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DEFORMING COMPLETE HERMITIAN METRICS WITH UNBOUNDED CURVATURE*

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Abstract. We produce solutions to the Kähler-Ricci flow emerging from complete initial metrics g_0 which are C^0 Hermitian limits of Kähler metrics. Of particular interest is when g_0 is Kähler with unbounded curvature. We provide such solutions for a wide class of $U(n)$-invariant Kähler metrics g_0 on $\mathbb{C}^n$, many of which having unbounded curvature. As a special case we have the following Corollary: The Kähler-Ricci flow has a smooth short time solution starting from any smooth complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with either non-negative or non-positive holomorphic bisectional curvature, and the solution exists for all time in the case of non-positive curvature.

Key words. Kähler-Ricci flow, parabolic Monge-Ampère equation, $U(n)$ invariant Kähler metrics.

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1. Introduction. Let (M^n, g_0) be a complete noncompact Riemannian manifold. The Ricci flow is the following evolution equation:

$$\tag{1.1} \begin{cases} \dfrac{\partial}{\partial t} g_{ij} = -2R_{ij} \\ g(0) = g_0. \end{cases}$$

In [S1] Shi proved that if the curvature of g_0 is bounded then (1.1) has a solution $g(t)$ up to some time $T > 0$ depending only on the curvature bound for g_0 and the dimension n of M such that the curvature is bounded in space-time. If in addition, (M^n, g_0) is a Kähler manifold with complex dimension n, then Shi [S2] proved that the solution $g(t)$ is also Kähler, and hence $g(t)$ satisfies Kähler-Ricci flow equation:

$$\tag{1.2} \begin{cases} \dfrac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \\ g(0) = g_0. \end{cases}$$

See Theorem 2.1 for more details.

There are many results of existence without assuming that the initial condition g_0 has bounded curvature. In [Si], Simon proved that starting from any sufficiently small C^0 perturbation g_0 of a complete Riemannian metric with bounded curvature, there is a short time solution of the Ricci harmonic heat flow. We also refer to the works [KL, SSS1] where the Ricci harmonic heat flow is solved starting with rough initial data obtained from a sufficiently small perturbation of the Euclidean metric on $\mathbb{R}^n$, and [SSS2] for a similar result for the hyperbolic metrics. In [CW], Cabezas-Rivas and Wilking obtained a short time existence result of the Ricci flow starting

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from any complete Riemannian metric with nonnegative complex sectional curvature. They do not assume the curvature is bounded and do not assume the initial metric is a small perturbation of a complete metric with bounded curvature. The solutions from [Si], [KL],[SSS1] and [SSS2] are complete and have bounded curvature when $t > 0$. In [CW], complete solutions are constructed where the curvature is bounded whenever $t > 0$ and examples are also given of complete solutions where the curvature is unbounded when $t > 0$.

For Kähler-Ricci flow, when $n = 1$, Giesen and Topping [GT] proved that (1.2) always has a solution starting from any smooth Kähler metric g_0 which may have unbounded curvature, and may even be incomplete. In fact, they also constructed solutions where $g(t)$ is complete with unbounded curvature for all $t \in [0, T)$. Using the construction of [CW], Yang and Zheng [YZ] proved that if g_0 is a $U(n)$ invariant complete Kähler metric with nonnegative sectional curvature, and with some technical assumptions on the solution $g(t)$ of (1.1), then $g(t)$ is Kähler for $t > 0$. Hence in this case Kähler-Ricci flow (1.2) has short time solution.

In this work we want to discuss the short time existence and long time existence of the Kähler-Ricci flow (1.2) in higher dimensions without the assumption that g_0 has bounded curvature. We obtain the following:

Theorem 1.1. *Let g_0 be a complete continuous Hermitian metric on a noncompact complex manifold M^n. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on M converging uniformly on compact subsets to g_0 and another complete Kähler metric $\hat{g}$ with bounded curvature on M such that $C^{-1}\hat{g} \le h_{k,0} \le C\hat{g}$ for some C independent of k. Then for some $T > 0$, the Kähler-Ricci flow* (1.2) *has a complete smooth solution $g(t)$ on $M \times (0, T)$ which has bounded curvature for all $t > 0$, and extends continuously to $M \times [0, T)$ with $g(0) = g_0$. Moreover, if g_0 is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M, then $g(t)$ extends to a smooth solution to* (1.2) *on $M \times [0, T)$ with $g(0) = g_0$.*

One can also estimate the existence time T and bounds of the norms of the curvature tensor and its covariant derivatives with respect to $g(t)$, see Theorem 4.2 for more details.

As a corollary, we obtain an estimate of T in Theorem 2.1 in terms of the upper bound of the holomorphic bisectional curvature. In fact, we can prove a more general result (see Corollary 4.2):

Let (M^n, g_0) be a complete noncompact Kähler manifold with bounded curvature. Suppose that $\hat{g} \le g_0 \le C\hat{g}$ for some complete Kähler metric $\hat{g}$ with bounded curvature and holomorphic bisectional curvatures bounded above by K. Let $T = 1/(2nK)$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow (1.2) *has a complete smooth solution $g(t)$ on $M \times [0, T)$ with $g(0) = g_0$. Moreover, the curvature of $g(t)$ is uniformly bounded on $M \times [0, T']$ for all $0 < T' < T$.*

Another corollary is that one can prove that the Kähler-Ricci flow (1.2) has a short time solution if g_0 is perturbation of a complete Kähler metric $\hat{g}$ with bounded curvature by a potential satisfying certain growth conditions. More precisely (see Corollary 4.1),

Let (M^n, g_0) be a complete noncompact Kähler manifold with bounded curvature. Suppose u is a C^2 function such that $|\nabla u|_{g_0}$ and $|u|$ are of sublinear growth and *such that $g_0 + \sqrt{-1}\partial\bar\partial u$ is uniformly equivalent to g_0. Then for some $T > 0$, the Kähler-Ricci flow* (1.2) *has a complete smooth solution $g(t)$ on $M \times (0,T)$ which has bounded curvature for all $t > 0$ and extends continuously to $M \times [0,T)$ with $g(0) = g_0 + \sqrt{-1}\partial\bar\partial u$.*

If the condition in Theorem 1.1 that $C^{-1}\hat{g} \le h_{k,0} \le C\hat{g}$ is relaxed to only assuming $h_{k,0} \ge C^{-1}\hat{g}$, we still can obtain a short time solution under some additional assumptions on $h_{k,0}$. See Theorem 4.1, for more details. However, in this case, we do not know if the curvature of the solution is bounded for $t > 0$.

Applying our general existence theorems to $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$, we obtain:

THEOREM 1.2. *Let g_0 be a complete smooth $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with either non-negative or non-positive holomorphic bisectional curvature. Then for some $T > 0$, the Kähler-Ricci flow* (1.2) *has a complete smooth $U(n)$-invariant solution $g(t)$ on $\mathbb{C}^n \times [0,T)$ with $g(0) = g_0$. Moreover, the solution exists for all time in case that g_0 has non-positive holomorphic bisectional curvature.*

REMARK 1. Building on the results here, the authors proved in [CLT] that the solution in Theorem 1.2 exists for all time when g_0 has non-negative bisectional curature as well (see also remark 5). We refer to [CLT] for this and more general longtime existence results.

This gives an affirmative answer to a question posed by Yang-Zheng [YZ]. In fact, one can prove results more general than the Theorem above. See Theorems 5.3, 5.4 and their corollaries for more details. We also obtain some long time existence results for g_0 with nonnegative holomorphic bisectional curvature, see Theorem 5.5

The organization of the paper is as follows. In section §2 we review some basic theory and estimates for (1.2), and in §3 we prove some further a priori estimates which we will need later. §4 contains our main existence theorems Theorems 4.2 and 4.1 and accompanying corollaries. In §5 we review Wu-Zheng's description in [WZ] of $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ and use this to apply our previous results to prove Theorems 5.3 and 5.4.

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2. Preliminaries. In this section, we review some well known results for the Kähler Ricci flow which we will use in the paper. We first recall Shi's short time existence theorem for (1.2) in [S2].

THEOREM 2.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold with curvature bounded by a constant K. Then for some $0 < T \le \infty$ depending only on K and the dimension n, there exists a smooth solution $g(t)$ to* (1.2) *on $M \times [0,T)$ with $g(0) = g_0$ such that*

(i) $g(t)$ is Kähler and equivalent to g_0 for all $t \in [0,T)$;

(ii) $g(t)$ has uniformly bounded curvature on $M \times [0,T']$ for all $0 < T' < T$. More precisely, for any $l \ge 0$ there exists a constant C_l depending only on l, K and the dimension n such that

$$sup_M |\nabla^l \mathrm{Rm}(g(t))|^2_{g(t)} \le \frac{C_l}{t^l},$$

on $M \times [0,T)$.

(iv) If $T<\infty$ and $\lim_{t\to T}\sup_M |Rm(x,t)|<\infty$, then $g(t)$ extends to a smooth solution to (1.2) *on $M\times[0,T_1)$ for some $T_1>T$ so that (ii) is still true with T replaced by T_1.*

The solution $g(t)$ in Theorem 2.1 has uniformly bounded curvature on $M\times[0,T')$ for any $0<T'<T$. From this it is easy to see that $g(0)$ and $g(t)$ are uniformly equivalent on $M\times[0,T')$. On the other hand, it is a well known fact that one can study the Kähler-Ricci flow (1.2) through the parabolic Monge-Ampère equation. This was originated in [C], and we refer to [CT] and references therein for further details on this fact. By the Evans-Krylov theory [Ev, Kr] for fully non-linear equations, which in the case of (1.2) takes the form in Theorem 2.2 below, one may conclude that if $g(0)$ and $g(t)$ are uniformly equivalent, then subsequent curvature bounds follow (see [SW], and also [Yu] for proofs using only the maximum principle). Actually, we need a more general version in the sense that we need local estimates where $g(t)$ is assumed to be uniformly equivalent to a fixed background metric $\hat{g}$.

Let us first fix some notations and terminology. $(M^n,\hat{g})$ is said to have bounded geometry of infinite order if the curvature tensor and all its covariant derivatives are uniformly bounded. In particular, the solution $g(t)$ in Theorem 2.1 has bounded geometry of infinite order for $t>0$.

Also, we will denote the geodesic ball with respect to the metric g with center at p and radius r by $B_g(p,r)$. The following theorem can be found in [SW].

THEOREM 2.2. *Let $(M^n,\hat{g})$ be a complete noncompact Kähler manifold with bounded geometry of infinite order. Let $h(t)$ be a solution of Kähler-Ricci* (1.2) *on $M\times[0,T)$ with initial condition h_0 which is a complete Kähler metric. For any $x\in M$, suppose there is a constant $N>0$, such that*

$$N^{-1}\hat{g}\le h(t)\le N\hat{g}$$

on $B_{\hat{g}}(x,1)\times[0,T)$. Let $\hat{\nabla}$ be the covariant derivative with respect to $\hat{g}$. Then

(i)

$$|\hat{\nabla}^k h|^2_{\hat{g}}\le\frac{C_k}{t^k}$$

on $B_{\hat{g}}(x,1/2)\times(0,T)$, for some constant C_k depending only on k, $\hat{g}$, n, T and N.

(ii) If we assume $|\hat{\nabla}^k h_0|^2_{\hat{g}}$ is bounded in $B_{\hat{g}}(x,1)$ by c_k, for $k\ge1$, then

$$|\hat{\nabla}^k h|^2_{\hat{g}}\le C_k,$$

on $B_{\hat{g}}(x,1/2)\times[0,T)$ for some constant C_k depending only on k, c_k, n, T and N.

Proof. Since $\hat{g}$ has bounded geometry of infinite order, by [TY], for any $x\in M$ there exists a local biholomorphism $\phi_x:D\to M$, where $D=D(1)$ is the open unit ball in $\mathbb{C}^n$, satisfying the following in D:

(a) $\phi_x(0)=x$, $\phi_x(D)\subset\hat{B}(x,1)$, $\phi_x(D)\supset\hat{B}(x,2\delta)$ for some $\delta>0$ which is independent of x.

(b) $C^{-1}\delta_{ij}\le(\phi_x^*(\hat{g}))_{i\bar{j}}\le C\delta_{ij}$ for some C independent of x.

(c) $\left|\dfrac{\partial^l(\phi_x^*(\hat{g}))_{i\bar{j}}}{\partial z^L}\right|\le C_l$ for any l,i,j and multi index L of length l for some constant C_l which is independent of x.

Consider $\phi_x^*(h(t))$, which clearly will solve (1.2) on $D(1) \times [0, T)$. By the Evans-Krylov theory [Ev, Kr] for fully non-linear elliptic and parabolic equations (see also [SW] for a maximum principle proof in the case of Kähler Ricci flow), the result follows. $\square$

We end this section with the following longtime existence Theorem from [CT] when we look at certain special solutions to (1.2) on $\mathbb{C}^n$ in Theorem 5.5.

THEOREM 2.3. *Let (M, g_0) be a complete non-compact Kähler manifold such that*

i) $|\mathrm{Rm}(x)| \to 0$ as $d(x) \to \infty$ where $d(x)$ is the distance function on M from some $p \in M$.

ii) The injectivity radius of (M, g_0) is uniformly bounded below by some constant $c > 0$.

iii) There exists a strictly pluri-subharmonic function F on M.

Then the Kähler-Ricci flow (1.2) has a complete smooth solution $g(t)$ on $M \times [0, \infty)$ with $g(0) = g_0$. Moreover, the curvature of $g(t)$ is bounded uniformly on $M \times [0, T]$ for all $T < \infty$.

3. Further estimates. In this section we prove some more estimates which we will need later. One main tool is Theorem 2.2. Hence we want to obtain C^0 estimate for solutions of Kähler-Ricci flow in terms of a background metric.

Recall that the holomorphic bisectional curvature of a Kähler manifold is said to be bounded above by K if

$$\frac{R(X, \bar{X}, Y, \bar{Y})}{\|X\|^2\|Y\|^2 + |\langle X, \bar{Y}\rangle|^2} \leq K \tag{3.1}$$

for any two nonzero (1,0)-vectors X, Y. The holomorphic bisectional curvature of a Kähler manifold being bounded below by K is defined similarly.

In the following, $\hat{\nabla}$ always denotes the covariant derivative of $\hat{g}$.

LEMMA 3.1. *Let $h(t)$ be a solution to (1.2) on $M^n \times [0, T_0)$ with $h(0) = h_0$ such that $h(t)$ has uniformly bounded curvature for on $M \times [0, T']$ for all $0 < T' < T_0$. Let $\hat{g}$ be another complete Kähler metric on M with bounded curvature such that the holomorphic bisectional curvature bounded above by $K \geq 0$. Let $T = \frac{1}{2nK}$ if $K > 0$, otherwise let $T = \infty$.*

(i) Suppose $h_0 \geq \hat{g}$. Then $h(t) \geq \left(\frac{1}{n} - 2Kt\right)\hat{g}$ on $M \times [0, \min\{T_0, T\})$.

(ii) Suppose in addition to (i) we have $h_0 \leq C\hat{g}$, that is, suppose $\hat{g} \leq h_0 \leq C\hat{g}$, then

$$(1 - w(t))\hat{g} \leq h(t) \leq (1 + w(t))\hat{g}$$

on $M \times [0, \min\{T_0, T\})$, where $w(t) = \sqrt{v_2(t)(v_1(t) + v_2(t) - 2n)}$, $v_1(t) = \frac{1}{\frac{1}{n} - 2Kt}$, $v_2(t) = nCe^{-2\kappa v_1(t)t}$ and κ is a lower bound on the bisectional curvature of $\hat{g}$. In particular, we have $w(0) = n\sqrt{C(C-1)}$.

Proof. (i) Let $\phi(t) := \mathrm{tr}_{h(t)}\hat{g}$. Let $\square = \frac{\partial}{\partial t} - \Delta$, where Δ is the Laplacian with respect to $h(t)$. Then as in [ST], we can calculate in a normal coordinate relative to $h(t)$ and use (1.2) to get

$$\begin{aligned} \square\phi =& ((h)^{i\bar{j}}\hat{g}_{i\bar{j}}) - h^{k\bar{l}}(h^{i\bar{j}}\hat{g}_{i\bar{j}})_{k\bar{l}} \\ =& (R^{i\bar{j}}\hat{g}_{i\bar{j}}) - (R^{i\bar{j}}\hat{g}_{i\bar{j}}) + h^{k\bar{l}}h^{i\bar{j}}\widehat{R}_{i\bar{j}k\bar{l}} - \widehat{g}^{p\bar{q}}h^{k\bar{l}}h^{i\bar{j}}\partial_k\widehat{g}_{i\bar{q}}\partial_{\bar{l}}\widehat{g}_{p\bar{j}} \\ \leq& 2K\phi^2. \end{aligned} \tag{3.2}$$

Now $v_1(t)$ is the positive solution to the ODE

$$\frac{dv_1(t)}{dt} = 2Kv_1^2(t); \;\; v_1(0) = n$$

for $t \in [0, T)$. Let $S \in (0, \min\{T_0, T\})$ be fixed. Since $h(t)$ has uniformly bounded curvature on $M \times [0, S]$ we have $h(t) \geq C_1 h_0 \geq C_1 \hat{g}$ for some $C_1 > 0$ and hence ϕ is a bounded function on $M \times [0, S]$. Moreover, v_1 is also a bounded function on $M \times [0, S]$. Let $A = \sup_{M\times[0,S]}(\phi + v_1)$. Then on $M \times [0, S]$

$$\begin{aligned}
&\Box \left(e^{-(2AK+1)t}(\phi - v_1)\right) \\
&\qquad\qquad \leq e^{-(2AK+1)t} \left[2K(\phi^2 - v_1^2) - (2AK+1)(\phi - v_1)\right] \\
&\qquad\qquad = e^{-(2AK+1)t} \;\left[2K(\phi + v_1) - (2AK+1)\right](\phi - v_1)
\end{aligned}$$

which is nonpositive at the points where $\phi - v_1 \geq 0$. Using the fact that $h(t)$ has uniformly bounded curvature on $M\times[0, S]$ and the fact that $e^{-(2AK+1)t}(\phi - v_1) \leq 0$ at $t = 0$, and is uniformly bounded on $M\times[0, S]$, we conclude that $e^{-(2AK+1)t}(\phi - v_1) \leq 0$ and thus $(\phi - v_1) \leq 0$ on $M \times [0, S]$ by the maximum principle, see [NT, Theorem 1.2] for example. This proves (i).

(ii) Let $\psi(t) := \mathrm{tr}_{\hat{g}} h(t)$. For any fixed $S \in [0, min\{T_0, T\})$, as in [C] we calculate in a normal coordinate relative to $\hat{g}$ and use (1.2) to get that on $M \times [0, S)$:

$$\begin{aligned}
\Box\psi =& (\hat{g}^{i\bar{j}}(h_t)_{i\bar{j}}) - h^{k\bar{l}}(\hat{g}^{i\bar{j}}h_{i\bar{j}})_{k\bar{l}} \\
=& -(\hat{g}^{i\bar{j}}R_{i\bar{j}}) - h^{k\bar{l}}(\hat{R}^{i\bar{j}}_{k\bar{l}}h_{i\bar{j}}) + (\hat{g}^{i\bar{j}}R_{i\bar{j}}) - \widehat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}}h_{k\bar{q}} \\
=& -h^{k\bar{l}}h_{i\bar{j}}\widehat{R}_{k\bar{l}}^{\;\bar{j}i} - \widehat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}}h_{k\bar{q}} \\
\leq& -2\kappa v_1(t)\psi \\
\leq& -2\kappa v_1(S)\psi
\end{aligned} \tag{3.3}$$

by (i). Let $w_S(t) = nCe^{-2cv_1(S)t}$ be the solution to the ODE

$$\frac{dw_S(t)}{dt} = -2cv_1(S)w_S(t); \;\; w_S(0) = nC.$$

Then arguing as above, we have $\psi \leq w_S$ on $M^n \times [0, S]$. In particular, we get $\psi(S) \leq w_S(S)$ for every $S \in [0, min\{T_0, T\})$.

So far, we have $\phi(t) \leq v_1(t)$, and $\psi(t) \leq v_2(t)$ on $M\times[0, min\{T_0, T\})$ where v_1, v_2 are as in the statement of the Lemma. Now we follow an idea from [S1]. At any point in $(p, t) \in M \times [0, min\{T_0, T\})$, let $\lambda_i's$ be the eigenvalues of h with respect to $\hat{g}$, and calculate at (p, t)

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\lambda_i}(1 - \lambda_i)^2 =& \sum_{i=1}^{n} \frac{1}{\lambda_i} + \lambda_i - 2 \\
=& \phi + \psi - 2n \\
\leq& v_1(t) + v_2(t) - 2n
\end{aligned} \tag{3.4}$$

and thus for any fixed i we have

$$-w(t) \leq \lambda_i - 1 \leq w(t) \tag{3.5}$$

where $w(t) = \sqrt{v_2(t)(v_1(t)+v_2(t)-2n)}$. The conclusion in (ii) then follows. □

The following lemma basically says that if a local solution $h(t)$ to (1.2) is a priori uniformly equivalent to a fixed metric $\hat{g}$ in space time, and close to $\hat{g}$ at time $t=0$, then it remains close to $\hat{g}$ in a uniform space time region. Note that in contrast to Lemma 3.1, the assumption here is on $h(t)$ for all t.

Lemma 3.2. *Let $h(t)$ be a smooth solution to* (1.2) *on $B(1)\times[0,T)$ with $h(0)=h_0$ where $B(1)$ is the unit Euclidean ball in $\mathbb{C}^n$. Let $\hat{g}$ be a smooth Kähler metric on $B(1)$. Suppose*

$$N^{-1}\hat{g} \le h(t) \le N\hat{g} \tag{3.6}$$

on $B(1)\times[0,T)$ for some $N>0$, and that

$$\hat{g} \le h_0 \le C\hat{g} \tag{3.7}$$

on $B(1)$. Then there exists a positive continuous function $a(t):[0,T)\to\mathbb{R}$ depending only on $\hat{g}, N, C$ and n such that

$$\frac{(1-a(t))}{C}h_0 \le h \le (1+a(t))h_0 \tag{3.8}$$

on $B(1/2)\times[0,T)$, where $a(0)=n\sqrt{C(C-1)}$.

Proof. As in the previous Lemma, let $\phi = \mathrm{tr}_h\hat{g}$, $\psi = \mathrm{tr}_{\hat{g}}h$ on $B(1)\times[0,T_0)$. Choose some smooth non-negative cutoff function on $\eta: B(1)\to\mathbb{R}$ satisfying $\eta|_{B(1/2)} = 1$, $\eta|_{(B(3/4))^c} = 0$, $|\hat{\nabla}\eta|^2 \le C_1\eta$, $|\partial\bar{\partial}\eta|_{\hat{g}} \le C_2$ on $B(1)$ for some constants C_1, C_2 depending only on $\hat{g}$. Using the fact that $h(t)\ge N^{-1}\hat{g}$, we have

$$|\nabla\eta|^2 = h^{i\bar{j}}\eta_i\eta_{\bar{j}} \le N|\hat{\nabla}\eta|^2 \le NC_1,$$

and

$$|\Delta\eta| = \left|h^{i\bar{j}}\eta_{i\bar{j}}\right| \le N|\partial\bar{\partial}\eta|_{\hat{g}} \le NC_2.$$

Now we consider the function $\eta\phi$ on $B(1)\times[0,T)$. Then in $B(1)\times[0,T)$ at the point where $\eta>0$, as in the proof of Lemma 3.1 (i) we obtain

$$\begin{aligned}(\partial_t-\Delta)(\eta\phi) &= \eta(\partial_t-\Delta)\phi - 2<\nabla\eta,\nabla\phi> -\phi\Delta\eta\\ &= \eta(\partial_t-\Delta)\phi - 2\frac{<\nabla\eta,\nabla(\eta\phi)>}{\eta} + \frac{2|\nabla\eta|^2}{\eta}\phi - \phi\Delta\eta\\ &\le \eta C_3\phi^2 - 2\frac{<\nabla\eta,\nabla(\eta\phi)>}{\eta} + 2NC_1\phi + NC_2\phi\\ &\le C_4 - 2\frac{<\nabla\eta,\nabla(\eta\phi)>}{\eta}\end{aligned} \tag{3.9}$$

for some constants C_3, C_4 depending only on $\hat{g}, N, C$ and n, where we have used the assumption (3.8). Since $\eta\phi$ is zero outside $B(3/4)$, applying the maximum principle to $\eta\phi - C_4t$ one can conclude that

$$\eta\phi \le n + C_4t =: \tilde{v_1}(t)$$

on $B(1)\times[0,T)$.

Now consider the function $\eta\psi$ on $B(1)\times[0,T)$. Using the proof of Lemma 3.1 (ii) and estimating as above we obtain

$$\eta\psi \le nC + C_5 t =: \tilde{v_2}(t) \tag{3.10}$$

on $B(1)\times[0,T)$ for some constant C_5 depending only on $\hat{g}, N, C$ and n.

Now at any point in $(p,t)\in B(1/2)\times[0,T)$, let $\lambda_i's$ be the eigenvalues of h with respect to $\hat{g}$. Then as in the proof of Lemma 3.1 (ii) we get that at (p,t)

$$-\tilde{w}(t) \le \lambda_i - 1 \le \tilde{w}(t) \tag{3.11}$$

where $\tilde{w}(t) = \sqrt{\tilde{v_2}(t)(\tilde{v_1}(t)+\tilde{v_2}(t)-2n)}$. Since $\tilde{v_1}(0) = n$ and $\tilde{v_2}(0) = nC$, the lemma follows easily from this. □

In contrast to the previous lemma, in the following lemmas we only assume a lower bound on a solution $h(x,t)$ to (1.2).

Lemma 3.3. *Let $h(x,t)$ be a smooth solution to* (1.2) *on $M\times[0,T)$ with $h(0)=h_0$. Let $p\in M$. Suppose there is a positive continuous function $\alpha(t):[0,T)\to\mathbb{R}$ such that*

$$h(t)\ge \alpha(t)\hat{g}.$$

where $\hat{g}$ is a complete Kähler metric with bounded curvature. Then, there exists a positive continuous function $\beta(r,t):[1,\infty)\times[0,T)\to\mathbb{R}$ $\beta(r,t)$ depending only on $\hat{g}$ the upper bound of $\mathrm{tr}_{\hat{g}}h_0$ in $B_{\hat{g}}(p,2r)$, the lower bound of scalar curvature $R(0)$ of $h(0)$ in $B_{\hat{g}}(p,2r)$, $\alpha(t)$ and the dimension n such that for $r\ge 1$

$$h(t)\le \beta(r,t)\hat{g}.$$

in $B_{\hat{g}}(p,r)\times[0,T)$.

Proof. Let $d(x)$ be the distance with respect to $\hat{g}$ from x to a fixed point $p\in M$. Since $\hat{g}$ has bounded curvature, by [S2] (see also [T]) there exists a smooth positive function $\rho(x)$ satisfying $d(x)+1\le\rho(x)\le d(x)+C$ on M for some $C>0$, with $|\hat{\nabla}\rho|$, $|\hat{\nabla}^2\rho|$ are bounded on M. Hence without loss of generality, we may assume for simplicity that $d(x)$ is in fact smooth with $|\hat{\nabla}d|, |\hat{\nabla}^2 d|$ bounded on M.

Let $\phi(s)$ be smooth function on $\mathbb{R}$ such that $\phi=1$ for $s\le 1$ and is zero for $s\ge 2$. Moreover, we assume $\phi'\le 0$, $(\phi')^2/\phi\le C_1$, $|\phi''|\le C_2$. Let R be the scalar curvature of $h(t)$. Then

$$\left(\frac{\partial}{\partial t}-\Delta\right)R \ge \frac{1}{n}R^2. \tag{3.12}$$

on $M\times[0,T)$. Let $\varphi(x)=\phi(d(x)/r)$. Then $\varphi(x)=0$ if $d(x)\ge 2r$. Fix some $T'<T$. Then as in the proof of the previous lemma, we compute

$$\begin{aligned}
|\nabla\varphi|^2 =& \frac{1}{r^2}(\phi')^2|\nabla d|^2\\
=& \frac{1}{r^2}(\phi')^2 h^{i\bar{j}} d_i d_{\bar{j}}\\
\le& \frac{1}{r^2\alpha(t)}(\phi')^2 \hat{g}^{i\bar{j}} d_i d_{\bar{j}}\\
\le& \frac{C_3}{r^2}(\phi')^2
\end{aligned}$$

on $B(2r) \times [0, T']$ for some constant C_3 depending only on $T', \alpha(t)$ and $\hat{g}$. Similarly,

$$\begin{aligned}|\Delta\varphi| =& |\frac{1}{r}\phi'\Delta d + \frac{1}{r^2}\phi''|\nabla d|^2| \\ \leq & C_4(\frac{1}{r} + \frac{1}{r^2})\end{aligned}$$

on $B(2r) \times [0, T']$ where C_4 depends on $C_1, C_2, T', \alpha(t)$ and $\hat{g}$.

Now

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(\varphi R) =& \varphi\left(\frac{\partial}{\partial t} - \Delta\right) R - R\Delta\varphi - 2\langle \nabla R, \nabla\varphi\rangle \\ \geq & \frac{1}{n}\varphi R^2 - C_5|R| - 2\langle\nabla R, \nabla\varphi\rangle\end{aligned} \tag{3.13}$$

on $B(2r) \times [0, T']$ where C_5 depends only on C_4 and r. Suppose the infimum of φR on $B(2r) \times [0, T']$ is attained at $t = 0$, then $R \geq \min\{0, \inf_{B_{\hat{g}}(p,2r)} R(h_0)\}$ on $B_{\hat{g}}(r)$. Suppose instead that φR attains a negative minimum at some $(x, t) \in B(2r) \times [0, T']$ where $t > 0$. Then at (x, t), $\nabla R = -\frac{R\nabla\varphi}{\phi}$. Hence at this point,

$$0 \geq \frac{1}{n}\varphi R^2 - C_6|R| \tag{3.14}$$

where C_6 depends only on C_6, C_3 and r. Hence

$$\varphi^2|R| \leq nC_6.$$

on $B(2r) \times [0, T']$ and we conclude that $R \geq -C_7$ on $B_{\hat{g}}(p, r) \times [0, T']$ for some C_7 depending only on $T', \hat{g}, r, \alpha(t)$. On the other hand,

$$\frac{\partial}{\partial t} \log\left(\frac{\det(h_{\alpha\bar{\beta}})(t)}{\det(h_{\alpha\bar{\beta}}(0))}\right) = -R \leq C_7.$$

So

$$\frac{\det(h_{\alpha\bar{\beta}})(t)}{\det(\hat{g}_{\alpha\bar{\beta}})} \leq e^{C_7 t}\frac{\det(h_{\alpha\bar{\beta}})(0)}{\det(\hat{g}_{\alpha\bar{\beta}})}$$

on $B_{\hat{g}}(p, r) \times [0, T']$. Let λ_i be eigenvalues of $h(t)$ with respect to $\hat{g}$. By assumption, $\lambda_i(x, T') \geq \alpha(T')$ for each i and $x \in B_{\hat{g}}(p, r)$, the above inequality then implies $\lambda_i(x, T') \leq \beta(r, T')$ for some $\beta(r, T')$ depending only on the those constants listed in the Lemma. Moreover, it is not hard to see that $\beta(r, T')$ can be chosen to depend continuously on r, T' as $\alpha(t)$ is continuous. The Lemma follows as $T' < T$ is arbitrary. □

Remark 2. Given only a local solution $h(t)$ to (1.2) on $B(1) \times [0, T)$ where $B(1)$ is the unit ball on $\mathbb{C}^n$, it is not hard to see from its proof that the conclusion of Lemma 3.3 will hold in $B(r) \times [0, T)$ for all $r \leq 1/2$.

4. Kähler Ricci flow: general existence Theorems. We are now ready to state and prove our main existence Theorems for (1.2) using the estimates in the previous section. Theorems 4.1 and 4.2 provide general existence Theorems for (1.2) when the initial Kähler metric is realized as a limit of a sequence of Kähler metrics satisfying certain properties. The curvature of the initial metric may be unbounded or even undefined. As an application, In Corollary 4.1 we obtain an existence result for Hermitian continuous metrics which are perturbations of a complete Kähler metric with bounded curvature. In Corollary 4.2 we apply the above Theorems to provide an estimate for the maximal existence time for (1.2) assuming that the curvature of the initial Kähler metric is bounded. In some cases, we have long time existence.

In the following, we say that a sequence of smooth metrics h_k converge smoothly to a metric g on a set U, if h_k converge to g in C^∞ norm on U.

Theorem 4.1. *Let g_0 be a complete continuous Hermitian metric on a noncompact complex manifold M^n. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on M converging uniformly on compact subsets to g_0 and there exists another complete Kähler metric $\hat g$ on M with bounded curvature and with holomorphic bisectional curvature bounded from above by $K \geq 0$ satisfying:*

(i) $h_{k,0} \geq \hat g$ for all k;

(ii) for every k, Kähler-Ricci flow (1.2) has smooth solution $h_k(t)$ with initial value $h_{k,0}$ on $M \times [0, T')$ for some $T' > 0$ independent of k such that the curvature of $h_k(t)$ is uniformly bounded on $M \times [0, T_1]$ for all $0 < T_1 < T'$;

(iii) for some fixed point $p \in M$, for any $r > 0$ there exists a constant $C_r > 0$ such that the scalar curvature R_k of $h_{k,0}$ satisfies $R_k \geq -C_r$ on $B_{\hat g}(p, r)$ for all k.

Let $T = \min\{T', \frac{1}{2nK}\}$ if $K > 0$, otherwise let $T = T'$. Then the Kähler-Ricci flow (1.2) has a complete smooth solution $g(t)$ on $M \times (0, T)$ which extends continuously to $M \times [0, T)$ with $g(0) = g_0$ and satisfies $g(t) \geq (1/n - 2Kt)\hat g$ on $M \times (0, T)$.

Moreover, if g_0 is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M, then $g(t)$ extends to a smooth solution to (1.2) on $M \times [0, T)$ with $g(0) = g_0$.

Proof. By Lemma 3.1, we have

$$h_k(t) \geq \left(\frac{1}{n} - 2Kt\right)\hat g \tag{4.1}$$

as long as $t < T_0 = 1/(2nK)$. By Theorem 2.1, let $\hat g(t)$ be the solution Kähler-Ricci flow in the theorem with initial condition $\hat g$. Then for any $1 > \epsilon > 0$ small, choose $0 < t_0$ small enough so that $(1-\epsilon)\hat g(t_0) \leq \hat g \leq (1+\epsilon)\hat g(t_0)$. Then we have

$$h_k(t) \geq \left(\frac{1}{n} - 2Kt\right)(1-\epsilon)\hat g(t_0) \tag{4.2}$$

and $\hat g(t_0)$ has bounded geometry of infinite order. By Lemma 3.3, for there is a positive continuous function $\beta(r, t) : [1, \infty) \times [0, T_0) \to \mathbb{R}$ such that for $r \geq 1$

$$h_k(t) \leq \beta(r, t)\hat g(t_0) \tag{4.3}$$

in $\hat B(p, r) \times [0, T)$ where $T = \min\{T', \frac{1}{2nK}\}$ and $p \in M$ is a fixed point. We conclude from Theorem 2.2 (i), that passing to some subsequence, the $h_k(t)$'s converge to a solution $g(t)$ of Kähler-Ricci on $M \times (0, T)$ so that (4.1) is true with $h_k(t)$ replaced with $g(t)$. Moreover, if g_0 is smooth and $\{h_k\}$ converges smoothly and uniformly to g_0 on compact sets, then we see from Theorem 2.2 (ii) that in fact $g(t)$ extends to a smooth solution on $M \times [0, T)$ such that $g(0) = g_0$.

We now prove $g(t)$ converge uniformly on compact set to g_0 as $t \to 0$ when g_0 is only assumed to be continuous. Fix any $x \in M$ and a local biholomorphism

$\phi : B(1) \to M$ where $B(1)$ is the open unit ball in $\mathbb{C}^n$ with image inside $B_{\hat{g}}(x,1)$, and $\phi(0) = x$. Consider the pullbacks $\phi^* h_k(t)$, $\phi^* h_{k,0} = \phi^* h_k(0)$, $\phi^* \hat{g}$, which by abuse of notation we will simply denote by $h_k(t)$, $h_{k,0}$, $\hat{g}$, respectively, for the remainder of proof. In particular, $h_k(t)$ solves Kähler-Ricci flow (1.2) on $B(1) \times [0,T)$.

Now by our hypothesis on the convergence of h_k, given any $\delta > 0$ we may find k_0 such that $|h_{k_0,0} - g_0|_{\hat{g}} \le \delta$ and

$$(1-\delta)h_{k_0,0} \le h_{k,0} \le (1+\delta)h_{k_0,0} \tag{4.4}$$

for all $k \ge k_0$ on $B(1)$. On the other hand, by (4.2) and (4.3) we can find $N > 0$ such that

$$N^{-1}h_{k_0,0} \le h_k(t) \le N h_{k_0,0} \tag{4.5}$$

in $B(1) \times [0, T/2)$ for all $k \ge k_0$. Then by Lemma 3.2, there exists a continuous function $a(t)$ depending on N, h_{k_0} and δ such that

$$(1-a(t))\frac{(1-\delta)^2}{(1+\delta)}h_{k_0,0} \le h_k(t) \le (1+a(t))(1+\delta)h_{k_0,0}$$

in $B(\frac{1}{2}) \times [0, T/2)$ with $a(0) = n\sqrt{C(C-1)}$, with $C = (1+\delta)/(1-\delta)$. Note that $a(t)$ is independent of k. Letting $k \to \infty$ gives

$$(1-a(t))\frac{(1-\delta)^2}{(1+\delta)}h_{k_0,0} \le g(t) \le (1+a(t))(1+\delta)h_{k_0,0} \tag{4.6}$$

in $B(\frac{1}{2}) \times (0, T/2)$. We then get

$$\begin{aligned}
&\limsup_{t\to 0} |g(t) - g_0|_{\hat{g}} \\
&\quad \le \limsup_{t\to 0} \left(|g(t) - h_{k_0,0}|_{\hat{g}} + |h_{k_0,0} - g_0|_{\hat{g}}\right) \\
&\quad \le \left[\left|1 - (1-a(0))\frac{(1-\delta)^2}{(1+\delta)}\right| + |(1+a(0))(1+\delta) - 1|\right] |h_{k_0,0}|_{\hat{g}} \\
&\qquad + \delta |h_{k_0,0}|_{\hat{g}}
\end{aligned}$$

uniformly on $B(\frac{1}{2})$. Then letting $\delta \to 0$ above, and using the fact that $a(0) \to 0$ as $\delta \to 0$, and (4.2) and (4.3) we conclude that

$$\limsup_{t\to 0} |g(t) - g_0|_{\hat{g}} = 0$$

uniformly on $B(\frac{1}{2})$. Hence $g(t)$ converge to g_0 uniformly on compact sets as $t \to 0$. □

We do not have any bound on the curvature of solution $g(t)$ in the previous theorem. Also in the previous theorem, we assume that the Kähler-Ricci flow (1.2) has solution with initial condition $h_{k,0}$ on a fixed time interval independent of k. We want to remove this assumption and obtain curvature bound for the solutions. In order to do this, we assume $h_{k,0}$ also has an uniform upper bound.

Theorem 4.2. *Let g_0 be a complete continuous Hermitian metric on a noncompact complex manifold M^n. Suppose there exists a sequence $\{h_{k,0}\}$ of smooth complete Kähler metrics with bounded curvature on M converging uniformly on compact subsets to g_0 and there exists another complete Kähler metric $\hat{g}$ on M with bounded curvature and holomorphic sectional curvature bounded from above by $K \ge 0$ such that* *(i)* $C^{-1}\hat{g} \le h_{k,0} \le C\hat{g}$ *for some* C *independent of* k;

(ii) h_k *has bounded curvature for every* k.

Let $T = 1/(2CnK)$ *if* $K > 0$, *otherwise let* $T = \infty$. *Then the Kähler-Ricci flow* (1.2) *has a smooth solution* $g(t)$ *on* $M \times (0, T)$ *such that*

(a) $(1/(nC) - 2Kt)\hat{g} \le g(t) \le B(t)\hat{g}$ *on* $M \times (0, T)$ *for some positive continuous function* $B(t)$ *depending only on* C, $\hat{g}$ *and* n.

(b) $g(t)$ *has bounded curvature for* $t > 0$. *More precisely, for any* $0 < T' < T$ *and for any* $l \ge 0$ *there exists a constant* C_l *depending only on* C, l, T', $\hat{g}$ *and the dimension* n *such that*

$$sup_M |\nabla^l \mathrm{Rm}(g(t))|^2_{g(t)} \le \frac{C_l}{t^{l+2}},$$

(c) $g(t)$ *converges uniformly on compact subsets to* g_0 *as* $t \to 0$.

Moreover, if g_0 *is smooth and* $\{h_{k,0}\}$ *converges smoothly and uniformly on compact subsets of* M, *then* $g(t)$ *extends to a smooth solution on* $M \times [0, T)$ *with* $g(0) = g_0$.

Proof. For each k, let $h_k(t)$ be the solution to (1.2) with initial condition h_k from Theorem 2.1 which is defined on $M \times [0, T_k)$ for some $T_k > 0$. We first claim that we can choose T_k such that $T_k \ge T$ for some positive T for all k. By Lemma 3.1, there is a positive continuous function $B(t) : [0, T) \to \mathbb{R}$ independent of k such that

$$(1/n - 2CKt)\hat{g} \le h_k(t) \le B(t)\hat{g}$$

in $M \times [0, \min\{T_k, T\})$ where $T = 1/(2nCK)$. As before, we may assume that $\hat{g}$ has bounded geometry of infinite order. By Theorem 2.2, we conclude that if $T_k < T$, then $|\mathrm{Rm}(h_k(t))|_{h_k(t)}$ are bounded in $M \times [0, T_k)$. By Theorem 2.1, we see that one can extend $h_k(t)$ so that $T_k \ge T$ for all k as claimed. Given upper and lower bounds on $h_k(t)$ as above, we may conclude from Theorem 2.2, as in the proof of Theorem 4.1, that there is a smooth solution to the Kähler-Ricci flow $g(t)$ on $M \times (0, T)$ satisfying condition (a) and (c) from which we conclude, by Theorem 2.2 (i), that condition (b) is also satisfied. □

Corollary 4.1. *Let* $(M^n, \hat{g})$ *be a complete Kähler manifold with bounded curvature. Suppose* u *is real* C^2 *function on* M *such that* $|\hat{\nabla} u| + |u| = o(r)$ *and*

$$A^{-1}\hat{g} \le \hat{g} + \sqrt{-1}\partial\bar{\partial}u \le A\hat{g}$$

for some $A > 1$, *where* $\hat{\nabla}$ *is the covariant derivative with respect to* $\hat{g}$. *Then for some* $T > 0$, *the Kähler-Ricci flow* (1.2) *has a complete solution* $g(t)$ *on* $M \times [0, T)$ *with* $g(0) = g_0 + \sqrt{-1}\partial\bar{\partial}u$ *and satisfying the conclusion of Theorem 4.2.*

(Here $f = o(r^k)$ represents a positive function on M such that $f(x)/d_p^k(x)$ approaches 0 as $d_p(x) \to \infty$ where $d(x)$ is the distance function from some fixed point in M relative to $\hat{g}$).

Proof. Let $d(x)$ be the distance with respect to $\hat{g}$ from x to a fixed point $p \in M$. As in the proof of Lemma 3.3, we may assume without loss of generality that $d(x)$ is smooth with $|\hat{\nabla} d|, |\hat{\nabla}^2 d|$ bounded on M. Let ϕ be a smooth function on $\mathbb{R}$ such that $0 \le \phi \le 1$, $\phi(s) = 1$ for $s \le 1$ and $\phi(s) = 0$ for $s \ge 2$, and $|\phi'| + |\phi''| \le c_1$ for some c_1. For any $k \ge 1$, let $\eta_k(x) = \phi(d(x)/k)$. Then $|\hat{\nabla}\eta_k|, |\hat{\nabla}^2\eta_k| \le c_2/k$ on M for some constant c_2 independent of k. Now let $\{u_k\}$ be a sequence of smooth functions on M which converging to u, uniformly on compact subsets of M in the C^2 norm. For each

k we have

$$\begin{aligned}\partial\bar\partial(\eta_k u_j) =&\eta_k\partial\bar\partial u_j + u_j\partial\bar\partial\eta_k + \partial u_j\wedge\bar\partial\eta_k + \partial\eta_k\wedge\bar\partial u_j\\ &\to \eta_k\partial\bar\partial u + u\partial\bar\partial\eta_k + \partial u\wedge\bar\partial\eta_k + \partial\eta_k\wedge\bar\partial u\end{aligned}\tag{4.7}$$

uniformly on M as $j\to\infty$. Since $\partial\eta_k$ and $\partial\bar\partial\eta_k$ vanish on $B(k)$ and outside $B(2k)$, and $|\hat\nabla u| + |u| = o(r)$, for any $\epsilon > 0$ we have $|\partial u_j \wedge \bar\partial\eta_k|_{\hat g} + |u_j\partial\bar\partial\eta_k|_{\hat g} \le \epsilon$ if j is large enough. Hence, for any k, we can find u_{j_k} with $j_k\to\infty$ as $k\to\infty$ such that $h_k = \hat g + \sqrt{-1}\partial\bar\partial(\eta_k u_{j_k})$ is a Kähler metric such that

$$(2A)^{-1}\hat g \le h_k \le 2A\hat g.$$

In particular, h_k is complete, outside a compact set $h_k = \hat g$ and thus has bounded curvature, and h_k converges to g_0 uniformly on compact sets in C^0. The corollary now follows from Theorem 4.2 □

Remark 3. Note that if $|\hat\nabla u| = o(1)$, then $|u| = o(r)$. This will imply that $|\hat\nabla u|+|u| = o(r)$. Also from the proof of the theorem, if $\hat g$ has bounded curvature and u is a smooth function on M such that $\partial\bar\partial u$, u and $\hat\nabla u$ are bounded, then for $\epsilon > 0$ small enough, the Kähler-Ricci flow with initial condition $\hat g + \epsilon\sqrt{-1}\partial\bar\partial u$ has a short time solution.

By Theorem 4.2, one may obtain estimates for maximal time interval of existence of the Kähler-Ricci flow constructed in Theorem 2.1.

Corollary 4.2. *Let M^n be a complex noncompact manifold and let g_0, $\hat g$ be complete Kähler metrics with bounded curvature on M. Suppose the holomorphic bisectional curvature of $\hat g$ is bounded above by $K \ge 0$ and that $\hat g \le g_0 \le C\hat g$ for some $C \ge 1$. Let $T = 1/2nK$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow (1.2) has a complete smooth solution $g(t)$ on $M \times [0, T)$ with $g(0) = g_0$ such that for all $t \in [0, T)$, $g(t)$ has bounded curvature and*

$$(1/n - 2Kt)\,\hat g \le g(t).\tag{4.8}$$

In particular, the Kähler-Ricci flow has a long time solution if the initial condition is a complete Kähler metric with non-positive and bounded holomorphic bisectional curvatures.

5. Kähler Ricci flow of U(n) invariant metrics on $\mathbb{C}^n$. In this section we apply Theorems 4.1, 4.2 to $U(n)$ invariant metrics on $\mathbb{C}^n$.

5.1. Wu-Zheng's construction. We recall Wu-Zheng's construction in [WZ] of smooth $U(n)$ invariant metrics on $\mathbb{C}^n$. Begin with a smooth function $\xi : [0,\infty) \to \mathbb{R}$ with $\xi(0) = 0$, and define functions $h, f : [0,\infty) \to \infty$ by

$$h(r) := He^{\int_0^r -\frac{\xi(t)}{t}dt};\quad f(r) := \frac{1}{r}\int_0^r h(t)dt\tag{5.1}$$

where $h(0) = H > 0$ and $f(0) = h(0)$.

Now define a $U(n)$ invariant metric g on $\mathbb{C}^n$ by

$$g_{i\bar j} = f(r)\delta_{ij} + f'(r)\overline{z_i}z_j,\tag{5.2}$$

where $g_{i\bar{j}}$ are the components of g in the standard coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$ and $r = |z|^2$. Notice that a different choice of $h(0)$ simply corresponds to scaling the metric g above. In the following, we always take $H = 1$, i.e. $h(0) = 1$. Wu-Zheng [WZ] proved:

THEOREM 5.1. *[Wu-Zheng]*
(i) The metric g above is complete if

$$\int_0^\infty \frac{\sqrt{h}}{\sqrt{t}} dt = \infty. \tag{5.3}$$

Conversely, up to scaling by a constant factor, every complete smooth $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ can be generated in this way.

(ii) At the point $z = (z_1, 0, \ldots, 0)$, relative to the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \ldots, \frac{1}{\sqrt{f}}\partial_{z_n}\}$ with respect to $g_{i\bar{j}}$, we have

(a) $A = R_{1\bar{1}1\bar{1}} = \dfrac{\xi'}{h}$,

(b) $B = R_{1\bar{1}i\bar{i}} = \dfrac{1}{(rf(r))^2} \displaystyle\int_0^r \xi'(t) \left(\int_0^t h(s) ds \right) dt$,

(c) $C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \dfrac{2}{(rf(r))^2} \displaystyle\int_0^r h(t)\xi(t) dt$,

where $2 \leq i \neq j \leq n$ and these are the only non-zero components of the curvature tensor at z except those obtained from A, B or C by the symmetric properties of R.

In this section, C always denotes the quantity in the above theorem. By the above construction, Wu-Zheng [WZ] proved the correspondence below for positively curved metrics, while Yang later showed in [Y] that this extends to a correspondence for non-negatively curved metrics.

THEOREM 5.2. *[Wu-Zheng, Yang] There is a one to one correspondence between the set of all smooth complete $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with non-negative holomorphic bisectional curvature (modulo scaling by a constant factor) and the set of all smooth functions $\xi : [0, \infty) \to \mathbb{R}$ satisfying*

$$\xi(0) = 0, \quad \xi' \geq 0, \quad \xi \leq 1. \tag{5.4}$$

REMARK 4. One direction of the above correspondence is immediately obvious from Theorem 5.1 (ii)a. In particular, it is obvious that if g has non-negative (non-positive) holomorphic bisectional curvature then $\xi' \geq 0$ ($\xi' \leq 0$).

5.2. Applications of Theorems 4.1 and 4.2 to $U(n)$ invariant metrics. We now apply Theorems 4.1 and 4.2 to $U(n)$ invariant metrics. First we have the following lemma.

LEMMA 5.1. *Let g be a complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ generated by ξ.*

(i) If $\left|\frac{\xi'}{h}\right|$ is uniformly bounded, then the curvature of g is uniformly bounded.

(ii) If $\lim_{r\to\infty} \left|\frac{\xi'(r)}{h(r)}\right| = 0$, and $\lim_{r\to\infty} rf(r) = \infty$ then the curvature of g approaches to zero as $r \to \infty$.

Proof. (i) It is sufficient to prove that the holomorphic bisectional curvature is uniformly bounded under the assumption that $\left|\frac{\xi'}{h}\right|$ is uniformly bounded by c, say. By Theorem 5.1, in the notations of the theorem it is sufficient to prove that $|A|, |B|, |C|$ are uniformly bounded. It is obviously $|A| \le c$. Now

$$\begin{aligned}|B| \le& \frac{1}{r^2 f^2}\int_0^r ch(t)dt\left(\int_0^t h(s)ds\right)dt\\ \le& \frac{c}{r^2 f^2}\left(\int_0^r h(t)dt\right)^2\\ =& c\end{aligned}$$

because $h > 0$ and $rf(r) = \int_0^r h(t)dt$. Similarly, since

$$|\xi(r)| \le \int_0^r |\xi'(t)|dt \le c\int_0^r h(t)dt,$$

we have

$$|C| \le 2c.$$

(ii) If $\lim_{r\to\infty}\left|\frac{\xi'(r)}{h(r)}\right| = 0$, then $\lim_{r\to\infty} A = 0$. On the other hand, for any $\epsilon > 0$, there is r_0 such that $\left|\frac{\xi'(r)}{h(r)}\right| \le \epsilon$ for $r \ge r_0$. Then

$$|B| \le \frac{1}{r^2 f^2}\int_0^{r_0} |\xi'|(t)\left(\int_0^t h(s)ds\right)dt + \epsilon.$$

Since $rf(r) \to \infty$ as $r \to \infty$, it is easy to see that $\lim_{r\to\infty}|B| = 0$. Also

$$|\xi|(r) \le \int_0^{r_0} |\xi'|(t)dt + \epsilon\int_{r_0}^r h(t)dt$$

if $r \ge r_0$. Hence

$$|C| \le \frac{1}{rf}\int_0^{r_0} |\xi'|(t)dt + \epsilon,$$

and one can conclude that $\lim_{r\to\infty}|C| = 0$. From these (ii) follows. □

LEMMA 5.2. *Let $\xi : [0,\infty)$ be a smooth function with $\xi(0) = 0$. Suppose $\xi(r) = a$ for some constant $a \le 1$ for all $r \ge r_0$. Then ξ generates a complete $U(n)$ invariant metric g such that the curvature of g approaches 0 as $x \to \infty$ on $\mathbb{C}^n$.*

Proof. For $r \ge r_0$,

$$\int_0^r \frac{\xi(t)}{t}dt = \int_0^{r_0}\frac{\xi(t)}{t}dt + a\log(\frac{r}{r_0}).$$

Hence $h(r) = c_1 r^{-a}$ for some constant $c_1 > 0$ for all $r \ge r_0$. Since $a \le 1$, it is easy to see that

$$\int_0^\infty \frac{\sqrt{h}(r)}{\sqrt{r}}dr = \infty.$$

Hence g is complete by Theorem 5.1. Also $\xi' = 0$ near infinity, and

$$rf(r) = \int_0^r h(t)dt \geq c_2 + c_3 \log r$$

for some constants c_2, c_3 with $c_3 > 0$ because $a \leq 1$. The result follows from Lemma 5.1. □

THEOREM 5.3. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ generated by a smooth function $\xi : [0, \infty) \to \mathbb{R}$ with $\xi(0) = 0$. Suppose there exists $\hat{\xi} : [0, \infty) \to \mathbb{R}$ with $\hat{\xi}(0) = 0$ which generates a smooth complete $U(n)$ invariant Kähler metric $\hat{g}$ with bounded curvature and holomorphic bisectional curvature bounded above by $K \geq 0$, such that for all $r \geq 0$*

$$\int_0^r \frac{\xi - \hat{\xi}}{t} dt \leq c$$

for some $c > 0$ independent of r. Let $T = 1/(2nKe^c)$ if $K > 0$, otherwise let $T = \infty$. Then the Kähler-Ricci flow (1.2) has a smooth complete $U(n)$ invariant solution $g(t)$ on $M \times [0, T)$ with $g(0) = g_0$.

Proof. As ξ and $\hat{\xi}$ are smooth, for each $k \geq 0$ there exists a $\delta_k > 0$ and a smooth "cutoff" function $\eta_k : (-\infty, \infty) \to \mathbb{R}$ satisfying

$$\eta_k(r) : \begin{cases} = 1 & \text{if } -\infty < r \leq k, \\ 0 < \eta_k(r) < 1 & \text{if } k < r < k + \delta_k, \\ = 0 & \text{if } k + \delta_k \leq r < \infty, \end{cases} \tag{5.5}$$

and

$$\int_k^{k+\delta_k} \left| \frac{(\xi - \hat{\xi})}{t} \right| dt \leq 1/k \tag{5.6}$$

for all k. Fix such a choice of $\eta_k' s$, and consider the sequence of functions $\{\xi_k\}$: $[0, \infty) \to \infty$ defined by

$$\xi_k(r) = \eta_k \xi + (1 - \eta_k)\hat{\xi}$$

and let ω_k be the corresponding sequence of smooth $U(n)$ invariant Kähler metrics. Then

$$\begin{aligned} \int_0^r \frac{\xi_k(t) - \hat{\xi}(t)}{t} dt &= \int_0^r \frac{\eta_k(\xi - \hat{\xi})}{t} dt \\ &= \begin{cases} \int_0^r \frac{\xi - \hat{\xi}}{t} dt, & \text{if } r \leq k; \\ \int_0^k \frac{\xi - \hat{\xi}}{t} dt + \alpha_k, & \text{if } r > k \end{cases} \end{aligned}$$

where

$$|\alpha_k| \leq \int_k^{k+\delta_k} \left| \frac{\xi - \hat{\xi}}{t} \right| dt \leq \frac{1}{k},$$

Hence

$$\int_0^r \frac{\xi_k(t)-\hat{\xi}(t)}{t}dt \le c+\frac{1}{k}.$$

where C is the constant in the hypothesis. This implies, by (5.1) and (5.2), that

$$\exp(-c-\frac{1}{k})\hat{g} \le \omega_k.$$

In particular, ω_k is complete. Also, from (5.1) and (5.2), we have

$$\omega_k \le c_k \hat{g}$$

where $c_k = \exp\left(\int_0^{k+\delta_k}\left|\frac{\xi-\hat{\xi}}{t}\right| dt\right)$. It is also easy to see that ω_k has bounded curvature for each k, and thus by Corollary 4.2, for each k there exists a solution $g_k(t)$ to (1.2) on $M\times[0,T_k)$ where $T_k = 1/(2nK\exp(c+\frac{1}{k}))$. By uniqueness [CZ], $g_k(t)$ is $U(n)$ invariant for all t. The result now follows from Theorem 4.1. ☐

By Theorem 5.3, we have

Corollary 5.1. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric g_0 on $\mathbb{C}^n$ generated by a smooth function $\xi:[0,\infty)\to\mathbb{R}$ with $\xi(0)=0$. If $\xi(r)\le 1$, then for some $T>0$ the Kähler-Ricci flow* (1.2) *has a complete $U(n)$ invariant smooth solution $g(t)$ on $\mathbb{C}^n\times[0,T)$ with $g(0)=g_0$. If in fact $\xi(r)\le 0$, in particular if $\xi'\le 0$, then the solution exists on $\mathbb{C}^n\times[0,\infty)$.*

Proof. Let $\hat{\xi}$ be a smooth function on $[0,\infty)$ with $\hat{\xi}(0)=0$ and $\hat{\xi}(r)=1$ for $r\ge 1$. Then $\hat{\xi}$ generates a complete $U(n)$ invariant Kähler metric with bounded curvature by Lemma 5.2. The first result follows from Theorem 5.3.

If $\xi\le 0$, then we can choose $\hat{\xi}=0$ which generates the standard Euclidean metric. The second result also follows from Theorem 5.3. ☐

We do not have any curvature bound on the solution in Theorem 5.3. In the next theorem, the solution also has some bounds on the curvature and its derivatives.

Theorem 5.4. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ generated by a smooth function $\xi:[0,\infty)\to\mathbb{R}$ with $\xi(0)=0$. Suppose there exist $\alpha\le 0$ and β such that for all $0<a<r$,*

$$\int_a^r \frac{(\alpha-\xi)}{t}dt,\quad \int_a^r \frac{(\xi-1)}{t}dt \le \beta. \tag{5.7}$$

Then for some $T>0$ the Kähler-Ricci flow (1.2) *has a complete smooth $U(n)$ invariant solution $g(t)$ on $\mathbb{C}^n\times[0,T)$ with $g(0)=g_0$. Moreover, for every $l\ge 0$ there exists a constant c_l depending only on α, β, l and n such that*

$$\sup_{p\in\mathbb{C}^n}\|\nabla^l \mathrm{Rm}(p,t)\|_t^2 \le \frac{c_l}{t^{l+2}} \tag{5.8}$$

on $\mathbb{C}^n\times(0,T)$.

If in addition,

$$\int_0^r \frac{\xi}{t} < \sigma \tag{5.9}$$

for some constant σ independent of r, then the above solution to Kähler-Ricci flow is defined on $\mathbb{C}^n \times [0, \infty)$ and satisfies (5.8) *on $\mathbb{C}^n \times (0, T')$ for some $T' > 0$.*

REMARK 5. In [CLT] it was proved that if $\beta < 1$ in Theorem 5.4 then $T = \infty$. While (5.8) holds on $\mathbb{C}^n \times (0, T')$ for some $T' > 0$, it is unclear if this is true for all T'.

COROLLARY 5.2. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric g_0 on $\mathbb{C}^n$ generated by a smooth function $\xi : [0, \infty) \to \mathbb{R}$ with $\xi(0) = 0$. If $\alpha \le \xi(r) \le 1$ for some $\alpha \le 0$, in particular if $\xi' \ge 0$ so that g_0 has nonnegative holomorphic bisectional curvature, then for some $T > 0$ the Kähler-Ricci flow* (1.2) *has a smooth solution on $\mathbb{C}^n \times [0, T)$ with $g(0) = g_0$ and satisfies* (5.8)*. Moreover, the solution $g(t)$ has nonnegative holomorphic bisectional curvature for $t \in (0, T)$.*

If in fact $c \le \xi \le 0$ for all r, then the solution exists for all time and satisfies (5.8) *on $\mathbb{C}^n \times (0, T')$ for some T' depending only on c and n.*

Proof. If $c \le \xi(r) \le 1$ (or $c \le \xi(r) \le 0$) for some c, then the conditions of Theorem 5.4 clearly hold. In case g_0 has non-negative holomorphic bisectional curvature, the fact that $g(t)$ has non-negative bisectional curvature for all $t \in [0, T)$ was proved in [YZ]. □

REMARK 6. Let $\hat{\xi} : [0, \infty) \to \mathbb{R}$ be smooth with $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1 + 1/\ln r$ for $r \ge 1$ say. Then from the proof of Proposition 5.1, it is not hard to see that the corresponding $\hat{g}$ is complete with bounded curvature. Now it is easy to construct a smooth function $\xi \ge \hat{\xi}$ satisfying the assumptions in Theorem 5.3, where the corresponding g is complete with unbounded curvature. Thus ξ satisfies the assumptions in Theorem 5.3, while it is also easy to see that ξ does not satisfy the assumptions in Theorem 5.4.

5.3. Proof of Theorem 5.4. By Theorem 4.2, Theorem 5.4 will follow once we produce a sequence ξ_k and function $\hat{\xi}$ such that the corresponding $U(n)$ invariant Kähler metrics h_k, g_0 and $\hat{g}$ satisfy the hypothesis of Theorem 4.2. We begin by proving the existence of such a function $\hat{\xi}$:

PROPOSITION 5.1. *Under assumptions* (5.7) *of Theorem 5.4 on ξ, there exists $\hat{\xi}$ such that the corresponding $U(n)$ invariant metric $\hat{g}$ has bounded curvature and*

$$c^{-1}\hat{g} \le g_0 \le c\hat{g} \tag{5.10}$$

on $\mathbb{C}^n$ for some constant $c > 0$. If in addition, (5.9) *is true, then $\hat{\xi}$ can be chosen to be nonpositive.*

Proof. Assume (5.7) is true. We consider three different cases.

Case 1: Suppose there is $c' > 0$ such that $\int_1^r \frac{\xi - 1}{t} dt \ge c'$ for all $r \ge 1$. Let $\hat{\xi}$ be a fixed smooth function on $[0, \infty)$ such that $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1$ for $r \ge 1$. Let $\hat{g}$ be the complete $U(n)$ invariant metric generated by $\hat{\xi}$. Then there is c'' such that for any $1 \ge r > 0$,

$$\left| \int_0^r \frac{\xi - \hat{\xi}}{t} dt \right| \le \left| \int_0^r \frac{\xi}{t} dt \right| + \left| \int_0^r \frac{\hat{\xi}}{t} dt \right| \le c''$$

for some c''. For $r \geq 1$

$$\int_1^r \frac{\xi - \hat{\xi}}{t} dt = \int_1^r \frac{\xi - 1}{t} dt \leq \beta$$

and

$$\int_1^r \frac{\xi - \hat{\xi}}{t} dt = \int_1^r \frac{\xi - 1}{t} dt \geq c'.$$

Hence by (5.1) and (5.2), the $U(n)$ invariant Kähler metric $\hat{g}$ generated by $\hat{\xi}$ satisfies the required conditions in the Proposition.

Case 2: Suppose there is $c' > 0$ such that $\int_1^r \frac{\alpha - \xi}{t} dt \geq c'$ for $r \geq 1$. Let $\hat{\xi}$ be a fixed smooth function on $[0, \infty)$ such that $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = \alpha$ for $r \geq 1$. Then as in the previous case, the $U(n)$ invariant Kähler metric $\hat{g}$ generated by $\hat{\xi}$ satisfies the required conditions in the Proposition. Note that in this case, $\hat{\xi}$ can be chosen to be nonpositive.

Case 3: Suppose as a function of r, $\int_1^r \frac{\xi - 1}{t} dt$ is not bounded from below and $\int_1^r \frac{\xi - \alpha}{t} dt$ is not bounded from above. We want to find $\hat{\xi}$ and $1 \leq a_0 < a_1 < a_2 \cdots \to \infty$ such that $\hat{\xi}$ generates a complete $U(n)$ metric $\hat{g}$ such that

$$\int_{a_{2i}}^{a_{2(i+1)}} \frac{\xi - \hat{\xi}}{t} dt = 0 \tag{5.11}$$

for all $i \geq 0$;

$$\left| \int_{a_{2i}}^{r} \frac{\xi - \hat{\xi}}{t} dt \right| \leq c_1 \tag{5.12}$$

for some c_1 for all $i \geq 0$ and for all $r \in [a_{2i}, a_{2(i+1)})$; and

$$\left| \frac{\hat{\xi}'(r)}{\hat{h}(r)} \right| \leq c_2 \tag{5.13}$$

for some c_2 for all $r \geq 0$. Then by Lemma 5.1, (5.1) and (5.2), we can conclude that $\hat{g}$ satisfies the conditions of the Proposition.

Fix a smooth function ρ on $\mathbb{R}$, such that

$$\rho(t) = \begin{cases} 1, & \text{if } t \leq 1 + \epsilon; \\ \alpha, & \text{if } t \geq 3 - \epsilon, \end{cases}$$

and $\rho' \leq 0$, where $\epsilon > 0$ is small enough so that $1 + \epsilon < 3 - \epsilon$. Then $\alpha \leq \rho \leq 1$.

Let $\hat{\xi}$ be a smooth function on $[0, 1]$ with $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1$ near $r = 1$ such that $0 \leq \hat{\xi} \leq 1$. We are going to find a_i and $\hat{\xi}(r)$ on $[a_i, a_{i+1}]$ inductively. Let $a_0 = 1$.

$$\int_{a_0}^{3a_0} \frac{\xi - \rho}{t} dt = \int_{a_0}^{3a_0} \frac{\xi(t) - 1 + 1 - \rho(t)}{t} dt \leq \beta + (1 - \alpha) \log 3.$$

Since $\int_{3a_0}^r \frac{\xi - \alpha}{t} dt$ is not bounded from above, there is a *first* $a_1 > 3a_0$ such that

$$\int_{a_0}^{3a_0} \frac{\xi - \rho}{t} dt + \int_{3a_0}^{a_1} \frac{\xi - \alpha}{t} dt = c_3$$

where $c_3 = \beta + (1-\alpha)\log 3 + 1$. On the other hand,

$$\int_{a_1}^{3a_1} \frac{\xi - (1+\alpha-\rho(\frac{t}{a_1}))}{t} dt \geq -\beta - (1-\alpha)\log 3.$$

Since $\int_{3a_1}^{r} \frac{\xi-1}{t} dt$ is not bounded from below, there exists a *first* $a_2 > 3a_1$, such that

$$\int_{a_1}^{3a_1} \frac{\xi - (1+\alpha-\rho(\frac{t}{a_1}))}{t} dt + \int_{3a_1}^{a_2} \frac{\xi - 1}{t} dt = -c_3.$$

Define

$$\hat{\xi}(r) = \begin{cases} \rho(r), & \text{if } a_0 \leq r \leq 3a_0; \\ \alpha, & \text{if } 3a_0 < r \leq a_1; \\ 1+\alpha-\rho(\frac{r}{a_1}), & \text{if } a_1 < r \leq 3a_1; \\ 1, & \text{if } 3a_1 < r \leq a_2. \end{cases}$$

It is easy to see that $\hat{\xi}$ is smooth on $[0, a_2]$ with $\hat{\xi}(r) = 1$ near a_2. Moreover, $\alpha \leq \hat{\xi} \leq 1$ on $[1, a_2]$, and

$$\int_{a_0}^{a_2} \frac{\xi - \hat{\xi}}{t} dt = 0,$$

so (5.11) is true for $i = 0$. It is easy to see that

$$|\xi'| \leq \frac{c_4}{r}$$

where $c_4 = 3\max|\rho'|$.

For $a_0 \leq r \leq a_1$, by the definition of a_1 we have

$$\int_{a_0}^{r} \frac{\xi - \hat{\xi}}{t} dt \leq c_3.$$

For $a_1 < r \leq a_2$,

$$\begin{aligned} \int_{a_0}^{r} \frac{\xi - \hat{\xi}}{t} dt &= \left(\int_{a_0}^{a_1} + \int_{a_1}^{r}\right) \frac{\xi - \hat{\xi}}{t} dt \\ &\leq c_3 + \int_{a_1}^{r} \frac{\xi - 1 + 1 - \hat{\xi}}{t} dt \\ &\leq c_3 + \beta + (1-\alpha)\log 3. \end{aligned}$$

Hence for $a_0 \leq r \leq a_2$,

$$\int_{a_0}^{r} \frac{\xi - \hat{\xi}}{t} dt \leq 2c_3.$$

Similarly, one can prove that

$$\int_{a_0}^{r} \frac{\xi - \hat{\xi}}{t} dt \geq -2c_3.$$

To summarize, we have find $\hat{\xi}(r)$ and $a_0 < a_1 < a_2$ such that $\hat{\xi}$ is smooth and defined on $[0, a_2]$ with $\alpha \leq \hat{\xi} \leq 1$ on $[a_0, a_2]$, satisfying (5.11) with $i = 0$, (5.12) with $i = 0$, $c_1 = 2c_3$, and $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, a_2]$. Moreover, $\hat{\xi}(r) = 1$ near $r = a_2$.

From the above construction, it is easy to see that one can continue and find $a_2 < a_3 < a_4 \cdots \to \infty$ and $\hat{\xi}$ with $\alpha \leq \hat{\xi}(r) \leq 1$ for $r \geq a_0$, satisfying (5.11) with and (5.12) with $c_1 = 2c_3$, and $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, \infty)$.

Since $\hat{\xi} \leq 1$,

$$\hat{h}(r) \geq c_5 \exp(-\int_1^r \frac{1}{t} dt) \geq \frac{c_5}{r}$$

for some $c_5 > 0$ for all $r \geq 1$. Combing with the fact that $|\hat{\xi}'| \leq \frac{c_4}{r}$ on $[a_0, \infty)$, we conclude that (5.13) is also true.

Suppose in addition ξ satisfies (5.9). If $\int_0^r \frac{\xi}{t} dt$ is uniformly bounded from below, then one can take $\hat{\xi} \equiv 0$. If $\int_1^r \frac{\xi}{t} dt$ is not bounded from below and $\int_1^r \frac{\xi - \alpha}{t} dt$ is not bounded from above, then one can proceed as in the proof of Case 3 in the above, by taking $\rho = 0$ near $r = 1$ instead. Then one can get $\hat{\xi}$ to be nonpositive. This completes the proof of the Proposition. □

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let $\hat{g}$ be the $U(n)$ invariant Kähler metric with bounded curvature generated by $\hat{\xi}$ defined in Proposition 5.1, so that

$$c_1^{-1} \hat{g} \leq g_0 \leq c_1 \hat{g} \tag{5.14}$$

for some $c_1 > 0$ as in Proposition 5.1. As in the proof of Theorem 5.3, choose $\delta_k > 0$ and smooth "cutoff" functions $\eta_k : (-\infty, \infty) \to \mathbb{R}$ satisfying

$$\eta_k(r) : \begin{cases} = 1 & \text{if } -\infty < r \leq k \\ 0 < \eta_k(r) < 1 & \text{if } k < r < k + \delta_k \\ = 0 & \text{if } k + \delta_k \leq r < \infty. \end{cases} \tag{5.15}$$

and

$$\int_k^{k+\delta_k} \left| \frac{(\xi - \hat{\xi})}{t} \right| dt \leq 1 \tag{5.16}$$

for all k. Let $\{\xi_k\} : [0, \infty) \to \infty$ be defined by

$$\xi_k(r) = \eta_k \xi + (1 - \eta_k) \hat{\xi}.$$

Then as in the proof of Theorem 5.3, each ξ_k generates a $U(n)$ invariant Kähler metric ω_k so that

$$c_2^{-1} \hat{g} \leq \omega_k \leq c_2 \hat{g} \tag{5.17}$$

for some constant $c_2 > 0$, for all k. Now recall that the curvature of $\hat{g}$ is bounded by a constant K as in Proposition 5.1, and thus by Theorem 2.1 we may assume without loss of generality that $\hat{g}$ has bounded geometry of order infinity. In particular, the formula of curvature in Theorem 5.1 implies that each ω_k also has bounded curvature. We also clearly have $\omega_k \to g_0$ uniformly and smoothly on compact subsets of M. By

Theorem 4.2, there is a solution $g(t)$ of the Kähler-Ricci flow with initial condition g_0 on $M \times [0, T)$ for some $T > 0$ so that

$$||\mathrm{Rm}(g(t))||^2_{g(t)} \le \frac{c_3}{t^2}$$

for some $c_3 > 0$ and for all $0 < t < T$. The estimates for $||\nabla^l \mathrm{Rm}||$ for each $l \ge 0$ then follows from the general results of [S1].

If in addition that

$$\int_0^\infty \frac{\xi}{t} dt < \sigma$$

for some σ for all r, then $\hat{\xi}$ can be chosen to be nonpositive. Then ξ_k is nonpositive near infinity. Therefore the Kähler-Ricci flow with initial condition h_k has longtime solution $h_k(t)$ by Theorem 5.3. On the other hand, $g_0 \ge c_4 g_e$ for some constant $c_4 > 0$, where g_e is the Euclidean metric on $\mathbb{C}^n$. By (5.14) and (5.17),

$$h_k \ge c_5 g_e$$

for some constant $c_5 > 0$ for all k. By Theorem 4.1, there exists a longtime solution $g(t)$ to (1.2) with initial condition g_0. On $(0, T)$ from the previous paragraph, $g(t)$ is the same as before. Hence $g(t)$ also satisfies (5.8) on $\mathbb{C}^n \times (0, T)$. □

The long time existence results in Theorem 5.4 are basically for $U(n)$ invariant metrics with non-positive curvature. The following Theorem gives a longtime existence result for $U(n)$ invariant metrics with non-negative curvature.

Theorem 5.5. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ generated by a smooth function $\xi : [0, \infty) \to \mathbb{R}$ with $\xi(0) = 0$. Suppose $\xi(r) = a$ for r sufficiently large where $a \le 1$. Then the Kähler-Ricci flow has a smooth complete $U(n)$ invariant solution $g(t)$ on $\mathbb{C}^n \times [0, \infty)$ with $g(t) = g_0$. In general, if there is $C > 0$ such that*

$$-C \le \int_1^r \frac{\xi - a}{t} dt \le C \tag{5.18}$$

for some $a \le 1$ for all $r > 1$ and such that $|\xi'| = o(r^{-a})$, then the Kähler-Ricci flow has a smooth complete $U(n)$ invariant solution $g(t)$ on $\mathbb{C}^n \times [0, \infty)$ with $g(t) = g_0$ such that the curvature of $g(t)$ is uniformly bounded on $M \times [0, T]$ for all $T < \infty$.

Remark 7. If $a \le 0$, then we have long time solution by Theorem 5.4. However, there is no curvature bound obtained for all t in that theorem. In that theorem, we can only conclude that the curvature of the solution is uniformly bounded in $M \times [0, T]$ for some $T > 0$.

Proof. Suppose (5.18) is true. We want to prove that the curvature of g tends to zero as $x \to \infty$. Consider the case that $a < 1$, then

$$h \ge c_1 r^{-a}$$

for large r for some $c_1 > 0$. Hence $rf \ge c_2 r^{1-a}$ for r large for some $c_2 > 0$ and $rf(r) \to \infty$ as $r \to \infty$. $|\xi'| = o(r^{-a})$ implies $|\frac{\xi'}{h}| = o(1)$. By Lemma 5.1, the curvature of g_0 approaches to zero at infinity.

Suppose $a = 1$, then there is $c_3 > 0$ such that

$$h \geq \frac{c_3}{r}$$

for r large. So

$$rf \geq c_4 \log r$$

for some $c_4 > 0$ if r is large. By Lemma 5.1, the curvature of g_0 also approaches to zero at infinity.

The Theorem now follows from the above curvature decay estimates, Lemma 5.3 below which implies the injectivity radius of g is bounded below on $\mathbb{C}^n$, and Theorem 2.3 because $\mathbb{C}^n$ has a strictly pluri-subharmonic function. □

LEMMA 5.3. *Let $\xi(r) = a$ for all r sufficiently large and $a \leq 1$. Let g be the corresponding $U(n)$ invariant Kähler metric on $\mathbb{C}^n$. Then the injectivity radius of g is bounded below by a positive constant on $\mathbb{C}^n$*

Proof. We begin by assuming $a < 1$. Indeed, this will be sufficient for our applications. By the estimate in [CGT] and by the fact that the curvature of g is bounded by Lemma 5.2, in order to prove the injectivity radius of g is positive on $\mathbb{C}^n$ it is sufficient to prove there is a constant $c > 0$ such that

$$V_g(B_g(x, 1)) \geq c$$

for all x where $B_g(x, 1)$ is the geodesic ball of radius 1 with center at x with respect to g. Let τ be the geodesic distance from the origin, then for $a < 1$ and $r = |z|^2 > r_0$.

$$\tau(z) = \int_0^r \frac{\sqrt{h}}{2\sqrt{s}} ds = c_1 + c_2 r^{\frac{1}{2}(1-a)} \tag{5.19}$$

for some constants c_1, c_2 with $c_2 > 0$. So

$$rf(r) = c_3 + c_4(\tau - c_1)^2$$

with $c_4 > 0$.

$$V(B_g(0; \tau)) = c_n(rf)^n = \left(c_3 + c_4(\tau - c_1)^2\right)^n,$$

where $\tau = \tau(r)$ is given by (5.19). Hence if τ is large, then

$$V_g(B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)) \geq c_5 \tau^{2n-1} \tag{5.20}$$

for some $c_5 > 0$ independent of τ. Let $\mathcal{F}$ be a maximal disjoint family of $B_g(x, 1)$ with $x \in \partial B_g(0, \tau)$. Let $\mathcal{C} = \{x |\ B_g(x, 1) \in \mathcal{F}$ and let $N = N(\tau) = \#(\mathcal{C})$. We claim that $\bigcup_{x \in \mathcal{C}} B_g(x, 3) \supset B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)$. In fact, if $y \in B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)$, then there is $y' \in \partial B_g(0; \tau)$ such that $d_g(y, y') < 1$. On the other hand, there is $x \in \mathcal{C}$ with $d_g(x, y') < 2$. From these the claim followers.

Since g is $U(n)$ invariant, $v = v(\tau) = V_g(B_g(x, 3))$ is constant for $x \in \partial B_g(0, \tau)$. Hence we have

$$Nv \geq c_5 \tau^{2n-1}$$

and

$$v \geq \frac{c_5}{N}\tau^{2n-1}. \tag{5.21}$$

By the expressions of h and f, on $\partial B_g(0,\tau) = \partial B_0(0,\sqrt{r})$, $c_6^{-1}r^{-a}g_0 \leq g \leq c_6 r^{-a}g_0$ for some $c_6 > 0$ if r is large, where $B_0(0,\sqrt{r})$ is the Euclidean ball with radius $\sqrt{r}$ and center at the origin. Let $B_g^\tau(x,\rho)$ be the geodesic ball with respect to the intrinsic distance of $\partial B_g(0,\tau)$. Define $B_0^\tau(x,\rho)$ similarly with respect to g_0.

Since $B_g(x,1) \supset B_g^\tau(x,1)$, and $B_0^\tau(x, c_6^{-1}r^{\frac{a}{2}}) \subset B_g^\tau(x,1)$. Hence $\{B_0^\tau(x,c_6^{-1}r^{\frac{a}{2}})|x \in \mathcal{C}\}$ is a disjoint family and

$$NV_{g_0}(B_0^\tau(x,c_6^{-1}r^{\frac{a}{2}})) \leq V_{g_0}(\partial B_0(0,\sqrt{r})) = c_n r^{\frac{2n-1}{2}},$$

where c_n is the volume of the unit sphere in $\mathbb{C}^n$. Let $\rho = r^{\frac{1}{2}}$, then the volume of the geodesic ball of radius s_0 in $\partial B_0(0,\rho)$ is

$$c_n\rho^{n-2}\int_0^{s_0} \sin^{2n-2}\frac{s}{\rho}ds.$$

Let $s_0 = c_6^{-1}r^{\frac{a}{2}}$. Then $s_0/\rho \to 0$ as $r \to \infty$. Hence for r large,

$$\begin{aligned} V_{g_0}(B_0^\tau(x,c_6^{-1}r^{\frac{a}{2}})) \geq & c_7\int_0^{s_0} s^{2n-2}ds \\ = & c_8 s_0^{2n-1}. \end{aligned}$$

Hence

$$\begin{aligned} v \geq & c_5 N^{-1}\tau^{2n-1} \\ \geq & c_n^{-1}c_5c_8\tau^{2n-1}r^{-\frac{2n-1}{2}}s_0^{2n-1} \\ \geq & c_9 \end{aligned}$$

for some positive constant c_9 independent of τ.

We now consider the case when $a = 1$. Consider Cao's cigar soliton $\tilde{g}$ which is a complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$. It was shown in [WZ] $\tilde{g}$ has positive sectional curvatures and is generated by $\widetilde{\xi}$ satisfying

$$\int_0^\infty \frac{\xi - \widetilde{\xi}}{t}dt < \infty \tag{5.22}$$

since $\xi(r) = 1$ for sufficiently large r (see Theorem 3 in [WZ]).

In particular, by (5.1) and (5.2) it follows that g and $\tilde{g}$ are uniformly equivalent and thus

$$V_g(B_g(p,1)) \geq CV_{\tilde{g}}(B_{\tilde{g}}(p,1))$$

for some $C > 0$ for all $p \in \mathbb{C}^n$ and for some constant C independent of p. To bound the injectivity radius of g from below, it suffices to prove that the volume in the RHS above is uniformly bounded below. This follows from [GM] since $\tilde{g}$ is complete, and has bounded positive sectional curvatures. For completeness, we include a proof below that $\tilde{g}$ has bounded curvature. By Wu-Zheng:

Let $\tilde\phi = r\tilde f$ and $t = \log r$. $\tilde\phi' = r\tilde h$. Hence $\tilde\phi > 0$, $\tilde\phi' > 0$ for $t > -\infty$. Here all primes on $\tilde\phi$ are with respect to t. Since $A, B, C > 0$, we only need to prove that A, B, C are bounded from above. It is sufficient to prove that A, B, C are bounded from above for $t \geq 0$. For $t \geq 0$, by [WZ, §4]

$$A = n(1 + \frac{n-1}{\tilde\phi}) - \tilde\phi' \left(1 + \frac{2(n-1)}{\tilde\phi} + \frac{n(n-1)}{\tilde\phi^2}\right) \leq n(1 + \frac{n-1}{\tilde\phi(0)}),$$

because $\tilde\phi' > 0$, $\tilde\phi > 0$. So A is bounded.

$$B = \frac{1}{(r\tilde f)^2} \int_0^r \frac{d\widetilde{\xi}}{dr} \left(\int_0^t \tilde h(s) ds\right) dt \leq \frac{1}{r\tilde f}$$

because $\frac{d\widetilde{\xi}}{dr} > 0$, $\widetilde{\xi}(r) \leq 1$. On the other hand by (5.22), $\tilde h(r) \geq c r^{-1}$ for $r \geq 1$. Hence $r\tilde f \sim c \log r$. So B is bounded. Similarly, C is also bounded. □

REMARK 8. In case $1 > a \geq 0$, we may simply compare g with a metric $\tilde g$ with nonnegative bisectional curvature generated by $\widetilde{\xi}$ with $\widetilde{\xi} = a$ near infinity. In this case, $\tilde g$ has maximum volume growth by [WZ]. Hence each geodesic ball of radius 1 is bounded below by a constant which is uniform for all points. So this is also true for g.

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