called a *vector field along f*.

The Levi-Civita connection ∇ of G induces a unique connection ∇^f on f^*TG which satisfies the condition

$$\nabla^f_X(V \circ f) = (\nabla_{df(X)}V) \circ f,$$

for all vector fields X on M and $V \in \Gamma(TG)$, see [25, p. 4].

Next assume that M is a Riemannian 2-manifold with a Riemannian metric ds^2_M. Then the *second fundamental form* ∇df of f is defined by

$$(\nabla df)(X, Y) = \nabla^f_X df(Y) - df(\nabla^M_X Y), \quad X, Y \in \mathfrak{X}(M). \tag{1.6}$$

Here ∇^M is the Levi-Civita connection of (M, ds^2_M). The *tension field* $\tau(f)$ of f is a section of f^*TG defined by $\tau(f) = \mathrm{tr}(\nabla df)$.

1.3. For a smooth map $f : (M, ds^2_M) \to (G, ds^2)$, the *energy* of f is defined by

$$E(f) = \int_M \frac{1}{2}|df|^2 \, dA.$$

A smooth map f is a *harmonic map* provided that f is a critical point of the energy under compactly supported variations. It is well known that f is a harmonic map if and only if its tension field $\tau(f)$ is equal to zero, that is,

$$\tau(f) = \mathrm{tr}(\nabla df) = 0.$$

It should be remarked that the harmonicity of f is invariant under conformal transformations of M. Thus the harmonicity makes sense for maps from Riemann surfaces.

1.4. Let $f : M \to G$ be a conformal immersion of a Riemann surface M into G. Take a local complex coordinate $z = x + iy$ and represent the induced metric by $e^u dz d\bar{z}$. It is a fundamental fact that the tension field of f is related to the *mean curvature vector field* $\boldsymbol{H}$ by:

$$\tau(f) = 2\boldsymbol{H}. \tag{1.7}$$

This formula shows that a conformal immersion $f : M \to G$ is a minimal surface if and only if it is harmonic. Since the metric is left-invariant the equation above can be rephrased, using the vector fields $\Phi = f^{-1} f_z$ and $\bar{\Phi} = f^{-1} f_{\bar{z}}$, as

$$\Phi_{\bar{z}} + \bar{\Phi}_z + \{\Phi, \bar{\Phi}\} = e^u f^{-1} \boldsymbol{H}, \tag{1.8}$$

where $\{\cdot, \cdot\}$ denotes the bilinear symmetric map defined by

$$\{X, Y\} = \nabla_X Y + \nabla_Y X \tag{1.9}$$

for $X, Y \in \mathfrak{g}$. By (1.8), the harmonic map equation can be computed as [1]

$$\Phi_{\bar{z}} + \bar{\Phi}_z + \{\Phi, \bar{\Phi}\} = 0. \tag{1.10}$$

Thus the Maurer-Cartan equation (1.5) together with the harmonic map equation (1.10) is equivalent to

$$2\Phi_{\bar{z}} + \{\Phi, \bar{\Phi}\} = [\Phi, \bar{\Phi}]. \tag{1.11}$$

We summarize the above discussion as the following theorem.

[1]Let $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denote the symmetric bilinear map defined by

$$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle, \quad X, Y, Z \in \mathfrak{g}.$$

Then $2U(\Phi, \bar{\Phi})$ and $\{\Phi, \bar{\Phi}\}$ are the same, since

$$\langle \nabla_X Y + \nabla_Y X, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle + X\langle Z, Y\rangle - 2Z\langle X, Y\rangle - 2\langle [X, Z], Y\rangle - 2\langle [Y, Z], X\rangle)$$

and the left invariance of the vector fields implies that $X\langle Y, Z\rangle = Y\langle X, Z\rangle = X\langle Z, Y\rangle = Z\langle X, Y\rangle = 0$. Moreover, since

$$\langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle = -\langle X, \mathrm{ad}(Y)Z\rangle - \langle Y, \mathrm{ad}(X)Z\rangle = -\langle \mathrm{ad}^*(Y)X, Z\rangle - \langle \mathrm{ad}^*(X)Y, Z\rangle,$$

$2U(\Phi, \bar{\Phi})$, $\{\Phi, \bar{\Phi}\}$ and $-\mathrm{ad}^*(\Phi)\bar{\Phi} - \mathrm{ad}^*(\bar{\Phi})\Phi$ are the same (see [2, Section 2.1] for another formulation of harmonic maps into Lie groups with left-invariant metric).

THEOREM 1.2. *Let $f : M \to G$ be a conformal minimal immersion. Then $\alpha = f^{-1}df = \Phi dz + \bar{\Phi} d\bar{z}$ satisfies* (1.1) *and* (1.11). *Conversely, let $\mathbb{D}$ be a simply connected domain and $\alpha = \Phi dz + \bar{\Phi} d\bar{z}$ a $\mathfrak{g}$-valued 1-form on $\mathbb{D}$ satisfying* (1.1) *and* (1.11). *Then for any initial condition in G there exist a conformal minimal immersion $f : \mathbb{D} \to G$ such that $f^{-1}df = \alpha$.*

Proof. Let $\alpha = \Phi dz + \bar{\Phi} d\bar{z}$ be a $\mathfrak{g}$-valued 1-form satisfying (1.1) and (1.11). Then subtraction and addition of the complex conjugate of (1.11) to itself gives the integrability condition (1.5) and the harmonicity condition (1.10), respectively. Hence Proposition 1.1 implies that there exists a conformal immersion f such that $f^{-1}df = \alpha$. Since f is harmonic, it is minimal. □

In the study of harmonic maps of Riemann surfaces into compact semi-simple Lie groups equipped with a bi-invariant Riemannian metric, the zero curvature representation is the starting point of the loop group approach, see Segal [41], Uhlenbeck [45]. In case the metric on the target Lie group is only left invariant we need to require the additional condition

$$\{\Phi, \bar{\Phi}\} = 0,$$

the so-called *admissibility condition.*

It should be remarked that all the examples of minimal surfaces in Nil_3 studied in this paper do not satisfy the admissibility condition. Thus we can not expect to generalize the Uhlenbeck-Segal approach for harmonic maps into compact semi-simple Lie groups to maps into more general Lie groups in a straightforward manner.

2. The three-dimensional Heisenberg group Nil_3.

2.1. We define a 1-parameter family $\{\mathrm{Nil}_3(\tau)\}_{\tau \in \mathbb{R}}$ of 3-dimensional Lie groups

$$\mathrm{Nil}_3(\tau) = (\mathbb{R}^3(x_1, x_2, x_3), \cdot)$$

with multiplication:

$$(x_1, x_2, x_3) \cdot (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, x_3 + \tilde{x}_3 + \tau(x_1\tilde{x}_2 - \tilde{x}_1 x_2)).$$

The unit element id of $\mathrm{Nil}_3(\tau)$ is $(0, 0, 0)$. The inverse element of (x_1, x_2, x_3) is $-(x_1, x_2, x_3)$. Obviously, $\mathrm{Nil}_3(0)$ is the abelian group $(\mathbb{R}^3, +)$. The groups $\mathrm{Nil}_3(\tau)$ and $\mathrm{Nil}_3(\tau')$ are isomorphic if $\tau\tau' \neq 0$.

2.2. The Lie algebra $\mathfrak{nil}_3(\tau)$ of $\mathrm{Nil}_3(\tau)$ is $\mathbb{R}^3$ with commutation relations:

$$[e_1, e_2] = 2\tau e_3, \quad [e_2, e_3] = [e_3, e_1] = 0 \tag{2.1}$$

with respect to the natural basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The formulas (2.1) imply that $\mathfrak{nil}_3(\tau)$ is nilpotent. The respective left translated vector fields of e_1, e_2 and e_3 are

$$E_1 = \partial_{x_1} - \tau x_2 \partial_{x_3}, \; E_2 = \partial_{x_2} + \tau x_1 \partial_{x_3} \text{ and } E_3 = \partial_{x_3}.$$

We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{nil}_3(\tau)$ so that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Then the resulting left-invariant Riemannian metric $ds_\tau^2 = \langle \cdot, \cdot \rangle_\tau$ on $\mathrm{Nil}_3(\tau)$ is

$$ds_\tau^2 = (dx_1)^2 + (dx_2)^2 + \omega_\tau \otimes \omega_\tau, \tag{2.2}$$

where

$$\omega_\tau = dx_3 + \tau(x_2 dx_1 - x_1 dx_2). \tag{2.3}$$

The 1-form ω_τ satisfies $d\omega_\tau \wedge \omega_\tau = -2\tau\, dx_1 \wedge dx_2 \wedge dx_3$. Thus ω_τ is a *contact form* on $\mathrm{Nil}_3(\tau)$ if and only if $\tau \neq 0$.

The homogeneous Riemannian 3-manifold $(\mathrm{Nil}_3(\tau), ds^2_\tau)$ is called the three-dimensional *Heisenberg group* if $\tau \neq 0$. Note that $(\mathrm{Nil}_3(0), ds^2_0)$ is the Euclidean 3-space $\mathbb{E}^3$. The homogeneous Riemannian 3-manifold $(\mathrm{Nil}_3(1/2), ds^2_{1/2})$ is frequently referred to as the model space Nil_3 of the nilgeometry in the sense of Thurston, [43].

Remark 2.1. For $\tau \neq 0$, $(\mathrm{Nil}_3(\tau), \omega_\tau)$ is a contact manifold, and the unit Killing vector field E_3 is the *Reeb vector field* of this contact manifold. In particular $\mathrm{Nil}_3(1)$ is isometric to the *Sasakian space form* $\mathbb{R}^3(-3)$ in the sense of contact Riemannian geometry, [12, 13].

We orient $\mathrm{Nil}_3(\tau)$ so that $\{E_1, E_2, E_3\}$ is a positive orthonormal frame field. Then the volume element dv_τ of the oriented Riemannian 3-manifold $\mathrm{Nil}_3(\tau)$ with respect to the metric ds^2_τ is $dx_1 \wedge dx_2 \wedge dx_3$. The *vector product operation* $\times$ with respect to this orientation is defined by

$$\langle X \times Y, Z\rangle_\tau = dv_\tau(X, Y, Z)$$

for all vector fields X, Y and Z on $\mathrm{Nil}_3(\tau)$.

2.3. The nilpotent Lie group $\mathrm{Nil}_3(\tau)$ is realized as a closed subgroup of the general linear group $\mathrm{GL}_4\mathbb{R}$. In fact, $\mathrm{Nil}_3(\tau)$ is imbedded in $\mathrm{GL}_4\mathbb{R}$ by $\iota : \mathrm{Nil}_3(\tau) \to \mathrm{GL}_4\mathbb{R}$;

$$\iota(x_1, x_2, x_3) = e^{x_1} E_{11} + \sum_{i=2}^{4} E_{ii} + 2\tau x_1 E_{23} + (x_3 + \tau x_1 x_2) E_{24} + x_2 E_{34},$$

where E_{ij} are 4 by 4 matrices with the ij-entry 1, and all others 0. Clearly ι is an injective Lie group homomorphism. Thus $\mathrm{Nil}_3(\tau)$ is identified with $\{\iota(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \mathrm{Nil}_3(\tau)$. The Lie algebra $\mathfrak{nil}_3(\tau)$ corresponds to

$$\{u_1 E_{11} + 2\tau u_1 E_{23} + u_3 E_{24} + u_2 E_{34} \mid u_1, u_2, u_3 \in \mathbb{R}\}.$$

The orthonormal basis $\{e_1, e_2, e_3\}$ is identified with

$$e_1 = E_{11} + 2\tau E_{23},\ \ e_2 = E_{34}\ \text{ and }\ e_3 = E_{24}.$$

The exponential map $\exp : \mathfrak{nil}_3(\tau) \to \mathrm{Nil}_3(\tau)$ is given explicitly by

$$\exp(x_1 e_1 + x_2 e_2 + x_3 e_3) = e^{x_1} E_{11} + \sum_{i=2}^{4} E_{ii} + 2\tau x_1 E_{23} + (x_3 + \tau x_1 x_2) E_{24} + x_2 E_{34}. \tag{2.4}$$

This shows that exp is a diffeomorphism. Moreover the inverse mapping $\exp^{-1}$ can be identified with the global coordinate system (x_1, x_2, x_3) of $\mathrm{Nil}_3(\tau)$. The coordinate system (x_1, x_2, x_3) is called the *exponential coordinate system* of $\mathrm{Nil}_3(\tau)$. In this coordinate system the exponential map is the identity map.

2.4. The Levi-Civita connection ∇ of ds^2_τ is given by

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = \tau\, e_3, \quad \nabla_{e_1} e_3 = -\tau\, e_2, \\ &\nabla_{e_2} e_1 = -\tau\, e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \tau\, e_1, \\ &\nabla_{e_3} e_1 = -\tau\, e_2, \quad \nabla_{e_3} e_2 = \tau\, e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned} \tag{2.5}$$

The Riemannian curvature tensor R defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is given by

$$\begin{aligned} R(X,Y)Z = &-3\tau^2 \left(\langle Y,Z\rangle_\tau X - \langle Z,X\rangle_\tau Y\right) \\ &+ 4\tau^2 \left(\omega_\tau(Y)\omega_\tau(Z)X - \omega_\tau(Z)\omega_\tau(X)Y\right) \\ &+ 4\tau^2 \left(\omega_\tau(X)\langle Y,Z\rangle_\tau - \omega_\tau(Y)\langle Z,X\rangle_\tau\right) e_3. \end{aligned}$$

The Ricci tensor field Ric is given by

$$\mathrm{Ric} = -\tau^2\langle\cdot,\cdot\rangle_\tau + 4\tau^2\omega_\tau \otimes \omega_\tau.$$

The scalar curvature of $\mathrm{Nil}_3(\tau)$ is τ^2. The symmetric bilinear map $\{\cdot,\cdot\}$ defined in (1.8) is explicitly given by

$$\{e_1, e_2\} = 0, \quad \{e_1, e_3\} = -2\tau e_2 \quad \text{and} \quad \{e_2, e_3\} = 2\tau e_1. \tag{2.6}$$

Note that $\{\cdot,\cdot\}$ measures the non right-invariance of the metric. In fact $\{\cdot,\cdot\} = 0$ if and only if ds^2_τ is right invariant (and hence bi-invariant). The formulas (2.6) imply that ds^2_τ is bi-invariant only when $\tau = 0$.

3. Surface theory in Nil_3.

3.1. Hereafter we study $\mathrm{Nil}_3(1/2)$ for simplicity and denote the space by Nil_3. The metric $ds^2_{1/2}$ is simply denoted by $ds^2 = \langle\cdot,\cdot\rangle$ and $\omega_{1/2}$ of (2.3) by ω.

Let $f : M \to \mathrm{Nil}_3$ be an immersion of a 2-manifold. Our main interests are surfaces of constant mean curvature (and in particular minimal surfaces). In the case where a surface has nonzero constant mean curvature, we can assume without loss of generality that M is orientable and f is a conformal immersion of a Riemann surface. In the minimal surface case, if necessary, taking a double covering, we may assume that f is an orientable conformal immersion of a Riemann surface. Clearly, Nil_3 is a three-dimensional Riemannian spin manifold. Thus f induces a spin structure on M. Hereafter we will always use the induced spin structure on M.

As in Section 1, we consider the 1-form Φdz on a simply connected domain $\mathbb{D} \subset M$ that takes values in the complexification $\mathfrak{nil}_3^{\mathbb{C}}$ of the Lie algebra $\mathfrak{nil}_3$. With respect to the natural basis $\{e_1, e_2, e_3\}$ of $\mathfrak{nil}_3$, we expand Φ as $\Phi = \sum_{k=1}^3 \phi_k e_k$ and assume that (1.3) and (1.4) are satisfied. Then there exist complex valued functions ψ_1 and ψ_2 such that [2]

$$\phi_1 = (\overline{\psi_2})^2 - \psi_1^2, \quad \phi_2 = i((\overline{\psi_2})^2 + \psi_1^2), \quad \phi_3 = 2\psi_1\overline{\psi_2}, \tag{3.1}$$

[2] In [8, (8)], the spinor representation

$$\phi_1 = \frac{i}{2}(\overline{\psi_2}^2 + \psi_1^2), \quad \phi_2 = \frac{1}{2}(\overline{\psi_2}^2 - \psi_1^2), \quad \phi_3 = \psi_1\overline{\psi_2}$$

is used. The correspondence to ours representation is

$$\psi_j \to \sqrt{2}\psi_j, \quad \text{and} \quad e_1 \to e_2, \quad e_2 \to e_1.$$

Thus the sign of the unit normal also changes.

where $\overline{\psi_2}$ denotes the complex conjugate of ψ_2. It is easy to check that $\psi_1(dz)^{1/2}$ and $\psi_2(d\bar{z})^{1/2}$ are well defined on M. More precisely, $\psi_1(dz)^{1/2}$ and $\psi_2(d\bar{z})^{1/2}$ are respective sections of the spin bundles Σ and $\bar{\Sigma}$ over M, see Appendix C. The sections $\psi_1(dz)^{1/2}$ and $\psi_2(d\bar{z})^{1/2}$ are called the *generating spinors* of the conformally immersed surface f in Nil_3. The coefficient functions ψ_1 and ψ_2 are also called the generating spinors of f, see [8]. Note that after a change of coordinates the new generating spinors φ_1, φ_2 are $\varphi_1(w) = \sqrt{z_w}\psi_1(z(w))$ and $\varphi_2(w) = \sqrt{\bar{z}_w}\psi_2(z(w))$.

We would like to note that from this representation of Φ it is straightforward to verify f has a branch point, that is Φ and $\bar{\Phi}$ are linearly dependent, if and only if $\psi_1 = \psi_2 = 0$ at the point. Sometimes, we consider conformal immersions with branch points. Since we are interested in immersions, we will only admit a nowhere dense set of branch points, if any.

The conformal factor e^u of the induced metric $\langle df, df\rangle$ can be expressed by the spinors ψ_1, ψ_2 via formula (1.4):

$$e^u = 4(|\psi_1|^2 + |\psi_2|^2)^2. \tag{3.2}$$

Remark 3.1. Let $f : M \to \mathrm{Nil}_3$ be a conformal immersion. Then $\phi_3 = 2\psi_1\overline{\psi_2}$ can not vanish identically on any open subset of M. In fact, if $\phi_3 = 0$, then f is normal to E_3 everywhere. Namely, f is an integral surface of the contact distribution defined by the equation $\omega = 0$. However, since ω is a contact form on Nil_3, this is impossible. (The maximum dimension of an integral manifold is one.) In particular, for every conformal immersion $f : M \to \mathrm{Nil}_3$, there exists an open and dense subset M_f on which $\psi_1 \neq 0$ and $\psi_2 \neq 0$.

Example 1 (Vertical plane). Let Π be an affine plane in Nil_3 defined by

$$\Pi = \{(x_1, x_2, x_3) \mid ax_1 + bx_2 + c = 0\}$$

for some constants a, b and c. Such a plane is called a *vertical plane* in Nil_3. One can see that every vertical plane is minimal in Nil_3. Vertical planes are homogeneous and minimal Hopf cylinders. See Proposition B.1 and Theorem A.1. Vertical planes are minimal and flat, but not totally geodesic. It should be emphasized that there are no totally umbilical surfaces in Nil_3, see [40, 33].

3.2. Let N denote the positively oriented unit normal vector field along f. We then define an unnormalized normal vector field L by

$$L = e^{u/2}N. \tag{3.3}$$

Note that $L(dz)^{1/2}(d\bar{z})^{1/2}$ is well defined on M. We call this section the *normal* of f. We also note that $e^{u/2}L = e^u N$ is given by the vector product $f_x \times f_y$.

Moreover, from (3.1), the left translated vector field $f^{-1}N$ of the unit normal N to $\mathfrak{nil}_3$ can be represented by the spinors ψ_1 and ψ_2:

$$f^{-1}N = \frac{1}{|\psi_1|^2 + |\psi_2|^2}\left(2\,\mathrm{Re}(\psi_1\psi_2)e_1 + 2\,\mathrm{Im}(\psi_1\psi_2)e_2 + (|\psi_1|^2 - |\psi_2|^2)e_3\right), \tag{3.4}$$

where Re and Im denote the real and the imaginary parts of a complex number. Accordingly, the left translation of the unnormalized normal $f^{-1}L = e^{u/2}f^{-1}N$ can be computed as

$$f^{-1}L = 4\,\mathrm{Re}(\psi_1\psi_2)e_1 + 4\,\mathrm{Im}(\psi_1\psi_2)e_2 + 2(|\psi_1|^2 - |\psi_2|^2)e_3. \tag{3.5}$$

We define a function h by

$$h = \langle f^{-1}L, e_3\rangle = 2(|\psi_1|^2 - |\psi_2|^2).$$

Then we get a section $h(dz)^{1/2}(d\bar{z})^{1/2}$ of $\Sigma \otimes \bar{\Sigma}$. This section is called the *support* of f. The coefficient function h is called the *support function* of f with respect to z.

REMARK 3.2. Let us denote by ϑ the angle between N and the Reeb vector field E_3, then h is represented as $h = e^{u/2}\cos\vartheta$. The angle function ϑ is called the *contact angle* of f. One can check that $h(dz)^{1/2}(d\bar{z})^{1/2} = \cos\vartheta|df|$. Here $|df| = e^{u/2}(dz)^{1/2}(d\bar{z})^{1/2}$ is a half density on M.

From (3.5), we obtain the following Proposition.

PROPOSITION 3.3. *For a surface $f : \mathbb{D} \to \mathrm{Nil}_3$, the following properties are equivalent*:

1. *f has the support function equal zero at p in $\mathbb{D}$, that is, the support function h of f vanishes at p, $h(p) = 0$.*
2. *E_3 is tangent to f at p.*

Let $\pi : \mathrm{Nil}_3 \to \mathbb{R}^2$ be the natural projection defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$. We define a *Hopf cylinder* by the inverse image of a plane curve under the projection π. Hopf cylinders are flat and its mean curvature is half of the curvature of the base curve.

It is clear from the definition that surfaces tangent to E_3 are Hopf cylinders, [5]. Thus a surface which has zero support, that is $h \equiv 0$, is a Hopf cylinder.

For later purpose we list some notion: A surface is called *vertical at p* in M if E_3 is tangent to f at p in M. A surface is *vertical*, if it is vertical at all points p in M. A surface is called *nowhere vertical* if it is nowhere tangent to E_3.

3.3. Conformal immersions into Nil_3 are characterized by the integrability condition (1.5) and the structure equation (1.8). Note, since the target space Nil_3 is three-dimensional, the mean curvature vector field $\boldsymbol{H}$ in (1.8) can be represented as

$$\boldsymbol{H} = HN,$$

where H is the mean curvature and N is the unit normal. These equations are given by six equations for the functions ϕ_1, ϕ_2 and ϕ_3 or, equivalently, for the generating spinors ψ_1 and ψ_2, see [8, (18)]. Then the equations (1.5) and (1.8) are equivalent to the following *nonlinear Dirac equation*, that is,

$$\not{D}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} := \begin{pmatrix}\partial_z\psi_2 + \mathcal{U}\psi_1\\ -\partial_{\bar{z}}\psi_1 + \mathcal{V}\psi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{3.6}$$

where

$$\mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h. \tag{3.7}$$

Here H, e^u and h are the mean curvature, the conformal factor and the support function for f respectively. More precisely by Remark 3.1, we have $\psi_1\psi_2 \neq 0$ on an open dense subset M_f, and on this subset, we show that (1.5) and (1.8) together are equivalent with the nonlinear Dirac equation. Thus we extend to M by continuity. The complex function $\mathcal{U}(= \mathcal{V})$ is called the *Dirac potential* of the nonlinear Dirac operator $\not{D}$.

Remark 3.4.

1. The above equivalence can be seen explicitly as follows: The coefficients of e_1, e_2 and e_3 in (1.5) and (1.8) give six equations. The equations given by the respective coefficients of e_1 and e_2 in (1.5) and (1.8) together are equivalent to the nonlinear Dirac equation. Conversely, the equations given by the coefficients of e_3 in (1.5) and (1.8) follow from the nonlinear Dirac equation. Therefore the nonlinear Dirac equation is equivalent to the integrability equation (1.5) together with the structure equation (1.8).
2. To prove the equations (1.5) and (1.8) from the nonlinear Dirac equation, we can choose a real-valued function H freely, however, $e^{u/2}$ and h, which are the functions in the Dirac potential $\mathcal{U}(=\mathcal{V})$, (3.7), and solutions $\psi_j, (j=1,2)$ of the nonlinear Dirac equation need to satisfy the special relation:

$$e^{u/2} = 2(|\psi_1|^2 + |\psi_2|^2) \text{ and } h = 2(|\psi_1|^2 - |\psi_2|^2).$$

Under this special condition, we derive the equations (1.5) and (1.8). Moreover, up to an initial condition, there exists an immersion into Nil_3 such that the conformal factor, the mean curvature and support function are $e^{u/2}$, H and h, respectively.

3.4. The *Hopf differential* $A\,dz^2$ is the $(2,0)$-part of the second fundamental form for $f: M \to \mathrm{Nil}_3$ defined by

$$A = \langle \nabla^f_{\partial_z} f_z, N\rangle.$$

It is easy to see that A can be expanded as

$$A = \langle \nabla_{f_z} f_z, N\rangle = \langle (f^{-1}f_z)_z, f^{-1}N\rangle + \langle \textstyle\sum_{k,j} \phi_k\phi_j \nabla_{e_k} e_j, f^{-1}N\rangle,$$

where ϕ_k, ϕ_j are defined in (3.1). Then using the formulas in (2.5) and the $f^{-1}N$ in (3.4), the coefficient function A can be given explicitly as

$$A = 2(\psi_1(\overline{\psi_2})_z - \overline{\psi_2}(\psi_1)_z) + 4i\psi_1^2(\overline{\psi_2})^2. \tag{3.8}$$

Next, define B as the complex valued function

$$B = \frac{1}{4}(2H+i)\tilde{A}, \quad \text{where} \quad \tilde{A} = A + \frac{\phi_3^2}{2H+i}. \tag{3.9}$$

Here A and ϕ_3 are the Hopf differential and the e_3-component of $f^{-1}f_z$ for f in Nil_3, respectively.

The complex quadratic differential $\tilde{A}\,dz^2$ will be called the *Berdinsky-Taimanov differential* [42, Lemma 1]. Next we recall the *Abresch-Rosenberg differential* of a surface $f: M \to \mathrm{Nil}_3(\tau)$. It is the quadratic differential Qdz^2 given by [27, 1]:

$$Qdz^2 = 2(H+i\tau)Adz^2 + 4\tau^2\phi_3^2\,dz^2.$$

It is clear that for $\tau = 1/2$, the quadratic differential $4Bdz^2$ is the Abresch-Rosenberg differential. [3]

[3] In [8, (20)], A is defined as

$$A = (\overline{\psi_2}\psi_{1z} - \psi_1\overline{\psi_2}_z) + i\psi_1^2(\overline{\psi_2})^2.$$

In some papers, $A_{AR} := \tilde{A}dz^2$ is called Abresch-Rosenberg differential. The differential Bdz^2 with $B = (2H+i)\tilde{A}/4$ is also called Abresch-Rosenberg differential, see e.g., [26].

3.5. We are mainly interested in conformal immersions of constant mean curvature into Nil_3. Namely, our main interest is the case, where both the Berdinsky-Taimanov differential and the Abresch-Rosenberg differential are holomorphic. However, these differentials do not enter the nonlinear Dirac equations (3.6) and (3.7). It is therefore fortunate that Berdinsky [7] found another system of partial differential equations for the spinor field $\tilde{\psi} = (\psi_1, \psi_2)$ of a surface f which is actually equivalent with the nonlinear Dirac equations (for a proof see [42]) and where the quadratic differentials enter. We define a function w using the Dirac potential $\mathcal{U}(= \mathcal{V})$ as

$$e^{w/2} = \mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h. \tag{3.10}$$

Here, to define the complex function w, we need assume that the mean curvature H and the support function h do not have any common zero. For nonzero constant mean curvature surfaces this is no restriction, however, for minimal surfaces, this assumption is equivalent to that h never vanish, that is, surfaces are nowhere vertical. By Proposition 3.3, minimal immersions with $h = 0$ everywhere are exactly the vertical planes. In what follows, we will thus always exclude vertical planes from our discussions, and usually also work on open sets not including any point where vertical. We can therefore use the representation (3.10).

THEOREM 3.5 ([7]). *Let $\mathbb{D}$ be a simply connected domain in $\mathbb{C}$ and $f : \mathbb{D} \to \mathrm{Nil}_3$ a conformal immersion and w is a complex function defined in* (3.10). *Then the vector $\tilde{\psi} = (\psi_1, \psi_2)$ satisfies the system of equations*

$$\tilde{\psi}_z = \tilde{\psi}\tilde{U}, \quad \tilde{\psi}_{\bar{z}} = \tilde{\psi}\tilde{V}, \tag{3.11}$$

where

$$\begin{aligned} \tilde{U} &= \begin{pmatrix} \frac{1}{2}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -e^{w/2} \\ Be^{-w/2} & 0 \end{pmatrix}, \\ \tilde{V} &= \begin{pmatrix} 0 & -\bar{B}e^{-w/2} \\ e^{w/2} & \frac{1}{2}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}}e^{-w/2+u/2} \end{pmatrix}. \end{aligned} \tag{3.12}$$

Conversely, every vector solution $\tilde{\psi}$ to (3.11) *with* (3.10) *and* (3.12) *is a solution to the nonlinear Dirac equation* (3.6) *with* (3.7).

Sketch of proof. Taking the derivative of the potential $\mathcal{U} = e^{w/2}$ with respect to z, we have

$$\partial_z e^{w/2} = -\frac{H_z}{2}e^{u/2} - \frac{H}{2}(e^{u/2})_z + \frac{i}{4}h_z.$$

Using the explicit formulas for $e^{u/2}$ and h by ψ_1 and ψ_2, we have

$$\frac{w_z}{2}e^{w/2} = -\frac{H_z}{2}e^{u/2} - \frac{2H+i}{2}\left(\psi_{2z}\overline{\psi_2} + \psi_2(\overline{\psi_2})_z\right) - \frac{2H-i}{2}\left(\psi_{1z}\overline{\psi_1} + \psi_1(\overline{\psi_1})_z\right).$$

From the nonlinear Dirac equation we can rephrase this as

$$\frac{w_z}{2}e^{w/2} = -\frac{H_z}{2}e^{u/2} - \frac{2H+i}{2}\psi_2(\overline{\psi_2})_z - \frac{2H-i}{2}(\psi_1)_z\overline{\psi_1} - 2iH\psi_1\overline{\psi_2}|\psi_2|^2.$$

By multiplying the equation above by ψ_1 and using the equation $e^{w/2} = -He^{u/2}/2 + ih/4 = -(2H+i)|\psi_2|^2/2 - (2H-i)|\psi_1|^2/2$, we derive

$$\psi_{1z} = \left(\frac{w_z}{2} + \frac{H_z}{2}e^{-w/2+u/2}\right)\psi_1 + Be^{-w/2}\psi_2.$$

Similarly, ${\psi_2}_z, {\psi_1}_{\bar z}$ and ${\psi_2}_{\bar z}$ can be computed by the nonlinear Dirac equation.

Conversely, let $\tilde\psi$ be a vector solution of (3.11). Then the second column in $\tilde U$ and the first column in $\tilde V$ produce the nonlinear Dirac equation. □

The compatibility condition for (3.11) gives the Gauss-Codazzi equations of a surface $f : \mathbb{D} \to \mathrm{Nil}_3$. We will use in the rest of this paper the stronger condition $\tilde U_{\bar z} - \tilde V_z + [\tilde V, \tilde U] = 0$ which we will continue to call "Gauss-Codazzi equations". This stronger condition holds for all surfaces of constant mean curvature. These are four equations, one for each matrix entry. We obtain

$$\frac{1}{2}w_{z\bar z} + e^w - |B|^2 e^{-w} + \frac{1}{2}(H_{z\bar z} + p)e^{-w/2+u/2} = 0, \tag{3.13}$$

where p is $H_z(-w/2 + u/2)_{\bar z}$ for the $(1,1)$-entry and $H_{\bar z}(-w/2 + u/2)_z$ for the $(2,2)$-entry, respectively. Moreover, the remaining two equations are

$$\begin{aligned} \bar B_z e^{-w/2} &= -\frac{1}{2}\bar B H_z e^{-w+u/2} - \frac{1}{2}H_{\bar z}e^{u/2}, \\ B_{\bar z} e^{-w/2} &= -\frac{1}{2}B H_{\bar z} e^{-w+u/2} - \frac{1}{2}H_z e^{u/2}. \end{aligned} \tag{3.14}$$

The Codazzi equations (3.14) imply that B is holomorphic if the surface is of constant mean curvature. However, we should emphasize that the holomorphicity of B does not imply the constancy of the mean curvature. This situation is very different from the case of space forms. For a precise statement we refer to Appendix A.

REMARK 3.6. Let w, B, H be solutions to the Gauss-Codazzi equations (3.13) and (3.14). To obtain the immersion into Nil_3, a vector solution $\tilde\psi = (\tilde\psi_1, \tilde\psi_2)$ of (3.11) is not enough; the complex function $e^{w/2}$ also needs to satisfy

$$e^{w/2} = -H(|\tilde\psi_1|^2 + |\tilde\psi_2|^2) + \frac{i}{2}(|\tilde\psi_1|^2 - |\tilde\psi_2|^2).$$

In Proposition 4.5 for constant mean curvature surfaces, this will be rephrased in terms of equations for w, H and B.

4. Constant mean curvature surfaces in Nil_3.

4.1. Let f be a conformal immersion in Nil_3 as in the preceding section and $\tilde\psi = (\psi_1, \psi_2)$ and $e^{w/2} = \mathcal{U} = \mathcal{V} \neq 0$ the spinors generating f and the Dirac potential, respectively. Then we have the equations $\tilde\psi_z = \tilde\psi \tilde U$ and $\tilde\psi_{\bar z} = \tilde\psi \tilde V$ as before. Take a fundamental system $\tilde F$ of solutions to this system, we obtain the matrix differential equations

$$\tilde F_z = \tilde F \tilde U, \quad \tilde F_{\bar z} = \tilde F \tilde V. \tag{4.1}$$

It will be convenient for us to replace this system of equations by some gauged system. Consider the $\mathrm{GL}_2\mathbb{C}$ valued function $G = \mathrm{diag}(e^{-w/4}, e^{-w/4})$ and put $F := \tilde F G$, where $\mathrm{diag}(a, b)$ denotes the diagonal $\mathrm{GL}_2\mathbb{C}$ matrix with entries a, b. Then the complex matrix F satisfies the equations

$$F_z = FU, \quad F_{\bar z} = FV, \tag{4.2}$$

where $U = G^{-1}\tilde U G + G^{-1}G_z$ and $V = G^{-1}\tilde V G + G^{-1}G_{\bar z}$.

4.2. We define a family of Maurer-Cartan forms α^λ, parametrized by U and V and the spectral parameter $\lambda \in \mathbb{C}^\times (:= \mathbb{C} \setminus \{0\})$ as follows:

$$\alpha^\lambda := U^\lambda dz + V^\lambda d\bar{z}, \tag{4.3}$$

where

$$\begin{aligned} U^\lambda &= \begin{pmatrix} \frac{1}{4}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -\lambda^{-1}e^{w/2} \\ \lambda^{-1}Be^{-w/2} & -\frac{1}{4}w_z \end{pmatrix}, \\ V^\lambda &= \begin{pmatrix} -\frac{1}{4}w_{\bar{z}} & -\lambda\bar{B}e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}}e^{-w/2+u/2} \end{pmatrix}. \end{aligned} \tag{4.4}$$

We note that $U^\lambda|_{\lambda=1} = U$ and $V^\lambda|_{\lambda=1} = V$. Similar to what happens in space forms, a surface f in Nil_3 of constant mean curvature can be characterized as follows:

THEOREM 4.1. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion and α^λ the 1-form defined in (4.3). Then the following statements are mutually equivalent:*
1. *f has constant mean curvature.*
2. *$d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \mathrm{GL}_2\mathbb{C}$.*
3. *The map $\mathrm{Ad}(F)\sigma_3$ from $\mathbb{D}$ to the semi-Riemannian symmetric space $\mathrm{GL}_2\mathbb{C}/\operatorname{diag}$ is harmonic.*

Here σ_3 denotes the diagonal matrix with entries $1, -1$ and diag *denotes the diagonal subgroup $\mathrm{GL}_1\mathbb{C} \times \mathrm{GL}_1\mathbb{C}$ of $\mathrm{GL}_2\mathbb{C}$.*

Proof. We start by writing out the conditions describing that $d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \mathrm{GL}_2\mathbb{C}$. It is straightforward to see that $d + \alpha^\lambda$ is flat for all $\lambda \in \mathbb{C}^\times$ if and only if the equation

$$(U^\lambda)_{\bar{z}} - (V^\lambda)_z + [V^\lambda, U^\lambda] = 0. \tag{4.5}$$

is satisfied for all $\lambda \in \mathbb{C}^\times$. The coefficients of λ^{-1}, λ^0 and λ of (4.5) can be computed explicitly as follows:

$$\lambda^{-1}\text{-part: } \frac{1}{2}H_{\bar{z}}e^{u/2} = 0, \quad B_{\bar{z}} + \frac{1}{2}BH_{\bar{z}}e^{-w/2+u/2} = 0, \tag{4.6}$$

$$\lambda^0\text{-part: } \frac{1}{2}w_{z\bar{z}} + e^w - |B|^2e^{-w} + \frac{1}{2}(H_{z\bar{z}} + p)e^{-w/2+u/2} = 0, \tag{4.7}$$

$$\lambda\text{-part: } \bar{B}_z + \frac{1}{2}\bar{B}H_z e^{-w/2+u/2} = 0, \quad \frac{1}{2}H_z e^{u/2} = 0, \tag{4.8}$$

where p is $H_z(-w/2 + u/2)_{\bar{z}}$ for the $(1,1)$-entry and $H_{\bar{z}}(-w/2 + u/2)_z$ for the $(2,2)$-entry, respectively. Since the equations in (4.7) are structure equations for the immersion f, these are always satisfied, which in fact is equivalent to (3.13).

(1) $\Rightarrow$ (2): Assume now that f has constant mean curvature. Then, as already mentioned earlier, $\tilde{A} = (A + \phi_3^2/(2H + i))$ is holomorphic, [8, Corollary2, Proposition3]. Thus $B = (2H + i)\tilde{A}/4$ is holomorphic as well. Clearly now, all equations characterizing the flatness of $d + \alpha^\lambda$ are satisfied.

(2) $\Rightarrow$ (1): Assume now that $d + \alpha^\lambda$ is flat. Then it is easy to see that this implies H is constant.

(1) $\Leftrightarrow$ (3): Assume now that the first two statements of the theorem are satisfied. Then the coefficient matrices U^λ and V^λ actually have trace $= 0$ and α^λ describes

the Maurer-Cartan form of a harmonic map of the symmetric space $\mathrm{SL}_2\mathbb{C}/\operatorname{diag}$ as in [24, Proposition3.3]. Conversely, if $\mathrm{Ad}(F)\sigma_3$ is a harmonic map into $\mathrm{GL}_2\mathbb{C}/\operatorname{diag} = \mathrm{SL}_2\mathbb{C}/\operatorname{diag}$, then [24] shows that $d+\alpha^\lambda$ is flat. Note that the diagonal subgroup diag of $\mathrm{SL}_2\mathbb{C}$ is $\{\operatorname{diag}(\gamma, \gamma^{-1}) \mid \gamma \in \mathbb{C}^\times\}$ that is isomorphic to $\mathrm{GL}_1\mathbb{C}$. □

Remark 4.2. The semi-Riemannian symmetric space $\mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$ is identified with the space of all oriented geodesics in the hyperbolic 3-space $\mathbb{H}^3$. The pairwise hyperbolic Gauss maps of constant mean curvature surfaces in $\mathbb{H}^3$ are Lagrangian harmonic maps into the indefinite Kähler symmetric space $\mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$, [22].

From the list of equations characterizing the flatness of $d + \alpha^\lambda$, we obtain the following.

Corollary 4.3. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion. If f has constant mean curvature, then B is holomorphic and*

$$w_{z\bar{z}} + 2e^w - 2|B|^2 e^{-w} = 0 \tag{4.9}$$

holds. The equation (4.9) *is the Gauss equation of the constant mean curvature surface.*

Remark 4.4.
1. By what was said earlier, the converse is almost true. Actually there is only one counter example. In particular, the converse to the statement in the corollary is true, if the spinors ψ_1 and ψ_2 satisfy $|\psi_1| \neq |\psi_2|$.
2. If, in the setting of the corollary above, we assume B to be holomorphic, then the solution to the elliptic equation (4.9) produces an analytic solution w. Inserting this into (4.1) we see that the spinors ψ_j, $j = 1, 2$ are analytic. Therefore, the condition in (1) above follows, if ψ_1 and ψ_2 have a different absolute value at least at one point of the domain $\mathbb{D}$.

4.3. If f is a constant mean curvature immersion into Nil_3, then the corresponding maps $\mathrm{Ad}(F)\sigma_3$, where F satisfies $F^{-1}dF = \alpha^\lambda, \lambda \in \mathbb{S}^1$, into $\mathrm{SL}_2\mathbb{C}/\operatorname{diag}$ all are harmonic. They form the *associated family* of harmonic maps. Tracing back all steps carried out so far, starting from f and ending up at F, one can define maps $f^\lambda : \mathbb{D} \to \mathrm{Nil}_3^{\mathbb{C}}$ into the complexification $\mathrm{Nil}_3^{\mathbb{C}}$ of Nil_3. We call the 1-parameter family of maps $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ the *associated family* of f. From the definition it is clear that the associated family f^λ has the invariant support function $e^{w/2}$ and the varying Abresch-Rosenberg differential $4B^\lambda dz^2$, that is, $B^\lambda = \lambda^{-2}B$. One could hope that all these maps in the associated family actually have values in Nil_3 and are of constant mean curvature. This turns out not to be the case.

The reason is that for constant mean curvature surfaces into Nil_3 there exists a special formula, due to Berdinsky [6, 7].

Proposition 4.5 ([6, 7]). *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion of constant mean curvature. Then the imaginary part of w is constant or one of the following three formulas holds:*

$$e^{(\bar{w}-w)/2} = \frac{2H+i}{2H-i}, \quad e^{(\bar{w}-w)/2} = \frac{2H-i}{2H+i}$$

or

$$\left| r + \bar{r}\frac{B}{|e^w|}\right|^2 = -st\left(1 - \frac{|B|^2}{|e^{2w}|}\right)^2, \tag{4.10}$$

where B and w are as above and $r = -\frac{1}{2}(2H+i)(\bar{w}-w)_z$, $s = (2H+i)e^{\bar{w}/2} - (2H-i)e^{w/2}$ *and* $t = (2H+i)e^{w/2} - (2H-i)e^{\bar{w}/2}$.

Sketch of proof. Let us consider the equation

$$e^{w/2} = -H(|\psi_1|^2 + |\psi_2|^2) + \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2).$$

This equation is rewritten equivalently in the form

$$I = \bar{\psi}^t \hat{h}\psi \ \text{ with } \ \hat{h} = \begin{pmatrix} \frac{1}{2}(-2H+i)e^{-w/2} & 0 \\ 0 & \frac{1}{2}(-2H-i)e^{-w/2} \end{pmatrix}. \tag{4.11}$$

This equation is differentiated for z and for $\bar{z}$. The resulting equations are

$$\operatorname{tr}\left(\frac{1}{2}e^{-w/2}\begin{pmatrix} 0 & \frac{tB}{|e^w|} \\ t & r \end{pmatrix}\begin{pmatrix} |\psi_1|^2 & \psi_1\overline{\psi_2} \\ \overline{\psi_1}\psi_2 & |\psi_2|^2 \end{pmatrix}\right) = 0, \tag{4.12}$$

$$\operatorname{tr}\left(\frac{1}{2}e^{-w/2}\begin{pmatrix} -\bar{r} & s \\ \frac{s\bar{B}}{|e^w|} & 0 \end{pmatrix}\begin{pmatrix} |\psi_1|^2 & \psi_1\overline{\psi_2} \\ \overline{\psi_1}\psi_2 & |\psi_2|^2 \end{pmatrix}\right) = 0, \tag{4.13}$$

where $r = -\frac{1}{2}(2H+i)(\bar{w}-w)_z$, $t = (2H+i)e^{w/2} - (2H-i)e^{\bar{w}/2}$ and $s = (2H+i)e^{\bar{w}/2} - (2H-i)e^{w/2}$. Conversely, from these latter two equations one obtains $ce^{w/2} = -H(|\psi_1|^2 + |\psi_2|^2) + i(|\psi_1|^2 - |\psi_2|^2)/2$, where c is a complex constant. One can normalize things or manipulate things so that this constant can be removed. Thus (4.11) is equivalent with (4.12) and (4.13). The equations (4.12) and (4.13) are then reformulated equivalently in the form:

$$\frac{tB}{|e^w|}\xi + t\bar{\xi} + r|\xi|^2 = 0 \ \text{ and } \ \frac{sB}{|e^w|}\xi + s\bar{\xi} + r = 0,$$

where $\xi = \psi_2/\psi_1$, which can be done without loss of generality since otherwise $\psi_1 = 0$ identically and the Berdinsky system would also yield $\psi_2 = 0$. The next conclusion in [7] requires to assume that s does not vanish identically. Therefore we need to admit the case $s = 0$ and continue with the assumption $s \neq 0$. Inserting the second equation into the first yields $r(|\xi|^2 - t/s) = 0$. We thus need to admit the case $r = 0$. It is easy to see that this latter condition implies that the imaginary part of w is constant. This same statement follows from $s = 0$. Now we assume $s \neq 0$ and $r \neq 0$ and obtain (as claimed in [7]) $|\xi|^2 = t/s$. Inserting this into the first of the last two equations above we see that these two equations are complex conjugates of each other now if $t \neq 0$. But $t = 0$ implies again that the imaginary part of w is constant. Finally, solving for ξ in the second equation above and inserting into the first one now obtains the equation

$$\left(1 - \frac{|B|^2}{|e^w|^2}\right)\bar{\xi} = -\frac{1}{s}\left(r + \bar{r}\frac{B}{|e^w|}\right).$$

Taking absolute values here yields the last of our equations. □

As a corollary to this result we obtain the following.

Corollary 4.6. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion of constant mean curvature. If the associated family f^λ of immersions into $\mathrm{Nil}_3^{\mathbb{C}}$ as defined above actually has values in Nil_3 for all $\lambda \in \mathbb{S}^1$, then $\bar{w} - w$ is constant and the surfaces are minimal.*

Proof. Assume the last of the equations is satisfied. Introducing λ as in this paper can also be interpreted, like for constant mean curvature surfaces in $\mathbb{R}^3$ by Bonnet, as replacing B by $\lambda^{-2}B$, $\lambda \in \mathbb{S}^1$. This is an immediate consequence of (4.9). Let ψ_j be the λ-dependent solutions to the equations (4.2). Then f^λ is defined from these ψ_j as f was defined in the case $\lambda = 1$. Thus the Berdinsky system associated with f^λ is a constant mean curvature system and the immersions f^λ all are constant mean curvature immersions into Nil_3. Therefore, all the quantities associated with these immersions satisfy the Berdinsky equation (4.10). As a consequence of our assumptions, this equation needs to be satisfied for all B^λ. But replacing B by $\lambda^{-2}B = B^\lambda$ in (4.10) we see that the required equality only holds for all $\lambda \in \mathbb{S}^1$ if and only if the function r occurring in (4.10) vanishes identically. Moreover, the vanishing of r implies that $\bar{w} - w$ is antiholomorphic, and since this function only attains values in $i\mathbb{R}$, it is constant. Let us consider next the Gauss equation (4.9). Since the imaginary part $\operatorname{Im} w = \theta_0$ of w is constant, the term $w_{z\bar{z}}$ is real. Therefore the imaginary part of $e^w - B\bar{B}e^{-w}$ vanishes. A simple computation shows that this implies $(e^{2\operatorname{Re} w} + B\bar{B})\sin(\theta_0) = 0$, whence $\sin(\theta_0) = 0$ and θ_0 is an integral multiple of π. As a consequence, $e^{w/2} = e^{\operatorname{Re} w/2}e^{ik\pi/2}$. If k is odd, then $e^{w/2}$ is purely imaginary and $H = 0$ follows. If k is even, then $e^{w/2}$ is real. This implies $|\psi_1|^2 = |\psi_2|^2$, which shows that it is a surface of non constant mean curvature, see Appendix A. □

REMARK 4.7. We have just shown that associated families of "real" constant mean curvature surfaces in Nil_3 can only be minimal. We will show in the next section that actually every minimal surface is a member of an associated family of minimal surfaces in Nil_3.

5. Characterizations of minimal surfaces in Nil_3.

5.1. We recall the beginning of section 4. In particular, we consider the family of Maurer-Cartan forms α^λ

$$\alpha^\lambda := U^\lambda dz + V^\lambda d\bar{z}, \quad \lambda \in \mathbb{S}^1, \tag{5.1}$$

where U^λ and V^λ are defined in (4.4). For surfaces of constant mean curvature these expressions have a particularly simple form:

$$\begin{aligned} U(\lambda)(:= U^\lambda) &= \begin{pmatrix} \frac{1}{4}w_z & -\lambda^{-1}e^{w/2} \\ \lambda^{-1}Be^{-w/2} & -\frac{1}{4}w_z \end{pmatrix}, \\ V(\lambda)(:= V^\lambda) &= \begin{pmatrix} -\frac{1}{4}w_{\bar{z}} & -\lambda\bar{B}e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}w_{\bar{z}} \end{pmatrix}. \end{aligned} \tag{5.2}$$

Minimal surfaces can be easily characterized among all constant mean curvature surfaces in the following manner.

LEMMA 5.1. *Let f be a surface of constant mean curvature in* Nil_3. *Then the following statements are mutually equivalent*:

1. *f is a minimal surface.*
2. *$e^{w/2} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h$ is purely imaginary.*
3. *The matrices $U(\lambda)$ and $V(\lambda)$ satisfy*

$$V(\lambda) = -\sigma_3 \overline{U(1/\bar{\lambda})}^t \sigma_3, \quad \textit{where } \sigma_3 = \operatorname{diag}(1, -1). \tag{5.3}$$

In particular, for a constant mean curvature surface f, the Maurer-Cartan form α^λ takes values in the real Lie subalgebra $\mathfrak{su}_{1,1}$ of $\mathfrak{sl}_2\mathbb{C}$ if and only if f is minimal;

$$\mathfrak{su}_{1,1} = \left\{ \begin{pmatrix} ai & b \\ \bar{b} & -ai \end{pmatrix} \;\middle|\; a \in \mathbb{R},\, b \in \mathbb{C} \right\}.$$

5.2. As is well known, constancy of the mean curvature of surfaces in three-dimensional space forms is equivalent to the holomorphicity of the Hopf differential. Moreover constancy of the mean curvature is characterized by harmonicity of appropriate Gauss maps.

To obtain another characterization of minimal surfaces we will introduce the notion of Gauss map for surfaces in Nil_3. Let N be the unit normal vector field along the surface f and $f^{-1}N$ the left translation of N.

We identify the Lie algebra $\mathfrak{nil}_3$ of Nil_3 with Euclidean 3-space $\mathbb{E}^3$ via the natural basis $\{e_1, e_2, e_3\}$. Under this identification, the map $f^{-1}N$ can be considered as a map into the unit two-sphere $\mathbb{S}^2 \subset \mathfrak{nil}_3$. We now consider the *normal Gauss map* g of the surface f in Nil_3, [31, 20]: The map g is defined as the composition of the stereographic projection π from the south pole with $f^{-1}N$, that is, $g = \pi \circ f^{-1}N : \mathbb{D} \to \mathbb{C} \cup \{\infty\}$ and thus, applying the stereographic projection to $f^{-1}N$ defined in (3.4), we obtain

$$g = \frac{\underline{\psi_2}}{\overline{\psi_1}}\,. \tag{5.4}$$

Note that the unit normal N is represented in terms of the normal Gauss map g as

$$f^{-1}N = \frac{1}{1+|g|^2}\left(2\,\mathrm{Re}(g)e_1 + 2\,\mathrm{Im}(g)e_2 + (1-|g|^2)e_3\right). \tag{5.5}$$

The formula (5.5) implies that f is nowhere vertical if and only if $|g| < 1$ or $|g| > 1$.

Remark 5.2.

1. If $|g| > 1$, then the e_3-component of $f^{-1}N$ has a negative sign. Therefore such surfaces are called "downward". Analogously $|g| < 1$ the surfaces are called "upward".
2. The normal Gauss map of a vertical plane satisfies $|g| \equiv 1$. Conversely if the normal Gauss map g of a conformal minimal immersion satisfies $|g| \equiv 1$, then it is a vertical plane.

5.3. We have seen in Theorem 4.1 and Remark 4.2, there exist harmonic maps into the semi-Riemannian symmetric space $\mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$ associated to constant mean curvature surface in Nil_3. In view of Lemma 5.1, one would expect that minimal surfaces can be characterized by harmonic maps into semi-Riemannian symmetric spaces associated to the real Lie subgroup

$$\mathrm{SU}_{1,1} = \left\{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{SL}_2\mathbb{C} \right\}$$

of $\mathrm{SL}_2\mathbb{C}$ with Lie algebra $\mathfrak{su}_{1,1}$. For this purpose we recall a Riemannian symmetric space representation of the hyperbolic 2-space $\mathbb{H}^2$. Note that the symmetric space $\mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$ is regarded as a "complexification" of $\mathbb{H}^2$. Since $\mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C} = \mathrm{SU}_{1,1}^{\mathbb{C}}/\mathrm{U}_1^{\mathbb{C}}$.

Let us equip the Lie algebra $\mathfrak{su}_{1,1}$ with the following Lorentz scalar product $\langle \cdot, \cdot \rangle_m$

$$\langle X, Y \rangle_m = 2\mathrm{tr}\,(XY), \quad X, Y \in \mathfrak{su}_{1,1}.$$

Then $\mathfrak{su}_{1,1}$ is identified with Minkowski 3-space $\mathbb{E}^{2,1}$ as an indefinite scalar product space. The hyperbolic 2-space $\mathbb{H}^2$ of constant curvature -1 is realized in $\mathfrak{su}_{1,1}$ as a quadric

$$\mathbb{H}^2 = \{X \in \mathfrak{su}_{1,1} \mid \langle X, X \rangle_m = -1, \ \ \langle X, i\sigma_3 \rangle_m < 0\}.$$

The Lie group $\mathrm{SU}_{1,1}$ acts transitively and isometrically on $\mathbb{H}^2$ via the Ad-action. The isotropy subgroup at $i\sigma_3/2$ is U_1 that is the Lie subgroup of $\mathrm{SU}_{1,1}$ consisting of diagonal matrices. The resulting homogeneous Riemannian 2-space $\mathbb{H}^2 = \mathrm{SU}_{1,1}/\mathrm{U}_1$ is a Riemannian symmetric space with involution $\sigma = \mathrm{Ad}(\sigma_3)$.

Next we recall the stereographic projection from the hyperbolic 2-space $\mathbb{H}^2 \subset \mathbb{E}^{2,1}$ onto the Poincaré disc $\mathcal{D} \subset \mathbb{C}$. We identify $\mathfrak{su}_{1,1}$ with Minkowski 3-space $\mathbb{E}^{2,1}$ by the correspondence:

$$\frac{1}{2}\begin{pmatrix} ri & -p-qi \\ -p+qi & -ri \end{pmatrix} \in \mathfrak{su}_{1,1} \longleftrightarrow (p, q, r) \in \mathbb{E}^{2,1}.$$

Under this identification, the stereographic projection $\pi_h : \mathbb{H}^2 \to \mathcal{D}$ with base point $-i\sigma_3/2$ is given explicitly by

$$\pi_h(p, q, r) = \frac{1}{1+r}(p + qi). \tag{5.6}$$

The inverse mapping of π_h^{-1} is computed as

$$\pi_h^{-1}(z) = \frac{1}{1-|z|^2}\left(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), 1 + |z|^2\right), \quad |z| < 1.$$

5.4. Minimal surfaces in Nil_3 are characterized in terms of the normal Gauss map as follows.

THEOREM 5.3. *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion which is nowhere vertical and α^λ the* 1*-form defined in* (4.3)*. Moreover, assume that the unit normal $f^{-1}N$ defined in* (3.4) *is upward. Then the following statements are equivalent*:

1. *f is a minimal surface.*
2. *$d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \mathrm{SU}_{1,1}$.*
3. *The normal Gauss map g for f is a non-conformal harmonic map into the hyperbolic* 2*-space $\mathbb{H}^2$.*

Proof. The equivalence of (1) and (2) follows immediately from Theorem 4.1 in view of Lemma 5.1.

Next we consider (2) $\Rightarrow$ (3). Since α^λ takes values in $\mathfrak{su}_{1,1}$, there exists a solution matrix to (4.2) which is contained in $\mathrm{SU}_{1,1}$. We express this matrix in terms of spinors ψ_1 and ψ_2. First we recall that the vector $\psi = e^{-w/4}(\psi_1, \psi_2)$ solves this equation. Since in our case now $e^{w/2}$ is purely imaginary, it is straightforward to verify that also the vector $\psi^* = \overline{e^{-w/4}}(\overline{\psi_2}, \overline{\psi_1})$ solves the same system of differential equations.

Let S be the fundamental system of solutions to the system (4.2) which has the vector ψ as its first row and the vector ψ^* as its second row. Since the coefficient matrices have trace $= 0$, we know $\det S =$ constant. From the form of S we infer $\det S = |e^{-w/4}|^2(|\psi_1|^2 - |\psi_2|^2)$.

By assumption, the normal is "upward", whence $\det S > 0$, see (3.4). As a consequence, after multiplying S by some positive real constant c (actually, $c = 1/\sqrt{2}$) we can assume $\det(cS) = 1$. Taking into account the form of S and cS we see that cS is a solution to (4.2) which takes values in $\mathrm{SU}_{1,1}$.

Let F be a family of maps such that $F^{-1}dF = \alpha^\lambda$ with $F|_{\lambda=1} = cS$ and define a map N_m by

$$N_m = \frac{i}{2}\,\mathrm{Ad}(F)\sigma_3|_{\lambda=1}.$$

Clearly, N_m takes values in $\mathbb{H}^2 \subset \mathfrak{su}_{1,1}$. Let $\mathfrak{su}_{1,1} = \mathfrak{u}_1 \oplus \mathfrak{p}$ denote the Cartan decomposition of the Lie algebra $\mathfrak{su}_{1,1}$ induced by the derivative of $\sigma = \mathrm{Ad}(\sigma_3)$. Here the linear subspace $\mathfrak{p}$ is identified with the tangent space of $\mathbb{H}^2$ at the origin $i\sigma_3/2$. It is known, [24, Proposition 3.3] and Appendix D, that N_m is harmonic if and only if

$$F^{-1}dF = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \tag{5.7}$$

where $\alpha_0 : T\mathbb{D} \to \mathfrak{u}_1$ and $\alpha_1 : T\mathbb{D} \to \mathfrak{p}$ are $\mathfrak{u}_1$ and $\mathfrak{p}$ valued 1-forms respectively, and superscripts $'$ and $''$ denote $(1,0)$ and $(0,1)$-part respectively. It is easy to check that α^λ, as defined in (5.1) coincides with the right hand side of (5.7).

In terms of the generating spinors ψ_1 and ψ_2, the map N_m can be computed as

$$N_m = \frac{i}{2}\,\mathrm{Ad}(F)\sigma_3|_{\lambda=1} = \frac{i}{2(|\psi_1|^2 - |\psi_2|^2)}\begin{pmatrix} |\psi_1|^2 + |\psi_2|^2 & 2i\psi_1\psi_2 \\ 2i\overline{\psi_1\psi_2} & -|\psi_1|^2 - |\psi_2|^2 \end{pmatrix},$$

where we set

$$F|_{\lambda=1} = \frac{1}{\sqrt{|\psi_1|^2 - |\psi_2|^2}}\begin{pmatrix} \sqrt{i}^{-1}\psi_1 & \sqrt{i}^{-1}\psi_2 \\ \sqrt{i}\,\overline{\psi_2} & \sqrt{i}\,\overline{\psi_1} \end{pmatrix}. \tag{5.8}$$

Applying the stereographic projection $\pi_h : \mathbb{H}^2 \subset \mathbb{E}^{2,1} \to \mathcal{D} \subset \mathbb{C}$ as in (5.6) with base point $-i\sigma_3/2$ to N_m, we obtain

$$\pi_h \circ N_m = \frac{\overline{\psi_2}}{\overline{\psi_1}}.$$

As a consequence, the map $\pi_g \circ N_m$ is actually the normal Gauss map g given in (5.4) and we have $|g| < 1$, since we assumed that f is nowhere vertical and $f^{-1}N$ upward. Moreover, g can be considered as a harmonic map into $\mathbb{H}^2$ through the stereographic projection. Since the $(1,0)$-part of the upper right entry of $\alpha^\lambda|_{\lambda=1}$ is non-degenerate, the normal Gauss map g is non-conformal. Therefore, (2) implies (3).

Finally, we consider (3) $\Rightarrow$ (2). By assumption we know that the normal Gauss map g is harmonic. Therefore a loop group approach is applicable [24]. In particular, there is a moving frame F which takes values in $\mathrm{SU}_{1,1}$ from which g can be obtained by projection to $\mathrm{SU}_{1,1}/\mathrm{U}_1 = \mathbb{H}^2$. But now the result proven in the next section can be applied and the claim is proven. □

REMARK 5.4.

1. In the theorem above we have made two additional assumptions: "nowhere vertical", which means no branch points and "upward". The first condition is also equivalent with $|\psi_1| \neq |\psi_2|$. Hence the Gauss map does not reach the boundary of $\mathbb{H}^2$. The second condition implies that the Gauss map always stays inside the unit disk, that is, the upper hemisphere of $\mathbb{S}^2$ and never move across the unit circle to the lower hemisphere of $\mathbb{S}^2$.

2. The harmonicity of the normal Gauss map g for a minimal surface f can be seen from the partial differential equation for g, see [20, 31].
3. The minimal surface corresponding to a normal Gauss map g is uniquely determined as follows: Let f and $\hat{f}$ be two minimal surfaces with the same Gauss map g. Then the moving frames F and $\hat{F}$ to f and $\hat{f}$ are in the relation $F = \hat{F}K_0$ for some $K_0 \in \mathrm{U}_1$, and a straightforward computation shows that K_0 is constant. It is easy to see that the respective generating spinors ψ_i and $\hat{\psi}_i$ $(i = 1, 2)$ for f and $\hat{f}$ satisfy $\psi_i = k_0\hat{\psi}_i$ for some constant k_0, which means that ψ_i and $\hat{\psi}_i$ are the same up to a change of coordinates.

Definition 1. Let f be a minimal surface in Nil_3 and F as above the corresponding $\mathrm{SU}_{1,1}$-valued solution to the equation $F^{-1}dF = \alpha^\lambda, \lambda \in \mathbb{S}^1$, where α^λ is defined by (4.3) and $F|_{\lambda=1}$ is given in (5.8). Then F is called *extended frame* of the minimal surface f.

For later reference we express the extended frame associated with respect to the generating spinors ψ_1 and ψ_2 for a minimal surface;

$$F(\lambda) = \frac{1}{\sqrt{|\psi_1(\lambda)|^2 - |\psi_2(\lambda)|^2}} \begin{pmatrix} \sqrt{i}^{-1}\psi_1(\lambda) & \sqrt{i}^{-1}\psi_2(\lambda) \\ \sqrt{i}\,\overline{\psi_2(\lambda)} & \sqrt{i}\,\overline{\psi_1(\lambda)} \end{pmatrix}. \tag{5.9}$$

We would like to note that the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ in this expression are only determined up to some positive real function.

6. Sym formula. In this section, we present an immersion formula for minimal surfaces in Nil_3. This formula will be called the *Sym-formula*. It involves exclusively the extended frames of minimal surfaces. We will also explain the relation to another formula for f stated in [18].

6.1. We first identify the Lie algebra $\mathfrak{nil}_3$ of Nil_3 with the Lie algebra $\mathfrak{su}_{1,1}$ as a *real vector space*. In $\mathfrak{su}_{1,1}$, we choose the following basis:

$$\mathcal{E}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \mathcal{E}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ \text{and} \ \mathcal{E}_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{6.1}$$

One can see that $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ is an orthogonal basis of $\mathfrak{su}_{1,1}$ with timelike vector $\mathcal{E}_3$. A linear isomorphism $\Xi : \mathfrak{su}_{1,1} \to \mathfrak{nil}_3$ is then given by

$$\mathfrak{su}_{1,1} \ni x_1\mathcal{E}_1 + x_2\mathcal{E}_2 + x_3\mathcal{E}_3 \longmapsto x_1e_1 + x_2e_2 + x_3e_3 \in \mathfrak{nil}_3. \tag{6.2}$$

Note that the linear isomorphism Ξ is not a Lie algebra isomorphism. Next we consider the exponential map $\exp : \mathfrak{nil}_3 \to \mathrm{Nil}_3$ defined in (2.4). We define a smooth bijection $\Xi_{\mathrm{nil}} : \mathfrak{su}_{1,1} \to \mathrm{Nil}_3$ by $\Xi_{\mathrm{nil}} := \exp \circ \Xi$. In what follows we will take derivatives for functions of λ. Note that for $\lambda = e^{i\theta} \in \mathbb{S}^1$, we have $\partial_\theta = i\lambda\partial_\lambda$.

Theorem 6.1. *Assume the map* $F : \mathbb{D} \to (\Lambda \mathrm{SU}_{1,1})_\sigma$ *satisfies* (5.2) *and is written in the form* (5.9) *with* $-2ie^{w/2} = |\psi_1(\lambda)|^2 - |\psi_2(\lambda)|^2 > 0$, *and* m *and* N_m *are respectively the maps*

$$m = -i\lambda(\partial_\lambda F)F^{-1} - N_m \ \ \textit{and} \ \ N_m = \frac{i}{2}\,\mathrm{Ad}(F)\sigma_3. \tag{6.3}$$

Moreover, define a map $f^\lambda : \mathbb{D} \to \mathrm{Nil}_3$ *by* $f^\lambda := \Xi_{\mathrm{nil}} \circ \hat{f}^\lambda$ *with*

$$\hat{f}^\lambda = \left(m^o - \frac{i}{2}\lambda(\partial_\lambda m)^d\right)\Big|_{\lambda\in\mathbb{S}^1}, \tag{6.4}$$

where the superscripts "o" and "d" denote the off-diagonal and diagonal part, respectively. Then, for each $\lambda \in \mathbb{S}^1$*, the map* f^λ *is a minimal surface in* Nil_3 *and* N_m *is the normal Gauss map of* f^λ*. In particular, if* F *is the frame of some minimal surface* f *in* Nil_3*, then* $f^\lambda|_{\lambda=1}$ *gives the original minimal surface up to a rigid motion.*

Proof. Since m and $i\lambda(\partial_\lambda m^d)$ take values in the Lie algebra of $\mathrm{SU}_{1,1}$, the map f^λ takes values in Nil_3 via the bijection Ξ_{nil}. Let us express the extended frame $F(\lambda)$ by $\psi_1(\lambda)$ and $\psi_2(\lambda)$ as in (5.9). We note that $\psi_1(\lambda)$ and $\psi_2(\lambda)$ depend on λ and for each $\lambda \in \mathbb{S}^1$, the extended frame F takes values in $\mathrm{SU}_{1,1}$. Then a straightforward computation shows that

$$\begin{aligned}\partial_z m &= \mathrm{Ad}(F)\left(-i\lambda\partial_\lambda U^\lambda - \frac{i}{2}[U^\lambda, \sigma_3]\right) \\ &= -2i\lambda^{-1}e^{w/2}\,\mathrm{Ad}(F)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \\ &= \lambda^{-1}\begin{pmatrix}-\psi_1(\lambda)\overline{\psi_2(\lambda)} & -i\psi_1(\lambda)^2\\ -i\overline{\psi_2(\lambda)}^2 & \psi_1(\lambda)\overline{\psi_2(\lambda)}\end{pmatrix}.\end{aligned} \tag{6.5}$$

Thus

$$\partial_z m = \phi_1(\lambda)\mathcal{E}_1 + \phi_2(\lambda)\mathcal{E}_2 - i\phi_3(\lambda)\mathcal{E}_3 \tag{6.6}$$

with

$$\phi_1(\lambda) = \lambda^{-1}\left(\overline{\psi_2(\lambda)}^2 - \psi_1(\lambda)^2\right), \quad \phi_2(\lambda) = i\lambda^{-1}\left(\overline{\psi_2(\lambda)}^2 + \psi_1(\lambda)^2\right)$$

and

$$\phi_3(\lambda) = 2\lambda^{-1}\psi_1(\lambda)\overline{\psi_2(\lambda)}.$$

Thus using (6.5), the derivative of m with respect to z and λ can be computed as

$$\begin{aligned}\partial_z(i\lambda(\partial_\lambda m)) = i\lambda\partial_\lambda(\partial_z m) &= i\lambda\partial_\lambda\left(-2i\lambda^{-1}e^{w/2}\,\mathrm{Ad}(F)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right), \\ &= -i(\partial_z m) - [m + N_m, \partial_z m]\,.\end{aligned} \tag{6.7}$$

Here $[a, b]$ denotes the usual bracket of matrices, that is, $[a, b] = ab - ba$. Using (6.5), we have

$$[-N_m, \partial_z m]^d = -i(\partial_z m)^d$$

and

$$-[m, \partial_z m]^d = \left(\phi_1(\lambda)\int \phi_2(\lambda)\,dz - \phi_2(\lambda)\int \phi_1(\lambda)\,dz\right)\mathcal{E}_3.$$

Thus we have

$$\partial_z\left(-\frac{i\lambda\partial_\lambda m^d}{2}\right) = \left(\phi_3(\lambda) - \frac{1}{2}\phi_1(\lambda)\int \phi_2(\lambda)\,dz + \frac{1}{2}\phi_2(\lambda)\int \phi_1(\lambda)\,dz\right)\mathcal{E}_3. \tag{6.8}$$

Therefore, combining (6.6) and (6.8), we obtain

$$\partial_z \hat{f}^\lambda = \phi_1(\lambda)\mathcal{E}_1 + \phi_2(\lambda)\mathcal{E}_2 + \left(\phi_3(\lambda) - \frac{1}{2}\phi_1(\lambda)\int \phi_2(\lambda)\,dz + \frac{1}{2}\phi_2(\lambda)\int \phi_1(\lambda)\,dz\right)\mathcal{E}_3.$$

We now use the identification (6.2) with the left translation $(f^\lambda)^{-1}$, that is,

$$(f^\lambda)^{-1}\partial_z f^\lambda = \phi_1(\lambda)e_1 + \phi_2(\lambda)e_2 + \phi_3(\lambda)e_3. \tag{6.9}$$

Thus $\lambda^{-1/2}\psi_1(\lambda)$ and $\lambda^{1/2}\psi_2(\lambda)$ are spinors for f^λ for each $\lambda \in \mathbb{S}^1$. In particular, the function

$$\frac{i}{2}(|\lambda^{-1/2}\psi_1(\lambda)|^2 - |\lambda^{1/2}\psi_2(\lambda)|^2) = e^{w/2}$$

does not depend on λ and implies that the mean curvature H is equal to zero. Moreover, the conformal factor of the induced metric of f^λ is given by

$$e^u = 4(|\psi_1(\lambda)|^2 + |\psi_2(\lambda)|^2)^2.$$

This metric is non-degenerate, since F takes values in $\mathrm{SU}_{1,1}$ for each $\lambda \in \mathbb{S}^1$, that is, $|\psi_1(\lambda)|$ and $|\psi_2(\lambda)|$ are not simultaneously equal to zero. Thus the map f^λ actually defines a minimal surface in Nil_3 for each $\lambda \in \mathbb{S}^1$. By the same argument as in the proof of Theorem 4.1 for the spinors $\lambda^{-1/2}\psi_1(\lambda)$ and $\lambda^{1/2}\psi_2(\lambda)$, the map N_m is the normal Gauss map for the minimal surface f^λ. Then, at $\lambda = 1$, the minimal surface given by $f^\lambda|_{\lambda=1}$ and the original minimal surface have the same metric $e^u dz d\bar{z}$, the holomorphic differential Bdz^2 and the support $h(dz)^{1/2}(d\bar{z})^{1/2}$. Thus up to a rigid motion it is the same minimal surface. This completes the proof. □

Remark 6.2.

1. For each $\lambda \in \mathbb{S}^1$ the immersion m defined in (6.3) gives a spacelike surface of constant mean curvature in Minkowski 3-space $\mathbb{E}^{2,1} = \mathfrak{su}_{1,1}$, see [36, 11]. It is well known that the Sym formula for constant mean curvature surfaces in $\mathbb{E}^{2,1}$(or $\mathbb{E}^3$) involves the first derivative with respect to λ only, however, the formula for Nil_3 involves the second derivative with respect to λ as well. Purely technically the reason is the subtraction term. But there should be a better geometric reason.
2. Theorem 6.1 gives clear geometric meaning for the immersion formula for f. The Sym formula (6.4) for f was written down in [18] in a different way.

In the following Corollary, we compute the Abresch-Rosenberg differential $4B^\lambda dz^2$ for the 1-parameter family f^λ in Theorem 6.1 and it implies that the family f^λ actually defines the associated family.

Corollary 6.3. *Let f be a conformal minimal surface in* Nil_3 *and f^λ the family of surfaces defined by* (6.4). *Then f^λ preserves the mean curvature* $(= 0)$ *and the support. The Abresch-Rosenberg differential $4B^\lambda dz^2$ for f^λ is given by $4B^\lambda dz^2 = 4\lambda^{-2}Bdz^2$, where $4Bdz^2$ is the Abresch-Rosenberg differential for f. Therefore $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ is the associated family of the minimal surface f.*

Proof. From Theorem 4.1 it is clear that a minimal surface f in Nil_3 defines a 1-parameter family f^λ of the minimal immersion f such that $f^\lambda|_{\lambda=1} = f$, and f^λ is a minimal surface for each $\lambda \in \mathbb{S}^1$. The spinors for f^λ are given as the functions $\lambda^{-1/2}\psi_1(\lambda)$ and $\lambda^{1/2}\psi_2(\lambda)$, see the proof of Theorem 4.1. Using (5.1) and (5.9) we obtain

$$(\lambda^{-1/2}\psi_1(\lambda))_z = \frac{1}{2}w_z(\lambda^{-1/2}\psi_1(\lambda)) + (\lambda^{1/2}\psi_2(\lambda))(\lambda^{-2}B)e^{-w/2}. \tag{6.10}$$

Comparing (6.10) to (3.11), the Abresch-Rosenberg differential $4B^\lambda dz^2$ for f^λ is $4\lambda^{-2}Bdz^2$. Moreover B^λ is holomorphic, since B is holomorphic. We note that

surface	mean curvature	metric	holo. differential	support
$f(= f^{\lambda}\|_{\lambda=1})$	$H = 0$	$\exp(u) dz d\bar{z}$	$B dz^2$	$h(dz)^{1/2}(d\bar{z})^{1/2}$
f^{λ}	$H = 0$	$\exp(u^{\lambda}) dz d\bar{z}$	$\lambda^{-2} B dz^2$	$h(dz)^{1/2}(d\bar{z})^{1/2}$

Table 6.1
An original minimal surface f and the deformation family f^{λ}.

the support function for f^{λ} is given by $e^{w/2} = i(|\psi_1(\lambda)|^2 - |\psi_2(\lambda)|^2)/2$, which is invariant in this family. Therefore this 1-parameter family f^{λ} is the associated family as explained in Section 4.3. □

Remark 6.4. In general, the metric $e^u dz d\bar{z} = 4(|\psi_1(\lambda)|^2 + |\psi_2(\lambda)|^2)^2 dz d\bar{z}$ is not preserved in the associated family. This is in contrast to the case of an associated family of nonzero constant mean curvature surfaces in $\mathbb{E}^3$ or $\mathbb{E}^{2,1}$, where the metric is preserved.

7. Potentials for minimal surfaces. In this section, we show that pairs of meromorphic and anti-meromorphic 1-forms, the so-called *normalized potentials*, are obtained from the extended frames of minimal surfaces in Nil_3 via the Birkhoff decomposition of loop groups.

We first define the twisted $\mathrm{SL}_2\mathbb{C}$ loop group as a space of continuous maps from $\mathbb{S}^1$ to the Lie group $\mathrm{SL}_2\mathbb{C}$, that is,

$$\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma} = \{g : \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{C} \mid g(-\lambda) = \sigma g(\lambda)\},$$

where $\sigma = \mathrm{Ad}(\sigma_3)$. We restrict our attention to loops in $\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma}$ such that the associate Fourier series of the loops are absolutely convergent. Such loops determine a Banach algebra, the so-called *Wiener algebra*, and it induces a topology on $\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma}$, the so-called *Wiener topology*. From now on, we consider only $\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma}$ equipped with the Wiener topology.

Let $D^{\pm}$ denote respective the inside of unit disk and the union of outside of the unit disk and infinity. We define *plus* and *minus* loop subgroups of $\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma}$;

$$\Lambda^{\pm} \mathrm{SL}_2\mathbb{C}_{\sigma} = \{g \in \Lambda \mathrm{SL}_2\mathbb{C}_{\sigma} \mid g \text{ can be extended holomorphically to } D^{\pm}\}. \tag{7.1}$$

By $\Lambda_*^{+} \mathrm{SL}_2\mathbb{C}_{\sigma}$ we denote the subgroup of elements of $\Lambda^{+} \mathrm{SL}_2\mathbb{C}_{\sigma}$ which take the value identity at zero. Similarly, by $\Lambda_*^{-} \mathrm{SL}_2\mathbb{C}_{\sigma}$ we denote the subgroup of elements of $\Lambda^{-} \mathrm{SL}_2\mathbb{C}_{\sigma}$ which take the value identity at infinity.

We also define the $\mathrm{SU}_{1,1}$-loop group as follows:

$$(\Lambda \mathrm{SU}_{1,1})_{\sigma} = \left\{ g \in \Lambda \mathrm{SL}_2\mathbb{C}_{\sigma} \mid \sigma_3 \overline{g(1/\bar{\lambda})}^{t-1} \sigma_3 = g(\lambda) \right\}. \tag{7.2}$$

It is clear that extended frames of minimal surfaces in Nil_3 are elements in $(\Lambda \mathrm{SU}_{1,1})_{\sigma}$.

Theorem 7.1 (Birkhoff decomposition, [39]). *The respective multiplication maps*

$$\Lambda_*^{-} \mathrm{SL}_2\mathbb{C}_{\sigma} \times \Lambda^{+} \mathrm{SL}_2\mathbb{C}_{\sigma} \to \Lambda \mathrm{SL}_2\mathbb{C}_{\sigma} \quad \text{and} \quad \Lambda_*^{+} \mathrm{SL}_2\mathbb{C}_{\sigma} \times \Lambda^{-} \mathrm{SL}_2\mathbb{C}_{\sigma} \to \Lambda \mathrm{SL}_2\mathbb{C}_{\sigma} \tag{7.3}$$

are analytic diffeomorphisms onto open dense subsets of $\Lambda \mathrm{SL}_2\mathbb{C}_{\sigma}$.

It is easy to check that the extended frames F are elements in $\Lambda \mathrm{SL}_2\mathbb{C}_\sigma$, since U^λ and V^λ satisfy the twisted condition. Applying the Birkhoff decomposition of Theorem 7.1 to the extended frame F, we obtain a pair of meromorphic and anti-meromorphic 1-forms, that is, the pair of normalized potentials.

THEOREM 7.2 (Pairs of normalized potentials). *Let F be the extended frame of some minimal immersion in* Nil_3 *on some simply connected domain $\mathbb{D} \subset \mathbb{C}$ and decompose F as $F = F_- V_+ = F_+ V_-$ according to Theorem 7.1. Then F_- and F_+ are meromorphic and anti-meromorphic respectively. Moreover, the Maurer-Cartan forms $\xi_\pm$ of $F_\pm$ are given explicitly as follows:*

$$\begin{cases} \xi_-(z,\lambda) = F_-^{-1}(z,\lambda) dF_-(z,\lambda) = \lambda^{-1} \begin{pmatrix} 0 & -p \\ Bp^{-1} & 0 \end{pmatrix} dz, \\ \xi_+(z,\lambda) = F_+^{-1}(z,\lambda) dF_+(z,\lambda) = -\sigma_3 \overline{\xi_-^c(z, 1/\bar{\lambda})}^t \sigma_3 d\bar{z}, \end{cases} \tag{7.4}$$

where p is a meromorphic function on $\mathbb{D}$, $\xi_-^c(z,\lambda)$ denotes the coefficient matrix of $\xi_-(z,\lambda)$ and B is the holomorphic function on $\mathbb{D}$ defined in (3.9)*, which is the coefficient of the Abresch-Rosenberg differential.*

Proof. From the equality $F = F_- V_+$, the Maurer-Cartan form of F_- can be computed as

$$\xi_- = F_-^{-1} dF_- = V_+ F^{-1}(dF V_+^{-1} - F V_+^{-1} dV_+ V_+^{-1}) = \mathrm{Ad}(V_+)\alpha^\lambda - dV_+ V_+^{-1}.$$

Since the coefficient matrix of ξ_- is an element in the Lie algebra of $\Lambda_*^- \mathrm{SL}_2\mathbb{C}_\sigma$ and the lowest degree of entries of the right hand side with respect to λ is equal to -1, the 1-form ξ_- can be computed as

$$\xi_- = \lambda^{-1} \begin{pmatrix} 0 & -e^{w/2} v_+^2 \\ B e^{-w/2} v_+^{-2} & 0 \end{pmatrix} dz,$$

where $\mathrm{diag}(v_+, v_+^{-1})$ is the constant coefficient of the Fourier expansion of V_+ with respect to λ. Moreover from [24, Lemma 2.6], it is known that F_- is meromorphic on $\mathbb{D}$, and thus ξ_- is meromorphic on $\mathbb{D}$. Setting $p = e^{u/2} v_+^2$, we obtain the form ξ_- in (7.4). Similarly, by the equality $F = F_+ V_-$, the 1-form ξ_+ can be computed as

$$\xi_+ = F_+^{-1} dF_+ = V_- F^{-1}(dF V_-^{-1} - F V_-^{-1} dV_- V_-^{-1}) = \mathrm{Ad}(V_-)\alpha^\lambda - dV_- V_-^{-1}.$$

Since the coefficient matrix of ξ_+ is an element in the Lie algebra of $\Lambda^+ \mathrm{SL}_2\mathbb{C}_\sigma$ and the highest degree of entries of the right hand side with respect to λ is equal to 1, the 1-form ξ_+ can be computed as

$$\xi_+ = \lambda \begin{pmatrix} 0 & -\bar{B} e^{-w/2} v_-^2 \\ e^{w/2} v_-^{-2} & 0 \end{pmatrix} d\bar{z},$$

where $\mathrm{diag}(v_-, v_-^{-1})$ is the constant coefficient of the Fourier expansion of V_- with respect to λ. Similar to the case of ξ_-, from [24, Lemma 2.6] it is known that F_+ is anti-meromorphic on $\mathbb{D}$, and thus ξ_+ is anti-meromorphic on $\mathbb{D}$. Since F has the symmetry $F(\lambda) = \sigma_3 \overline{F(1/\bar{\lambda})}^{t-1} \sigma_3$,

$$F(\lambda) = \sigma_3 \overline{F_-(1/\bar{\lambda})}^{t-1} \sigma_3 \sigma_3 \overline{V_+(1/\bar{\lambda})}^{t-1} \sigma_3$$

is the second case in the Birkhoff decomposition of Theorem 7.1. Since the Birkhoff decomposition is unique, F_+ can be computed as

$$F_+(\lambda) = \sigma_3 \overline{F_-(1/\bar{\lambda})}^{t-1} \sigma_3.$$

Therefore, $\xi_+ = F_+^{-1} dF_+$ has the symmetry as stated in (7.4). □

DEFINITION 2. The pair of meromorphic and anti-meromorphic 1-forms $\xi_\pm$ defined in (7.4) is called the *pair of normalized potentials.*

8. Generalized Weierstrass type representation for minimal surfaces in Nil_3**.** In the previous section, a pair of normalized potentials was obtained from a minimal surface in Nil_3. In this section, we will conversely show the generalized Weierstrass type representation formula for minimal surfaces in Nil_3 from pairs of normalized potentials. It should be emphasized that every simply-connected non-vertical minimal surface can be obtained by the following procedure.

Step I. Let (ξ_-, ξ_+) be a pair of normalized potentials defined in (7.4). Solve the pair of ordinary differential equations:

$$dC_\pm = C_\pm \xi_\pm, \tag{8.1}$$

where $C_+(\bar{z}_*, \lambda) = \sigma_3 \overline{C_-(z_*, 1/\bar{\lambda})}^{t-1} \sigma_3$ and the initial condition $C_-(z_*, \lambda)$ is chosen such that $C_-^{-1}(z_*, \lambda) C_+(\bar{z}_*, \lambda)$ is Birkhoff decomposable in both ways of Theorem 7.1.

Step II. Applying the Birkhoff decomposition of Theorem 7.1 to the element $C_-^{-1} C_+$, we obtain for almost all z

$$C_-^{-1} C_+ = V_+ V_-^{-1}, \tag{8.2}$$

where $V_+ \in \Lambda_*^+ \mathrm{SL}_2\mathbb{C}_\sigma$ and $V_- \in \Lambda^- \mathrm{SL}_2\mathbb{C}_\sigma$.

REMARK 8.1.

1. If we change the initial condition of C_- from $C_-(z_*, \lambda)$ to $U(\lambda) C_-(z_*, \lambda)$ by an element $U(\lambda)$ in $(\Lambda \mathrm{SU}_{1,1})_\sigma$ which is independent of z, then the Birkhoff decomposition for $\tilde{C}_-^{-1} \tilde{C}_+$ with $\tilde{C}_-(z, \lambda) = U(\lambda) C_-(z, \lambda)$ and $\tilde{C}_+(z, \lambda) = \sigma_3 \overline{U(1/\bar{\lambda})}^{t-1} \sigma_3 C_+(z, \lambda)$ is the same as $C_-^{-1} C_+$, that is,
$$\tilde{C}_-^{-1} \tilde{C}_+ = C_-^{-1} C_+ = V_+ V_-^{-1},$$
since $U(\lambda)$ is in $(\Lambda \mathrm{SU}_{1,1})_\sigma$.
2. The Birkhoff decomposition (8.2) permits to define $\hat{F} = C_- V_+ = C_+ V_-$. The expressions $C_- = \hat{F} V_+^{-1}$ and $C_+ = \hat{F} V_-^{-1}$ look like Iwasawa decompositions. However, for this we need $\hat{F}$ to be $\mathrm{SU}_{1,1}$-valued. That one can replace $\hat{F}$ in some open subset $\mathbb{D}_0 \subset \mathbb{D}$ by some $F = \hat{F} k$, k diagonal, real and independent of λ such that $F \in (\Lambda \mathrm{SU}_{1,1})_\sigma$, will be shown below. Note however, that such a decomposition can be obtained in general only for $z \in \mathbb{D}_0$, since there are two open Iwasawa cells and if $C_-(z, \lambda)$ moves into the second open Iwasawa cell, then $C_-(z, \lambda) = \hat{F}(z, \lambda) \omega_0(\lambda) V_+^{-1}(z, \lambda)$ for some $\omega_0(\lambda)$, see [11].

THEOREM 8.2. *Let* $F = C_+ V_- = C_- V_+$ *be the loop defined by the Birkhoff decomposition in* (8.2). *Then* $V_-|_{\lambda=\infty}$ *is a* λ*-independent diagonal* $\mathrm{SL}_2\mathbb{C}$ *matrix* *with real entries. If its real diagonal entries are positive, then there exists a λ-independent diagonal $\mathrm{SL}_2\mathbb{C}$ matrix D such that $FD \in (\Lambda\mathrm{SU}_{1,1})_\sigma$ is the extended frame of some minimal surface in Nil_3 around the base point z_*. If the real diagonal entries of $V_-|_{\lambda=\infty}$ are negative, then there exists a λ-independent diagonal $\mathrm{SL}_2\mathbb{C}$ matrix D and $\omega_0 = \left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{smallmatrix}\right)$ such that $C_+ = FV_-^{-1} = (FD\omega_0)\omega_0(DV_-^{-1})$, where $FD\omega_0 \in (\Lambda\mathrm{SU}_{1,1})_\sigma, DV_-^{-1} \in \Lambda^-\mathrm{SL}_2\mathbb{C}_\sigma$ and $\omega_0 FD$ is the extended frame of some minimal surface in Nil_3 around the base point z_*.*

Proof. From **Step I**, the solution $C_+(\bar{z}, \lambda)$ in (8.1) satisfies the symmetry

$$C_+(\bar{z}, \lambda) = \sigma_3 \overline{C_-(z, 1/\bar{\lambda})}^{t-1} \sigma_3.$$

Therefore

$$\begin{aligned} V_+(z, \bar{z}, \lambda) V_-(z, \bar{z}, \lambda)^{-1} &= C_-(z, \lambda)^{-1} C_+(\bar{z}, \lambda) \\ &= \sigma_3 \overline{C_+(\bar{z}, 1/\bar{\lambda})^{-1} C_-(\bar{z}, 1/\bar{\lambda})}^{t-1} \sigma_3 \\ &= \sigma_3 \overline{V_-(z, \bar{z}, 1/\bar{\lambda})}^{t-1} \sigma_3 \sigma_3 \overline{V_+(z, \bar{z}, 1/\bar{\lambda})}^{t} \sigma_3. \end{aligned} \tag{8.3}$$

Thus by the uniqueness of the Birkhoff decomposition of Theorem 7.1, we have

$$V_+(z, \bar{z}, \lambda) = \sigma_3 \overline{V_-(z, \bar{z}, 1/\bar{\lambda})}^{t-1} \sigma_3 K(z, \bar{z}), \tag{8.4}$$

where K is some λ-independent $\mathrm{SL}_2\mathbb{C}$ diagonal matrix. Let $V_-|_{\lambda=\infty} = \mathrm{diag}(v_-, v_-^{-1})$ denote the λ-independent constant coefficient of the Fourier expansion of V_- with respect to λ. Then by (8.3) and (8.4), v_- takes values in $\mathbb{R}$ and $K = V_-|_{\lambda=\infty}$. Since at $z_* \in \mathbb{C}$, the loop $C_-^{-1}C_+$ is Birkhoff decomposable, $v_- > 0$ or $v_- < 0$ on some open subset $\mathbb{D} \subset \mathbb{C}$ containing z_*. Let us consider first the case of $v_- > 0$. The Maurer-Cartan form of F can be computed as

$$F^{-1}dF = \mathrm{Ad}(V_+^{-1})\xi_- + V_+^{-1}dV_+ = \mathrm{Ad}(V_-^{-1})\xi_+ + V_-^{-1}dV_-.$$

Since the lowest degree of entries of the middle term is λ^{-1} and the highest degree of entries of the right term is λ, we obtain

$$F^{-1}dF = \lambda^{-1} \begin{pmatrix} 0 & -p \\ Bp^{-1} & 0 \end{pmatrix} dz + \alpha_0 + \lambda \begin{pmatrix} 0 & \bar{B}\bar{p}^{-1}v_-^{-2} \\ -\bar{p}v_-^2 & 0 \end{pmatrix} d\bar{z},$$

where α_0 consists of the dz-part only and is computed from $V_-^{-1}dV_-$ as

$$\alpha_0 = \begin{pmatrix} v_-^{-1}(v_-)_z dz & 0 \\ 0 & -v_-^{-1}(v_-)_z dz \end{pmatrix}.$$

Let us consider the change of coordinates $w = \int_{z_*}^z p(t)dt$, that is, $dw = p(z)dz$ and $d\bar{w} = \overline{p(z)}d\bar{z}$. Then

$$F^{-1}dF = \lambda^{-1} \begin{pmatrix} 0 & -1 \\ Bp^{-2} & 0 \end{pmatrix} dw + \alpha_0 + \lambda \begin{pmatrix} 0 & \bar{B}\bar{p}^{-2}v_-^{-2} \\ -v_-^2 & 0 \end{pmatrix} d\bar{w}, \tag{8.5}$$

and α_0 is unchanged, since $v_-^{-1}(v_-)_z dz = v_-^{-1}(v_-)_w dw$. Finally choosing the gauge $D = \mathrm{diag}(v_-^{-1/2}, v_-^{1/2})$, we have

$$(FD)^{-1}d(FD) = \lambda^{-1} \begin{pmatrix} 0 & -v_- \\ \tilde{B}v_-^{-1} & 0 \end{pmatrix} dw + \tilde{\alpha}_0 + \lambda \begin{pmatrix} 0 & \bar{\tilde{B}}v_-^{-1} \\ -v_- & 0 \end{pmatrix} d\bar{w}, \tag{8.6}$$

with $\tilde{B} = Bp^{-2}$ and

$$\tilde{\alpha}_0 = \begin{pmatrix} \frac{1}{2}(\log v_-)_w dw - \frac{1}{2}(\log v_-)_{\bar{w}} d\bar{w} & 0 \\ 0 & -\frac{1}{2}(\log v_-)_w dw + \frac{1}{2}(\log v_-)_{\bar{w}} d\bar{w} \end{pmatrix}.$$

Thus the Maurer-Cartan form (8.6) has the form stated in (4.3). Moreover, using (8.4) and $D^{-2} = K$, we have

$$V_+(z,\bar{z},\lambda)D(z,\bar{z}) = \sigma_3 \overline{V_-(z,\bar{z},1/\bar{\lambda})D(z,\bar{z})}^{t-1}\sigma_3.$$

Therefore, $FD = C_- V_+ D = C_+ V_- D$ takes values in $(\Lambda \mathrm{SU}_{1,1})_\sigma$ and by Theorem 6.1 is the extended frame for some minimal surface in Nil_3. We now consider the case of $v_- < 0$. Then similar to the case of $v_- > 0$, the Maurer-Cartan equation of F is the same as in (8.5). Let $\omega_0 = \left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{smallmatrix}\right)$. Then choosing the gauge $D\omega_0 = \begin{pmatrix} 0 & \lambda |v_-|^{-1/2} \\ -\lambda^{-1}|v_-|^{1/2} & 0 \end{pmatrix}$ we have

$$V_+(z,\bar{z},\lambda)D(z,\bar{z})\omega_0(\lambda) = \sigma_3 \overline{V_-(z,\bar{z},1/\bar{\lambda})D(z,\bar{z})\omega_0(1/\bar{\lambda})}^{t-1}\sigma_3.$$

Therefore, $FD\omega_0 = C_- V_+ D\omega_0 = C_+ V_- D\omega_0$ takes values in $(\Lambda \mathrm{SU}_{1,1})_\sigma$. Moreover $\omega_0 FD = \omega_0^{-1}(FD\omega_0)\omega_0$ also takes values in $(\Lambda \mathrm{SU}_{1,1})_\sigma$ and its Maurer-Cartan form has the form stated in (4.3). Thus $\omega_0 FD$ is the extended frame of some minimal surface in Nil_3. □

Step III. In this final step, minimal surfaces in Nil_3 can be obtained by the Sym formula (see Theorem 6.1) for the extended frame F: Let $m = -i\lambda(\partial_\lambda F)F^{-1} - \frac{i}{2}F\sigma_3 F^{-1}$ and $\hat{f}^\lambda$ as

$$\hat{f}^\lambda = \left(m^o - \frac{i}{2}\lambda(\partial_\lambda m)^d \right)\Big|_{\lambda \in \mathbb{S}^1},$$

as in Theorem 6.1. Then via the identification in (6.2), for each λ, the map $f^\lambda = \Xi_{\mathrm{nil}} \circ \hat{f}^\lambda$ gives a minimal surface in Nil_3.

REMARK 8.3. The normalized potential ξ_- in (7.4) generating the harmonic map associated with a minimal surface can be explicitly computed from the geometric data, by the so-called *Wu's formula* as follows:

$$\xi_- = \lambda^{-1} \begin{pmatrix} 0 & -e^{\hat{w}(z) - \hat{w}(0)/2} \\ B(z)e^{-\hat{w}(z)+\hat{w}(0)/2} & 0 \end{pmatrix} dz, \tag{8.7}$$

where $4Bdz^2$ is the Abresch-Rosenberg differential and $e^{\hat{w}(z)}$ is the holomorphic extension of $e^{w(z,\bar{z})} = -h^2(z,\bar{z})/16$ around the base point $z = 0$ with the support function h. The proof of this formula is analogous to the original proof of Wu's formula for constant mean curvature surfaces in $\mathbb{E}^3$, see [47].

9. Examples. We exhibit some examples of minimal surfaces. In our general frame work, if one change the initial condition of C_- from $C_-(z_*,\lambda)$ to $U(\lambda)C_-(z_*,\lambda)$ for some $U(\lambda) \in (\Lambda \mathrm{SU}_{1,1})_\sigma$, then the corresponding harmonic maps into $\mathbb{H}^2$ are isometric. However, the associated minimal surfaces can differ substantially, since isometries of $\mathbb{H}^2$ do not correspond in general to isometries of Nil_3. Since $\mathrm{SU}_{1,1}$ is a three-dimensional Lie group, the initial conditions $U(\lambda) \in (\Lambda \mathrm{SU}_{1,1})_\sigma$ for each $\lambda \in \mathbb{S}^1$ could yield three-dimensional families of non-isometric minimal surfaces. However, choosing a $\mathrm{SU}_{1,1}$-diagonal matrix, which is an isometry of Nil_3 by rotation, the initial conditions in general give only two-dimensional families of non-isometric minimal surfaces.

9.1. Horizontal umbrellas. Let ξ_- be the normalized potential defined as

$$\xi_- = -\lambda^{-1}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} dz.$$

It is easy to compute the solution to $dC_- = C_-\xi_-$ with $C_-(z=0,\lambda) = \mathrm{diag}(\sqrt{i}^{-1}, \sqrt{i})$. It is given by

$$C_- = \begin{pmatrix} \sqrt{i}^{-1} & -i\sqrt{i}^{-1}\lambda^{-1}z \\ 0 & \sqrt{i} \end{pmatrix}.$$

Then the loop group decomposition $C_- = FV_+$ with $F \in (\Lambda \mathrm{SU}_{1,1})_\sigma$ and $V_+ \in \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma$ can be computed explicitly:

$$F = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} \sqrt{i}^{-1} & -i\sqrt{i}^{-1}\lambda^{-1}z \\ i\sqrt{i}\lambda\bar{z} & \sqrt{i} \end{pmatrix} \quad \text{and} \quad V_+ = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} 1 & 0 \\ -i\lambda\bar{z} & 1-|z|^2 \end{pmatrix}.$$

Hence

$$\hat{f}^\lambda = \frac{2}{1-|z|^2}\begin{pmatrix} 0 & -i\lambda^{-1}z \\ i\lambda\bar{z} & 0 \end{pmatrix}.$$

By the identification (6.2), we obtain

$$f^\lambda = -\frac{2}{1-|z|^2}(\lambda^{-1}z + \lambda\bar{z},\ i(\lambda\bar{z} - \lambda^{-1}z),\ 0).$$

In this case the associated family consists of different parametrizations of the same *horizontal plane*. It is easy to see that the Abresch-Rosenberg differential of a horizontal plane is zero.

Taking a different $(\Lambda \mathrm{SU}_{1,1})_\sigma$-initial condition for the above C_-, the resulting surfaces are non-vertical planes: Let $\mathcal{F}(x_1, x_2) = ax_1 + bx_2 + c$ be a linear function on the x_1x_2-plane. Then the graph of $\mathcal{F}$ is a minimal surface in Nil_3 with negative Gaussian curvature

$$K = -\frac{3 + 2(b - x_1/2)^2 + 2(a + x_2/2)^2}{4\{1 + 2(b - x_1/2)^2 + (a + x_2/2)^2\}^2} < 0.$$

Then the graph of $\mathcal{F}$ is called a *horizontal umbrella*.

9.2. Hyperbolic paraboloids. Let ξ_- be the normalized potential

$$\xi_- = -\frac{i}{4}\lambda^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz.$$

It is easy to compute the solution $dC_- = C_-\xi_-$ with $C_-(z=0,\lambda) = \mathrm{diag}(\sqrt{i}^{-1}, \sqrt{i})$, which is given by

$$C_- = \begin{pmatrix} \sqrt{i}^{-1}\cosh p & \sqrt{i}^{-1}\sinh p \\ \sqrt{i}\sinh p & \sqrt{i}\cosh p \end{pmatrix},$$

where $p = -i\lambda^{-1}z/4$. Then the loop group decomposition $C_- = FV_+$ with $F \in (\Lambda \mathrm{SU}_{1,1})_\sigma$ and $V_+ \in \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma$ can be computed explicitly:

$$F = \begin{pmatrix} \sqrt{i}^{-1}\cosh(p+p^*) & \sqrt{i}^{-1}\sinh(p+p^*) \\ \sqrt{i}\sinh(p+p^*) & \sqrt{i}\cosh(p+p^*) \end{pmatrix} \text{ and } V_+ = \begin{pmatrix} \cosh p^* & -\sinh p^* \\ -\sinh p^* & \cosh p^* \end{pmatrix},$$

where $p^* = i\lambda\bar{z}/4$. A direct computation shows that

$$\hat{f}^\lambda = \frac{1}{2}\begin{pmatrix} (p-p^*)\sinh(2(p+p^*)) & \sinh(2(p+p^*)) + 2(p-p^*) \\ \sinh(2(p+p^*)) - 2(p-p^*) & -(p-p^*)\sinh(2(p+p^*)) \end{pmatrix}.$$

By the identification (6.2), we obtain

$$f^\lambda = (-2i(p-p^*),\ -\sinh(2(p+p^*)),\ 2i(p-p^*)\sinh(2(p+p^*))).$$

This is an associated family of minimal surfaces in Nil_3 which actually parametrize the same hyperbolic paraboloid, that is, $x_3 = x_1x_2/2$. It is easy to see that the Abresch-Rosenberg differential of a hyperbolic paraboloid is $4Bdz^2 = \lambda^{-2}/4dz^2$. Note that a hyperbolic paraboloid $x_3 = x_1x_2/2$ can be written as

$$f(x_1, x_2) = \exp(x_1e_1) \cdot \exp(x_2e_2).$$

Taking a different $(\Lambda\mathrm{SU}_{1,1})_\sigma$-initial condition for the above C_-, the resulting surfaces are the *saddle-type examples* of [1]. They are the special case of the *translational-invariant examples*, see [34]. The *saddle-type minimal surfaces* were discovered by Bekkar [3], see also [32, Part II, Proposition 1.9, Remark 1.10]. The saddle-type one was also found by [29] as translation invariant minimal surfaces.

Remark 9.1. Let G be a compact semi-simple Lie group equipped with a bi-invariant Riemannian metric. Take linearly independent vectors X, Y in the Lie algebra. Then the map $f : \mathbb{R}^2 \to G$ defined by

$$f(x, y) = \exp(xX) \cdot \exp(yY)$$

is a harmonic map. Moreover one can see that f is of finite type (1-type in the sense of [15].)

9.3. Helicoids and catenoids. We first note that in place of normalized potentials $\xi_- = \lambda^{-1}\left(\begin{smallmatrix} 0 & p \\ q & 0 \end{smallmatrix}\right) dz$ with p, q meromorphic functions, one can also generate the same surface by *holomorphic potentials* η of the form

$$\eta = \sum_{i=-1}^{\infty} \lambda^i \eta_i,$$

where η_{2k-1} and η_{2k} are respectively off-diagonal and diagonal holomorphic $\mathfrak{sl}_2\mathbb{C}$-valued 1-forms, see [21].

Let η be a holomorphic potential of the form

$$\eta = Ddz, \text{ with } D = \begin{pmatrix} c & a\lambda^{-1} + b\lambda \\ -a\lambda - b\lambda^{-1} & -c \end{pmatrix}, \quad a = -b,\ c = \frac{1}{2}, \tag{9.1}$$

where $a \in \mathbb{R}^\times$. It is easy to compute that the solution C_- to $dC_- = C_-\eta$ with initial condition $C_-(z = 0, \lambda) = \mathrm{id}$ is

$$C_- = \exp(zD).$$

Let $C_- = FV_+$ be the loop group decomposition of C_- with $F \in (\Lambda \mathrm{SU}_{1,1})_\sigma$ and $V_+ \in \Lambda^+ \mathrm{SL}_2\mathbb{C}_\sigma$, where F takes values in the loop group of $\mathrm{SU}_{1,1}$, see [11, Section 5.1] for the explicit decomposition using elliptic functions. Let $z = x + iy$ be the complex coordinate and γ the $2\pi k$ translation in y-direction, that is, $\gamma^* z = z + 2\pi i k, k \in \mathbb{R}$. Then C_- changes as $\gamma^* C_- = \exp(2\pi i k D) C_-$. Since $M = \exp(2\pi i k D) \in (\Lambda \mathrm{SU}_{1,1})_\sigma$, the loop group decomposition for $\gamma^* C_-$ is computed as

$$\gamma^* C_- = (MF) \cdot (\gamma^* V_+), \quad MF \in (\Lambda \mathrm{SU}_{1,1})_\sigma.$$

Let $\hat{f}^\lambda$ be the immersion defined from F via the Sym formula (6.4). Then a straightforward computation shows that $\hat{f}^\lambda$ changes by γ as follows:

$$\gamma^* \hat{f}^\lambda = (\mathrm{Ad}(M)m - X)^o - \frac{1}{2} \left(\mathrm{Ad}(M)(i\lambda \partial_\lambda m) + [X, \mathrm{Ad}(M)m] - Y\right)^d, \tag{9.2}$$

where m is the map defined in (6.3),

$$X = i\lambda(\partial_\lambda M)M^{-1} \text{ and } Y = i\lambda\partial_\lambda X = i\lambda\partial_\lambda(i\lambda(\partial_\lambda M)M^{-1}).$$

A direct computation shows that $M|_{\lambda=1} = \mathrm{diag}(e^{\pi i k}, e^{-\pi i k})$,

$$X|_{\lambda=1} = \begin{pmatrix} 0 & 2ia(1-e^{2\pi i k}) \\ -2ia(1-e^{-2\pi i k}) & 0 \end{pmatrix}$$

and

$$Y|_{\lambda=1} = 4a^2 \begin{pmatrix} (e^{2\pi i k} - e^{-2\pi i k}) - 4\pi i k & 0 \\ 0 & -(e^{2\pi i k} - e^{-2\pi i k}) + 4\pi i k \end{pmatrix}.$$

By the identification (6.2), we see that this gives a helicoidal motion along the x_3-axis through the point $(-4a, 0, 0)$ with the angle $2\pi k$ and the pitch $8a^2$, see Appendix B for the isometry group of Nil_3. Thus the resulting surface $f^\lambda|_{\lambda=1}$ is a helicoid. It is easy to see that the Abresch-Rosenberg differential of a helicoid is $4B dz^2 = -4a^2\lambda^{-2} dz^2$, $a \in \mathbb{R}^\times$.

Choosing some appropriate $(\Lambda \mathrm{SU}_{1,1})_\sigma$-initial condition for C_- above will yield a surface of revolution, that is, a catenoid surface. Catenoids and helicoids are not isometric in Nil_3, even though their Gauss map are isometric in $\mathbb{H}^2$.

Remark 9.2.

1. If the parameter a in the potential (9.1) is chosen properly, then the resulting surface is the standard helicoid as in (B.1). It is a minimal helicoid in $\mathbb{E}^3$, see Appendix B.2.
2. The holomorphic potential defined in (9.1) produces, via the immersion m, a nonzero constant mean curvature surface m of revolution in Minkowski 3-space, [11, Section 5.1]. More precisely, the axis of this surface of revolution is timelike. It would be interesting to know what surfaces correspond to the above potential with the condition $(a+b)^2 - c^2$ negative with $a \neq -b$, positive or zero, that is, the axis is timelike which is not parallel to e_3, spacelike or lightlike in Minkowski 3-space, respectively.
3. To the best of our knowledge, the associated family of a helicoid gives a new family of minimal surfaces. All these surfaces have the same support function.

Appendix A. Surfaces with holomorphic Abresch-Rosenberg differential.

A.1. In this appendix, we determine all surfaces with holomorphic Abresch-Rosenberg differential.

THEOREM A.1. *Let f be a conformal immersion into* Nil_3 *and B its Abresch-Rosenberg differential defined in* (3.9)*. If B is holomorphic, then the surface f is one of the following*:

1. *A constant mean curvature surface.*
2. *A Hopf cylinder.*

Proof. The structure equations for a surface with holomorphic B can be phrased as

$$-BH_{\bar z}e^{-w/2+u/2} = H_z e^{u/2}, \tag{A.1}$$

$$\frac{1}{2}w_{z\bar z} + e^w - |B|^2 e^{-w} + \frac{1}{2}(H_{z\bar z} + p)e^{-w/2+u/2} = 0, \tag{A.2}$$

$$-\bar B H_z e^{-w/2+u/2} = H_{\bar z} e^{u/2}, \tag{A.3}$$

where p is $H_z(-w/2+u/2)_{\bar z}$ or $H_{\bar z}(-w/2+u/2)_z$ respectively. Since H is real, taking the complex conjugate of (A.1) and inserting it into (A.3), we obtain

$$\bar B H_z e^{-\bar w/2} = \bar B H_z e^{-w/2}. \tag{A.4}$$

This equation holds if $B = 0$ or $H = \mathrm{const}$ or $e^{-\bar w/2} = e^{-w/2}$. If $H = \mathrm{const}$, then we are in case (1). Let us assume now H not constant. If B is identically zero, then (A.1) and (A.3) show that H is constant. We assume now $B \neq 0$ and $H \neq \mathrm{const}$. Then the equation (A.4) implies $e^{w/2} = e^{\bar w/2}$. Using the identity $e^{w/2} = -He^{u/2}/2 + ih/4$, we obtain that the support function $h = 2(|\psi_1|^2 - |\psi_2|^2)$ is equal to zero, that is $|\psi_1| = |\psi_2|$. Thus the surface is tangent to E_3 by Proposition 3.3 and this condition is equivalent to that the surface is a Hopf cylinder. □

General Hopf cylinders are of constant mean curvature H if and only if the curvature of the base curve is constant and equal to $2H$. Therefore the only minimal Hopf cylinders are vertical planes. In the proof of the above theorem, we have seen the case where the holomorphic differential B vanishes identically. This describes in fact constant mean curvature surfaces. From [1], such surfaces are classified as follows.

PROPOSITION A.2 ([1]). *The surfaces with identically vanishing Abresch-Rosenberg differential are constant mean curvature surfaces and they are classified as follows:*

1. *For $H \neq 0$, they are spheres of revolution.*
2. *For $H = 0$, they are vertical planes or horizontal umbrellas.*

A.2. It is known that any two-dimensional Lie subgroup of Nil_3 is normal (see for example [38, Corollary 3.8]). Moreover all two-dimensional Lie subgroups belong to the 1-parameter family $\{G(t) \mid t \in \mathbb{R}P^1\}$ of normal subgroups defined by

$$G(t) = \{(x_1, tx_1, x_3) | x_1, x_3 \in \mathbb{R}\}. \tag{A.5}$$

Here the coordinate plane $x_1 = 0$, that is $\{(0, x_2, x_3) | x_2, x_3 \in \mathbb{R}\}$, is regarded as $G(\infty)$. Note that $G(0)$ is the coordinate plane $x_2 = 0$. For any $t \neq t'$, $G(t)$ and

$G(t')$ only intersect along the subgroup $\mathit{\Gamma} = \{(0,0,x_3) | x_3 \in \mathbb{R}\}$, that is, the x_3-axis. There are no more two-dimensional Lie subgroups [38, Theorem 3.6-(5)]. Each $G(t)$ is realized as a vertical plane in Nil_3. Every vertical plane is congruent to $G(t)$ for some $t \in \mathbb{R}P^1$.

Appendix B. Isometry group of three-dimensional Heisenberg group.

B.1. The identity component $\mathrm{Iso}_\circ(\mathrm{Nil}_3(\tau))$ of the isometry group of $\mathrm{Nil}_3(\tau)$ is the semi-direct product $\mathrm{Nil}_3(\tau) \rtimes \mathrm{U}_1$ if $\tau \neq 0$. Here U_1 is identified with $\mathbb{S}^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$.

The action of $\mathrm{Nil}_3(\tau) \rtimes \mathrm{U}_1$ is given by

$$((a,b,c), e^{i\theta}) \cdot (x_1, x_2, x_3) = (a,b,c) \cdot (\cos\theta x_1 - \sin\theta x_2, \sin\theta x_1 + \cos\theta x_2, x_3).$$

The Heisenberg group Nil_3 itself acts on Nil_3 by left translations and is represented by $(\mathrm{Nil}_3(\tau) \rtimes \mathrm{U}_1)/\mathrm{U}_1$ as a naturally reductive homogeneous space. One can see that this homogeneous space is not Riemannian symmetric.

B.2. The Lie algebra $\mathfrak{iso}(\mathrm{Nil}_3(\tau))$ is generated by four Killing vector fields E_1, E_2, E_3 and $E_4 = -x_2\partial_{x_1} + x_1\partial_{x_2}$. The commutation relations are:

$$[E_4, E_1] = E_2, \ [E_4, E_2] = -E_1, \ [E_1, E_2] = E_3.$$

The 1-parameter transformation group $\{\rho_\theta\}$ generated by E_4 consists of rotations $\rho_\theta = ((0,0,0), e^{i\theta})$ of angle θ along the x_3-axis. An isometry $\rho_t^{(\mu)} \in \mathrm{Nil}_3(\tau) \rtimes \mathrm{U}_1$ of the form $\rho_t^{(\mu)} = ((0,0,\mu t), e^{it})$ is called a *helicoidal motion with pitch* μ. In particular, a helicoidal motion with pitch 0 is nothing but a rotation ρ_t.

DEFINITION 3. A conformal immersion $f : M \to \mathrm{Nil}_3(\tau)$ is said to be a *helicoidal surface* if it is invariant under some helicoidal motion. In particular, f is said to be a *surface of revolution* if it is invariant under some rotation ρ_t.

The standard helicoid

$$f(x_1, x_2) = (x_1, x_2, \mu \tan^{-1}(x_2/x_1)) \tag{B.1}$$

is a helicoidal minimal surface in $\mathrm{Nil}_3(\tau)$. In fact this surface is invariant under helicoidal motions of pitch μ. Note that this helicoid is minimal in any $E(\kappa, \tau)$, [4].

Tomter [44] studied (non-minimal) constant mean curvature rotational surfaces in Nil_3. Caddeo, Piu and Ratto [16] studied rotational surfaces of constant mean curvature (including minimal surfaces) in Nil_3 by the equivariant submanifold geometry (in the sense of W. Y. Hsiang). Figueroa, Mercuri and Pedrosa [29] investigated surfaces of constant mean curvature invariant under some 1-parameter isometry groups.

B.3. In this subsection, we classify homogeneous surfaces in Nil_3.

DEFINITION 4. A surface $f : M \to \mathrm{Nil}_3$ is said to be *homogeneous* if there exists a connected Lie subgroup G of $\mathrm{Iso}_\circ(\mathrm{Nil}_3)$ which acts transitively on the surface.

Then we have the following classification of homogeneous surfaces in Nil_3, cf., [33].

PROPOSITION B.1. *Homogeneous surfaces in* Nil_3 *are congruent to one of the following surfaces*:

1. *An orbit of a normal subgroup* $G(t)$ *defined in* (A.5).

2. An orbit of the Lie subgroup $\{((0,0,s),e^{it}) \mid s,t \in \mathbb{R}\} \subset \mathrm{Nil}_3 \rtimes \mathrm{U}_1$.
In the former case, surfaces are vertical planes. Surfaces in the latter case are Hopf cylinders over circles. Thus the only homogeneous minimal surfaces in Nil_3 *are vertical planes.*

Proof. Let $f : M \to \mathrm{Nil}_3$ be a conformal homogeneous immersion with the transitive group G. We first show that without loss of generality the surface $f(M)$ contains the identity element id of Nil_3. Let $z_0 \in M$, then $\hat{f} = L_{f(z_0)^{-1}} \circ f$ is a conformal homogeneous immersion with group $\hat{G} = L_{f(z_0)^{-1}} G L_{f(z_0)}$. Since $\hat{f}(z_0) =$ id, the claim follows.

We next show that every conformal homogeneous immersion $f : M \to \mathrm{Nil}_3$ admits a two-dimensional transitive group. If the transitive group G has $\dim G = 4$, then $G = \mathrm{Iso}_\circ(\mathrm{Nil}_3)$. However then $f(M) = G.\,\mathrm{id} = \mathrm{Nil}_3$, which contradicts the fact $f(M)$ has dimension 2. If the transitive group G has $\dim G = 3$, then the isotropy subgroup G_{id} of G has dimension 1. Since $\mathrm{Iso}_\circ(\mathrm{Nil}_3) = \mathrm{Nil}_3 \rtimes \mathrm{U}_1$ with normal subgroup Nil_3, we can write every $g_o \in G_{\mathrm{id}}$ in the form $g_o = (n,s)$, where $n \in \mathrm{Nil}_3$ and $s \in \mathrm{U}_1$. Then $\mathrm{id} = g_o.\,\mathrm{id} = n$. Hence $g_o \in \mathrm{U}_1$. Therefore $G_{\mathrm{id}} = \mathrm{U}_1$. On the Lie algebra level, let $n_1 \oplus s_1$ and $n_2 \oplus s_2$, $0 \oplus 1$ be a basis for $\mathfrak{g} = \mathrm{Lie}\, G$. Then also $n_1 \oplus 0$, $n_2 \oplus 0$ and $0 \oplus 1$ are a basis of $\mathfrak{g}$. Moreover, $n_1 \oplus 0$ and $n_2 \oplus 0$ generate a two-dimensional subalgebra $\mathfrak{g}_o$. Let G_o denote the corresponding connected subgroup of G, then $G = G_o \cdot \mathrm{U}_1$ and $G_o.\,\mathrm{id} = G.\,\mathrm{id} = f(M)$. Hence the claim follows.

Next we consider the family $G(t)$, $(t \in \mathbb{R}P^1)$, of two-dimensional abelian normal subgroups of Nil_3 defined in Section A.2. It is known that the groups $G(t)$ are the only two-dimensional Lie subgroups of Nil_3.

We now classify the two-dimensional connected transitive Lie subgroups G_o of $\mathrm{Iso}_\circ(\mathrm{Nil}_3) = \mathrm{Nil}_3 \rtimes \mathrm{U}_1$. Let $a_1 = n_1 \oplus s_1$ and $a_2 = n_2 \oplus s_2$ be a basis of $\mathfrak{g}_o = \mathrm{Lie}\, G_o$. If $s_1 = s_2 = 0$, then $G_o \subset \mathrm{Nil}_3$ and the claim follows by what was said above. Assume $s_1 \neq 0$. Then after some subtraction and some scaling we can assume $a_1 = n_1 \oplus 1$ and $a_2 = n_2$. For the commutator of a_1 and a_2 we obtain

$$[a_1, a_2] = [n_1, n_2] + [1, n_2] \oplus 0. \tag{B.2}$$

In the following discussion, we choose the basis 1 of $\mathfrak{u}_1$ such that the left translation of 1 to be the Killing vector field E_4 defined in B.2.

Case 1, $[a_1, a_2] = 0$: We obtain $[n_1, n_2] + [1, n_2] = 0$. Since $[n_1, n_2]$ is a multiple of e_3, putting $n_2 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ and $[n_1, n_2] = \beta e_3$, we conclude $\alpha_1 = \alpha_2 = 0$ and $n_2 = \alpha_3 e_3$.

Case 2, $[a_1, a_2] \neq 0$: In this case we obtain $\alpha n_2 (= \alpha a_2) = [n_1, n_2] + [1, n_2]$ with $\alpha \neq 0$, since we do not have the $a_1 (= n_1 \oplus 1)$-component. Putting $n_2 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, $[n_1, n_2] = \beta e_3$, we obtain

$$\alpha(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = \beta e_3 + \alpha_1 e_2 - \alpha_2 e_1 + \alpha_3 e_3,$$

whence $\alpha\alpha_1 = -\alpha_2$, $\alpha\alpha_2 = \alpha_1$, $\beta + \alpha_3 = \alpha\alpha_3$. Thus $\alpha_1 = -\alpha^2\alpha_1$ and $\alpha_1 = 0$. Hence also $\alpha_2 = 0$ and $n_2 = e_3$ can be assumed. But then $[a_1, a_2] = [a_1, e_3] = 0$, which is a contradiction. Altogether we have obtained the following list of possible Lie algebras:

$$\mathfrak{g}(t) = \mathrm{Lie}\, G(t) \quad \text{and} \quad < n_1 \oplus 1, e_3 > .$$

This completes the proof. □

Remark B.2. Since homogeneous Riemannian spaces are complete, we find for any $g \in G$, there exists a unique $\gamma \in \mathrm{Aut}\,\widetilde{M}$ and $z \in \widetilde{M}$:

$$f(\gamma \cdot z) = g.f(z).$$

Here $\widetilde{M}$ denotes the universal cover of M.

Appendix C. Spin structure on Riemann surfaces. A spin structure on an oriented Riemannian n-manifold (M, g) is a certain principal fiber bundle over M with structure group Spin_n which is a 2-fold covering over the orthonormal frame bundle $\mathrm{SO}(M)$ of M. In the two-dimensional case, a spin structure can be defined in the following manner [37].

A *spin structure* on a Riemann surface M is a complex line bundle Σ over M together with a smooth surjective fiber-preserving map $\mu : \Sigma \to K_M$ to the holomorphic cotangent bundle K_M of M satisfying

$$\mu(\alpha s) = \alpha^2 \mu(s)$$

for any section s of Σ and any function α. One can see that $\Sigma \otimes \Sigma$ is isomorphic to K_M. The complex line bundle Σ is called the *spin bundle* and the section s of Σ is called a *spinor* of M. The squaring map μ is kept implicit by writing s^2 for $\mu(s)$ and st for $\{\mu(s+t) - \mu(s-t)\}/4$. Take a local complex coordinate z on M. Then there exist two sections of Σ whose images under μ are dz. Choose one of these sections, and refer to it consistently as $(dz)^{1/2}$. Under this notation, every spinor can be expressed locally in the form $\psi(dz)^{1/2}$.

Appendix D. Harmonic maps into reductive homogeneous spaces.

D.1. Let G/K be a reductive homogeneous space. We equip G/K with a G-invariant Riemannian metric which is derived from a left-invariant Riemannian metric on G.

Then the orthogonal complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{k}$ of K can be identified with the tangent space of G/K at the origin $o = K$. The Lie algebra $\mathfrak{g}$ is decomposed into the orthogonal direct sum:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of linear subspaces. We define a symmetric bilinear map $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ by

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{p}}\rangle + \langle Y, [Z,X]_{\mathfrak{p}}\rangle, \quad X, Y, Z \in \mathfrak{p},$$

where $[Z,Y]_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of $[Z,Y]$. A Riemannian reductive homogeneous space G/K is said to be *naturally reductive* if $U = 0$. In particular, when G is a compact semi-simple Lie group and the G-invariant Riemannian metric on G/K is derived from a bi-invariant Riemannian metric of G, then G/K is said to be a *normal Riemannian homogeneous space*. Normal Riemannian homogeneous spaces are naturally reductive. Note that in case $K = \{\mathrm{id}\}, G/K = G$, U is related to the symmetric bilinear map $\{\cdot,\cdot\}$ defined in (1.9) by $2U = \{\cdot,\cdot\}$.

D.2. A smooth map $f : M \to N$ of a Riemannian 2-manifold M into a Riemannian manifold N is said to be a *harmonic map* if it is a critical point of the energy

$$E(f) = \int \frac{1}{2}|df|^2\, dA$$

with respect to all compactly supported variations. It is well known that a map f is harmonic if and only if its tension field $\mathrm{tr}(\nabla df)$ vanishes. The harmonicity is invariant under conformal transformations of M. When the target space N is a Riemannian reductive homogeneous space G/K, the harmonic map equation for f has a particularly simple form. The harmonic map equation for maps into Lie groups was already discussed in Section 1. Therefore we assume now that $\dim K \geq 1$.

Let $f : \mathbb{D} \to G/K$ be a smooth map from a simply connected domain $\mathbb{D} \subset \mathbb{C}$ into a Riemannian reductive homogeneous space. Take a frame $F : \mathbb{D} \to G$ of f and put $\alpha := F^{-1}dF$. Then we have the identity (*Maurer-Cartan equation*):

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Decompose α along the Lie algebra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ in the form

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}, \quad \alpha_{\mathfrak{k}} \in \mathfrak{k}, \quad \alpha_{\mathfrak{p}} \in \mathfrak{p}.$$

We decompose $\alpha_{\mathfrak{p}}$ with respect to the conformal structure of $\mathbb{D}$ as

$$\alpha_{\mathfrak{p}} = \alpha'_{\mathfrak{p}} + \alpha''_{\mathfrak{p}},$$

where $\alpha'_{\mathfrak{p}}$ and $\alpha''_{\mathfrak{p}}$ are the $(1,0)$ and $(0,1)$ part of $\alpha_{\mathfrak{p}}$, respectively.

The harmonicity of f is equivalent to

$$d(*\alpha_{\mathfrak{p}}) + [\alpha_{\mathfrak{k}} \wedge *\alpha_{\mathfrak{p}}] = U(\alpha_{\mathfrak{p}} \wedge *\alpha_{\mathfrak{p}}), \tag{D.1}$$

where $*$ denotes the Hodge star operator of $\mathbb{D}$. The Maurer-Cartan equation is split into its $\mathfrak{k}$-component and $\mathfrak{p}$-component:

$$d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{k}} = 0, \tag{D.2}$$

$$d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] + d\alpha''_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{p}}] + [\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = 0. \tag{D.3}$$

Hence for a harmonic map $f : \mathbb{D} \to G/K$ with a framing F, the pull-back 1-form $\alpha = F^{-1}dF$ satisfies (D.1), (D.2) and (D.3). Combining (D.1) and (D.3), we obtain

$$d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] = -\frac{1}{2}[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} + U(\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}). \tag{D.4}$$

One can easily check that the harmonic map equation for f combined with the Maurer-Cartan equation is equivalent to the system (D.2) and (D.4).

Assume that

$$[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = 0, \quad U(\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}) = 0. \tag{D.5}$$

Then the harmonic map equation together with the Maurer-Cartan equation is reduced to the system of equations:

$$d\alpha'_{\mathfrak{p}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{p}}] = 0, \quad d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}] = 0.$$

This system of equations is equivalent to the following *zero-curvature representation*:

$$d\alpha^{\lambda} + \frac{1}{2}[\alpha^{\lambda} \wedge \alpha^{\lambda}] = 0,$$

where $\alpha^\lambda := \alpha_{\mathfrak{h}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\,\alpha''_{\mathfrak{p}}$ with $\lambda \in \mathbb{S}^1$.

PROPOSITION D.1. *Let $\mathbb{D}$ be a domain in $\mathbb{C}$ and $f : \mathbb{D} \to G/K$ a harmonic map which satisfies the admissibility condition* (D.5). *Then the loop of connections $d + \alpha^\lambda$ is flat for all λ. Namely*:

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0 \tag{D.6}$$

for all λ. Conversely assume that $\mathbb{D}$ is simply connected. Let $\alpha^\lambda = \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_{\mathfrak{p}}$ be an $\mathbb{S}^1$-family of $\mathfrak{g}$-valued 1*-forms satisfying* (D.6) *for all $\lambda \in \mathbb{S}^1$. Then there exists a* 1*-parameter family of maps $F^\lambda : \mathbb{D} \to G$ such that*

$$(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda \quad \textit{and} \quad f^\lambda = F^\lambda \bmod K : \mathbb{D} \to G/K$$

is harmonic for all λ.

The 1-parameter family $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ of harmonic maps is called the *associated family* of the original harmonic map $f = f^\lambda|_{\lambda=1}$ which satisfies the admissibility condition. The map F^λ is called an *extended frame* of f. When the target space G/K is a Riemannian symmetric space with a semi-simple G, then the admissibility condition is fulfilled automatically for any f, since $U = 0$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. In the case $G/K = \mathrm{Nil}_3 \rtimes \mathrm{U}_1/\mathrm{U}_1$, harmonic maps into Nil_3 do in general not satisfy the admissibility condition. Note that harmonic maps into a naturally reductive Riemannian homogeneous space G/K satisfying the admissibility condition are called *strongly harmonic maps* in [35]. Note that all the examples of minimal surfaces in $\mathrm{Nil}_3 \rtimes \mathrm{U}_1/\mathrm{U}_1$ discussed in this paper do not satisfy the admissibility condition.

REFERENCES

[1] U. ABRESCH AND H. ROSENBERG, *Generalized Hopf differentials*, Mat. Contemp., 28 (2005), pp. 1–28.

[2] V. BALAN AND J. DORFMEISTER, *A Weierstrass-type representation for harmonic maps from Riemann surfaces to general Lie groups*, Balkan J. Geom. Appl., 5:1 (2000), pp. 7–37.

[3] M. BEKKAR, *Exemples de surfaces minimales dans l'espace de Heisenberg*, Rend. Sem. Fac. Sci. Univ. Cagliari, 61:2, pp. 123–130.

[4] M. BEKKAR, F. BOUZIANI, Y. BOUKHATEMA, AND J. INOGUCHI, *Helicoids and axially symmetric minimal surfaces in 3-dimensional homogeneous spaces*, Differ. Geom. Dyn. Syst. (Electronic), 9 (2007), pp. 21–39.

[5] M. BELKHELFA, F. DILLEN, AND J. INOGUCHI, *Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces*, PDE's, submanifolds and affine differential geometry, Banach center publ. vol. 57, Polish Acad. Sci., pp. 67–87, 2002.

[6] D. A. BERDINSKIĬ, E-mail communication.

[7] D. A. BERDINSKIĬ, *Surfaces of constant mean curvature in the Heisenberg group (Russian)*, Mat. Tr., 13:2 (2010), pp. 3–9. English translation: Siberian Adv. Math., 22:2 (2012), pp. 75–79.

[8] D. A. BERDINSKIĬ AND I. A. TAĬMANOV, *Surfaces in three-dimensional Lie groups (Russian)*, Sibirsk. Mat. Zh., 46:6 (2005), pp. 1248–1264. English translation: Siberian Math. J., 46:6 (2005), pp. 1005–1019.

[9] D. A. BERDINSKIĬ AND I. A. TAĬMANOV, *Surfaces of revolution in the Heisenberg group and a spectral generalization of the Willmore functional*, Siberian Math. J., 48:3 (2007), pp. 395–407.

[10] L. BIANCHI, *Sugli sazi a tre dimensioni che ammettono un gruppo continuo di movimenti*, Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Tereza, Tomo XI (1898), pp. 267–352. English Translation: *On the three-dimensional spaces which admit a continuous group of motions*, General Relativity and Gravitation, 33:12 (2001), pp. 2171–2252.

[11] D. Brander, W. Rossman, and N. Schmitt, *Holomorphic representation of constant mean curvature surfaces in Minkowski space: consequences of non-compactness in loop group methods*, Adv. in Math., 223:3 (2010), pp. 949–986.

[12] C. P. Boyer, *The Sasakian geometry of the Heisenberg group*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 52(100):3 (2009), pp. 251–262.

[13] C. P. Boyer and K. Galicki, *Sasakian geometry,* Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[14] F. E. Burstall, D. Ferus, F. Pedit, and U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. Math.(2), 138 (1993), pp. 173–212.

[15] F. E. Burstall and F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J., 80 (1995), pp. 353–382.

[16] R. Caddeo, P. Piu, and A. Ratto, SO(2)-*invariant minimal and constant mean curvature surfaces in* 3-*dimensional homogeneous spaces*, Manuscripta Math., 87:1 (1995), pp. 1–12.

[17] E. Cartan, *Leçon sur la geometrie des espaces de Riemann,* Second Edition, Gauthier-Villards, Paris, 1946.

[18] S. Cartier, *Surfaces des espaces homogènes de dimension 3*, Thèse de Doctorat, Université Paris-Est Marne-la-Vallée, 2011. `http://perso-math.univ-mlv.fr/users/cartier.sebastien/`.

[19] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Comment. Math. Helv., 82:1 (2007), pp. 87–131.

[20] B. Daniel, *The Gauss map of minimal surfaces in the Heisenberg group*, Int. Math. Res. Not. IMRN, 3 (2011), pp. 674–695.

[21] J. Dorfmeister, *Generalized Weierstraß representation of surfaces*, Surveys on geometry and integrable systems, Adv. Stud. Pure Math. 51, Mathematical Society of Japan, Tokyo: pp. 55–111, 2008.

[22] J. F. Dorfmeister, J. Inoguchi, and S.-P. Kobayashi, *Constant mean curvature surfaces in hyperbolic 3-space via loop group*, J. Reine Angew. Math., 686 (2014), pp. 1–36.

[23] J. Dorfmeister, S.-P. Kobayashi, and F. Pedit, *Complex surfaces of constant mean curvature fibered by minimal surfaces*, Hokkaido Math. J., 39:1 (2010), pp. 1–55.

[24] J. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom., 6:4 (1998), pp. 633–668.

[25] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, Regional Conference Series in Math., 50, Amer. Math. Soc. 1983.

[26] I. Fernández and P. Mira, *A characterization of constant mean curvature surfaces in homogeneous 3-manifolds*, Differential Geom. Appl., 25:3 (2007), pp. 281–289.

[27] I. Fernández and P. Mira, *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space*, Trans. Amer. Math. Soc., 361:11 (2009), pp. 5737–5752.

[28] C. B. Figueroa, *On the Gauss map of a minimal surface in the Heisenberg group*, Mat. Contemp., 33 (2007), pp. 139–156.

[29] C. B. Figueroa, F. Mercuri, and R. H. L. Pedrosa, *Invariant surfaces of the Heisenberg group*, Ann. Mat. Pura Appl. (4), 177 (1999), pp. 173–194.

[30] N. J. Hitchin, *Harmonic maps from a* 2-*torus to the* 3-*sphere*, J. Differential Geom., 31:3 (1990), pp. 627–710.

[31] J. Inoguchi, *Minimal surfaces in the 3-dimensional Heisenberg group*, Differ. Geom. Dyn. Syst. (Electronic), 10 (2008), pp. 163–169.

[32] J. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama, *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces I, II*, Fukuoka Univ. Sci. Rep., 29:2 (1999), pp. 155–182; 30:1 (2000), pp. 17–47.

[33] J. Inoguchi, K. Kuwabara, and H. Naitoh, *Grassmann geometry on the* 3-*dimensional Heisenberg group*, Hokkaido Math. J., 34 (2005), pp. 375–391.

[34] J. Inoguchi, R. López, and M. I. Munteanu, *Minimal translation surfaces in the Heisenberg group* Nil_3, Geom. Dedicata, 161 (2012), pp. 221–231.

[35] I. Khemar, *Elliptic integrable systems: a comprehensive geometric interpretation*, Mem. Amer. Math. Soc., 1031 (2012).

[36] S.-P. Kobayashi, *Real forms of complex surfaces of constant mean curvature*, Trans. Amer. Math. Soc., 363:4 (2011), pp. 1765–1788.

[37] R. Kusner and N. Schmitt, *The spinor representation of surfaces in space*, `arXiv.dg-ga/9610005v1`, 1996.

[38] W. H. Meeks III and J. Pérez, *Constant mean curvature surfaces in metric Lie groups*, Contemp. Math., 570 (2012), pp. 25–110.

[39] A. Pressley and G. Segal, *Loop groups,* Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1986.

[40] A. Sanini, *Gauss map of a surface of the Heisenberg group*, Boll. Un. Mat. Ital. B (7), 11 (2, suppl.) (1997), pp. 79–93.

[41] G. B. Segal, *Loop groups and harmonic maps*, Advances in Homotopy Theory, (S. M. Salamon, B. F. Steer, W. A. Sutherland eds.), London Math. Soc. Lecture Note Series, 139, 1991.

[42] I. A. Taĭmanov, *Surfaces in three-dimensional Lie groups in terms of spinors*, RIMS Kokyuroku, 1605 (2008), pp. 133–150.

[43] W. M. Thurston (S. Levy ed.), *Three-dimensional geometry and topology, Vol. 1.* Princeton Math. Series, Vol. 35, Prenceton Univ. Press, 1997.

[44] P. Tomter, *Constant mean curvature surfaces in the Heisenberg group*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54 (1993), pp. 485–495.

[45] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Differential Geom., 30:1 (1989), pp. 1–50.

[46] G. Vranceanu, *Leçons de Géométrie Différentielle,* I, Ed. Acad. Rep. Pop. Roum., Bucarest, 1947.

[47] H. Wu, *A simple way for determining the normalized potentials for harmonic maps*, Ann. Global Anal. Geom., 17:2 (1999), pp. 189–199.