HOLOMORPHIC EXTENSION FROM A CONVEX HYPERSURFACE[*]

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Abstract. We discuss a general result of holomorphic extension of a real analytic function f defined on the boundary ∂D of a real analytic strictly convex subset $D \subset\subset \mathbb{C}^n$. We show that this follows from the hypothesis of separate holomorphic extension along stationary/extremal discs.

Key words. Analytic discs, Continuity principle, Separate analiticity.

AMS subject classifications. 32V10, 32N15, 32D10.

The problem of testing analyticity on a domain $D \subset \mathbb{C}^n$ by a family of discs has attracted a great deal of work. The first example of testing family was first observed by Agranovsky ans Valsky in [2] on the unit ball of $\mathbb{C}^n$. A significant generalization goes back to Stout [15] who uses as testing family all the straight lines. Reducing the testing family, Agranovsky and Semenov [3] use the lines which meet an open subset $D' \subset\subset D$. It is classical that the lines which meet a single point $z_o \in D$ do not suffice not even in the case of the sphere $\mathbb{B}^n$. Other testing families are considered, among others, by Globevnik [7], Globevnik and Stout [10], Rudin [14]. In the present paper, for a strictly convex real analytic domain D, we prove that the stationary discs passing through a boundary point ia a testing family. The stationary discs passing through an interior point is supplemented by another $(2n-2)$-parameter, generic set of discs are, alltogether, a testing family. In particular this second set can be chosen as the set of stationary discs through another point of D. This last result was already obtained in recent paper by Agranovsky [1] in the ball. We deal with stationary/extremal discs in the sense of Lempert [12]. We first introduce some terminology. A disc A is the holomorphic image of the standard disc Δ; $\mathbb{P}T^*\mathbb{C}^n$ is the cotangent bundle with projectivized fibers, and π the projection on the base point; $\mathbb{P}T^*_{\partial D}\mathbb{C}^n$ the projectivized conormal bundle to ∂D in $\mathbb{C}^n$.

DEFINITION 0.1. *A disc A of D is said to be stationary when it is endowed with a meromorphic lift $A^* \subset T^*\mathbb{C}^n$ with a simple pole attached to $T^*_{\partial D}\mathbb{C}^n$, that is, satisfying $\partial A^* \subset T^*_{\partial D}\mathbb{C}^n$.*

The meromorphic lifts to $T^*\mathbb{C}^n$ become holomorphic in $\mathbb{P}T^*\mathbb{C}^n$. We adopt this point of view (in accordance to [12]) but keep the same notation A^* for the lifts to $\mathbb{P}T^*\mathbb{C}^n$. We consider a $(2n-2)$-parameter family of stationary discs $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}^{2n-2}}$, the family $\mathcal{A}^* = \{A_t^*\}$ of their lifts and form the set in $\mathbb{P}T^*\mathbb{C}^n$ $M = \cup A_t^*$. The set M is generically a CR manifold of CR dimension 1 except at the points of a closed set; we denote by M^{reg} the complement of this set. We have a basic geometric statement

THEOREM 0.2. *Let $\mathcal{A}_{z_o}$ be the family of stationary discs which pass through a point $z_o \in \bar{D}$ and let M_{z_o} the union of their lifts; then*

$$(0.1) \qquad M_{z_o}^{reg} = M_{z_o} \setminus \pi^{-1}(z_o).$$

Proof. We first assume that D coincides with the unit ball $\mathbb{B}^n$ and z_o is in the interior. It is classical that the stationary discs are the straight lines. By a

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biholomorphic transformation of $\mathbb{B}^n$ we can displace z_o at 0. It is helpful to use the parametrization

$$\begin{array}{rcl} \partial\mathbb{B}^n \times (0,1) & \to & M_0 \\ (z,r) & \mapsto & (zr,[\bar{z}]), \end{array}$$

where square brackets denote projectivized coordinates. For fixed $r > 0$, this describes a totally real maximal manifold of $\mathbb{P}T^*\mathbb{C}^n$; thus $\dim_{CR} M_0 \leq 1$. On the other hand, M_0 is foliated by discs and therefore $\dim_{CR} M_0 = 1$.

Instead, for $r = 0$, we have $\pi^{-1}(0) \cap M_0 = \{0\} \times \mathbb{P}_{\mathbb{C}}^{n-1}$; thus any point of $\pi^{-1}(0) \cap M_0$ is CR singular since there the CR dimension jumps from 1 to n.

We pass now to a general strictly convex domain D. We know from [12] that there is a mapping $\Psi : \mathbb{B}^n \to D$ which interchanges 0 with z_o, is C^ω outside 0, transforms holomorphically the complex lines through 0 of $\mathbb{B}^n$ (denoted $\mathcal{A}_0$) into the stationary discs of D through z_o (denoted $\mathcal{A}_{z_o}$), and which fixes the tangent directions at the "centers". Therefore, Ψ lifts in a natural way to a mapping Ψ^* between the manifold M_0 (the union of the $\mathcal{A}_0$'s) to the corresponding manifold M_{z_o} (the union of $\mathcal{A}_{z_o}^*$'s). Denote by $\mathbb{B}_r^n$ the ball of radius r and put $D_r := \Psi(\mathbb{B}_r^n)$; we know from the theory of Lempert that

$$A_{D_r}^* = (A_D^*)|_{D_r}.$$

Since $(A_{D_r}^*)|_{\partial D_r} \subset \mathbb{P}T_{\partial D_r}^*\mathbb{C}^n$, it follows that $M_{z_o} \setminus \pi^{-1}(z_o) \subset \bigcup_r \mathbb{P}T_{\partial D_r}^*\mathbb{C}^n$. Thus, $\mathbb{P}T_{\partial D_r}^*\mathbb{C}^n$ being maximal totally real for any r, we conclude that M_{z_o} is a CR manifold except at points of $\pi^{-1}(z_o)$ and that it is CR-diffeomorphic, via Ψ^*, to $M_0 \setminus \pi^{-1}(0)$. If z_o is in the boundary the proof is the same but uses the boundary version of the Riemann-Lempert mapping Theorem as in Chang-Hu-Lee [6]. $\square$

Before introducing our main theorem we have to recall the theory by [12] for the parts that we need.

Stationary discs are stable under reparametrization. In particular, the pole can be displaced at any of their interior points. It is convenient to identify the lift A^* to its image in the projectivized bundle $\mathbb{P}T^*\mathbb{C}^n$ with coordinates $(z,[\zeta])$. We assume that D is strictly convex and C^{k+1}. In this situation, a stationary disc and its lift A^* are C^k up to $\partial\Delta$. Moreover, one has the following basic result for whose proof we refer to [12].

PROPOSITION 0.3. *For any point $(z,[\zeta]) \in \mathbb{P}T^*\mathbb{C}^n|_D$ there is unique, up to reparametrization, the stationary disc whose lift $A_{(z,[\zeta])}^*$ contains $(z,[\zeta])$. Moreover, the correspondence*

$$(0.2) \qquad\qquad (z,[\zeta]) \mapsto A_{(z,[\zeta])}^*, \qquad \mathbb{P}T^*\mathbb{C}^n|_D \to C^1(\bar{\Delta}),$$

is an immersion.

We consider now a C^ω function f in ∂D and suppose that it extends holomorphically along each disc of a certain family $\mathcal{A}$. The funtion f is not extended to a function over the union of the discs $A \in \mathcal{A}$ but it is naturally "lifted"' to a function F on the union of the A^*. This is defined by

$$F(z,[\zeta]) = f_{A_{(z,[\zeta])}}(z)$$

where $A_{(z,[\zeta])}$ is the unique stationary disc whose lift $A_{(z,[\zeta])}^*$ passes through $(z,[\zeta])$. The crucial point is that the A's may overlap on $\mathbb{C}^n$ but the A^*'s do not in $\mathbb{P}T^*\mathbb{C}^n$.

THEOREM 0.4. *Let $D \subset\subset \mathbb{C}^n$ be a strictly convex domain with C^ω boundary and f a C^ω function on ∂D. Suppose that f extends holomorphically along each disc of a $2n - 2$ parameter family $\mathcal{A}$. Then F extends holomorphically to a neighborhood of M^{reg}*

Proof. We first remark that at any point of M^{reg} the CR structure is fully formed by the discs. The set M is a manifold with boundary $E = \cup \partial A^*$ which is an open subset of the conormal bundle $\mathbb{P}T^*_{\partial D}\mathbb{C}^n$; in particular E is totally real. Since $f \in C^\omega(\partial D)$, then $F|_E \in C^\omega(E)$ and so F extends holomorphically to a full neighborhood of E in $\mathbb{P}T^*\mathbb{C}^n$. Outside the edge, at regular points, the CR structure of M is fully provided by the discs A^* by which it is foliated, that is, $T^{\mathbb{C}}_{(z,[\zeta])}M = T_{(z,[\zeta])}A^*$. In particular, since F is holomorphic along the A^*'s, than it is CR on M. By propagation of holomorphic extendibility on M^{reg} (cf. [11, 17])along the discs A^* we get the conclusion. $\square$

We focus our attention to the family $\mathcal{A}_{z_o}$ of discs through a point z_o.

THEOREM 0.5. *Let D be strictly convex with C^ω boundary and let $f \in C^\omega(\partial D)$. Suppose that one of the stationary discs through z_o, A_o, belongs to a second family $\mathcal{A}$, let $M = \bigcup_{A \in \mathcal{A}} A^*$ be the collection of their lifts and assume that $\pi^{-1}(z_o) \cap A^*_o \subset M^{reg}$. Suppose that f extends along the discs of the two families. Than f extends to a holomorphic function on D.*

COROLLARY 0.6. *Let D be a real analytic domain, $f \in C^\omega(\partial D)$ and suppose that f extends holomorphically along*
1. either the discs through two interior points
2. or the discs through a boundary point
Then f extends holomorphically.

REMARK 0.7. Note that in (1) the family of discs which pass through the second point is only used to cover the singular point of M_{z_o} over z_o; for this purpose, a much more general family than of discs through another point is suitable.

REMARK 0.8. Discs by two points of the ball are also present, as a testing family, in the papers [1] by Agranovsky and Globevnik [9]. Discs through one point in the case of the unit ball is also present in [4].

Proof of theorem 0.5. According to Theorem 0.4 F extends to a neighborhood of both M^{reg} and $M^{\mathrm{reg}}_{z_o}$. On the other hand, by Theorem 0.2, $M^{\mathrm{reg}}_{z_o} = M_{z_o} \setminus \pi^{-1}(z_o)$; Since we suppose that $\pi^{-1}(z_o) \cap A^*_o \subset M^{\mathrm{reg}}$ then

(0.3) F is holomorphic in a full neighborhood U^* of A^*_o

We prove now that (0.3) implies that

(0.4) F extends holomorphically on the whole $\mathbb{P}T^*\mathbb{C}^n|_D$.

To see this, we suppose $A^*_{(z,[\zeta])}(0) = (z, [\zeta])$ and define a function G by means of the Cauchy integral

(0.5) $$G(z, [\zeta]) := (2\pi i)^{-1} \int_{\partial \Delta} \frac{f \circ A_{(z,[\zeta])}(\tau)}{\tau} d\tau.$$

This is defined for any $(z, [\zeta]) \in \mathbb{P}T^*\mathbb{C}^n|_D$ and is real analytic. If $(z, [\zeta])$ is sufficiently close to A_o^*, then $A_{(z,\zeta)}^* \subset U^*$ where, according to 0.3, F is holomorphic. By Cauchy formula (noticing that $f \circ A_{(z,[\zeta])}|_{\partial\Delta} = F \circ A_{(z,[\zeta])}^*|_{\partial\Delta}$) we have that $G(z, [\zeta]) = F(z, [\zeta])$ then we have

$$G = F \qquad \text{in a neighborhood of } A_o^*.$$

Hence F, identified to G, extends holomorphically to the full $\mathbb{P}T^*\mathbb{C}^n|_D$. In particular since $\mathbb{P}T^*\mathbb{C}^n|_D$ has compact complex fibers it follows that F is constant in $[\zeta]$. Thus it is a function of z only, the holomorphic extension of f to D. □

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