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# VARIATIONAL PRINCIPLES FOR MINKOWSKI TYPE PROBLEMS, DISCRETE OPTIMAL TRANSPORT, AND DISCRETE MONGE-AMPERE EQUATIONS*

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**Abstract.** In this paper, we develop several related finite dimensional variational principles for discrete optimal transport (DOT), Minkowski type problems for convex polytopes and discrete Monge-Ampere equation (DMAE). A link between the discrete optimal transport, the discrete Monge-Ampere equation and the power diagram in computational geometry is established.

**Key words.** Monge-Ampere equation, Minkowski problem, Alexandrov problem, variational, power diagram.

**AMS subject classifications.** Primary 52B55; Secondary 52B11, 65M99.

## 1. Introduction.

**1.1. Statement of results.** The classical Minkowski problem for convex body has influenced the development of convex geometry and differential geometry throughout the twentieth century. In its simplest form, it states,

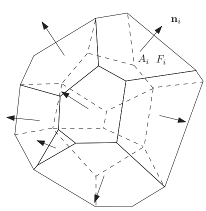


FIG. 1. *Minkowski problem*

PROBLEM 1 (Minkowski problem for compact polytopes in $\mathbf{R^n}$). *Suppose $n_1, ..., n_k$ are unit vectors which span $\mathbf{R^n}$ and $A_1, ..., A_k > 0$ so that $\sum_{i=1}^k A_i n_i = 0$. Find a compact convex polytope $P \subset \mathbf{R}^n$ with exactly k codimension-1 faces $F_1, ..., F_k$ so that $n_i$ is the outward normal vector to $F_i$ and the area of $F_i$ is $A_i$.*

Minkowski's famous solution to the problem says that the polytope $P$ exists and is unique up to parallel translation. Furthermore, Minkowski's proof is variational and suggests an algorithm to find the polytope. The Minkowski problem was generalized to the smooth case by Alexandrov where the area function is given by the Gauss curvature. Alexandrov [2] himself solved this generalized problem. The generalized Minkowski problem reduces to solving some Monge-Ampere equation. The regularity of the solution was studied by Pogorelov [12], Nirenberg [9] and Cheng and Yau [6]. In [6], Cheng and Yau gave a complete proof for the higher-dimensional Minkowski problem in Euclidean spaces.

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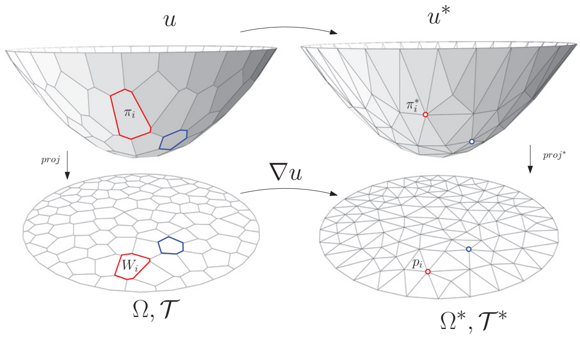


FIG. 2. *Discrete Optimal Transport Mapping (left to right): map $W_i$ to $p_i$. Discrete Monge-Ampere equation (right to left): $vol(W_i)$ is the discrete Hessian determinant of $p_i$.*

Minkowski problem for unbounded convex polytopes was considered and solved by A.D. Alexandrov and his student A. Pogorelov. Unlike the bounded case, one needs to deal with the infinite area unbounded faces. Their solutions are to prescribe the hyperplanes which contain the unbounded faces and phrase the Minkowski problem for bounded faces. Alexandrov solved the case where all unbounded faces are parallel to a given line and Pogorelov solved the remaining case in which the unbounded polytope contains a cone. In his influential book on convex polyhedra [1], Alexandrov proved the following fundamental theorem (Theorem 7.3.2 and theorem 6.4.2) which is one of the main foci of our investigation.

THEOREM 1.1. *(Alexandrov) Suppose $\Omega$ is a compact convex polytope with non-empty interior in $\mathbf{R^n}$, $p_1, ..., p_k \subset \mathbf{R^n}$ are distinct $k$ points and $A_1, ..., A_k > 0$ so that $\sum_{i=1}^k A_i = vol(\Omega)$. Then there exists a vector $h = (h_1, ..., h_k) \in \mathbf{R^k}$, unique up to adding the constant $(c, c, ..., c)$, so that the piecewise linear convex function*

$$u(x) = \max_{1 \le i \le k} \{x \cdot p_i + h_i\}$$

*satisfies $vol(\{x \in \Omega | \nabla u(x) = p_i\}) = A_i$.*

Note that Alexandrov stated his theorem in terms of the geometric language. The above theorem is stated in terms of convex functions. We call the functions $u$ and $\triangledown u(x)$ in the theorem the *Alexandrov potential* and the *Alexandrov map.* Alexandrov's proof is non-variational and non-constructive. Producing a variational proof of it was clearly in his mind. Indeed, on page 321 of [1], he asked if one can find a variational proof. One of the main results of the paper (theorem 1.2) gives a (finite dimensional) variational proof Alexandrov's Theorem 1.1. Indeed, we give a variational proof of a general version of Theorem 1.1 (Theorem 1.2 below) and produce an algorithm for finding the function $u$. Furthermore, our variational principle gives a new proof of the infinitesimal rigidity result of Alexandrov (corollary 3.2).

In recent surge of study on optimal transport, Theorem 1.1 is reproved and is a very special case of the work of Brenier [5] (see also for instance [15], Theorem 2.12(ii), and Theorem 2.32). Brenier proved, among other things, that the function $\triangledown u$ minimizes the quadratic cost $\int_\Omega |x - T(x)|^2 dx$ among all measure preserving maps (transport maps) $T : (\Omega, dx) \to (\mathbf{R^n}, \sum_{i=1}^k A_i \delta_{p_i})$. Here $\delta_p$ is the Dirac measure supported at the point $p$. In this special case of finite image measures $\sum_{i=1}^k A_i \delta_{p_i}$, an excellent proof that $\triangledown u$ minimizes the quadratic cost can also be found in ([4] Lemma

1). We reproduce their proof in §3.6 for completeness. Thus our work produces a finite dimensional variational principle and an algorithm for finding optimal transport maps with finite images.

**1.2. Variational principles.** Here is a simple framework which we will use to establish variational principles for solving equations in this paper. Suppose $X \subset \mathbf{R^k}$ is a simply connected open set and $A(x) = (A_1(x), ..., A_k(x)) : X \to \mathbf{R^k}$ is a smooth function so that $\frac{\partial A_i(x)}{\partial x_j} = \frac{\partial A_j(x)}{\partial x_i}$ for all $i, j$. Then for any given $B = (B_1, ..., B_k) \in \mathbf{R^k}$, solutions $x$ of the equation $A(x) = B$ are exactly the critical points of the function $E(x) = \int_a^x \sum_{i=1}^k (A_i(x) - B_i)dx_i$. Indeed, the assumption $\frac{\partial A_i(x)}{\partial x_j} = \frac{\partial A_j(x)}{\partial x_i}$ says the differential 1-form $\omega = \sum_{i=1}^k (A_i(x) - B_i)dx_i$ is closed in the simply connected domain $X$. Therefore the integral $E(x) = \int_a^x \omega$ is well defined independently of the choice of the path from $a$ to $x$. By definition, $\frac{\partial E(x)}{\partial x_i} = A_i(x) - B_i$, i.e., $\bigtriangledown E(x) = A(x) - B$. Thus $A(x) = B$ is the same as $\bigtriangledown E(x) = 0$.

We will use the above framework to give a variational proof of Alexandrov's theorem. The paper will mainly deal with piecewise linear (PL) convex functions. Here are the notations. Given $p_1, ..., p_k \in \mathbf{R^n}$ and $h = (h_1, ..., h_k) \in \mathbf{R^k}$, we use $u(x) = u_h(x)$ to denote the PL convex function

$$u_h(x) = \max_i \{x \cdot p_i + h_i\}.$$

Let $W_i(h) = \{x \in \mathbf{R^n} | \bigtriangledown u(x) = p_i\} = \{x | x \cdot p_i + h_i \geq x \cdot p_j + h_j \text{ for all } j\}$ be the closed convex polytope. Note that $W_i(h)$ may be empty or unbounded. One of the main result we will prove is,

THEOREM 1.2. *Let $\Omega$ be a compact convex domain in $\mathbf{R^n}$, $\{p_1, ..., p_k\}$ be a set of distinct points in $\mathbf{R^n}$ and $\sigma : \Omega \to \mathbf{R}$ be a positive continuous function. Then for any $A_1, ..., A_k > 0$ with $\sum_{i=1}^k A_i = \int_\Omega \sigma(x)dx$, there exists $b = (b_1, ..., b_k) \in \mathbf{R^k}$, unique up to adding a constant $(c, ..., c)$, so that $\int_{W_i(b)\cap\Omega} \sigma(x)dx = A_i$ for all $i$. The vectors $b$ are exactly minimum points of the convex function*

$$E(h) = \int_0^h \sum_{i=1}^k \int_{W_i(h)\cap\Omega} \sigma(x)dxdh_i - \sum_{i=1}^k h_i A_i \tag{1}$$

*on the open convex set $H = \{h \in \mathbf{R^k} \, | vol(W_i(h) \cap \Omega) > 0 \text{ for all } i\}$. Furthermore, $\nabla u_b$ minimizes the quadratic cost $\int_\Omega |x - T(x)|^2 \sigma(x)dx$ among all transport maps $T : (\Omega, \sigma dx) \to (\mathbf{R^n}, \sum_{i=1}^k A_i \delta_{p_i})$.*

We remark that Alexandrov's theorem corresponds to $\sigma \equiv 1$. The existence and the uniqueness of Theorem 1.2 are special case of the important work of Brenier [5] on optimal transport. Our main contribution is the variational formulation. The Hessian of the function $E(h)$ has a clear geometric meaning and is easy to compute (see equation (6)) based on the so-called power diagram from computational geometry [4], which enables one to efficiently compute the Alexandrov map $\nabla u_b$ using the Newton's method. See [14, 16, 7] for the applications of this algorithm in computer vision, computer graphics and visualization. Furthermore, as a consequence of our proof, we obtain a new proof of the infinitesimal rigidity theorem of Alexandrov that $\nabla E : H_0 \to \mathcal{A} = \{(A_1, ..., A_k) \in \mathbf{R^k} | A_i > 0, \sum_{i=1}^k A_i = \int_\Omega \sigma(x)dx\}$ is a local diffeomorphism (see Corollary 3.2). We remark that Aurenhammer et al. [4] also noticed the convexity of the function $E$, and they gave an elegant and simple proof of $\bigtriangledown u_b$ minimizing quadratic cost.

**1.3. Discrete Monge-Ampere equation (DMAE).** Closely related to the optimal transport problem is the Monge-Ampere equation (MAE). Let $\Omega$ be a compact domain in $\mathbf{R^n}$, $g : \partial\Omega \to \mathbf{R}$ and $A : \Omega \times \mathbf{R} \times \mathbf{R^n} \to \mathbf{R}$ be given. Then the Dirichlet problem for MAE is to find a function $w : \Omega \to \mathbf{R}$ so that

$$\begin{cases} det(Hess(w))(x) = A(x, w(x), \triangledown w(x)) \\ w|_{\partial\Omega} = g \end{cases} \tag{2}$$

where $Hess(w)$ is the Hessian matrix of $w$. There are vast literature and deep results known on the existence, uniqueness and regularity of the solution of MAE. We are interested in solving a discrete version of MAE in the simplest setting where $A(x, w, \triangledown w) = A(x) : \Omega \to \mathbf{R}$ so that $A(\Omega)$ is a finite set. By taking Fenchel-Lengendre dual of the Alexandrov potential function $u$, we produce a finite dimensional variational principle for solving a discrete Monge-Ampere equation.

In the discrete setting, one of the main tasks is to define the discrete Hessian determinant for piecewise linear function. We define,

Definition 1.3. *Suppose $(X, \mathcal{T})$ is a domain in $\mathbf{R^n}$ with a convex cell decomposition $\mathcal{T}$ and $w : X \to \mathbf{R}$ is a convex function which is linear on each cell (a PL convex function). Then the discrete Hessian determinant of $w$ assigns each vertex $v$ of $\mathcal{T}$ the volume of the convex hull of the gradients of $w$ at top-dimensional cells adjacent to $v$.*

One can define the discrete Hessian determinant of any piecewise linear function by using the signed volumes. This will not be discussed here. With the above definition of discrete Hessian determinant, following Pogorelov [11], one formulates the Dirichlet problem for discrete MAE (DMAE) as follows.

Problem 2 (Dirichlet problem for discrete MAE (DMAE)). *Suppose $\Omega = conv(v_1, ..., v_m)$ is the convex hull of $v_1, ..., v_m$ in $\mathbf{R^n}$. Let $p_1, ..., p_k$ be in $int(\Omega)$. Given any $g_1, ..., g_m \in \mathbf{R}$ and $A_1, ..., A_k > 0$, find a convex subdivision $\mathcal{T}$ of $\Omega$ with vertices exactly $\{v_1, ..., v_m, p_1, ..., p_k\}$ and a PL convex function $w : \Omega \to \mathbf{R}$ which is linear on each cell of $\mathcal{T}$ so that*

*(a) (Discrete Monge-Ampere Equation) the discrete Hessian determinant of $w$ at $p_i$ is $A_i$,*

*(b) (Dirichlet condition) $w(v_i) = g_i$.*

In [11], Pogorelov solved the above problem affirmatively. He showed that the PL function $w$ exists and is unique. However, his proof is non-variational. We improve Pogorelov's theorem to the following.

Theorem 1.4. *Suppose $\Omega = conv(v_1, ..., v_m)$ is an n-dimensional compact convex polytope in $\mathbf{R^n}$ so that $v_i \notin conv(v_1, ..., v_{i-1}, v_{i+1}, ..., v_k)$ for all $i$ and $p_1, ..., p_k$ are in the interior of $\Omega$. For any $g_1, ..., g_k \in \mathbf{R}$ and $A_1, ..., A_k > 0$, there exists a convex cell decomposition $\mathcal{T}$ having $v_i$ and $p_j$ as vertices and a piecewise linear convex function $w : (\Omega, \mathcal{T}) \to \mathbf{R}$ so that $w(v_i) = g_i, i = 1, ..., m$ and the discrete Hessian determinant of $w$ at $p_j$ is $A_j$, $j = 1, ..., k$. In fact, the solution $w$ is the Legendre dual of* $\max\{x \cdot p_j + h_j, x \cdot v_i - g_i | j = 1, ..., k, i = 1, ..., m\}$ *and $h$ is the unique minimal point of a strictly convex function.*

The paper is organized as follows. In §2, we recall briefly some basic properties of piecewise linear convex functions, their dual and power diagrams. Theorems 1.2 and 1.4 are proved in §3 and §4.

**2. Preliminary on PL convex functions, their duals and power diagrams.** We collect some well known facts about PL convex functions, their Legendre-Fenchel duals and their relations to power diagrams in this section. Most of the proofs are omitted. See Aurenhammer [3], Passera and Rullgård [10], Siersmas and van Manen [13] for details.

The following notations will be used. For $u, v, p_1, ..., p_k \in \mathbf{R^n}$, we use $u \cdot v$ to denote the dot product of $u, v$ and $conv(p_1, ..., p_k)$ to denote the convex hull of $\{p_1, ..., p_k\} \subset \mathbf{R^n}$. A convex polyhedron is the intersection of finitely many closed half spaces. A convex polytope is the convex hull of a finite set. The relative interior of a compact convex set $X$ will be denoted by $int(X)$.

**2.1. Legendre-Fenchel dual and PL convex functions.** The *domain* of a function $f : \mathbf{R^n} \to (-\infty, \infty]$, denoted by $D(f)$, is the set $\{x \in \mathbf{R^n} | f(x) < \infty\}$. A function $f$ is called *proper* if $D(f) \neq \emptyset$. For a proper function $f : \mathbf{R^n} \to (-\infty, \infty]$, the *Legendre-Fenchel duality* (or simply the dual) of $f$ is the proper function $f^* : \mathbf{R^n} \to (-\infty, \infty]$ defined by

$$f^*(y) = \sup\{x \cdot y - f(x) | x \in \mathbf{R^n}\}.$$

It is well known that $f^*$ is a proper, lower semi continuous convex function provided $f$ a proper convex function. For instance, for the linear function $f(x) = a \cdot x + b$, its dual $f^*$ has domain $D(f^*) = \{a\}$ so that $f^*(a) = -b$. The Legendre-Fenchel duality theorem says that for a proper lower semi continuous convex function $f$, $(f^*)^* = f$.

For $P = \{p_1, ..., p_k\} \subset \mathbf{R^n}$ and $h = (h_1, ..., h_k) \in \mathbf{R^k}$, we define the piecewise linear (PL) convex function $u_h(x)$ to be

$$u(x) = u_h(x) = u_{h,P}(x) = \max\{p_i \cdot x + h_i | i = 1, ..., k\} \tag{3}$$

The domain $D(u^*)$ of the dual $u^*$ is the convex hull $conv(p_1, ..., p_k)$ so that

$$u^*(y) = \min\{-\sum_{i=1}^{k} t_i h_i | t_i \geq 0, \sum_{i=1}^{k} t_i = 1, \sum_{t=1}^{k} t_i p_i = y\} \tag{4}$$

(See theorem 2.2.7 of Hörmander's book on Notions of Convexity [8]). In particular, $u^*$ is PL convex in the domain $D(u^*)$. For instance if $h = 0$, then $u^*(y) = 0$ for all $y \in D(u^*)$. Another useful consequence is,

Corollary 2.1. *If $p_i \notin conv(p_1, ..., p_{i-1}, p_{i+1}, ..., p_k)$, then*

$$u^*(p_i) = -h_i. \tag{5}$$

Indeed, the only way to express $p_i$ as a convex combination of $p_1, ..., p_k$ is $p_i = 1 \cdot p_i$. Thus (5) holds.

**2.2. PL convex functions, convex subdivisions and power diagrams.** A PL convex function $f$ defined on a closed convex polyhedron $K$ produces a convex subdivision (called natural subdivision) $\mathcal{T}$ of $K$. It is the same as the power diagram used in computational geometry. Let us recall briefly the definition (see for instance [10]). A *convex subdivision* of $K$ is a collection $\mathcal{T}$ of convex polyhedra (called cells) so that (a) $K = \cup_{\sigma \in \mathcal{T}} \sigma$, (b) if $\sigma, \tau \in \mathcal{T}$, then $\sigma \cap \tau \in \mathcal{T}$, and (c) if $\sigma \in \mathcal{T}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{T}$ if and only if $\tau$ is a face of $\sigma$. The collection $\mathcal{T}$ is determined by its

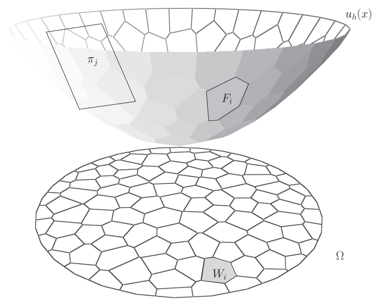


FIG. 3. *PL-convex function and its induced convex subdivision.*

top-dimensional cells. The set of all zero-dimensional cells in $\mathcal{T}$, denoted by $\mathcal{T}^0$, is called the vertices of $\mathcal{T}$.

If $f$ is a PL convex function defined on a convex polyhedron $K$, the natural convex subdivision $\mathcal{T}$ of $K$ associated to $f$ is the subdivision whose top-dimensional cells in $\mathcal{T}$ are the largest convex subsets on which $f$ are linear. The *vertices* of $f$ are defined to be the vertices of $\mathcal{T}$. Suppose $\{v_1, ..., v_m\}$ is the set of all vertices of $f$. Then $f$ is determined by its vertices $\{v_i\}$ and the values at the vertices $\{f(v_i)\}$. Indeed the graph of $f$ over $K$ is the lower boundary of the convex hull $conv((v_1, f(v_1)), ..., (v, f(v_m)))$ in $\mathbf{R}^n \times \mathbf{R}$. Recall that if $P$ is a convex polyhedron in $\mathbf{R^n} \times \mathbf{R}$, then the *lower faces* of $P$ are those faces $F$ of $P$ so that if $x \in F$, then $x - (0, ..., 0, \lambda)$ is not in $P$ for all $\lambda > 0$. The lower boundary of $P$ is the union of all lower faces of $P$. One can also describe $\mathcal{T}$ by using the epigraph. The epigraph $\{(x, t) \in K \times \mathbf{R} | t \geq f(x)\}$ of $f$ is naturally a convex polyhedron. Each cell in $\mathcal{T}$ is the vertical projection of a lower face of the epigraph.

Since the dual function $f^*$ is also PL convex on its domain $D(f^*)$, there is the associated convex subdivision $\mathcal{T}^*$ of $D(f^*)$. These two subdivisions $(D(f), \mathcal{T})$ and $(D(f^*), \mathcal{T}^*)$ are dual to each other in the sense that there exists a bijective map $\mathcal{T} \to \mathcal{T}^*$ denoted by $\sigma \to \sigma^*$ so that (a) $\sigma, \tau \in \mathcal{T}$ with $\tau \subset \sigma$ if and only if $\sigma^* \subset \tau^*$ and (b) if $\tau \subset \sigma$ in $\mathcal{T}$, then the $cone(\tau, \sigma)$ is dual to $cone(\sigma^*, \tau^*)$. Here the cone $cone(\tau, \sigma) = \{t(x - y) | x \in \sigma, y \in \tau, t \geq 0\}$ and dual of a cone $C$ is $\{x \in \mathbf{R^n} | \ y \cdot x \leq 0$ for all $y \in C\}$. See proposition 1 in section 2 of Passera and Rullgård [10].

For the PL convex function $u_h(x)$ given by (3), define the convex polyhedron $W_i = W_i(h) = \{x \in \mathbf{R^n} | \ x \cdot p_i + h_i \geq x \cdot p_j + h_j$ for all $j\}$. (Note that $W_i$ may be the empty set.) By definition, the convex subdivision $\mathcal{T}$ of $\mathbf{R^n}$ associated to $u_h$ is the union of all $W_i$'s and their faces. Identity (4) for $u^*$ says that the graph $\{(y, u^*(y)) | y \in conv(p_1, ..., p_k)\}$ of $u^*$ is the lower boundary of the convex hull $conv((p_1, -h_1), ..., (p_k, -h_l))$.

We summarize the convex subdivisions associated to PL convex functions as follows,

PROPOSITION 2.2. *(a) If $int(W_i(h)) \neq \emptyset$ and $p_1, ..., p_k$ are distinct, then $int(W_i(h)) = \{x \in \mathbf{R^n} | x \cdot p_i + h_i > \max_{j \neq i}\{x \cdot p_j + h_j\}\}$.*

*(b) If $p_i \notin conv(p_1, ..., p_{i-1}, p_{i+1}, ..., p_k)$, then $int(W_i(h)) \neq \emptyset$ and $W_i(h)$ is unbounded.*

*(c) If $conv(p_1, ..., p_{i-1}, p_{i+1}, ..., p_k)$ is n-dimensional and $p_i \in int(conv(p_1, ..., p_{i-1}, p_{i+1}, ..., p_k))$, then $W_i(h)$ is either bounded or empty.*

*(d) If $p_1, ..., p_k$ are distinct so that $int(W_i(h)) \neq \emptyset$ for all $i$, then the top-dimensional cells of $\mathcal{T}$ associated to $u_h$ are exactly $\{W_i(h) | i = 1, ..., k\}$, vertices of $u^*$*

*and the dual subdivision* $\mathcal{T}^*$ *of* $conv(p_1, ..., p_k)$ *are exactly* $\{p_1, ..., p_k\}$.

*(e) For any distinct* $p_1, ..., p_k \in \mathbf{R^n}$*, there is* $h \in \mathbf{R^k}$ *so that* $int(W_i(h)) \neq \emptyset$ *for all* $i$.

*Proof.* To see (a), by definition $\{x \in \mathbf{R^n} | x \cdot p_i + h_i > \max_{j \neq i}\{x \cdot p_j + h_j\}\}$ is open and is in $W_i(h)$. Hence it is included in the interior $int(W_i(h))$. Let $L_j(x) = x \cdot p_j + h_j$. By definition, $L_i(x) \geq L_j(x)$ for all $x \in W_i(h)$. It remains to show for each $p \in int(W_i)$, $L_i(p) > L_j(p)$ for $j \neq i$. Take a point $q \in int(W_i(h))$ so that $L_i(q) > L_j(q)$ (this is possible since $L_i \neq L_j$ for $j \neq i$). Choose a line segment $I$ from $q$ to $r$ in $int(W_i(h))$ so that $p \in int(I)$. Then using $L_i(q) > L_j(q), L_i(r) \geq L_j(r)$, $p = tq + (1-t)r$ for some $t \in (0,1)$ and linearity, we see that $L_i(p) > L_j(p)$. This establishes (a).

To see part (b) that $int(W_i) \neq \emptyset$, by the identity (4) for $u^*_{h,P'}$ with $P' = \{p_1, ..., p_k\} - \{p_i\}$, we have $u^*_{h,P'}(p_i) = \infty$. But $u^*_{h,P'}(p_i) = \sup\{x \cdot p_i - u_{h,P'}(x) | x \in \mathbf{R^n}\}$. Hence there exists $x$ so that $x \cdot p_i + h_i > u_{h,P'}(x) = \max_{j \neq i}(x \cdot p_j + h_j)$, i.e., $int(W_i(h)) \neq \emptyset$. Furthermore, $u^*_{h,P'}(p_i) = \infty$ implies $W_i(h)$ is non-compact, i.e., unbounded.

To see (c), suppose otherwise. Then the set $W_i(h)$ contains a ray $\{tv + a | t \geq 0\}$ for some non-zero vector $v$. Therefore, $(tv + a) \cdot p_i + h_i \geq (tv + a) \cdot p_j + h_j$ for all $j \neq i$. Divide the inequality by $t$ and let $t \to \infty$, we obtain $v \cdot p_i \geq v \cdot p_j$ for all $j \neq i$. This shows that the projection of $p_i$ to the line $\{tv | t \in \mathbf{R}\}$ is not in the interior of the convex hull of the projections of $\{p_1, ..., p_k\} - \{p_i\}$. This contradicts the assumption that $p_i$ is in the interior of the n-dimensional convex hull.

The first part of (d) follows from the definition. The duality theorem (proposition 1 in section 2 of [10]) shows the second part.

To see part (e), let us relabel the set $p_1, ..., p_k$ so that for all $i$ if $j > i$ then $p_j$ is not in the convex hull of $\{p_1, ..., p_i\}$. This is always possible due to the assumption that $p_1, ..., p_k$ are distinct. Indeed, choose a line $L$ so that the orthogonal projection of $p_i$'s to $L$ are distinct. Now relabel these points according to the linear order of the projections to $L$.

For this choice of ordering of $p_1, ..., p_k$, we construct $h_1, ..., h_k$ inductively so that $W_i(h)$ contains a non-empty open set. Let $h_1 = 0$, since $p_2 \neq p_1$, for any choice of $h_2$, both $vol(\{x | \bigtriangledown u_{(h_1,h_2)}(x) = p_i\}) > 0$ for $i = 1, 2$. Inductively, suppose $h_1, ..., h_i$ have been constructed so that $vol(\{x | \bigtriangledown u_{(h_1,...,h_i)}(x) = p_j\}) > 0$ for all $j = 1, 2, ..., i$. To construct $h_{i+1}$, first note that since $p_{i+1}$ is not in the convex hull of $p_1, ..., p_i$, by part (a), for any choice of $h_{i+1}$, $vol(W_{i+1}(h_1, ..., h_{i+1})) > 0$ and $W_{i+1}(h_1, ..., h_{i+1})$ is unbounded. Now by choosing $h_{i+1}$ very negative, we can make all $vol(W_j(h_1, ..., h_{i+1})) > 0$ for all $j = 1, 2, ..., i+1$. □

It is known that convex subdivisions associated to a PL convex function $u_h(x)$ on $\mathbf{R^n}$ are exactly the same as the power diagrams. See for instance [3], [13]. We recall briefly the power diagrams. Suppose $P = \{p_1, ..., p_k\}$ is a set of $k$ points in $\mathbf{R^n}$ and $w_1, ..., w_k$ are $k$ real numbers. The power diagram for the weighted points $\{(p_1, w_1), ..., (p_k, w_k)\}$ is the convex subdivision $\mathcal{T}$ defined as follows. The top-dimensional cells are $U_i = \{x \in \mathbf{R^n} | |x - p_i|^2 + w_i \leq |x - p_j|^2 + w_j \text{ for all } j\}$. Here $|x|^2 = x \cdot x$ is the square of the Euclidean norm and $|x - p_i|^2 + w_i$ is the *power distance* from $x$ to $(p_i, w_i)$. If all weights are zero, then $\mathcal{T}$ is the Voronoi decomposition associated to $P$. Since $|x - p_i|^2 + w_i \leq |x - p_j|^2 + w_j$ is the same as $x \cdot x - 2x \cdot p_i + |p_i|^2 + w_i \leq x \cdot x - 2x \cdot p_j + |p_j|^2 + w_j$ which is the same as $x \cdot p_i - \frac{1}{2}(|p_i|^2 + w_i) \geq x \cdot p_j - \frac{1}{2}(|p_j|^2 + w_j)$. We see that $U_i = \{x \in \mathbf{R^n} | x \cdot p_i + h_i \geq x \cdot p_j + h_j \text{ for all } j\}$ where $h_i = -\frac{1}{2}(|p_i|^2 + w_i)$. This shows the well known fact that,

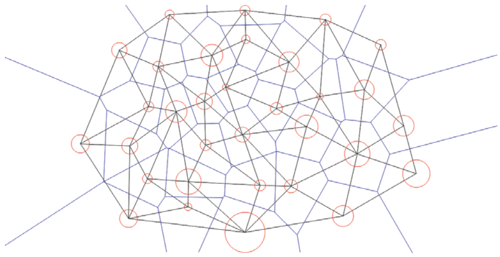

FIG. 4. *Power diagram and its dual weighted Delaunay triangulation.*

PROPOSITION 2.3. *The power diagram associated to* $\{(p_i, w_i)|i = 1, ..., k\}$ *is the convex subdivision associated to the PL convex function* $u_h$ *defined by (3) where* $h_i = -\frac{|p_i|^2 + w_i}{2}$.

**2.3. Variation of the volume of top-dimensional cells.** The following is the key technical proposition for us to establish variational principles.

PROPOSITION 2.4. *Suppose* $\sigma : \Omega \to \mathbf{R}$ *is continuous defined on a compact convex domain* $\Omega \subset \mathbf{R^n}$. *If* $p_1, ..., p_k \in \mathbf{R^n}$ *are distinct and* $h \in \mathbf{R^k}$ *so that* $vol(W_i(h) \cap \Omega) > 0$ *for all* $i$, *then* $w_i(h) = \int_{W_i(h)\cap\Omega} \sigma(x)dx$ *is a differentiable function in* $h$. *Furthermore, if* $W_i(h) \cap \Omega$ *and* $W_j(h) \cap \Omega$ *for* $j \neq i$ *share a codimension-1 face* $F$, *then we have*

$$\frac{\partial w_i(h)}{\partial h_j} = -\frac{1}{|p_i - p_j|} \int_F \sigma|_F(x)dA \tag{6}$$

*where* $dA$ *is the area form on* $F$, *and otherwise this partial derivative is zero. In particular, for any* $i, j$

$$\frac{\partial w_i(h)}{\partial h_j} = \frac{\partial w_j(h)}{\partial h_i}.$$

*Proof.* The proof is based on the following simple lemma.

LEMMA 2.5. *Suppose* $X$ *is a compact domain in* $\mathbf{R^n}$, $f : X \to \mathbf{R}$ *is a non-negative continuous function and* $\tau : \{(x, t) \in X \times \mathbf{R}|0 \leq t \leq f(x)\} \to \mathbf{R}$ *is continuous. For each* $t \geq 0$, *let* $f_t(x) = \min\{t, f(x)\}$. *Then* $W(t) = \int_X (\int_0^{f_t(x)} \tau(x, s)ds)dx$ *satisfies*

$$\lim_{t \to t_0^+} \frac{W(t) - W(t_0)}{t - t_0} = \int_{\{x|f(x)>t_0\}} \tau(x, t_0)dx \tag{7}$$

*and*

$$\lim_{t \to t_0^-} \frac{W(t) - W(t_0)}{t - t_0} = \int_{\{x|f(x)\geq t_0\}} \tau(x, t_0)dx. \tag{8}$$

*It is differentiable at* $t_0$ *if and only if* $\int_{\{x \in X|f(x)=t_0\}} \tau(x, t_0)dx = 0$.

*Proof.* Let $G_t(x) = \int_0^{f_t(x)} \tau(x, s)ds$ and $M$ be an upper bound of $|\tau(x, t)|$ in its domain. Since $|\min(a, b) - \min(a, c)| \leq |b - c|$, we have $|f_t(x) - f_{t'}(x)| \leq |t - t'|$. Now, for any $t \neq t'$,

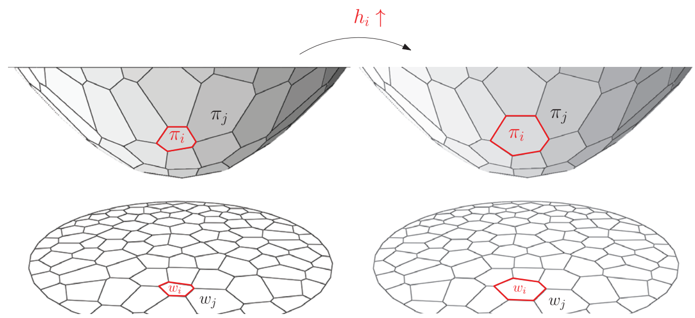


FIG. 5. *Variation of the volume of top-dimensional cells*

$$(9) \qquad |\frac{G_t(x)-G_{t'}(x)}{t-t'}| = \frac{1}{|t-t'|}|\int_{f_{t'}(x)}^{f_t(x)} \tau(x,s)ds| \le \frac{M}{|t-t'|}|f_t(x)-f_{t'}(x)| \le M.$$

Fix $t_0$ and $x \in X$. If $f(x) < t_0$, then for $t$ very close to $t_0$, $G_t(x) = \int_0^{f(x)} \tau(x,s)ds$. Hence $\lim_{t\to t_0} \frac{G_t(x)-G_{t_0}(x)}{t-t_0} = 0$. If $f(x) > t_0$, then for $t$ very close to $t_0$, $G_t(x) = \int_0^t \tau(x,s)ds$. Hence $\lim_{t\to t_0} \frac{G_t(x)-G_{t_0}(x)}{t-t_0} = \lim_{t\to t_0} \frac{1}{t-t_0}\int_{t_0}^t \tau(x,s)ds = \tau(x,t_0)$. If $f(x) = t_0$, then the above calculations shows $\lim_{t\to t_0^+} \frac{G_t(x)-G_{t_0}(x)}{t-t_0} = 0$ and $\lim_{t\to t_0^-} \frac{G_t(x)-G_{t_0}(x)}{t-t_0} = \tau(x,t_0)$. Therefore, by Lebesgue's dominated convergence theorem, we have

$$(10) \qquad \lim_{t\to t_0^+} \frac{W(t)-W(t_0)}{t-t_0} = \lim_{t\to t_0^+}\int_X \frac{G_t(x)-G_{t_0}(x)}{t-t_0}dx = \int_{\{x|f(x)>t_0\}} \tau(x,t_0)dx$$

and

$$(11) \qquad \lim_{t\to t_0^-} \frac{W(t)-W(t_0)}{t-t_0} = \lim_{t\to t_0^-}\int_X \frac{G_t(x)-G_{t_0}(x)}{t-t_0}dx = \int_{\{x|f(x)\ge t_0\}} \tau(x,t_0)dx$$

This establishes the lemma. □

Fix $a < b$, we call the domain $\{(x,t) \in X \times \mathbf{R} | a \le f(x), a \le t \le \min(f(x), b)\}$ a *cap domain* with base $\{x|f(x) \ge a\}$ and top $\{x|f(x) \ge b\}$ of *height* $(b-a)$ associated to the function $f$.

To prove the proposition 2.4, let $h' = (h_1, ..., h_{i-1}, h_i - \delta, h_{i+1}, ..., h_k)$. For small positive $\delta > 0$, by definition, $W_i(h') \subset W_i(h)$ and $W_j(h) \subset W_j(h')$. If $W_i(h) \cap W_j(h) \cap \Omega = \emptyset$, then $W_j(h) \cap \Omega = W_j(h') \cap \Omega$ for small $\delta$. Hence $\frac{\partial W_j(h)}{\partial h_i} = 0$. If $W_i(h) \cap \Omega$ and $W_j(h) \cap \Omega$ share a codimension-1 face $F$, then the closure $cl(W_j(h') \cap \Omega - W_j(h) \cap \Omega)$ is a cap domain with base $F$ associated to a convex function $f$ defined on $F$. The height of the cap domain is $\frac{1}{|p_i - p_j|}\delta$ and $f$ is PL convex so that the $(n-1)$-dimensional Lebesgue measure of set of the form $\{x \in F | f(x) = t\}$ is zero. Furthermore for $\delta > 0$, by definition

$$\begin{aligned}\frac{w_j(h') - w_j(h)}{\delta} &= \frac{1}{\delta}\int_{W_j(h')\cap\Omega - W_j(h)\cap\Omega} \sigma(x)dx \\ &= \frac{1}{\delta}\int_F \int_0^{f_t(y)} \tau(y,s)dsdy\end{aligned}$$

where $y \in F$ is the Euclidean coordinate and $\tau(y, s)$ is $\sigma$ expressed in the new coordinate. Thus, by lemma 2.5, we see

$$\lim_{\delta \to 0^+} \frac{w_j(h') - w_j(h)}{\delta} = \int_F \sigma|_F dA. \tag{12}$$

The same calculation shows that for $\delta < 0$ and close to 0, using the fact that $cl(W_j(h) \cap \Omega - W_j(h') \cap \Omega)$ is cap with top $F$, we see that (12) holds as well. Finally, if $W_i(h) \cap \Omega$ and $W_j(h) \cap \Omega$ share a face of dimension at most $n-2$, then the same calculation still works where the associate cap domain has either zero top area or zero base area. Thus the result holds. □

**3. A proof of Theorem 1.2.** Our proof is divided into several steps. In the first step, we show that the set $H = \{h \in \mathbf{R^k} | vol(W_i(h) \cap \Omega) > 0 \text{ for all } i\}$ is a non-empty open convex set. In the second step, we show that

$$E(h) = \int_a^h \sum_{i=1}^k \int_{W_i(h) \cap \Omega} \sigma(x) dx dh_i - \sum_{i=1}^k h_i A_i$$

is a well defined $C^1$-smooth function on $H$ so that $\frac{\partial E(h)}{\partial h_i} = \int_{W_i(h) \cap \Omega} \sigma(x) dx - A_i$ and $E(h + (c, ..., c)) = E(h)$ for all $c$. In the third step, we show that $E(h)$ is convex in $H$ and is strictly convex in $H_0 = H \cap \{h | \sum_{i=1}^k h_i = 0\}$. In the fourth step, we show that the map $E$ has a minimum point in $H$ by focusing on $E|_{H_0}$. Finally, for the completeness, we include a simple proof by Aurenhammer et al. (Lemma 1 in [4]) to show that $\nabla u_b$ is an optimal transport map minimizing the quadratic cost.

**3.1. Convexity of the domain $H$.** We begin with a simple observation that a compact convex set $X \subset \mathbf{R^n}$ has positive volume if and only if $X$ contains a non-empty open set, i.e., $X$ is n-dimensional. Therefore, $vol(W_i(h) \cap \Omega) > 0$ is the same as $W_i(h) \cap \Omega$ contains a non-empty open set in $\mathbf{R^n}$. The last condition, by the above proposition 2.2(a), is the same as there exists $x \in \Omega$ so that $x \cdot p_i > \max_{j \neq i}\{x \cdot p_j + h_j\}$.

Now to see that $H$ is convex, since $H = \cap_{i=1}^k H_i$ where $H_i = \{h \in \mathbf{R^k} | vol(W_i(h) \cap \Omega) > 0\}$, it suffices to show that $H_i$ is convex for each $i$. To this end, take $\alpha, \beta \in H_i$ and $t \in (0, 1)$. Then there exist two vectors $v_1, v_2 \in \Omega$ so that $v_1 \cdot p_i + \alpha_i > v_1 \cdot p_j + \alpha_j$ and $v_2 \cdot p_i + \beta_i > v_2 \cdot p_j + \beta_j$ for all $j \neq i$. Therefore, $(tv_1 + (1-t)v_2) \cdot p_i + (t\alpha_i + (1-t)\beta_i) > (tv_1 + (1-t)v_2) \cdot p_j + (t\alpha_j + (1-t)\beta_j)$ for all $j \neq i$. This shows that $t\alpha + (1-t)\beta$ is in $H_i$. Furthermore, each $H_i$ is non-empty. Indeed, given $h_1, ..., h_{i-1}, h_{i+1}, ..., h_k$, by taking $h_i$ very large, we see that $h = (h_1, ..., h_k)$ is in $H_i$. Also, from the definition, $H_i$ is an open set. Therefore, to show $H$ is an open convex set, it remains to show that $H$ is non-empty.

To see $H \neq \emptyset$, it suffices to show that there exists $h$ so that $vol(W_i(h)) > 0$ (which could be $\infty$) for all $i$. Indeed, after a translation, we may assume that 0 is in the interior of $\Omega$. Since $u_{\lambda h}(x) = \lambda u_h(x/\lambda)$ and $W_i(\lambda h) = W_i(h)/\lambda$, if $vol(W_i(h)) > 0$ for all $i$, then for $\lambda > 0$ large, $vol(W_i(\lambda h) \cap \Omega) > 0$. Now by proposition 2.2(e), we can find $h$ so that $vol(W_i(h)) > 0$ for all $i$. Thus $H \neq \emptyset$.

**3.2. The function $E(h)$ and its gradient.** By proposition 2.4, we see that the differential 1-form $\eta = \sum_{i=1}^k \int_{W_i(h) \cap \Omega} \sigma(x) dx dh_i - \sum_{i=1}^k A_i dh_i$ is a closed 1-form defined on the simply connected open set $H$. Therefore its integral $E(h) = \int_a^h \eta$ is a well defined $C^1$-smooth function independent of the choice of the path from $a$ to

$h$ in $H$. Furthermore, by definition, $\frac{\partial E(h)}{\partial h_i} = \int_{W_i(h)\cap\Omega} \sigma(x)dx - A_i$. The condition $E(h+(c,...,c)) = E(h)$ follows from the definition since $W_i(h+(c,...,c)) = W_i(h)$.

It can be shown that there is a constant $C$ so that

$$E(h) = \int_\Omega u_h(x)\sigma(x)dx - \sum_{i=1}^k A_i h_i + C.$$

**3.3. The convexity of $E(h)$.** We will show that the Hessian matrix of $E(h)$ is positive semi-definite and has a 1-dimensional null space spanned by the vector $(1,1,...,1)$. This implies that $E$ is convex in $H$ and is strictly convex when restricted to $H_0$.

Let $w_i(h) = \int_{W_i(h)\cap\Omega} \sigma(x)dx$. By the calculation above, $\partial E(h)/\partial h_i = w_i(h) - A_i$ and $\sum_{i=1}^k w_i(h) = \int_\Omega \sigma(x)da$. Therefore $\sum_{i=1}^k \partial^2 E/\partial h_i \partial h_j = 0$ for each $j$. By proposition 2.4, for $i \neq j$, $\frac{\partial w_i(h)}{\partial h_j} \leq 0$. Furthermore, if $W_i(h)$ and $W_j(h)$ share a codimension-1 face $F$ in $\Omega$, then $\frac{\partial w_i(h)}{\partial h_j} = -\frac{1}{|p_i - p_j|}\int_F \sigma|_F(x)dA < 0$. This implies that the Hessian matrix $Hess(E) = [\frac{\partial w_i(h)}{\partial h_j}]$ is diagonally dominated and therefore positive semi-definite (i.e., all diagonal entries are positive and all off diagonal entries are non-positive so that the sum of entries of each row is zero). Hence $E(h)$ is convex in $H$.

Corollary 3.1. *The Hessian matrix $Hess(E)$ of $E(h)$ is positive semi-definite with 1-dimensional null space generated by $(1,1,..,1)$. In particular, $E|_{H_0} : H_0 = \{h \in H | \sum_{i=1}^k h_i = 0\} \to \mathbf{R}$ is strictly convex and its gradient $\bigtriangledown(E|_{H_0})$ is a diffeomorphism.*

*Proof.* To see that $E|_{H_0}$ is strictly convex, we show that the null space of the Hessian $Hess(E) = [\frac{\partial w_i(h)}{\partial h_j}] = [a_{ij}]$ is generated by $(1,...,1)$. Obviously $(1,...,1)$ is in the null space. To see that the null space is 1-dimensional, suppose $v = (v_1,...,v_k)$ is a non-zero vector so that $Hess(E)v^t = 0$ ($v^t$ is the transpose of $v$). Let us assume without loss of generality that $|v_{i_1}| = \max_i\{|v_i|\}$ and $v_{i_1} > 0$. Using $a_{i_1 i_1}v_{i_1} = \sum_{j\neq i_1} a_{i_1 j}v_j$ and $a_{i_1 i_1} = -\sum_{j\neq i_1}|a_{i_1 j}|$, we see that $v_j = v_{i_1}$ for all indices $i$ with $a_{i_1 j} \neq 0$. It follows that index set $I = \{i | v_i = \max_j\{|v_j|\}\}$ has the following property. If $i_1 \in I$ and $a_{i_1 i_2} \neq 0$, then $i_2 \in I$. We claim that $I = \{1,2,...,k\}$, i.e., $v = v_1(1,1,...,1)$. Indeed, for any two indices $i \neq j$, since $\Omega$ is connected and $W_r(h)$ are convex, there exists a sequence of indices $i_1 = i, i_2, ..., i_m = j$ so that $W_{i_s}(h)\cap\Omega$ and $W_{i_{s+1}}(h)\cap\Omega$ share a codimension-1 face for each $s$. Therefore $a_{i_s i_{s+1}} \neq 0$. Translating this to the Hessian matrix $Hess(E) = [a_{ij}]$, it says that for any two diagonal entries $a_{ii}$ and $a_{jj}$, there exists a sequence of indices $i_1 = i, i_2, ..., i_m = j$ so that $a_{i_s i_{s+1}} <$ 0. This last condition together with the property of $I$ imply $I = \{1,2,...,k\}$, i.e., $dim(Ker(Hess(E))) = 1$. □

**3.4. Existence of minimal points for $E(h)$.** By the variational framework, the critical points of $E(h)$ in $H$ are the solutions $b$ in theorem 1.2. To show that $E(h)$ has critical points in $H$, due to the convexity of $E$ and $E(h+(c,...,c)) = E(h)$, it suffices to show that $E(h)$ has a minimal points in $H_0$.

First we claim that $H_0$ is bounded. Otherwise, there exists a sequence of vectors $h^{(m)}$ in $H_0$ so that $\lim_{m\to\infty} h_i^{(m)} = \infty$ and $\lim_{m\to\infty} h_j^{(m)} = -\infty$ for some indices $i,j$. Since $\Omega$ is compact, we see that for $m$ large, for all points $x \in \Omega$,

$$x \cdot p_i + h_i^{(m)} > x \cdot p_j + h_j^{(m)}.$$

This shows that $W_j(h^{(m)}) \cap \Omega = \emptyset$ which contradicts the assumption that $h^{(m)} \in H_0$.

The function $E(h)$ can be extended continuously to the compact closure $X$ of $H_0$ using the same expression. Therefore, it has a minimal point $q \in X$. We claim that $q \in H_0$. If otherwise, $q \in \partial X$, i.e., there are indices $j$ so that $vol(W_j(q) \cap \Omega) = 0$, we derive a contraction as follows.

Due to convexity, there exists a non-zero vector $v \in \mathbf{R^k}$ so that $q + tv \in H_0$ for small $t > 0$. Now for any $c \in \mathbf{R}$, we have $q + t(v + c(1, ..., 1)) \in H$ for small $t > 0$. Therefore, by taking $c$ large, we may assume that all $v_i > 0$ and $q + tv \in H$ for small $t > 0$. Define $\delta \in \mathbf{R^k}$ so that

$$\delta_i = v_i$$

if $w_i(q) = 0$ and

$$\delta_i = 0$$

if $w_i(q) > 0$. Note that $v_i \geq \delta_i$ for all $i$. We claim that for small $t > 0$, $q + t\delta \in H$, i.e., $w_i(q + t\delta) > 0$ for all $i$. Indeed, if $w_i(q) > 0$, then $w_i(q + t\delta) > 0$ for small $t > 0$ due to continuity. Next, if $w_i(q) = 0$, we claim $W_i(q + tv) \subset W_i(q + t\delta)$. This implies $w_i(q + t\delta) > 0$ since $w_i(q + tv) > 0$. To see the claim, take $x \in W_i(q + tv) = \{x | x \cdot p_i + (q_i + tv_i) \geq x \cdot p_j + (q_j + tv_j), \text{ all } j\}$. Then $x \cdot p_i + q_i + t\delta_i = x \cdot p_i + q_i + tv_i \geq x \cdot p_j + q_j + tv_j \geq x \cdot p_j + q_j + t\delta_j$, i.e., $x \in W_i(q + t\delta)$.

Since $E(h + (c, ..., c)) = E(h)$ and every point $x \in H$ is of the form $h + (c, ..., c)$ for some $h \in H_0, c \in \mathbf{R}$, the point $q$ is also a minimal point of $E$ defined on the closure of $H$. Now by the construction of $\delta$, we have $E(q + t\delta) \geq E(q)$ for small $t > 0$. By taking derivative with respect to $t$ at $t = 0$, we obtain, $\bigtriangledown E(q) \cdot \delta \geq 0$, i.e.,

$$\sum_{i=1} (w_i(q) - A_i)\delta_i \geq 0. \tag{13}$$

But by the construction of $\delta$, the right-hand-side above is $-\sum_{i \in J} A_i v_i < 0$ where $J = \{i | w_i(q) = 0\}$ and $A_i > 0$. The contradiction shows that $q$ is in $H_0$.

**3.5. Infinitesimal rigidity.** With above preparations, we now show the map gradient map sending $h$ to $\Phi(h) = (w_1(h), ..., w_k(h))$ is a diffeomorphism from $H_0$ onto $\mathcal{A} = \{(A_1, ..., A_k) \in \mathbf{R^k} | A_i > 0, \sum_{i=1}^k A_i = \int_\Omega \sigma(x)dx\}$. Indeed, by the calculation above, $\Phi(h)$ is $\nabla \widetilde{E|_{H_0}}$ where $\widetilde{E}(h) = E(h) + \sum_{i=1}^k A_i h_i$ is a strictly convex function with positive definite Hessian on $H_0$. In particular, its gradient $\Phi$ is an injective local diffeomorphism from $H_0$ to $\mathcal{A}$. On the other hand, for any choice of $A \in \mathcal{A}$, the existence of critical points of $\widetilde{E}(h) - \sum_{i=1}^k A_i h_i$ proved in §3.4 shows that $\Phi$ is onto.

As a consequence of the proof, we obtained a new proof of the infinitesimal rigidity theorem of Alexandrov.

Corollary 3.2. *(Alexandrov) The map $\bigtriangledown \widetilde{E} : H_0 \to W$ sending the normalized heights $h$ to the area vector $(w_1(h), ..., w_k(h))$ is a local diffeomorphism.*

**3.6. $\nabla u_b$ is an optimal transport map.** We reproduce the elegant proof by Aurenhammer et al. [4] here for completeness. Notice the quadratic transport cost of $\nabla u_b$ is $\sum_{i=1}^k \int_{W_i} |p_i - x|^2 \sigma(x)dx$. From Proposition 2.3, $\{W_1, ..., W_k\}$ is the power diagram associated to $\{(p_i, w_i = |p_i|^2 - 2b_i)\}$ in $R^d$. Suppose $\{U_1, ..., U_k\}$ is any partition of $R^d$ so that $\int_{U_i} \sigma(x)dx = \int_{W_i} \sigma(x)dx, i = 1, 2, \cdots, k$. By the definition of the power diagram, we have

$$\sum_{i=1}^{k} \int_{W_i} (|x-p_i|^2 + w_i)\sigma(x)dx = \sum_{i,j=1}^{k} \int_{W_i \cap U_j} (|x-p_i|^2 + w_i)\sigma(x)dx$$
$$\leq \sum_{i,j=1}^{k} \int_{W_i \cap U_j} (|x-p_j|^2 + w_j)\sigma(x)dx = \sum_{j=1}^{k} \int_{U_j} (|x-p_j|^2 + w_j)\sigma(x)dx.$$

Using $\int_{W_i} \sigma(x)dx = \int_{U_i} \sigma(x)dx$, we have

$$\sum_{i=1}^{k} \int_{W_i} |x-p_i|^2 \sigma(x)dx) \leq \sum_{i=1}^{k} \int_{U_i} |x-p_i|^2 \sigma(x)dx.$$

This shows that $\nabla u_b$ minimizes the quadratic transport cost.

**4. A proof of Theorem 1.4.** We fix $g_1, ..., g_m$ throughout the proof. For simplicity, let $p_{k+j} = v_j$ and $h_{k+j} = -g_j$ for $j = 1, ..., m$ and let

$$W_i(h) = \{x \in \mathbf{R}^\mathbf{n} | x \cdot p_i + h_i \geq x \cdot p_j + h_j, j = 1, ..., k+m\}.$$

Define

$$H = \{h \in \mathbf{R}^\mathbf{k} | vol(W_i(h)) > 0, i = 1, ..., k+m\}.$$

LEMMA 4.1. *(a) $H$ is a non-empty open convex set in $\mathbf{R}^\mathbf{k}$.*
*(b) For each $h \in H$ and $i = 1, ..., k$ and $j = 1, ..., m$, $W_i(h)$ is a non-empty bounded convex set and $W_{k+j}(h)$ is a non-empty unbounded set.*

*Proof.* The proof of convexity of $H$ is exactly the same as that of §2.1. We omit the details. Also, by definition $H$ is open. To show that $H$ is non-empty, using proposition 2.2(e), there exists $\bar{h} \in \mathbf{R}^\mathbf{k}$ so that for all $i = 1, ..., k$, $vol(W_i(\bar{h})) > 0$. We claim for $t > 0$ large the vector $h = \bar{h} + (t, ..., t) \in H$. Indeed, let $B$ be a large compact ball so that $B \cap W_i(\bar{h}) \neq \emptyset$ for all $i = 1, ..., k$. Now choose $t$ large so that

$$\min_{x \in B}\{x \cdot p_i + h_i | i = 1, 2, ..., k\} > \max_{x \in B}\{x \cdot v_j + g_j | j = 1, ..., m\}.$$

For this choice of $h$, by definition, $W_i(\bar{h}) \cap B \subset W_i(h)$.

Part (b) follows from proposition 2.2 (b) and (c). □

For $h \in H$ and $i = 1, ..., k$, let $w_i(h) = vol(W_i(h)) > 0$. For each $h \in H$, by proposition 2.4 applied to a large compact domain $X$ whose interior contains $\cup_{i=1}^k W_i(h)$, we see that $w_i(h)$ is a differentiable function so that $\frac{\partial w_i}{\partial h_j} = \frac{\partial w_j}{\partial h_i}$ for all $i, j = 1, ..., k$. Thus the differential 1-form $\eta = \sum_{i=1}^k w_i(h)dh_i$ is a closed 1-form on the open convex set $H$. Since $H$ is simply connected, there exists a $C^1$-smooth function $E(h) : H \to \mathbf{R}$ so that $\frac{\partial E}{\partial h_i} = w_i(h)$.

LEMMA 4.2. *The Hessian matrix $Hess(E)$ of $E$ is positive definite for each $h \in H$. In particular, $E$ is strictly convex and $\nabla E : H \to \mathbf{R}^\mathbf{k}$ is a smooth embedding.*

*Proof.* By the same proof as in §2.3, we have for $i \neq j$, $\partial w_i(h)/\partial h_j = -\frac{1}{|p_i - p_j|} Area(F) < 0$ if $W_i(h)$ and $W_j(h)$ share a codimension-1 face $F$ and it is zero otherwise. Furthermore, for each $j = 1, ..., k$, $\sum_{i=1}^{k} \partial w_i/\partial h_j = \frac{\partial(\sum_{i=1}^{k} w_i(h))}{\partial h_j} > 0$ if $W_j(h)$ and one of $W_{\mu+k}(h)$ share a codimension-1 face. It is zero otherwise. This shows the Hessian matrix $Hess(E) = [a_{ij}]$ is diagonally dominated so that $a_{ij} \leq 0$ for all $i \neq j$ and $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$. Thus $Hess(E)$ is positive semi-definite. To show that it has no kernel, we proceed with the same argument as in the proof of corollary 3.1. The same argument shows that if $b = (b_1, ..., b_k)$ is a null vector for $[a_{ij}]$, then $b_1 = b_2 = ... = b_k$. On the other hand, there is an index $i$ so that $W_i(h)$ and one of $W_{j+k}(h)$ share a codimension-1 face, i.e, $a_{ii} > \sum_{i=1}^{k} |a_{ij}|$. Using $\sum_{j=1}^{k} a_{ij} b_1 = 0$, we see that $b_1 = 0$, i.e., $b = 0$. This establishes the lemma. □

Now we prove theorem 1.4 as follows. Let $\mathcal{A} = \{A = (A_1, ..., A_k) | A_i > 0\}$ and consider the strictly convex function $E(h) : H \to \mathbf{R}$ with $\partial E(h)/\partial h_i = w_i(h)$. Our goal is to prove that $E(h) - \sum_{i=1}^{k} A_i h_i$ has a minimal point in $H$ by showing $A = \bigtriangledown E(h)$ for some $h \in H$.

Let $\Phi = \bigtriangledown E : H \to \mathcal{A}$ be the gradient map. By lemma 4.2, $\Phi$ is an injective local diffeomorphism from $H$ to $\mathcal{A}$. In particular, due to $dim(H) = dim(\mathcal{A})$, $\Phi(H)$ is open in $\mathcal{A}$. To finish the proof that $\Phi(H) = \mathcal{A}$, since $\mathcal{A}$ is connected, it suffices to prove that $\Phi(H)$ is closed in $\mathcal{A}$. Take a sequence of points $h^{(i)}$ in $H$ so that $\Phi(h^{(i)})$ converges to a point $a \in \mathcal{A}$. We claim that $a \in \Phi(H)$. After taking a subsequence, we may assume that $h^{(i)}$ converges to a point in $[-\infty, \infty]^k$. We first show that $\{h^{(i)}\}$ is a bounded set in $\mathbf{R^k}$. If otherwise, there are three possibilities: (a) there is $j$ so that $h_j^{(i)} \to -\infty$ as $i \to \infty$, (b) there are two indices $j_1$ and $j_2$ so that $\lim_{i\to\infty} h_{j_1}^{(i)} = \infty$ and $\{h_{j_2}^{(i)}\}$ is bounded, and (c) for all indices $j$, $\lim_{i\to\infty} h_j^{(i)} = \infty$. In the first case (a), due to $p_j \in int(conv(v_1, ..., v_m))$ and $h_j^{(i)}$ is very negative, $x \cdot p_j + h_j^{(i)} < \max\{x \cdot v_{j'} + g_{j'} | j' = 1, ..., m\}$ for $i$ large for all $x$. This implies for $i$ large $W_j(h^{(i)}) = \emptyset$ which contradicts the assumption that $\lim_i \Phi(h^{(i)}) = a \in \mathcal{A}$. In the case (b) that $\{h_{j_2}^{(i)}\}$ is bounded, then $W_{j_2}(h^{(i)})$ lies in a compact set $B$. For $i$ large, $x \cdot p_{j_1} + h_{j_1}^{(i)} \geq \max\{x \cdot v_j + g_j, x \cdot p_{j_2} + h_{j_2}^{(i)})\}$ for all $x \in B$. This implies that $W_{j_2}(h^{(i)}) = \emptyset$ for large $i$ which contradicts the assumption that $\lim_i \Phi(h^{(i)}) = a \in \mathcal{A}$. In the last case (c), since for each $j$, $\lim_{i\to\infty} h_j^{(i)} = \infty$, for any compact set $B$, there is an index $i$ so that $B \subset \cup_{\mu=1}^{k} W_\mu(h^{(i)})$. This implies that the sum of the volumes $\sum_{\mu=1}^{k} vol(W_\mu(h^{(i)})$ tends to infinity which again contradicts the assumption $\lim_i \Phi(h^{(i)}) = a \in \mathcal{A}$.

Now that $h^{(i)}$ is convergent to a point $h$ in $\mathbf{R^k}$, by the continuity of the map sending $h$ to $(w_1(h), ..., w_k(h))$ on $\mathbf{R^k}$, we see that $\Phi(h) = a$. This shows $h \in H$ and $a \in \Phi(H)$, i.e., $\Phi(H)$ is closed in $\mathcal{A}$.

Hence, given any $(A_1, ..., A_k) \in \mathcal{A}$, there exists a unique $h \in H$ so that $\Phi(h) = (A_1, ..., A_k)$. Let $u = \max\{x \cdot p_i + h_i | i = 1, ..., k + m\}$ be the PL convex function on $\mathbf{R^n}$ and $w$ be its dual. By corollary 2.1, we conclude that the vertices of $w$ are exactly $\{v_i, p_j | i, j\}$ with $w(v_i) = g_i$ and $w(p_j) = -h_j$ so that the discrete Hessian of $w$ at $p_i$, which is $w_i(h) = A_i$. Furthermore, by proposition 2.2, the associated convex subdivision of $w$ on $\Omega$ has exactly the vertex set $\{v_1, ..., v_k, p_1, ..., p_k\}$.

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