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ON LOW-DIMENSIONAL SOLVMANIFOLDS*

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1. Introduction. In this note we want to study compact homogeneous spaces G/Γ, where G is a connected and simply-connected Lie group and Γ a discrete subgroup in G. It is well known that the existence of such a Γ implies the unimodularity of the Lie group G. Recall that a Lie group G is called *unimodular* if for all $X \in \mathfrak{g}$ holds $\operatorname{tr} \operatorname{ad} X = 0$, where $\mathfrak{g}$ denotes the Lie algebra of G.

If we further demand G/Γ to be symplectic (when G is even-dimensional), a result of Chu [11] shows that G has to be solvable.

Therefore, we regard compact quotients of connected and simply-connected solvable Lie groups by discrete subgroups, so called *solvmanifolds*

The notion of nilpotent and solvable Lie algebras (and groups) is well known.

1.1. Formality. A *differential graded algebra (DGA)* is a graded $\mathbb{R}$-algebra $A = \bigoplus_{i\in\mathbb{N}} A^i$ together with an $\mathbb{R}$-linear map $d\colon A \to A$ such that $d(A^i) \subset A^{i+1}$ and the following conditions are satisfied:

(i) The $\mathbb{R}$-algebra structure of A is given by an inclusion $\mathbb{R} \hookrightarrow A^0$.

(ii) The multiplication is graded commutative, i.e. for $a \in A^i$ and $b \in A^j$ one has $a \cdot b = (-1)^{i\cdot j} b \cdot a \in A^{i+j}$.

(iii) The Leibniz rule holds: $\forall_{a\in A^i}\forall_{b\in A}\ d(a\cdot b) = d(a)\cdot b + (-1)^i a \cdot d(b)$

(iv) The map d is a differential, i.e. $d^2 = 0$.

Further, we define $|a| := i$ for $a \in A^i$.

The *i-th cohomology of a DGA* (A, d) is the algebra

$$H^i(A, d) := \frac{\ker(d\colon A^i \to A^{i+1})}{\operatorname{im}(d\colon A^{i-1} \to A^i)}.$$

If (B, d_B) is another DGA, then a $\mathbb{R}$-linear map $f\colon A \to B$ is called *morphism* if $f(A^i) \subset B^i$, f is multiplicative, and $d_B \circ f = f \circ d_A$. Obviously, any such f induces a homomorphism $f^*\colon H^*(A, d_A) \to H^*(B, d_B)$. A morphism of differential graded algebras inducing an isomorphism on cohomology is called *quasi-isomorphism*.

Definition 1.1.1. A DGA $(\mathcal{M}, d)$ is said to be *minimal* if

(i) there is a graded vector space $V = \left(\bigoplus_{i\in\mathbb{N}_+} V^i\right) = \operatorname{Span}\{a_k \mid k \in I\}$ with homogeneous elements a_k, which we call the generators,

(ii) $\mathcal{M} = \bigwedge V$,

(iii) the index set I is well ordered, such that $k < l \;\Rightarrow |a_k| \le |a_l|$ and the expression for da_k contains only generators a_l with $l < k$.

We shall say that $(\mathcal{M}, d)$ is a *minimal model for a differential graded algebra* (A, d_A) if $(\mathcal{M}, d)$ is minimal and there is a quasi-isomorphism of DGAs $\rho\colon (\mathcal{M}, d) \to (A, d_A)$, i.e. it induces an isomorphism $\rho^*\colon H^*(\mathcal{M}, d) \to H^*(A, d_A)$ on cohomology.

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The importance of minimal models is reflected by the following theorem, which is taken from Sullivan's work [53, Section 5].

THEOREM 1.1.2. *A differential graded algebra* (A, d_A) *with* $H^0(A, d_A) = \mathbb{R}$ *possesses a minimal model. It is unique up to isomorphism of differential graded algebras.*

We quote the existence-part of Sullivan's proof, which gives an explicit construction of the minimal model. Whenever we are going to construct such a model for a given algebra in this article, we will do it as we do it in this proof.

Proof of the existence. We need the following algebraic operations to "add" resp. "kill" cohomology.

Let $(\mathcal{M}, d)$ be a DGA. We "add" cohomology by choosing a new generator x and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(x), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(x) = 0,$$

and "kill" a cohomology class $[z] \in H^k(\mathcal{M}, d)$ by choosing a new generator y of degree $k-1$ and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(y), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(y) = z.$$

Note that z is a polynomial in the generators of $\mathcal{M}$.

Now, let (A, d_A) a DGA with $H^0(A, d_A) = \mathbb{R}$. We set $\mathcal{M}_0 := \mathbb{R}$, $d_0 := 0$ and $\rho_0(x) = x$.

Suppose now $\rho_k \colon (\mathcal{M}_k, d_k) \to (A, d_A)$ has been constructed so that ρ_k induces isomorphisms on cohomology in degrees $\leq k$ and a monomorphism in degree $(k+1)$.

"Add" cohomology in degree $(k+1)$ to get morphism of differential graded algebras $\rho_{(k+1),0} \colon (\mathcal{M}_{(k+1),0}, d_{(k+1),0}) \to (A, d_A)$ which induces an isomorphism $\rho^*_{(k+1),0}$ on cohomology in degrees $\leq (k+1)$. Now, we want to make the induced map $\rho^*_{(k+1),0}$ injective on cohomology in degree $(k+2)$.

We "kill" the kernel on cohomology in degree $(k+2)$ (by non-closed generators of degree (k+1)) and define $\rho_{(k+1),1} \colon (\mathcal{M}_{(k+1),1}, d_{(k+1),1}) \to (A, d_A)$ accordingly. If there are generators of degree one in $(\mathcal{M}_{(k+1),0}, d_{(k+1),0})$ it is possible that this killing process generates new kernel on cohomology in degree $(k+2)$. Therefore, we may have to "kill" the kernel in degree $(k+2)$ repeatedly.

We end up with a morphism $\rho_{(k+1),\infty} \colon (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty}) \to (A, d_A)$ which induces isomorphisms on cohomology in degrees $\leq (k+1)$ and a monomorphism in degree $(k+2)$. Set $\rho_{k+1} := \rho_{(k+1),\infty}$ and $(\mathcal{M}_{k+1}, d_{k+1}) := (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty})$.

Inductively we get the minimal model $\rho \colon (\mathcal{M}, d) \to (A, d_A)$. □

A *minimal model* $(\mathcal{M}_M, d)$ *of a connected smooth manifold* M is a minimal model for the de Rahm complex $(\Omega(M), d)$ of differential forms on M. The last theorem implies that every connected smooth manifold possesses a minimal model which is unique up to isomorphism of differential graded algebras.

Now, we are able to introduce the notion of formality. Endowed with the trivial differential, the cohomology of a minimal DGA is a DGA itself, and therefore it also possesses a minimal model. In general, this two minimal models need not to be isomorphic.

A minimal differential graded algebra $(\mathcal{M}, d)$ is called *formal* if there is a morphism of differential graded algebras

$$\psi \colon (\mathcal{M}, d) \to (H^*(\mathcal{M}, d), d_H = 0)$$

that induces the identity on cohomology.

This means that $(\mathcal{M}, d)$ and $(H^*(\mathcal{M}, d), d_H = 0)$ share their minimal model. The following theorem gives an equivalent characterisation.

Theorem 1.1.3 ([46, Theorem 1.3.1]). *A minimal model $(\mathcal{M}, d)$ is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N, and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.* □

This allows us to give a weaker version of the notion of formality.

Definition 1.1.4. A minimal model $(\mathcal{M}, d)$ is called *s-formal*, $s \in \mathbb{N}$, if we can write $\mathcal{M} = \bigwedge V$ and for each $i \leq s$ the space V^i generated by generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$ with $d(C^i) = 0$, d is injective on N^i and such that every closed element in the ideal $I(\bigoplus_{i \leq s} N^i)$ generated by $\bigoplus_{i \leq s} N^i$ in $\bigwedge \big(\bigoplus_{i \leq s} V^i \big)$ is exact in $\bigwedge V$.

Obviously, formality implies s-formality for every s.

The following theorem is an immediate consequence of the last definition.

Theorem 1.1.5. *Let $(\mathcal{M}, d)$ be a minimal model, where $\mathcal{M} = \bigwedge V$, $V = C \oplus N$ with $d(C) = 0$ and d is injective on N.*

Assume that there exist $r, s \in \mathbb{N}_+$, $n \in N^r$ and $x \in \bigwedge \big(\bigoplus_{i \leq s} V^i \big)$ such that holds

$$\forall_{c \in C^r} \ (n + c)\, x \text{ is closed and not exact.}$$

Then $(\mathcal{M}, d)$ is not $\max\{r, s\}$-formal. □

A connected smooth manifold is called *formal* (resp. *s-formal*) if its minimal model is formal (resp. s-formal).

The next theorem shows the reason for defining s-formality: in certain cases s-formality is sufficient for a manifold to be formal.

Theorem 1.1.6 ([20, Theorem 3.1]). *Let M be a connected and orientable compact smooth manifold of dimension $2n$ or $(2n - 1)$.*

Then M is formal if and only if it is $(n - 1)$-formal. □

Example ([20, Corollary 3.3])

(i) Every connected and simply-connected compact smooth manifold is 2-formal.
(ii) Every connected and simply-connected compact smooth manifold of dimension seven or eight is formal if and only if it is 3-formal. □

Proposition 1.1.7 ([20, Lemma 2.11]). *Let M_1, M_2 be connected smooth manifolds. They are both formal (resp. s-formal) if and only if $M_1 \times M_2$ is formal (resp. s-formal).* □

An important tool for detecting non-formality is the concept of Massey products: As we shall see below, the triviality of the Massey products is necessary for formality.

Let (A, d) be a differential graded algebra and $a_i \in H^{p_i}(A)$, $p_i > 0$, $1 \leq i \leq 3$, satisfying $a_j \cdot a_{j+1} = 0$ for $j = 1, 2$. Take elements α_i of A with $a_i = [\alpha_i]$ and write $\alpha_j \cdot \alpha_{j+1} = d\xi_{j,j+1}$ for $j = 1, 2$. The *(triple-)Massey product* $\langle a_1, a_2, a_3 \rangle$ of the classes a_i is defined as

$$[\alpha_1 \cdot \xi_{2,3} + (-1)^{p_1+1} \xi_{1,2} \cdot \alpha_3] \in \frac{H^{p_1+p_2+p_3-1}(A)}{a_1 \cdot H^{p_2+p_3-1}(A) + H^{p_1+p_2-1}(A) \cdot a_3}.$$

Remark. The definition of the triple-Massey product as an element of a quotient space is well defined, see e.g. [46, Section 1.6].

The next lemma shows the relation between formality and Massey products.

Lemma 1.1.8 ([46, Theorem 1.6.5]). *For any formal minimal differential graded algebra all Massey products vanish.* □

Corollary 1.1.9. *If the de Rahm complex* $(\Omega(M), d)$ *of a smooth manifold* M *possesses a non-vanishing Massey product, then* M *is not formal.* □

Fernández and Muñoz considered in [21] the geography of non-formal compact manifolds and obtained the following theorem:

Theorem 1.1.10. *Given* $m \in \mathbb{N}_+$ *and* $b \in \mathbb{N}$*, there are compact oriented* m*-dimensional smooth manifolds with* $b_1 = b$ *which are non-formal if and only if one of the following conditions holds:*
(i) $m \geq 3$ *and* $b \geq 2$*,*
(ii) $m \geq 5$ *and* $b = 1$*,*
(iii) $m \geq 7$ *and* $b = 0$*.* □

1.2. Symplectic, Kähler and Lefschetz manifolds. The main examples of formal spaces are Kähler manifolds. By definition, a Kähler manifold possesses a Riemannian, a symplectic and a complex structure that are compatible. The notion is well known

In [43], Newlander and Nirenberg proved their famous result that an almost complex structure J on a smooth manifold M is integrable if and only if $N_J \equiv 0$, where the *Nijenhuis tensor* N_J is defined as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for all vector fields X, Y on M.

The difficulty to prove non-existence of any Kähler structure is obvious. Nowadays, two easily verifiable necessary conditions for Kähler manifolds are known. First, we have the main theorem from the work [13] of Deligne, Griffiths, Morgan and Sullivan.

Theorem 1.2.1 ([13, p. 270]). *Compact Kähler manifolds are formal.* □

We say that a symplectic manifold (M^{2n}, ω) is *Lefschetz* if the homomorphism

$$\begin{array}{rccc} L^k\colon & H^{n-k}(M,\mathbb{R}) & \longrightarrow & H^{n+k}(M,\mathbb{R}) \\ & [\alpha] & \longmapsto & [\alpha \wedge \omega^k] \end{array}$$

is surjective for $k = n - 1$. If L^k is surjective for $k \in \{0, \ldots, n - 1\}$, then (M, ω) is called *Hard Lefschetz*.

Note that for compact M the surjectivity of L^k implies its injectivity.

Obviously, the Lefschetz property depends on the choice of the symplectic form. But as mentioned above, if there is a symplectic form satisfying the Lefschetz property, we have the following consequence that is purely topological.

Theorem 1.2.2. *The odd degree Betti numbers of a Hard Lefschetz manifold are even.*

Proof. Let (M^{2n}, ω) be a symplectic manifold satisfying the Lefschetz property. We us the same idea as in [26, p. 123]. For each $i \in \{0, \dots, n-1\}$ one has a non-degenerated skew-symmetric bilinear form

$$\begin{array}{ccc} H^{2i+1}(M,\mathbb{R}) \times H^{2i+1}(M,\mathbb{R}) & \longrightarrow & \mathbb{R}, \\ ([\alpha],[\beta]) & \longmapsto & [\alpha \wedge \beta \wedge \omega^{n-2i-1}] \end{array}$$

i.e. $H^{2i+1}(M,\mathbb{R})$ must be even-dimensional. □

Obviously, this also proves the next corollary.

Corollary 1.2.3. *The first Betti number of a Lefschetz manifold is even.* □

Finally, the following shows that the statement of the last theorem holds for Kähler manifolds:

Theorem 1.2.4 ([26, p. 122]). *Compact Kähler manifolds are Hard Lefschetz.* □

2. Nilmanifolds. We give a brief review of known results about a special kind of solvmanifolds, namely nilmanifolds.

A *nilmanifold* is a compact homogeneous space G/Γ, where G is a connected and simply-connected nilpotent Lie group and Γ a *lattice* in G, i.e. a discrete co-compact subgroup.

In contrast to arbitrary solvable Lie groups, there is an easy criterion for nilpotent ones which enables one to decide whether there is a lattice or not.

Recall that the exponential map $\exp\colon \mathfrak{g} \to G$ of a connected and simply-connected nilpotent Lie group is a diffeomorphism. We denote its inverse by $\log\colon G \to \mathfrak{g}$.

Theorem 2.1 ([48, Theorem 2.12]). *A simply-connected nilpotent Lie group G admits a lattice if and only if there exists a basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\mathfrak{g}$ of G such that the structure constants C_{ij}^k arising in the brackets*

$$[X_i, X_j] = \sum_k C_{ij}^k X_k$$

are rational numbers.

More precisely we have:

(i) Let $\mathfrak{g}$ have a basis with respect to which the structure constants are rational. Let $\mathfrak{g}_{\mathbb{Q}}$ be the vector space over $\mathbb{Q}$ spanned by this basis.
Then, if $\mathcal{L}$ is any lattice of maximal rank in $\mathfrak{g}$ contained in $\mathfrak{g}_{\mathbb{Q}}$, the group generated by $\exp(\mathcal{L})$ is a lattice in G.

(ii) If Γ is a lattice in G, then the $\mathbb{Z}$-span of $\log(\Gamma)$ is a lattice $\mathcal{L}$ of maximal rank in the vector space $\mathfrak{g}$ such that the structure constants of $\mathfrak{g}$ with respect to any basis contained in $\mathcal{L}$ belong to $\mathbb{Q}$. □

For a given lattice Γ in a connected and simply-connected nilpotent Lie group G, the subset $\log(\Gamma)$ need not to be an additive subgroup of the Lie algebra $\mathfrak{g}$.

Example. Consider the nilpotent Lie group $G := \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \}$.

Its Lie algebra is $\mathfrak{g} := \{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R}\}$, and the logarithm is given by

$$\log(\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & x & z - \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The set of integer matrices contained in G forms a lattice Γ in G and $\log(\Gamma)$ is not a subgroup of $\mathfrak{g}$.

If Γ is a lattice such that $\log(\Gamma)$ is a subgroup of the Lie algebra, we call Γ a *lattice subgroup*.

For later uses, we quote the following two results.

PROPOSITION 2.2 ([12, Lemma 5.1.4 (a)]). *Let G be a locally compact group, H a closed normal subgroup and Γ a discrete subgroup of G. Moreover, denote by $\pi\colon G \to G/H$ the natural map.*

If $\Gamma \cap H$ is a lattice in H, and Γ is a lattice in G, then $\pi(\Gamma)$ is a lattice in G/H and $\Gamma H = H\Gamma$ is a closed subgroup of G. □

THEOREM 2.3 ([12, p. 208]). *Let G be a connected and simply-connected nilpotent Lie group with lattice Γ and $k \in \mathbb{N}$.*

Then $\Gamma \cap D^{(k)}G$, $\Gamma \cap G^{(k)}$ resp. $\Gamma \cap G_{(k)}$ are lattices in $D^{(k)}G$, $G^{(k)}$ resp. $G_{(k)}$. Note, $G_{(1)}$ is the center $Z(G)$ of G. □

Note that we can associate a DGA to each Lie algebra $\mathfrak{g}$ as follows:

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ and denote by $\{x_1, \ldots, x_n\}$ the dual basis of $\mathfrak{g}^*$. The *Chevalley-Eilenberg complex* of $\mathfrak{g}$ is the differential graded algebra $(\bigwedge \mathfrak{g}^*, \delta)$ with δ given by

$$\delta(x_k) = -\sum_{i<j} C_{ij}^k \, x_i \wedge x_j,$$

where C_{ij}^k are the structure constants of $\{X_1, \ldots, X_n\}$.

THEOREM 2.4 ([44], [46, Theorem 2.1.3]). *Let G/Γ be a nilmanifold and denote by $\Omega_{l.i.}(G)$ the vector space of left-invariant differential forms on G.*

Then the natural inclusion $\Omega_{l.i.}(G) \to \Omega(G/\Gamma)$ induces an isomorphism on cohomology.

Moreover, the minimal model of G/Γ is isomorphic to the Chevalley-Eilenberg complex of the Lie algebra of G. □

COROLLARY 2.5. *Any nilmanifold satisfies $b_1 \geq 2$.*

Proof. Let $\mathfrak{g}$ be a nilpotent Lie algebra. By [59, Theorem 7.4.1] we have $H^1(\bigwedge \mathfrak{g}^*, \delta) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. By [16] any nilpotent Lie algebra $\mathfrak{g}$ satisfies the inequality $\dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \geq 2$ which then implies $b_1(\mathfrak{g}) \geq 2$. Hence the claim follows from the preceding theorem. □

We now quote some results that show that it is easy to decide whether a nilmanifold is formal, Kählerian or Hard Lefschetz.

THEOREM 2.6 ([28, Theorem 1]). *A nilmanifold is formal if and only if it is a torus.* □

THEOREM 2.7 ([46, Theorem 2.2.2]). *If a nilmanifold is Kählerian, then it is a torus.* □

This theorem follows from Theorem 2.6. Another proof was given by Benson and Gordon in [4]. In fact they proved the following:

THEOREM 2.8 ([4, pp. 514 et seq.]). *A symplectic non-toral nilmanifold is not Lefschetz.* □

COROLLARY 2.9. *A symplectic nilmanifold is Hard Lefschetz if and only if it is a torus, independent of the special choice of the symplectic form.* □

3. Solvmanifolds in general. A *solvmanifold* is a compact homogeneous space G/Γ, where G is a connected and simply-connected solvable Lie group and Γ a *lattice* in G, i.e. a discrete co-compact subgroup.

Every connected and simply connected solvable Lie group is diffeomorphic to $\mathbb{R}^m$ (see e.g. [57]), hence solvmanifolds are aspherical and their fundamental group is isomorphic to the considered lattice. Since [48, Theorem 3.6] directly imlies the following Theorem, the fundamental group plays an important role in the study of solvmanifolds.

THEOREM 3.1. *Two solvmanifolds with isomorphic fundamental groups are diffeomorphic.* □

Unfortunately, there is no simple criterion for the existence of a lattice in a connected and simply-connected solvable Lie group. We shall quote some necessary criteria.

PROPOSITION 3.2 ([37, Lemma 6.2]). *If a connected and simply-connected solvable Lie group admits a lattice then it is unimodular.* □

THEOREM 3.3 ([39],[46, Theorem 3.1.2]). *Let G/Γ be a solvmanifold that is not a nilmanifold and denote by N the nilradical of G.*

Then $\Gamma_N := \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is a closed subgroup in G and $G/(N\Gamma)$ is a torus. Therefore, G/Γ can be naturally fibred over a non-trivial torus with a nilmanifold as fiber:

$$N/\Gamma_N = (N\Gamma)/\Gamma \longrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = T^k$$

This bundle will be called the Mostow bundle. □

REMARK. The structure group action of the Mostow bundle is given by left translations

$$N\Gamma/\Gamma_0 \times N\Gamma/\Gamma \longrightarrow N\Gamma/\Gamma,$$

where Γ_0 is the largest normal subgroup of Γ which is normal in $N\Gamma$. (A proof of the topological version of this fact can be found in [52, Theorem I.8.15]. The proof for the smooth category is analogous.)

In view of Theorem 3.3, we are interested in properties of the nilradical of a solvable Lie group. The following proposition was first proved in [40].

PROPOSITION 3.4. *Let G be a solvable Lie group and N its nilradical.*
Then $\dim N \geq \frac{1}{2}\dim G$. □

In some cases, we will be able to apply the next theorem to the situation of Theorem 3.3. It then gives a sufficient condition for the Mostow bundle to be a principal bundle.

THEOREM 3.5. *Let G be a connected and simply-connected solvable Lie group and Γ a lattice in G. Suppose that $\{e\} \neq H \subsetneqq G$ is a closed normal abelian Lie subgroup of G with $H \subset N(\Gamma)$, the normalizer of Γ. (For example the latter is satisfied if H is central.) Assume further that $\Gamma_H := \Gamma \cap H$ is a lattice in H.*

Then $H/\Gamma_H = H\Gamma/\Gamma$ is a torus and

$$H/\Gamma_H \longrightarrow G/\Gamma \longrightarrow G/H\Gamma \tag{1}$$

is a principal torus bundle over a solvmanifold.

Proof. By assumption, H is a closed normal abelian subgroup of G and Γ_H is a lattice in H. We have for $h_1\gamma_1, h_2\gamma_2 \in H\Gamma$ with $h_i \in H$, $\gamma_i \in \Gamma$ the equivalence

$$(h_1\gamma_1)^{-1}(h_2\gamma_2) \in \Gamma \Leftrightarrow h_1^{-1}h_2 \in \Gamma_H,$$

i.e. $H/\Gamma_H = H\Gamma/\Gamma$. Therefore, Proposition 2.2 implies that (1) is a fibre bundle whose fibre is clearly a torus and its base a solvmanifold. The structure group action is given by the left translations

$$H\Gamma/\Gamma_0 \times H\Gamma/\Gamma \longrightarrow H\Gamma/\Gamma,$$

where Γ_0 is the largest normal subgroup of Γ which is normal in $H\Gamma$. (This can be seen analogous as in the Remark on page 205.) Since H is contained in $N(\Gamma) = \{g \in G \,|\, g\Gamma g^{-1} = \Gamma\}$, we have for each $h \in H$ and $\gamma, \gamma_0 \in \Gamma$

$$(h\gamma)\gamma_0(h\gamma)^{-1} = h\gamma\gamma_0\gamma^{-1}h^{-1} \in h\Gamma h^{-1} = \Gamma,$$

i.e. $\Gamma = \Gamma_0$ and the theorem follows. □

DEFINITION 3.6. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

(i) G and $\mathfrak{g}$ are called *completely solvable* if the linear map $\operatorname{ad} X\colon \mathfrak{g} \to \mathfrak{g}$ has only real roots[1] for all $X \in \mathfrak{g}$.

(ii) If G is simply-connected and $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism, then G is called *exponential*.

A nilpotent Lie group or algebra is completely solvable, and it is easy to see that completely solvable Lie groups or algebras are solvable. Moreover, the two propositions below show that simply-connected completely solvable Lie groups are exponential, and exponential Lie groups are solvable. Note that the second proposition is simply a reformulation of results of Saitô and Dixmier, see [49, Théorèmes II.1 et I.1] and [17, Théorème 3].

PROPOSITION 3.7 ([47, Theorem 2.6.3]). *Any exponential Lie group is solvable.* □

PROPOSITION 3.8. *A connected and simply-connected solvable Lie group G with Lie algebra $\mathfrak{g}$ is exponential if and only if the linear map $\operatorname{ad} X\colon \mathfrak{g} \to \mathfrak{g}$ has no purely imaginary roots for all $X \in \mathfrak{g}$.* □

[1] By a root of a linear map, we mean a (possibly non-real) root of the characteristic polynomial.

Let a lattice in a connected and simply-connected solvable Lie group be given. Then Theorem 3.3 stated that its intersection with the nilradical is a lattice in the nilradical. In the case of completely solvable Lie groups, we have an analogue for the commutator.

PROPOSITION 3.9 ([25, Proposition 1]). *Let G be a connected and simply-connected completely solvable Lie group and Γ a lattice in G.*

Then $[\Gamma,\Gamma]$ is a lattice in $[G,G]$. In particular, $\Gamma \cap [G,G]$ is a lattice in $[G,G]$. □

We formulate the result that enables us to compute the minimal model of solvmanifolds which are built by dividing a lattice out of a completely solvable group. The main part of the next theorem is due to Hattori [32].

THEOREM 3.10. *Let G/Γ be a solvmanifold. Denote by $(\bigwedge \mathfrak{g}^*, \delta)$ the Chevalley-Eilenberg complex of G and recall that $\mathfrak{g}^*$ is the set of left-invariant differential 1-forms on G. Then the following holds:*

- *(i) The natural inclusion $(\bigwedge \mathfrak{g}^*, \delta) \to (\Omega(G/\Gamma), d)$ induces an injection on cohomology.*
- *(ii) If G is completely solvable, then the inclusion in (i) is a quasi-isomorphism, i.e. it induces an isomorphism on cohomology. Therefore, the minimal model $\mathcal{M}_{G/\Gamma}$ is isomorphic to the minimal model of the Chevalley-Eilenberg complex.*
- *(iii) If $\mathrm{Ad}\,(\Gamma)$ and $\mathrm{Ad}\,(G)$ have the same Zariski closures[2], then the inclusion in (i) is a quasi-isomorphism.* □

Proof. (i) is [46, Theorem 3.2.10] and (iii) is [48, Corollary 7.29].

ad (ii): Denote the mentioned inclusion by $i\colon (\bigwedge \mathfrak{g}^*, \delta) \to (\Omega(G/\Gamma), d)$. By Hattori's Theorem (see [46, p. 77]), this is a quasi-isomorphism. It remains to show that the minimal model $\rho\colon (\mathcal{M}_{CE}, \delta_{CE}) \to (\bigwedge \mathfrak{g}^*, \delta)$ of $(\bigwedge \mathfrak{g}^*, \delta)$ is the minimal model of $(\Omega(G/\Gamma), d)$. Since the minimal model is unique and the map $i \circ \rho\colon (\mathcal{M}_{CE}, \delta_{CE}) \to (\Omega(G/\Gamma), d)$ is a quasi-isomorphism, this is obvious. □

We have seen in the last section that the first Betti number of a nilmanifold is greater than or equal to two. For arbitrary solvmanifolds this is no longer true. Below, we shall see various examples of solvmanifolds with $b_1 = 1$. The following corollary shows that $b_1 = 0$ cannot arise.

COROLLARY 3.11. *Any solvmanifold satisfies $b_1 \geq 1$.*

Proof. Let $\mathfrak{g}$ be a solvable Lie algebra. As in the nilpotent case we have $b_1(\bigwedge \mathfrak{g}^*, \delta) = \dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, and $\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \geq 1$ by solvability. The claim now follows from Theorem 3.10 (i). □

To end this section, we shortly discuss the existence problem for Kähler structures on solvmanifolds. The only Kählerian nilmanifolds are tori, but in the general context we have the hyperelliptic surfaces, which are non-toral Kählerian solvmanifolds, see Section 6 below. Benson and Gordon [5] conjectured in 1990 that the existence of a Kähler structure on a solvmanifold G/Γ with G completely solvable forces G/Γ to be toral and this is true. In fact, Hasegawa proved the following:

[2] A basis for the Zariski topology on $\mathrm{GL}(m,\mathbb{C})$ is given by the sets

$$U_p := \mathrm{GL}(m,\mathbb{C}) \setminus p^{-1}(\{0\}),$$

where $p\colon \mathrm{GL}(m,\mathbb{C}) \cong \mathbb{C}^{(m^2)} \to \mathbb{C}$ ranges over polynomials.

THEOREM 3.12 ([31]). *A solvmanifold G/Γ is Kählerian if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus.*

If G is completely solvable, then G/Γ is Kählerian if and only if it is a complex torus. □

4. Semidirect products. In order to define semidirect products, we recall the construction of the Lie group structure of the group of Lie group automorphisms of a simply-connected Lie group in the following theorem. It collects results that can be found in [58, pp. 117 et seq.].

THEOREM 4.1.

(i) Let $\mathfrak{h} = \big(|\mathfrak{h}| = \mathbb{R}^h, [\ldots,\ldots]\big)$ be an h-dimensional Lie algebra. Then the set $\mathrm{A}(\mathfrak{h})$ of Lie algebra isomorphisms of $\mathfrak{h}$ is a closed Lie subgroup of the automorphism group $\mathrm{Aut}(|\mathfrak{h}|)$ of the h-dimensional vector space $|\mathfrak{h}|$. The Lie algebra of $\mathrm{A}(\mathfrak{h})$ is

$$\mathfrak{d}(\mathfrak{h}) = \{\varphi \in \mathrm{End}(|\mathfrak{h}|) \mid \varphi \text{ derivation with respect to } [\ldots,\ldots]\}.$$

(ii) Let H be a connected and simply-connected Lie group with neutral element e and Lie algebra $\mathfrak{h}$. The Lie group structure of $\mathrm{A}(H)$, the group of Lie group automorphisms of H, is given by the following group isomorphism:

$$\mathrm{A}(H) \longrightarrow \mathrm{A}(\mathfrak{h}) \ , \ f \longmapsto d_e f$$

Moreover, if H is exponential, its inverse is the map

$$\mathrm{A}(\mathfrak{h}) \longrightarrow \mathrm{A}(H) \ , \ \varphi \longmapsto \exp^H \circ \varphi \circ \log^H .$$

□

For given (Lie) groups G, H and a (smooth) action $\mu\colon G \times H \to H$ by (Lie) group automorphisms, one defines the *semidirect product of G and H via μ* as the (Lie) group $G \ltimes_\mu H$ with underlying set (manifold) $G \times H$ and group structure defined as follows:

$$\forall_{(g_1,h_1),(g_2,h_2)\in G\times H}\ (g_1,h_1)(g_2,h_2) = \big(g_1 g_2, \mu(g_2^{-1}, h_1)h_2\big).$$

Note that for $(g,h) \in G \ltimes_\mu H$ we have $(g,h)^{-1} = \big(g^{-1}, \mu(g, h^{-1})\big)$.

If the action μ is trivial, i.e. $\forall_{g\in G,\, h\in H}\ \mu(g,h) = h$, one obtains the ordinary direct product. In the case of Lie groups G and H, the exponential map $\exp^{G\times H}$ is known to be the direct product of $\exp^G$ and $\exp^H$. If the action is not trivial, the situation becomes a little more complicated:

THEOREM 4.2. *Let G, H be connected Lie groups and $\mu\colon G \times H \to H$ a smooth action by Lie group automorphisms. Denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ and let $\phi := (d_{e_G}\mu_1)\colon \mathfrak{g} \to \mathfrak{d}(\mathfrak{h})$, where $\mu_1\colon G \to \mathrm{A}(\mathfrak{h})$ is given by $\mu_1(g) = d_{e_H}\mu(g,\ldots) = \mathrm{Ad}_g^{G\ltimes_\mu H}$.*

(i) The Lie algebra of $G \ltimes_\mu H$ is $\mathfrak{g} \ltimes_\phi \mathfrak{h}$. This Lie algebra is called semidirect product of $\mathfrak{g}$ and $\mathfrak{h}$ via ϕ. Its underlying vector space is $\mathfrak{g} \times \mathfrak{h}$ and the bracket for $(X_1,Y_1),(X_2,Y_2) \in \mathfrak{g} \times \mathfrak{h}$ is given by

$$[(X_1,Y_1),(X_2,Y_2)] = \big([X_1,X_2]_{\mathfrak{g}}, [Y_1,Y_2]_{\mathfrak{h}} + \phi(X_1)(Y_2) - \phi(X_2)(Y_1)\big).$$

In the sequel we shall identify $X \equiv (X,0)$ and $Y \equiv (0,Y)$.

(ii) For $(X,Y) \in \mathfrak{g} \ltimes_\phi \mathfrak{h}$ one has $\exp^{G \ltimes_\mu H}((X,Y)) = (\exp^G(X), \gamma(1))$, where $\gamma\colon \mathbb{R} \to H$ is the solution of

$$\dot\gamma(t) = (d_{e_H} R_{\gamma(t)})\big(\exp^{A(\mathfrak{h})}(-t\,\mathrm{ad}(X)|_{\mathfrak{h}})(Y)\big), \quad \gamma(0) = e_H.$$

Here R_a denotes the right translation by an element $a \in H$.

Proof. The proof of (i) can be found in [57]. We give a proof of (ii). Given a Lie group homomorphism f between Lie groups, we denote its differential at the neutral element by f_*.

For $(g_0,h_0),(g,h) \in G \ltimes_\mu H$ we have $R_{(g_0,h_0)}(g,h) = (R_{g_0}(g), R_{h_0}(\mu(g_0^{-1},h))$, and this yields for $(X,Y) \in \mathfrak{g} \ltimes_\phi \mathfrak{h}$

$$(R_{(g_0,h_0)})_*\big((X,Y)\big) = \big((R_{g_0})_*(X), (R_{h_0})_*\big(\mu_1(g_0^{-1})(Y)\big)\big).$$

Since $\big(\gamma_1(t),\gamma_2(t)\big) := \exp^{G\ltimes_\mu H}\big(t\,(X,Y)\big)$ is the integral curve through the identity of both the right- and left-invariant vector fields associated to (X,Y), the last equation implies that $\big(\gamma_1(t),\gamma_2(t)\big)$ is the solution of the following differential equations:

$$\gamma_1(0) = e_G, \dot\gamma_1(t) = (R_{\gamma_1(t)})_*(X), \tag{2}$$

$$\gamma_2(0) = e_H, \dot\gamma_2(t) = (R_{\gamma_2(t)})_*(\mu_1(\gamma_1(-t))(Y)). \tag{3}$$

$\gamma_1(t) = \exp^G(t\,X)$ is the solution of (2), and this implies

$$\mu_1(\gamma_1(-t)) = \mathrm{Ad}^{G\ltimes_\mu H}_{\gamma_1(-t)}|_{\mathfrak{h}} = \exp^{A(\mathfrak{h})}(-t\,\mathrm{ad}(X)|_{\mathfrak{h}}),$$

i.e. (3) is equivalent to $\gamma_2(0) = e_H,\ \dot\gamma_2(t) = (R_{\gamma_2(t)})_*(\exp^{A(\mathfrak{h})}(-t\,\mathrm{ad}(X)|_{\mathfrak{h}})(Y))$. So the theorem is proven. □

A connected and simply-connected solvable Lie group G with nilradical N is called *almost nilpotent* if it can be written as $G = \mathbb{R} \ltimes_\mu N$. Moreover, if N is abelian, i.e. $N = \mathbb{R}^n$, then G is called *almost abelian*.

Let $G = \mathbb{R} \ltimes_\mu N$ be an almost nilpotent Lie group. Since N has codimension one in G, we can consider μ as a one-parameter group $\mathbb{R} \to \mathrm{A}(N)$. By Theorem 4.1, there exists $\varphi \in \mathfrak{d}(\mathfrak{n})$ with

$$\forall_{t\in\mathbb{R}}\ \mu(t) = \exp^N \circ \exp^{Aut(|\mathfrak{n}|)}(t\varphi) \circ \log^N.$$

Choosing a basis of $|\mathfrak{n}|$, we can identify $\mathrm{Aut}(|\mathfrak{n}|)$ with a subset of $\mathfrak{gl}(n,\mathbb{R})$ and get

$$\forall_{t\in\mathbb{R}}\ d_e\big(\mu(t)\big) \in \exp^{GL(n,\mathbb{R})}\big(\mathfrak{gl}(n,\mathbb{R})\big).$$

Note, if N is abelian, the exponential map $\exp^N\colon \mathfrak{n} \to N$ is the identity. These considerations make it interesting to examine the image of $\exp^{GL(n,\mathbb{R})}$.

Theorem 4.3 ([45, Theorem 6]). *M is an element of $\exp^{GL(n,\mathbb{R})}(\mathfrak{gl}(n,\mathbb{R}))$ if and only if the real Jordan form of M contains in the form of pairs the blocks belonging to real negative eigenvalues λ_i^-, whenever there exist real negative eigenvalues λ_i^- of M. I.e. the block belonging to such a λ_i^- is of the following form*

$$\bigoplus_{j=1}^{n_i} \begin{pmatrix} J_{n_{ij}} & 0 \\ 0 & J_{n_{ij}} \end{pmatrix}$$

with

$$J_{n_{ij}} = \begin{pmatrix} \lambda_i^- & 1 & & 0 \\ & \lambda_i^- & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i^- \end{pmatrix} \in \mathrm{M}(n_{ij}, n_{ij}; \mathbb{R}).$$

□

We are now going to derive some facts that follow from the existence of a lattice in an almost nilpotent Lie group.

THEOREM 4.4 ([55]). *Let $G = \mathbb{R} \ltimes_\mu N$ be an almost nilpotent and completely solvable Lie group containing a lattice Γ.*

Then there is a one-parameter group $\nu\colon \mathbb{R} \to \mathrm{A}(N)$ such that $\nu(k)$ preserves the lattice $\Gamma_N := \Gamma \cap N$ for all $k \in \mathbb{Z}$. Γ is isomorphic to $\mathbb{Z} \ltimes_\nu \Gamma_N$ and G/Γ is diffeomorphic to $\big(\mathbb{R} \ltimes_\nu N\big)/\big(\mathbb{Z} \ltimes_\nu \Gamma_N\big)$.

Moreover, there are $t_1 \in \mathbb{R} \setminus \{0\}$ and an inner automorphism $I_{n_1} \in \mathrm{A}(N)$ such that $\nu(1) = \mu(t_1) \circ I_{n_1}$.

Proof. We know that Γ_N is a lattice in N and $\mathrm{im}(\Gamma \to G/N) \cong \Gamma/\Gamma_N$ is a lattice in $G/N \cong \mathbb{R}$. Therefore, $\Gamma/\Gamma_N \cong \mathbb{Z}$ is free, and the following exact sequence is split:

$$\{1\} \longrightarrow \Gamma_N \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow \{0\},$$

i.e. there is a group-theoretic section $s\colon \mathbb{Z} \to \Gamma$. [49, Théorème II.5] states that a group homomorphism from a lattice of completely solvable Lie group into another completely solvable Lie group uniquely extends to a Lie group homomorphism of the Lie groups. Hence, s extends uniquely to a one-parameter group $s\colon \mathbb{R} \to G$. Therefore,

$$\nu\colon \mathbb{R} \longrightarrow \mathrm{A}(N), \quad \nu(t)(n) = s(t) \cdot n \cdot s(t)^{-1}$$

is a one-parameter group with $\forall_{k\in\mathbb{Z}}\ \nu(k)(\Gamma_N) = \Gamma_N$, the lattice Γ is isomorphic to $\mathbb{Z} \ltimes_\nu \Gamma_N$ and G/Γ is diffeomorphic to $\big(\mathbb{R} \ltimes_\nu N\big)/\big(\mathbb{Z} \ltimes_\nu \Gamma_N\big)$.

Let $\gamma_1 := s(1) \in (\Gamma \setminus \Gamma_N) \subset \mathbb{R} \ltimes_\mu N$. There are unique $t_1 \in \mathbb{R} \setminus \{0\}$, $n_1 \in N$ with $\gamma_1 = t_1 \cdot n_1$, where we identify $t_1 \equiv (t_1, e_N) \in G$ and $n_1 \equiv (0, n_1) \in G$. Since $G = \mathbb{R} \ltimes_\nu N$ and $G = \mathbb{R} \ltimes_\mu N$ with the same normal subgroup N of G, one has for all $n \in N$

$$\nu(1)(n) = \gamma_1 \cdot n \cdot \gamma_1^{-1} = t_1 \cdot n_1 \cdot n \cdot n_1^{-1} \cdot t_1^{-1} = \mu(t_1)(n_1 \cdot n \cdot n_1^{-1}) = \mu(t_1)(I_{n_1}(n)),$$

from where the theorem follows. □

COROLLARY 4.5. *Let $G = \mathbb{R} \ltimes_\mu N$ be an almost nilpotent (not necessary completely solvable) Lie group containing a lattice Γ. Again, denote by $\Gamma_N := \Gamma \cap N$ the induced lattice in the nilradical of G.*

Then there exist $t_1 \in \mathbb{R} \setminus \{0\}$, a group homomorphism $\nu\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma_N)$, and an inner automorphism I_{n_1} of N such that $\Gamma \cong \mathbb{Z} \ltimes_\nu \Gamma_N$ and $\nu(1) = \mu(t_1) \circ I_{n_1}$.

If G is almost abelian, then a basis transformation yields $\Gamma \cong t_1\mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^n}} \mathbb{Z}^n$.

Proof. We argue as in the last proof. But we do not use [49, Théorème 5] and get only a group homomorphism $\nu\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma_N)$ (defined on $\mathbb{Z}$ instead of $\mathbb{R}$). For general N, the calculation at the end of the proof implies the claim.

Since an abelian group has only one inner automorphism, in the almost abelian case this yields $\nu(1) = \mu(t_1)|_{\Gamma_N}$, so ν can be extended to $\nu\colon \mathbb{R} \to \mathrm{A}(\mathbb{R}^n)$ via $\nu(t) := \mu(t \cdot t_1)$. Further, by Theorem 3.1, we have $\Gamma_N \cong \mathbb{Z}^n$. □

Hence we have seen, that the existence of a lattice in an almost nilpotent Lie group implies that a certain Lie group automorphism must preserve a lattice in the (nilpotent) nilradical. The next theorem deals with such automorphisms.

THEOREM 4.6. *Let N be a connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, $f_* \in \mathrm{A}(\mathfrak{n})$, and $f := \exp^N \circ f_* \circ \log^N \in \mathrm{A}(N)$, i.e. $d_e f = f_*$. Assume that f preserves a lattice Γ in N.*

Then there exists a basis $\mathfrak{X}$ of $\mathfrak{n}$ such that $M_{\mathfrak{X}}(f_*) \in \mathrm{GL}(n,\mathbb{Z})$, where $M_{\mathfrak{X}}(f_*)$ denotes the matrix of f_* with respect to $\mathfrak{X}$.

Moreover, if there are a one-parameter group $\mu\colon \mathbb{R} \to \mathrm{A}(N)$ and $t_0 \neq 0$ such that $\mu(t_0) = f$, i.e. $d_e(\mu(t_0)) = f_*$, then $\det\big(d_e(\mu(\ldots))\big) \equiv 1$.

Proof. By Theorem 2.1 (ii),

$$\mathcal{L} := \langle \log^N(\Gamma)\rangle_{\mathbb{Z}} = \{\sum_{i=1}^{m} k_i\, V_i \,|\, m \in \mathbb{N}_+, k_i \in \mathbb{Z}, V_i \in \log^N(\Gamma)\}$$

is a lattice in $\mathfrak{n}$. Therefore, there exists a basis $\mathfrak{X} = \{X_1, \ldots, X_n\}$ of $\mathfrak{n}$ such that $\mathcal{L} = \langle \mathfrak{X}\rangle_{\mathbb{Z}}$.

Since $f(\Gamma) \subset \Gamma$, we have $f_*\big(\log^N(\Gamma)\big) \subset \log^N(\Gamma)$. This implies $f_*(\mathcal{L}) \subset \mathcal{L}$ and hence, $M_{\mathfrak{X}}(f_*) \in \mathrm{GL}(n,\mathbb{Z})$.

Further, if $\mu(t_0) = f$ with μ, $t_0 \neq 0$ as in the statement of the theorem, then the map $\Delta := \det \circ d_e(\mu(\ldots))\colon (\mathbb{R},+) \to (\mathbb{R}\setminus\{0\},\cdot)$ is a continuous group homomorphism with $\Delta(0) = 1$ and $\Delta(t_0) = \pm 1$, i.e. $\Delta \equiv 1$. □

Obviously, a one-parameter group μ in the automorphism group of an abelian Lie group with $\mu(t_0)$ integer valued for $t_0 \neq 0$ defines a lattice in $\mathbb{R} \ltimes_\mu \mathbb{R}^n$. It is easy to compute the first Betti number of the corresponding solvmanifold, as the next proposition will show. Before stating it, we mention that the situation becomes more complicated in the case of a non-abelian and nilpotent group N.

Let a one-parameter group $\mu\colon \mathbb{R} \to \mathrm{A}(N)$ be given and $t_0 \neq 0$ such that $d_e(\mu(t_0))$ is an integer matrix with respect to a basis $\mathfrak{X}$ of the Lie algebra $\mathfrak{n}$ of N. In general, this does not enable us to define a lattice in $\mathbb{R} \ltimes_\mu N$. But if $\Gamma_N := \exp^N(\langle \mathfrak{X}\rangle_{\mathbb{Z}})$ is a lattice in N, i.e. Γ_N is a lattice group, then this is possible.

PROPOSITION 4.7. *Let $\mu\colon \mathbb{R} \to \mathrm{SL}(n,\mathbb{R})$ be a one-parameter group such that $\mu(1) = (m_{ij})_{i,j} \in \mathrm{SL}(n,\mathbb{Z})$.*

Then $M := (\mathbb{R} \ltimes_\mu \mathbb{R}^n)/(\mathbb{Z} \ltimes_\mu \mathbb{Z}^n)$ is a solvmanifold with

$$\pi_1(M) = \langle e_0, e_1, \ldots, e_n \;|\; \forall_{i \in \{1,\ldots,n\}}\; e_0 e_i e_0^{-1} = e_1^{m_{1i}} \cdots e_n^{m_{ni}} \quad \forall_{i,j \in \{1,\ldots,n\}}\; [e_i, e_j] = 1 \rangle$$

and $b_1(M) = n + 1 - \mathrm{rank}\big(\mu(1) - \mathrm{id}\big)$.

Proof. The statement about the fundamental group is clear. Therefore, we get

$$H_1(M,\mathbb{Z}) = \langle e_0, e_1, \ldots, e_n \;|\; \forall_{i \in \{1,\ldots,n\}}\; e_1^{m_{1i}} \cdots e_i^{m_{ii}-1} \cdots e_n^{m_{ni}} = 1 \quad \forall_{i,j \in \{0,\ldots,n\}}\; [e_i, e_j] = 1 \rangle$$

and this group is the abelianisation of

$$\mathbb{Z} \oplus \langle e_1, \dots, e_n \,|\, \forall_{i \in \{1,\dots,n\}}\, e_1^{m_{1i}} \cdots e_i^{m_{ii}-1} \cdots e_n^{m_{ni}} = 1 \rangle.$$

Now, the proof of the theorem about finitely generated abelian groups (see e.g. [7]) shows $H_1(M,\mathbb{Z}) = \mathbb{Z}^{n-k+1} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{d_i}$, where $d_1, \dots, d_k \in \mathbb{N}_+$ denote the elementary divisors of $\mu(1) - \mathrm{id}$. The proposition follows. □

We finally mention a result of Gorbatsevich. In view of Theorem 3.10 (iii), it enables us to compute the minimal model of a wide class of solvmanifolds which are discrete quotients of almost abelian Lie groups.

THEOREM 4.8 ([25, Theorem 4]). *Let $\mu\colon \mathbb{R} \to \mathrm{SL}(n,\mathbb{R})$ be a one-parameter group such that $\mu(1) = \exp^{SL(n,\mathbb{R})}(\dot{\mu}(0)) \in \mathrm{SL}(n,\mathbb{Z})$. Denote by $\lambda_1, \dots, \lambda_n$ the (possibly not pairwise different) roots of $\dot{\mu}(0)$. Then $\Gamma := (\mathbb{Z} \ltimes_\mu \mathbb{Z}^n)$ is a lattice in $G := (\mathbb{R} \ltimes_\mu \mathbb{R}^n)$.*

The Zariski closures of $\mathrm{Ad}\,(\Gamma)$ and $\mathrm{Ad}\,(G)$ coincide if and only if the number πi is not representable as a linear combination of the numbers λ_k with rational coefficients. □

5. Three-dimensional solvmanifolds.

PROPOSITION 5.1 ([3]). *Every 3-dimensional connected and simply-connected solvable non-nilpotent Lie group G that possesses a lattice Γ has a 2-dimensional nilradical. The Lie group can be written as $G = \mathbb{R} \ltimes_\mu \mathbb{R}^2$ and the lattice as $\Gamma = \mathbb{Z} \ltimes_\mu \mathbb{Z}^2$.*

Proof. This is a direct consequence of Proposition 3.4 and Corollary 4.5. □

THEOREM 5.2. *A three-dimensional solvmanifold G/Γ is non-formal if and only if $b_1(G/\Gamma) = 2$. In this case, G/Γ is diffeomorphic to a nilmanifold.*

Proof. By Theorem 2.6, it suffices to consider the case when G is solvable and non-nilpotent. The last proposition implies that there is a map $\nu\colon \mathbb{Z} \to \mathrm{SL}(2,\mathbb{Z})$ such that $\Gamma = \mathbb{Z} \ltimes_\nu \mathbb{Z}^2$.

If none of the roots of $\nu(1)$ equals 1, Proposition 4.7 implies $b_1 = 1$, so G/Γ is formal by Theorem 1.1.10.

Assume that $\nu(1)$ possesses the double root 1. Then Proposition 4.7 implies $b_1 = 3$ if $\nu(1)$ is diagonalisable and $b_1 = 2$ if $\nu(1)$ is not diagonalisable.

Case A: $\nu(1)$ is diagonalisable

Recall that a solvmanifold is uniquely determined by its fundamental group. Therefore, we can assume $G = \mathbb{R} \ltimes_\mu \mathbb{R}^2$ and $\Gamma = \mathbb{Z} \ltimes_\mu \langle v_1, v_2 \rangle_\mathbb{Z}$ with linearly independent $v_1, v_2 \in \mathbb{R}^2$ and $\mu(t) \equiv \mathrm{id}$. In this case, G/Γ is a torus which is formal.

Case B: $\nu(1)$ is not diagonalisable

In this case, we can assume $G = \mathbb{R} \ltimes_\mu \mathbb{R}^2$ as well as $\Gamma = \mathbb{Z} \ltimes_\mu \langle v_1, v_2 \rangle_\mathbb{Z}$ with linearly independent $v_1, v_2 \in \mathbb{R}^2$ and

$$\mu(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} = \langle T, X, Y \,|\, [T, Y] = X \rangle$ of G is nilpotent, so G/Γ is a nilmanifold with $b_1 = 2$. Therefore, it cannot be a torus and is not formal by Theorem 2.6. □

In [3, Chapter III ?] the three-dimensional solvmanifolds which have no nilmanifold structure are examined. This, together with the last theorem, yields a "cohomological" classification of three-dimensional solvmanifolds.

TABLE 5.1: *3-dimensional solvmanifolds*

	$b_1(G/\Gamma)$	G/Γ formal	Nilmfd. [3]	c.s. [4]
a)	3	yes	Torus	yes
b)	2	no	yes	yes
c)	1	yes	no	yes
d)	1	yes	no	no

THEOREM 5.3. *Every* 3*-dimensional solvmanifold* G/Γ *is contained in Table 5.1. In particular,* G/Γ *is non-formal if and only if it is a non-toral nilmanifold.* □

THEOREM 5.4. *Every lattice in the unique* 3*-dimensional connected and simply-connected non-abelian nilpotent Lie group*

$$U_3(\mathbb{R}) := \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}\}$$

is isomorphic to $\Gamma_{3,n} := \Gamma_{3,n}(\mathbb{Z}) := \{ \begin{pmatrix} 1 & x & \frac{z}{n} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z}\}$ *with* $n \in \mathbb{N}_+$.

Therefore, any three-dimensional nilmanifold with $b_1 = 2$ *is of the form* $U_3(\mathbb{R})/\Gamma_{3,n}(\mathbb{Z})$.

$\Gamma_{3,n}(\mathbb{Z})$ *is presented by* $\langle e_1, e_2, e_3 \,|\, [e_1, e_2] = e_3^n$ *and* e_3 *central* $\rangle$.

Proof. The proof follows from [3, Chapter III §7]. □

Sometimes, we shall write (x, y, z) for the corresponding matrix in $U_3(\mathbb{R})$.

PROPOSITION 5.5.

(i) $[U_3(\mathbb{R}), U_3(\mathbb{R})] = Z(U_3(\mathbb{R})) = \{(0, 0, z) \,|\, z \in \mathbb{R}\}$, $U_3(\mathbb{R})/Z(U_3(\mathbb{R})) \cong \mathbb{R}^2$

(ii) Every Lie group homomorphism $f\colon U_3(\mathbb{R}) \to U_3(\mathbb{R})$ *induces natural Lie group homomorphisms*

$$f_Z\colon Z(U_3(\mathbb{R})) \longrightarrow Z(U_3(\mathbb{R}))$$

and

$$\begin{array}{rcl} \overline{f}\colon\ U_3(\mathbb{R})/Z(U_3(\mathbb{R})) & \longrightarrow & U_3(\mathbb{R})/Z(U_3(\mathbb{R})). \\ {[(x, y, 0)] = [(x, y, z)]} & \longmapsto & [f\big((x, y, z)\big)] = [(f_1(x, y, 0), f_2(x, y, 0), 0)] \end{array}$$

$\overline{f}$ *uniquely determines* f_Z*, and* $\overline{f}$ *is an automorphism if and only if* f *is such.*

(iii) Let $\gamma_1 = (a_1, b_1, \frac{c_1}{n}), \gamma_2 = (a_2, b_2, \frac{c_2}{n}) \in \Gamma_{3,n}$*. Then there is a unique homomorphism* $g\colon \Gamma_{3,n} \to \Gamma_{3,n}$ *such that* $g\big((1, 0, 0)\big) = \gamma_1$ *and* $g\big((0, 1, 0)\big) = \gamma_2$*. Moreover,* $g\big((0, 0, \frac{1}{n})\big) = \big(0, 0, \frac{1}{n}(a_1 b_2 - a_2 b_1)\big)$.
One has $\Gamma_{3,n}/Z(\Gamma_{3,n}) \cong \mathbb{Z}^2$*, and* g *is an isomorphism if and only if*

$$\overline{g}\colon\ \Gamma_{3,n}/Z(\Gamma_{3,n}) \longrightarrow \Gamma_{3,n}/Z(\Gamma_{3,n})$$

is an isomorphism, i.e. $a_1 b_2 - a_2 b_1 = \pm 1$.

[3] possesses the structure of a solvmanifold as quotient of a nilpotent Lie group

[4] possesses the structure of a solvmanifold as quotient of a **c**ompletely **s**olvable Lie group

Proof. (i) is trivial.

ad (ii): Let $f\colon U_3(\mathbb{R}) \to U_3(\mathbb{R})$ be a Lie group homomorphism. Then

$$f\big((0,0,z)\big) = [f\big((z,0,0)\big), f\big((0,1,0)\big)] \in Z(U_3(\mathbb{R})), \tag{4}$$

i.e. $f\big(Z(U_3(\mathbb{R}))\big) \subset Z(U_3(\mathbb{R}))$. Moreover, one has for $(a,b,c) := f\big((x,y,0)\big)$

$$(a,b,0)^{-1} \cdot (a,b,c) = (-a,-b,-ab)\cdot(a,b,c) = (0,0,-2ab+c) \in Z(U_3(\mathbb{R})),$$

and therefore $[(a,b,0)] = \overline{f}([(x,y,0)])$. Now, (4) implies that f_Z is uniquely determined by $\overline{f}$.

Assume, f is an isomorphism. Then (4) also holds for f^{-1} and we get $f\big(Z(U_3(\mathbb{R}))\big) = Z(U_3(\mathbb{R}))$, i.e. f_Z is an isomorphism of the additive group $\mathbb{R}$. Since f is continuous, there exists $m \in \mathbb{R}\setminus\{0\}$ such that $f_Z\big((0,0,z)\big) = (0,0,mz)$. Denote by $(f_{ij})_{1\le i,j\le 2}$ the matrix of $\overline{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ of the vector space $\mathbb{R}^2$. One calculates

$$\begin{aligned}(0,0,\det(f_{ij})) &= [(f_{11},f_{21},0),(f_{12},f_{22},0)] = [f\big((1,0,0)\big), f\big((0,1,0)\big)]\\ &\overset{(4)}{=} (0,0,m),\end{aligned}$$

so $\overline{f}$ is an automorphism, since $m \neq 0$.

Conversely, if $\overline{f}$ is an automorphism, then the homomorphism f_Z is given by $f_Z\big((0,0,z)\big) = (0,0,\det(\overline{f})z)$ which is even an automorphism. Therefore, the 5-Lemma implies that f is an automorphism.

ad (iii): Let γ_1, γ_2 be as in (iii). Then $[\gamma_1,\gamma_2] = \big(0,0,\frac{1}{n}(a_1b_2 - a_2b_1)\big)^n$ and this implies the existence of the (unique) homomorphism g with the mentioned properties.

If g is an isomorphism, then $g(Z(\Gamma_{3,n})) = Z(\Gamma_{3,n}) = \{(0,0,\frac{z}{n}) \mid z \in \mathbb{Z}\}$, and therefore $|a_1b_2 - a_2b_1| = 1$. Since the matrix of $\overline{g}$ has determinant $a_1b_2 - a_2b_1$, $\overline{f}$ is an isomorphism.

Again, the converse is trivial. □

Theorem 5.6. *As a set, the group of Lie group automorphisms* $\mathrm{A}(U_3(\mathbb{R}))$ *equals* $\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2$*, the group law is given by*

$$(A,a)\circ(B,b) \longmapsto \big(AB, \det(B)B^{-1}a + \det(A)b\big), \tag{5}$$

and for $f = (A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}u\\v\end{pmatrix}) \in \mathrm{A}(U_3(\mathbb{R}))$ *and* $(x,y,z) \in U_3(\mathbb{R})$ *we have*

$$\begin{aligned}f\big((x,y,z)\big) =&\big(\alpha x + \beta y, \gamma x + \delta y,\\ &\qquad \det(A)z + \beta\gamma xy + \frac{\alpha\gamma}{2}x^2 + \frac{\beta\delta}{2}y^2 + uy - vx\big).\end{aligned} \tag{6}$$

Proof. Let $f \in \mathrm{A}(U_3(\mathbb{R}))$ and $(x,y,z) \in U_3(\mathbb{R})$ be given. We have to show that there is $(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}u\\v\end{pmatrix}) \in \mathrm{GL}(2,\mathbb{R})\times\mathbb{R}^2$ such that $f\big((x,y,z)\big)$ satisfies (6). Then a short computation yields (5).

Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ be the matrix of $\overline{f}$ with respect to the canonical basis of $\mathbb{R}^2$. We showed in the last proof $f\big((0,0,z)\big) = (0,0\det(\overline{f})z)$.

There exist smooth functions $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ with

$$f\big((x,0,0)\big) = (\alpha x, \gamma x, f_1(x)),$$
$$f\big((0,y,0)\big) = (\beta y, \delta y, f_2(y)).$$

We set $u := f_2'(0)$ and $v := -f_1'(0)$. The homomorphism property of f implies

$$\frac{1}{h}(f_1(x+h) - f_1(x)) = \frac{f_1(x) - f_1(0)}{h} + \alpha\gamma x,$$
$$\frac{1}{h}(f_2(y+h) - f_2(y)) = \frac{f_2(y) - f_2(0)}{h} + \beta\delta y,$$

and this yields

$$f_1(x) = -vx + \frac{\alpha\gamma}{2}x^2,$$
$$f_2(y) = uy + \frac{\beta\delta}{2}y^2.$$

Using $(x,y,z) = (0,y,0)(x,0,0)(0,0,z)$, one computes (6). □

Corollary 5.7. *$f = (A, \begin{pmatrix} u \\ v \end{pmatrix}) \in \mathrm{A}(U_3(\mathbb{R}))$ with $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ lies on a one-parameter group of $\mathrm{A}(U_3(\mathbb{R}))$ if and only if A lies one a one-parameter group of $GL(2,\mathbb{R})$.*

If $\nu_t = \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}$ denotes a one-parameter group with $\nu_1 = A$, then the map $\mu_t\colon \mathbb{R} \to \mathrm{A}(U_3(\mathbb{R}))$ defined by

$$\begin{aligned}\mu_t\big((x,y,z)\big) \;=\; &\big(\alpha_t x + \beta_t y, \gamma_t x + \delta_t y,\\ &\underbrace{(\alpha_t\delta_t - \beta_t\gamma_t)}_{=1} z + \beta_t\gamma_t xy + \tfrac{\alpha_t\gamma_t}{2}x^2 + \tfrac{\beta_t\delta_t}{2}y^2 + tuy - tvx\big)\end{aligned}$$

is a one-parameter group with $\mu_1 = f$.

Proof. The only claim that is not obvious is the fact that μ_t defines a one-parameter group. Using $\nu_{t+s} = \nu_t \circ \nu_s$, this can be seen by a short calculation. □

6. Four-dimensional solvmanifolds.

Proposition 6.1. *Every 4-dimensional connected and simply-connected solvable non-nilpotent Lie group G that possesses a lattice Γ has a 3-dimensional nilradical N which is either $\mathbb{R}^3$ or $U_3(\mathbb{R})$. Therefore, G/Γ fibers over S^1 (this is the Mostow bundle) and the Lie group can be written as $G = \mathbb{R} \ltimes_\mu N$. If N is abelian, a basis transformation yields $\Gamma = \mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^3}} \mathbb{Z}^3$. Otherwise, Γ is isomorphic to $\mathbb{Z} \ltimes_\nu \Gamma_{3,n}$, where $\nu\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma_{3,n})$ is a group homomorphism with*

$$\begin{aligned}\nu(1)(x,y,\tfrac{z}{n}) \;=\; &\big(a_1x + a_2y, \quad b_1x + b_2y, \quad a_2b_1xy + a_1b_1\tfrac{x(x-1)}{2} + a_2b_2\tfrac{y(y-1)}{2}\\ &+ \tfrac{1}{n}(c_1x + c_2y + (a_1b_2 - a_2b_1)z)\big),\end{aligned}$$

where $c_1, c_2, \in \mathbb{Z}$, and $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ is the matrix of $\overline{\nu(1)}$ with respect to the canonical basis of the $\mathbb{Z}$-module $\mathbb{Z}^2 = \Gamma_{3,n}/Z(\Gamma_{3,n})$. Moreover, $\overline{\nu}(1)$ lies on a one-parameter group $\mathbb{R} \to \mathrm{A}(U_3(\mathbb{R})/Z(U_3(\mathbb{R}))) = \mathrm{GL}(2,\mathbb{R})$, i.e. $\overline{\nu}(1) \in \mathrm{SL}(2,\mathbb{R})$.

Proof. From [46, Theorem 3.1.10] follows $\dim N = 3$ and $G = \mathbb{R} \ltimes_\mu N$. If N is abelian, Corollary 4.5 implies that we can assume $\Gamma = \mathbb{Z} \ltimes_{\mu|_{\mathbb{Z}^3}} \mathbb{Z}^3$.

Assume now that N is not abelian, i.e. $N = U_3(\mathbb{R})$. $\Gamma_N = \Gamma \cap N$ is a lattice in N and by Theorem 5.4, we have $\Gamma_N = \Gamma_{3,n}$. By Corollary 4.5, there is a homomorphism $\nu\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma_{3,n})$ with $\Gamma \cong \mathbb{Z} \ltimes_\nu \Gamma_{3,n}$. By Proposition 5.5(iii), $(a_1, b_1, \frac{c_1}{n}) := \nu(1)\big((1,0,0)\big)$ and $(a_2, b_2, \frac{c_2}{n}) := \nu(1)\big((0,1,0)\big) \in \Gamma_{3,n}$ determine $\nu(1)$. Since $(x, y, \frac{z}{n}) = (0,1,0)^y(1,0,0)^x(0,0,\frac{1}{n})^z$, a short computation yields the claimed formula for $\nu(1)\big((x, y, \frac{z}{n})\big)$.

Further, Corollary 5.7 implies that $\overline{\nu}(1)$ lies on a one-parameter group. □

THEOREM 6.2. *Every 4-dimensional solvmanifold G/Γ is contained in Table 6.1. In particular, G/Γ is non-formal if and only if it is a non-toral nilmanifold.*

TABLE 6.1: 4-*dimensional solvmanifolds*

	$b_1(G/\Gamma)$	G/Γ formal	symplectic	complex	Kähler	Nilmfd. [5]	c.s. [6]
a)	4	yes	yes	Torus	yes	Torus	yes
b)	3	no	yes	PKS [7]	no	yes	yes
c)	2	yes	yes	no	no	no	yes
d)	2	yes	yes	HS [8]	yes	no	no
e)	2	no	yes	no	no	yes	yes
f)	1	yes	no	no	no	no	yes
g)	1	yes	no	IS^0 [9]	no	no	no
h)	1	yes	no	IS^+ [10]	no	no	yes
i)	1	yes	no	SKS [11]	no	no	no

Proof. Apart from the column on formality the theorem follows from works of Geiges [23] and Hasegawa [29]. (Attention: In [29] a more general notion of solvmanifold is used!)

A decomposable four-dimensional connected and simply-connected nilpotent Lie group is abelian or has a two-dimensional center. The only connected and simply-connected indecomposable nilpotent Lie group of dimension four has a two-dimensional commutator. By Propositions 2.3 and 2.2, the corresponding nilmanifolds have the structure of orientable T^2-bundles over T^2. (The orientability follows from the total spaces' orientability.)

¿From a result of Geiges [23, Theorems 1 and 3] follows that they are contained in Table 6.1. (Recall that a nilmanifold is formal if and only if it is a torus.) In particular, every four-dimensional nilmanifold is symplectic.

Now, we regard a lattice $\Gamma = \mathbb{Z} \ltimes_\nu \Gamma_N$, $\Gamma_N \in \{\mathbb{Z}^3, \Gamma_{3,n}(\mathbb{Z})\}$, in a Lie group $G = \mathbb{R} \ltimes_\mu N$ as in the last proposition.

We expand Hasegawa's argumentation in [29] by the aspect of formality and consider the "roots" of $\nu(1)$. Recall, Theorem 3.1 implies that a solvmanifold is

[5] possesses the structure of a solvmanifold as quotient of a nilpotent Lie group
[6] possesses the structure of a solvmanifold as quotient of a **c**ompletely **s**olvable Lie group
[7] **P**rimary**K**odaira **S**urface
[8] **H**yperelliptic **S**urface
[9] **I**noue Surface of Type $\mathbf{S^0}$
[10] **I**noue Surface of Type $\mathbf{S^+}$
[11] **S**econdary **K**odaira **S**urface

determined by its fundamental group. Below, we shall use this fact several times.

Case A.: $\Gamma_N = \mathbb{Z}^3$
By Proposition 6.1, ν extends to a one-parameter group $\mathbb{R} \to \mathrm{SL}(3,\mathbb{R})$. Denote by $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ the roots of $\nu(1) \in \mathrm{SL}(3,\mathbb{Z})$, i.e. $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1$. We get from Theorem 4.3 and Lemma B.4 that the following subcases can occur:

- A.1.) $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$
  - A.1.1.) $\exists_{i_0}\ \lambda_{i_0} = 1$ (w.l.o.g. $\lambda_1 = 1$)
    - A.1.1.1.) $\lambda_2 = \lambda_3 = 1$
    - A.1.1.2.) $\lambda_2 = \lambda_3^{-1} \in \mathbb{R} \setminus \{1\}$
  - A.1.2.) $\forall_i\ \lambda_i \neq 1$
    - A.1.2.1.) $\nu(1)$ is diagonalisable
    - A.1.2.2.) $\nu(1)$ is not diagonalisable
- A.2.) $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$ and $\nu(1)$ is diagonalisable
- A.3.) $\exists_{i_0}\ \lambda_{i_0} \in \mathbb{C} \setminus \mathbb{R}$ (w.l.o.g. $\lambda_2 = \overline{\lambda_3} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_1 \in \mathbb{R}_+$)
  - A.3.1.) $\lambda_1 = 1$
  - A.3.2.) $\lambda_1 \neq 1$

One can now check that the cases give the mentioned propertys. □

Below, we give examples for each of the nine types of four-dimensional solvmanifolds. The Lie algebras of the connected and simply-connected four-dimensional solvable Lie groups that admit lattices are listed in Table A.1 in Appendix A.

Example. The following manifolds belong to the corresponding row in Table 6.1.

a) $\mathbb{R}^4/\mathbb{Z}^4$

b) $(\mathbb{R} \ltimes_{\mu_b} \mathbb{R}^3)/(\mathbb{Z} \ltimes_{\mu_b} \mathbb{Z}^3)$, $\mu_b(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

c) $(\mathbb{R} \ltimes_{\mu_c} \mathbb{R}^3)/\Gamma_c$ with

$$\Gamma_c = \mathbb{Z} \ltimes_{\mu_c} \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{18+8\sqrt{5}}{7+3\sqrt{5}} \\ \frac{2}{3+\sqrt{5}} \end{pmatrix} \rangle_{\mathbb{Z}},$$

$t_1 = \ln(\frac{3+\sqrt{5}}{2})$ and $\mu_c(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t\,t_1} & 0 \\ 0 & 0 & e^{-t\,t_1} \end{pmatrix}$

d) $(\mathbb{R} \ltimes_{\mu_d} \mathbb{R}^3)/(\pi\,\mathbb{Z} \ltimes_{\mu_d} \mathbb{Z}^3)$, $\mu_d(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$

e) $(\mathbb{R} \ltimes_{\mu_e} \mathbb{R}^3)/(\mathbb{Z} \ltimes_{\mu_e} \mathbb{Z}^3)$, $\mu_e(t) = \begin{pmatrix} 1 & t & \frac{1}{2}(t^2 - t) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

f) Consider $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -11 \\ 0 & 1 & 8 \end{pmatrix} \in \mathrm{SL}(3,\mathbb{Z})$. A has $X^3 - 8X^2 + 11X - 1$ as characteristic polynomial which possesses three pairwise different real roots $t_1 \approx 6,271$, $t_2 \approx 1,631$ and $t_3 \approx 0,098$. Therefore, A is conjugate to $\mu_f(1)$, where $\mu_f(t) = \begin{pmatrix} e^{t\,\ln(t_1)} & 0 & 0 \\ 0 & e^{t\,\ln(t_2)} & 0 \\ 0 & 0 & e^{t\,\ln(t_3)} \end{pmatrix}$, and this implies the existence of a lattice Γ_f in the completely solvable Lie group $\mathbb{R} \ltimes_{\mu_f} \mathbb{R}^3$.

g) Let $A := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix} \in \mathrm{SL}(3,\mathbb{Z})$. The characteristic polynomial of A is $X^3 - 4X^2 + 8X - 1$ which has three pairwise different roots $t_1 \approx 0,134$ and $t_{2,3} = (1/\sqrt{t_1})\,(\cos(\varphi) \pm i \sin(\varphi)) \approx 1,933 \pm 1,935\,i$. So A is conjugate to $\mu_g(1)$, where $\mu_g(t) = \begin{pmatrix} e^{t\,\ln(t_1)} & 0 & 0 \\ 0 & e^{t\,\ln(|t_2|)}\cos(t\,\varphi) & -e^{t\,\ln(|t_2|)}\sin(t\,\varphi) \\ 0 & e^{t\,\ln(|t_2|)}\sin(t\,\varphi) & e^{t\,\ln(|t_2|)}\cos(t\,\varphi) \end{pmatrix}$, and this implies the existence of a lattice Γ_g in the Lie group $\mathbb{R} \ltimes_{\mu_g} \mathbb{R}^3$.

h) Using Theorem 2.1, one shows that

$$\begin{aligned} \gamma_1 &:= (1, 1, -\frac{1+\sqrt{5}}{3+\sqrt{5}}), \\ \gamma_2 &:= (-\frac{2(2+\sqrt{5})}{3+\sqrt{5}}, \frac{1+\sqrt{5}}{3+\sqrt{5}}, -\frac{11+5\sqrt{5}}{7+3\sqrt{5}}), \\ \gamma_3 &:= (0, 0, \sqrt{5}) \end{aligned}$$

generate a lattice Γ in $U_3(\mathbb{R})$ with $[\gamma_1, \gamma_2] = \gamma_3$ and γ_3 central. Define the one-parameter group $\mu_h \colon \mathbb{R} \to \mathrm{A}(U_3(\mathbb{R}))$ by

$$\mu_h(t)\big((x, y, z)\big) = (e^{-t\,t_1}x, e^{t\,t_1}y, z),$$

where $t_1 := \ln(\frac{3+\sqrt{5}}{2})$. Then $\mu_h(1)$ preserves the lattice Γ with

$$\mu_h(1)(\gamma_1) = \gamma_1^2\gamma_2,\ \mu_h(1)(\gamma_2) = \gamma_1\gamma_2,\ \mu_h(1)(\gamma_3) = \gamma_3$$

and therefore, $\mathbb{Z} \ltimes_{\mu_h} \Gamma$ is a lattice in $\mathbb{R} \ltimes_{\mu_h} U_3(\mathbb{R})$.

i) Consider the Lie group $\widetilde{G}$ and the one-parameter group $\widetilde{\mu}$ of Case B.2 from the proof of the last theorem. Setting $\gamma_1 = (1, 0, 0)$, $\gamma_2 = (0, 1, 0)$ as well as $\gamma_3 = (0, 0, 1)$, $n = 1$ and $c_1 = c_2 = 0$, one explicitly gets an example.

The manifolds of type c) show that formal spaces with the same minimal model as a Kähler manifold need not be Kählerian. This was proved by Fernández and Gray.

Theorem 6.3 ([18]). *Let M be one of the symplectic solvmanifolds of type c) in the last theorem, i.e. M is formal and possesses no complex structure. M has the same minimal model as the Kähler manifold $T^2 \times S^2$.* □

7. Five-dimensional solvmanifolds.

7.1. Nilpotent and decomposable solvable Lie algebras. There are nine classes of nilpotent Lie algebras in dimension five, see Table A.2. Each of them has a basis with rational structure constants. By Theorem 2.1, the corresponding connected and simply-connected Lie groups admit lattices and accordingly to Theorem 2.6, the associated nilmanifolds are formal if and only if they are tori. For $i \in \{4, 5, 6\}$ the connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}_{5.i}$ possesses the left-invariant contact form x_1 (where x_1 is dual to the basis element $X_1 \in \mathfrak{g}_{\mathfrak{i}}$ as in Table A.2). Therefore, the corresponding nilmanifolds are contact.

The eight classes of decomposable unimodular non-nilpotent solvable Lie algebras are listed in Table A.3. Except for $\mathfrak{g}_{4.2} \oplus \mathfrak{g}_1$, the corresponding connected and simply-connected Lie groups admit lattices since both of their factors admit lattices.

THEOREM 7.1.1. *The connected and simply-connected Lie group $G_{4.2} \times \mathbb{R}$ with Lie algebra $\mathfrak{g}_{4.2} \oplus \mathfrak{g}_1$ possesses no lattice.*

Proof. Write G for $G_{4.2} \times \mathbb{R}$ and

$$\mathfrak{g} = \langle X_1, \ldots, X_5 \,|\, [X_1, X_4] = -2X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = X_2 + X_3 \rangle$$

for its Lie algebra which has $\mathfrak{n} = \mathbb{R}^4_{X_1,X_2,X_3,X_5}$ as nilradical. Therefore, G can be written as almost abelian Lie group $\mathbb{R} \ltimes_\mu \mathbb{R}^4$ with

$$\mu(t) = \exp^{GL(n,\mathbb{R})}(t\,\mathrm{ad}(X_4)) = \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{-t} & -te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Corollary 4.5, the existence of a lattice in G would imply that there is $t_1 \in \mathbb{R} \setminus \{0\}$ such that $\mu(t_1)$ is conjugate to an element of $\mathrm{SL}(4,\mathbb{Z})$. Clearly, the characteristic polynomial of $\mu(t_1)$ is $P(X) = (X-1)\,\widetilde{P}(X)$, where the polynomial $\widetilde{P}(X) = X^3 - kX^2 + lX - 1 \in \mathbb{Z}[X]$ has the double root e^{-t_1}. Lemma B.4 then implies $e^{-t_1} = 1$, i.e. $t_1 = 0$ which is a contradiction. ☐

PROPOSITION 7.1.2. *If Γ is a lattice in a five-dimensional completely solvable non-nilpotent connected and simply-connected decomposable Lie group G, then G/Γ is formal.*

Proof. Let G, Γ be as in the proposition. As usual, we denote by $\mathfrak{g}$ the Lie algebra of G. We have $\mathfrak{g} = \mathfrak{h} \oplus k\mathfrak{g}_1$ with $k \in \{1,2\}$ and a certain $(5-k)$-dimensional completely solvable non-nilpotent Lie algebra $\mathfrak{h}$, see Tables A.3 and A.1. By completely solvability and Theorem 3.10 (ii), G/Γ and the Chevalley-Eilenberg complex of $\mathfrak{h} \oplus k\mathfrak{g}_1$ share their minimal model $\mathcal{M}$. The lower dimensional discussion above shows that for all $\mathfrak{h}$ which can arise in the decomposition of $\mathfrak{g}$ the algebras $\mathcal{M}_{(\bigwedge \mathfrak{h}^*, \delta_\mathfrak{h})}$ and $\mathcal{M}_{(\bigwedge k\mathfrak{g}_1^*, \delta=0)} = (\bigwedge k\mathfrak{g}_1^*, \delta = 0)$ are formal. This implies the formality of $\mathcal{M} = \mathcal{M}_{(\bigwedge \mathfrak{h}^*, \delta_\mathfrak{h})} \otimes \mathcal{M}_{(\bigwedge k\mathfrak{g}_1^*, \delta=0)}$. ☐

7.2. Indecomposable non-nilpotent Lie algebras. There are 19 classes of indecomposable non-nilpotent Lie algebras in dimension five which are unimodular. These are listed in Tables A.4 – A.7. Instead of the small German letters for the Lie algebras in the mentioned tables, we use capital Latin letters (with the same subscripts) for the corresponding connected and simply-connected Lie groups.

Almost abelian algebras. We now consider the almost abelian Lie groups $G_{5.i} = \mathbb{R} \ltimes_{\mu_i} \mathbb{R}^4$. We write $\mu(t) = \mu_i(t) = \exp^{GL(4,\mathbb{R})}(t\,\mathrm{ad}(X_5))$, where $X_5 \in \mathfrak{g}_{5.i}$ is as in Table A.4 (X_5 depends on i). We know by Corollary 4.5, Theorem 4.6 and Proposition 4.7 that there is a lattice Γ in $G_{5.i}$ if and only if there exists $t_1 \neq 0$ such that $\mu(t_1)$ is conjugate to $\widetilde{\mu}(1) \in \mathrm{SL}(4,\mathbb{Z})$ and $\Gamma = \mathbb{Z} \ltimes_{\widetilde{\mu}} \mathbb{Z}^4$. This will be used in the proof of the following propositions. In most cases, we construct the lattice by giving explicit such an integer matrix $\widetilde{\mu}(1)$. Using Mathematica-software, one can check that the mentioned matrices are really conjugated. We will not write down the transformation matrix $T \in \mathrm{GL}(4,\mathbb{R})$ with $T\widetilde{\mu}(1)T^{-1} = \mu(t_1)$.

Methods to obtain integer matrices with given characteristic polynomial and necessary conditions for their existence are given in Appendix B.

PROPOSITION 7.2.1. *Let $p,q,r \in \mathbb{R}$ with $-1 \leq r \leq q \leq p \leq 1$, $pqr \neq 0$ and $p+q+r = -1$. If the completely solvable Lie group $G_{5.7}^{p,q,r}$ admits a lattice and M*

denotes the corresponding solvmanifold, then M is formal, $b_1(M) = 1$ and one of the following conditions holds:

(i) $b_2(M) = 0$,

(ii) $b_2(M) = 2$, *i.e.* $r = -1,\ p = -q \in]0, 1[$ *or*

(iii) $b_2(M) = 4$, *i.e.* $r = q = -1,\ p = 1$.

Moreover, there exist p, q, r as above satisfying (i), (ii) resp. (iii) such that $G^{p,q,r}_{5.7}$ admits a lattice.

Proof. We suppress the sub- and superscripts of G and $\mathfrak{g}$.

a) Assume, there is a lattice in G and denote the corresponding solvmanifold by M. Since $\mathfrak{g}$ is completely solvable, the inclusion of the Chevallier-Eilenberg complex $\big(\bigwedge(x_1, \ldots, x_5), \delta\big)$ into the forms on M induces an isomorphism on cohomology. Moreover, the minimal model of $\big(\bigwedge(x_1, \ldots, x_5), \delta\big)$ is isomorphic to the minimal model of M.

δ is given by

$$\delta x_1 = -x_{15},\ \delta x_2 = -p\, x_{25},\ \delta x_3 = -q\, x_{35},\ \delta x_4 = -r\, x_{45},\ \delta x_5 = 0.$$

(Here we write x_{ij} for $x_i x_j$.) This implies $b_1(M) = 1$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{array}{lll} \delta x_{12} = (1+p)\, x_{125}, & \delta x_{13} = (1+q)\, x_{135}, & \delta x_{14} = (1+r)\, x_{145}, \\ \delta x_{23} = (p+q)\, x_{235}, & \delta x_{24} = (p+r)\, x_{245}, & \delta x_{34} = (q+r)\, x_{345}. \end{array}$$

$-1 \le r \le q \le p \le 1$, $pqr \neq 0$ and $p + q + r = -1$ implies $p \neq -1$ and $q \neq -r$ and a short computation yields that either (i), (ii) or (iii) holds.

In each case, a determination of the 2-minimal model, i.e. the minimal model up to generators of degree two, shows that these generators are closed. By Definition 1.1.4, the minimal model then is 2-formal and Theorem 1.1.6 implies the formality of M.

b) Now, we show that there are examples for each of the three cases. In case (i), the proof is done in [27].

In case (ii), regard the matrix $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & 10 \end{pmatrix}$ which is conjugate to $\mu(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{-pt_1} & 0 & 0 \\ 0 & 0 & e^{pt_1} & 0 \\ 0 & 0 & 0 & e^{t_1} \end{pmatrix}$ for $t_1 = 2\ln(\frac{3+\sqrt{5}}{2})$ and $p = \frac{1}{2}$ since both matrices have the same characteristic polynomial which has four distinct real roots.

In case (iii), regard the matrix $\begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ which is conjugate to $\mu(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1} & 0 \\ 0 & 0 & 0 & e^{t_1} \end{pmatrix}$ for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$ since both diagonalisable matrices have the same minimal polynomial by Proposition B.8 (ii). ◻

We have seen that a non-formal solvmanifold is a non-toral nilmanifold in dimensions three and four. In higher dimensions this is no longer true as the following proposition shows:

PROPOSITION 7.2.2. *The completely solvable Lie group $G_{5.8}^{-1}$ admits a lattice. Moreover, for each lattice Γ the corresponding solvmanifold $M = G_{5.8}^{-1}/\Gamma$ has $b_1(M) = 2$ and is not formal.*

Proof. Again, we suppress the sub- and superscripts. G admits a lattice since $\mu(t) = \exp^{GL(4,\mathbb{R})}(t\,\mathrm{ad}(X_5)) = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{t} \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ are conjugated for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$.

Now, let Γ be an arbitrary lattice in G. By completely solvability and Theorem 3.10 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model $\mathcal{M}$ of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. The latter is given by

$$\delta x_1 = -x_{25},\ \delta x_2 = 0,\ \delta x_3 = -x_{35},\ \delta x_4 = x_{45},\ \delta x_5 = 0,$$

which implies $b_1(M) = 2$. Further, the minimal model $\rho\colon (\bigwedge V, d) \to (\bigwedge \mathfrak{g}^*, \delta)$ must contain two closed generators y_1, y_2 which map to x_2 and x_5. Then we have $\rho(y_1 y_2) = x_{25} = -\delta x_1$ and the minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree one such that $\rho(u) = -x_1$ and $du = y_1 y_2$. Since $\rho(uy_1) = -x_{12}$ and $\rho(uy_2) = -x_{15}$ are closed and non-exact, there are no further generators of degree one in V. But this implies that $(u + c)\, y_1$ is closed and non-exact in $\mathcal{M}$ for each closed element c of degree one. Using the notation of Theorem 1.1.5, we have $u \in N^1, y_1 \in V^1$ and $\mathcal{M}$ is not formal. ∎

PROPOSITION 7.2.3. *The completely solvable Lie group $G_{5.9}^{p,-2-p}$, $p \geq -1$, does not admit a lattice.*

Proof. The first half of the proof is taken from [27]. Assume there is a lattice. $\mu(t) = \begin{pmatrix} e^{-t} & -te^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-tp} & 0 \\ 0 & 0 & 0 & e^{t(2+p)} \end{pmatrix}$ is conjugate to an element of $\mathrm{SL}(4,\mathbb{Z})$ for $t = t_1 \neq 0$ and has roots $e^{-t_1}, e^{-t_1}, e^{-t_1 p}$ and $e^{t_1(2+p)}$. By Proposition B.6, this can occur if and only if $p = -1$. Therefore, for the remainder of the proof we assume $p = -1$.

The Jordan form of $\mu(t_1)$ is $\begin{pmatrix} e^{-t} & 1 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & e^{t} \end{pmatrix}$, i.e. the characteristic and the minimal polynomial of $\mu(t_1)$ are

$$\begin{aligned} P(X) &= (X - e^{-t_1})^2 (X - e^{t_1})^2 \\ &= X^4 - 2(e^{-t_1} + e^{t_1})X^3 + (e^{-2t_1} + e^{2t_1} + 4)X^2 - 2(e^{-t_1} + e^{t_1})X + 1, \\ m(X) &= (X - e^{-t_1})^2 (X - e^{t_1}) \\ &= X^3 - (2e^{-t_1} + e^{t_1})X^2 + (e^{-2t_1} + 2)X - e^{-t_1}. \end{aligned}$$

Since $\mu(t_1)$ is conjugate to an integer matrix, we have $P(X), m(X) \in \mathbb{Z}[X]$ by Theorem B.3. This is impossible for $t_1 \neq 0$. ∎

PROPOSITION 7.2.4 ([27]). *The completely solvable Lie group $G_{5.11}^{-3}$ does not admit a lattice.*

Proof. If the group admits a lattice, there exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that the characteristic polynomial of $\mu(t_1) = \begin{pmatrix} e^{-t_1} & -t_1 e^{-t_1} & \frac{t_1^2}{2} e^{-t_1} & 0 \\ 0 & e^{-t_1} & -t_1 e^{-t_1} & 0 \\ 0 & 0 & e^{-t_1} & 0 \\ 0 & 0 & 0 & e^{3t_1} \end{pmatrix}$ is a monic integer polynomial with a three-fold root e^{-t_1} and a simple root e^{3t_1}. By Proposition B.6, this is impossible for $t_1 \neq 0$. □

PROPOSITION 7.2.5 ([27]). *There are $q, r \in \mathbb{R}$ with $-1 \leq q < 0$, $q \neq -\frac{1}{2}$, $r \neq 0$ such that $G_{5.13}^{-1-2q,q,r}$ admits a lattice.* □

PROPOSITION 7.2.6. *There exists $r \in \mathbb{R} \setminus \{0\}$ such that $G_{5.13}^{-1,0,r}$ admits a lattice.*

Proof. Let $t_1 = \ln(\frac{3+\sqrt{5}}{2})$, $r = \pi/t_1$ and $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Then A is conjugate to $\mu_{0,r}(t_1) = \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 \\ 0 & e^{t_1} & 0 & 0 \\ 0 & 0 & \cos(rt_1) & \sin(rt_1) \\ 0 & 0 & \sin(rt_1) & \cos(rt_1) \end{pmatrix}$ and this implies the existence of a lattice. □

PROPOSITION 7.2.7. *$G_{5.14}^{0}$ admits a lattice.*

Proof. We have $\mu(t) = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}$. Let $t_1 = \frac{\pi}{3}$, then $\mu(t_1)$ is conjugate to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, so there is a lattice. □

PROPOSITION 7.2.8. *If there is a lattice Γ in the Lie group $G := G_{5.14}^{0}$ such that $b_1(G/\Gamma) = 2$, then G/Γ is not formal.*

Proof. By Theorem 3.10(i), the natural inclusion of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta) \to (\Omega(G/\Gamma), d)$ induces an injection on cohomology. $(\bigwedge \mathfrak{g}^*, \delta)$ is given by

$$\delta x_1 = -x_{25},\ \delta x_2 = 0,\ \delta x_3 = -x_{45},\ \delta x_4 = x_{35},\ \delta x_5 = 0.$$

This implies $b_1(\bigwedge \mathfrak{g}^*, \delta) = 2$, hence $H^1(G/\Gamma, d) = \langle [x_2], [x_5] \rangle$. Therefore

$$[x_2] \cdot H^1(G/\Gamma) + H^1(G/\Gamma) \cdot [x_5] = \langle [x_{25}] \rangle = \langle [\delta x_1] \rangle = 0,$$

and in the Massey product $\langle [x_2], [x_2], [x_5] \rangle = [-x_{15}]$ is no indeterminacy. Since x_{15} is closed and not exact, G/Γ cannot be formal. □

PROPOSITION 7.2.9. *The completely solvable Lie group $G_{5.15}^{-1}$ admits a lattice. For each lattice the corresponding solvmanifold satisfies $b_1 = 1$ and is non-formal.*

Proof. The construction of a lattice is given in [27]. The proof of the other statements is similar to the proof of Proposition 7.2.2. □

PROPOSITION 7.2.10 ([27]). *$G^{-1,q}_{5.16}$, $q \neq 0$, does not admit a lattice.*

Proof. If the group admits a lattice, there exists $t_1 \in \mathbb{R} \setminus \{0\}$ such that the characteristic polynomial of $\mu(t_1) = \begin{pmatrix} e^{-t_1} & -t_1 e^{-t_1} & 0 & 0 \\ 0 & e^{-t_1} & 0 & 0 \\ 0 & 0 & e^{t_1}\cos(t_1 q) & -e^{t_1}\sin(t_1 q) \\ 0 & 0 & e^{t_1}\sin(t_1 q) & e^{t_1}\cos(t_1 q) \end{pmatrix}$ is a monic integer polynomial with simple roots $e^{t_1}(\cos(t_1 q) \pm i \sin(t_1 q))$ and a double root e^{-t_1}. By Proposition B.6, this is impossible for $t_1 \neq 0$. □

PROPOSITION 7.2.11 ([27]). *There are $p, r \in \mathbb{R}$, $p \neq 0$, $r \notin \{0, \pm 1\}$, such that $G^{p,-p,r}_{5.17}$ admits a lattice.* □

PROPOSITION 7.2.12. *There exists $p \in \mathbb{R} \setminus \{0\}$ such that $G^{p,-p,\pm 1}_{5.17}$ admits a lattice.*

Proof. Let $p := \frac{1}{\pi}\ln(\frac{3+\sqrt{5}}{2})$, $t_1 := \pi$ and $A := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{pmatrix}$. Then $\mu(t_1) = \begin{pmatrix} e^{-t_1 p}\cos(t_1) & -e^{-t_1 p}\sin(t_1) & 0 & 0 \\ e^{-t_1 p}\sin(t_1) & e^{-t_1 p}\cos(t_1) & 0 & 0 \\ 0 & 0 & e^{t_1 p}\cos(\pm t_1) & -e^{t_1 p}\sin(\pm t_1) \\ 0 & 0 & e^{t_1 p}\sin(\pm t_1) & e^{t_1 p}\cos(\pm t_1) \end{pmatrix}$ is conjugate to A and this implies the existence of a lattice. □

PROPOSITION 7.2.13. *There exists $r \in \mathbb{R} \setminus \{0, \pm 1\}$ such that $G^{0,0,r}_{5.17}$ admits a lattice.*

Proof. Let $r \in \{2, 3\}$. Then $\mu(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(tr) & \sin(tr) \\ 0 & 0 & \sin(tr) & \cos(tr) \end{pmatrix}$ is an integer matrix for $t = \pi$. This implies the existence of a lattice. □

PROPOSITION 7.2.14. *$G^{0,0,\pm 1}_{5.17}$ admits a lattice.*

Proof. $\mu(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(\pm t) & \sin(\pm t) \\ 0 & 0 & \sin(\pm t) & \cos(\pm t) \end{pmatrix}$ is an integer matrix for $t = \pi$. This implies the existence of a lattice. □

PROPOSITION 7.2.15 ([27]). *$G^{0}_{5.18}$ admits a lattice.* □

Algebras with nilradical $\mathfrak{n} := \mathfrak{g}_{3.1} \oplus \mathfrak{g}_1 = \langle X_1, \dots, X_4 \,|\, [X_2, X_3] = X_1 \rangle$. We now regard the unimodular almost-nilpotent Lie groups $G_{5.i}$ with nilradical $N := U_3(\mathbb{R}) \times \mathbb{R}$, i.e. $i \in \{19, 20, 23, 25, 26, 28\}$. We can identify N with $\mathbb{R}^4$ as a manifold and the group law given by

$$(a, b, c, r) \cdot (x, y, z, w) = (a + x + bz\,,\, b + y\,,\, c + z\,,\, r + w).$$

The Lie algebras of the unimodular Lie groups $G_{5.i} = \mathbb{R} \ltimes_{\mu_i} N$ with nilradical N are listed in Table A.5. We have $\mu_i(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t\,\mathrm{ad}(X_5)) \circ \log^N$, where X_5 depends on i.

Assume there is a lattice Γ in $G_{5.i}$. By Corollary 4.5, there are $t_1 \neq 0$ and an inner automorphism I_{n_1} of N such that $\nu_i := \mu_i(t_1) \circ I_{n_1}, \nu_i^{-1} \in \mathrm{A}(N)$ preserve the lattice $\Gamma_N := \Gamma \cap N$ in N. For $n_1 = (a, b, c, r)$ one calculates

$$I_{n_1}(x, y, z, w) = (x + bz - yc\,,\ y\,,\ z\,,\ w). \tag{7}$$

$\Gamma_{N'} := \Gamma_N \cap N' \cong \mathbb{Z}$ is a lattice in $N' := [N, N] = \{(x, 0, 0, 0) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ by Theorem 2.3 and since $\nu_i(\Gamma_{N'}), \nu_i^{-1}(\Gamma_{N'}) \subset \Gamma_{N'}$, we have $\nu_i|_{\Gamma_{N'}} \in \mathrm{Aut}(\mathbb{Z})$. This implies $\nu_i|_{\Gamma_{N'}} = \pm\mathrm{id}$ and hence $\mu_i(t_1)|_{[N,N]} = \pm\mathrm{id}$ (a cause of (7) and the shape of $[N, N]$). Moreover, we have $[\mathfrak{n}, \mathfrak{n}] = \langle X_1 \rangle$ and since $\exp^{\mathbb{R}}$ is the identity,

$$\pm\mathrm{id} = \mu_i(t_1)|_{[N,N]} = \exp^{A(\mathfrak{n})}(t_1\, \mathrm{ad}(X_5)|_{\langle X_1\rangle})|_{[N,N]}.$$

(Note that $\exp^N([\mathfrak{n}, \mathfrak{n}]) = [N, N]$ by [57, Theorem 3.6.2].) Therefore, $t_1[X_5, X_1]$ has no component in $\langle X_1 \rangle$ and since $t_1 \neq 0$, this means that $[X_1, X_5]$ has no component in X_1-direction. The list of Lie algebras in Table A.5 implies:

Proposition 7.2.16. *The only connected and simply-connected solvable Lie groups with nilradical $U_3(\mathbb{R}) \times \mathbb{R}$ that can contain a lattice are $G^{-1}_{5.20}$ and $G^{0,\pm1}_{5.26}$.* □

Remark. In a previous version of this article, the group $G^{-1}_{5.20}$ is absent. It was added, after the author had read [15].

Proposition 7.2.17. *The completely solvable Lie group $G^{-1}_{5.20}$ admits a lattice. For each lattice the corresponding solvmanifold admits a contact form, is formal and has $b_1 = 2$.*

Proof. Using Theorem 2.1, one shows that

$$\begin{aligned}
\gamma_1 &:= \Big(\frac{20 + 9\sqrt{5}}{9 + 4\sqrt{5}}, 0, 0, 0\Big),\\
\gamma_2 &:= \Big(\frac{181 + 81\sqrt{5}}{47 + 21\sqrt{5}}, \frac{18 + 8\sqrt{5}}{7 + 3\sqrt{5}}, \frac{2}{3 + \sqrt{5}}, 0\Big),\\
\gamma_3 &:= \Big(\frac{181 + 81\sqrt{5}}{47 + 21\sqrt{5}}, 1, 1, 0\Big),\\
\gamma_4 &:= \Big(0, 0, 0, -\frac{20 + 9\sqrt{5}}{(9 + 4\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}\Big)
\end{aligned}$$

generate a lattice Γ_N in N with $[\gamma_2, \gamma_3] = \gamma_1$ and γ_1, γ_4 central.

A short calculation yields that $\mu(t)\big((x, y, z, w)\big) = (x - tw, e^{-t}y, e^t z, w)$ defines a one-parameter group in $\mathrm{A}(N)$. Moreover, for $t_1 = \ln(\frac{3+\sqrt{5}}{2})$ holds $\mu(t_1)(\gamma_1) = \gamma_1$, $\mu(t_1)(\gamma_2) = \gamma_3$, $\mu(t_1)(\gamma_3) = \gamma_2^{-1}\gamma_3^3$ and $\mu(t_1)(\gamma_4) = \gamma_1\gamma_4$.

This implies the existence of a lattice in $G := G^{-1}_{5.20} = \mathbb{R} \ltimes_\mu N$.

Let Γ be an arbitrary lattice in G. By completely solvability and Theorem 3.10 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model $\mathcal{M}$ of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. The latter is given by

$$\delta x_1 = -x_{23} - x_{45},\ \delta x_2 = -x_{25},\ \delta x_3 = x_{35},\ \delta x_4 = \delta x_5 = 0,$$

which implies $b_1(M) = 2$. Moreover, x_1 defines a left-invariant contact form on G/Γ.

The proof of the formality is similar to the proof of Proposition 7.2.2. □

PROPOSITION 7.2.18. *$G^{0,\varepsilon}_{5.26}$ admits a lattice for $\varepsilon = \pm 1$. For each lattice the corresponding solvmanifold is contact and has $b_1 \geq 2$.*

Proof. One calculates that $\mu\colon \mathbb{R} \to \mathrm{A}(N)$ defined by

$$\begin{aligned}&\mu(t)\big((x,y,z,w)\big)\\ &\quad = \big(x + h_t(y,z) - \varepsilon t w, \cos(t\pi)\, y - \sin(t\pi)\, z, \sin(t\pi)\, y + \cos(t\pi)\, z, w\big),\end{aligned}$$

where $h_t(y,z) = \frac{1}{2}\sin(t\pi)\Big(cos(t\pi)\big(y^2 - z^2\big) - 2\sin(t\pi) yz\Big)$, is a one-parameter group.

Then we have $G := G^{0,\varepsilon}_{5.26} = \mathbb{R} \ltimes_\mu N$ and $\mathbb{Z} \ltimes_\mu \{(x,y,z,w) \in N \,|\, x,y,z,w \in \mathbb{Z}\}$ is a lattice in G since $\mu(1)\big((x,y,z,w)\big) = (x - \varepsilon w, -y, -z, w)$.

Using $d_e\big(\mu(t)\big) = \log^N \circ \mu(t) \circ \exp^N$, we obtain the Lie algebra $\mathfrak{g}$ of G as

$$\langle X_1, \dots X_5 \,|\, [X_2, X_3] = X_1,\ [X_2, X_5] = X_3,\ [X_3, X_5] = -X_2,\ [X_4, X_5] = \varepsilon X_1 \rangle.$$

Denote $\{x_1, \dots, x_5\}$ the basis of $\mathfrak{g}^*$ which is dual to $\{X_1, \dots, X_5\}$, i.e. the x_i are left-invariant 1-forms on G. One calculates that x_1 is a left-invariant contact form on G, so it descends to a contact form on the corresponding solvmanifold.

The statement about the first Betti number follows from Theorem 3.10(i). □

Algebras with nilradical $\mathfrak{g}_{4.1} = \langle X_1, \dots, X_4 \,|\, [X_2, X_4] = X_1,\ [X_3, X_4] = X_2 \rangle$.

PROPOSITION 7.2.19. *No connected and simply-connected solvable Lie group $G_{5.i}$ with nilradical $N := G_{4.1}$ admits a lattice.*

Proof. There is only one unimodular connected and simply-connected solvable Lie group with nilradical $G_{4.1}$, namely the completely solvable group $G := G^{-\frac{4}{3}}_{5.30}$. We show that it admits no lattice.

The group N is $\mathbb{R}^4$ as a manifold with multiplication given by

$$(a,b,c,r) \cdot (x,y,z,w) = (a + x + wb + \frac{1}{2} w^2 c\,,\ b + y + wc\,,\ c + z\,,\ r + w),$$

and one calculates for $n_1 = (a,b,c,r)$

$$I_{n_1}(x,y,z,w) = (x + wb + \frac{1}{2} w^2 c - ry - rwc + \frac{1}{2} r^2 z\,,\ y + wc - rz\,,\ z\,,\ w).$$

Let $G = \mathbb{R} \ltimes_\mu N$, where $\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t\, \mathrm{ad}(X_5)) \circ \log^N$ and assume there is a lattice Γ in G. By Corollary 4.5, there are $t_1 \neq 0$ and $n_1 \in N$ such that $\nu := \mu(t_1) \circ I_{n_1} \in \mathrm{A}(N)$ preserves the lattice $\Gamma_N := \Gamma \cap N$ in N.

$\Gamma_{N'} := N' \cap \Gamma_N$ is a lattice in $N' := [N,N] = \{(x,y,0,0) \in N \,|\, x,y \in \mathbb{R}\} \cong \mathbb{R}^2$ by Theorem 2.3, and since $\nu(N') \subset N'$, this lattice is preserved by $\nu|_{N'}$. This and $\exp^{\mathbb{R}^2} = \mathrm{id}$ imply

$$\pm 1 = \det(\nu|_{N'}) = \det\big(\exp^{A(\mathfrak{n})}(t_1\, \mathrm{ad}(X_5)|_{[\mathfrak{n},\mathfrak{n}]})|_{[N,N]}\big) \cdot \underbrace{\det(I_{n_1}|_{N'})}_{=1},$$

i.e. $\mathrm{ad}(X_5)|_{[\mathfrak{n},\mathfrak{n}]}$ has trace equal to zero. This contradicts $\mathfrak{g}^{-\frac{4}{3}}_{5.30}$, see Table A.6. □

Non-almost nilpotent algebras. Now, there remain two unimodular connected and simply-connected solvable Lie groups in dimension five, namely $G_{5.33}^{-1,-1}$ and $G_{5.35}^{-2,0}$. In April 2009, Diatta and Foreman [15] proved that $G_{5.33}^{-1,-1}$ possesses a lattice. Note, Harshavardhan's argumentation in [27, p. 33] is not sufficient. One easily proves the following Proposition.

Proposition 7.2.20. *Consider an arbitrary lattice in $G_{5.33}^{-1,-1}$. Then the corresponding solvmanifold admits a contact form (since $G_{5.33}^{-1,-1}$ possesses the left-invariant contact form $x_1 + x_2 + x_3$ with x_i dual to $X_i \in \mathfrak{g}_{5.33}^{-1,-1}$ as in Table A.7, see [14]), is formal and has $b_1 = 2$.* □

Proposition 7.2.21. *$G_{5.35}^{-2,0}$ contains a lattice. For each lattice the corresponding solvmanifold is contact and has $b_1 \geq 2$.*

Proof. A lattice and a contact form were constructed by Geiges in [24]. One has the left-invariant contact form $x_1 + x_2$ on the Lie group, where x_1, x_2 are dual to the left-invariant vector fields as in Table A.7. Hence the form descends to each compact quotient by a discrete subgroup.

The statement about the first Betti number follows from Theorem 3.10(i). □

Conclusion. We have seen that each connected and simply-connected 5-dimensional solvable Lie group admits a lattice if it is nilpotent or decomposable with the exception of $G_{4.2} \times \mathbb{R}$. If an indecomposable non-nilpotent group $G_{5.i}$ gives rise to a solvmanifold it is contained in Table 7.1. Recall, by Theorem 3.10, we always have a lower bound for the solvmanifold's Betti numbers and in some cases the exact value. These can be read of in the second and the third column. The last column refers to the examples that we have constructed above. "yes" means that we have such for certain parameters that satisfy the conditions of the column "Comment". Except for $i = 33$ we have explicit examples for all possible values of i.

Assuming that there is a lattice in one the non-completely solvable Lie groups $G_{5.i}$, i.e. $i \in \{13, 14, 17, 18, 26, 35\}$, such that the inequalities in the above table are equalities, then one can calculate that such quotients are formal for $i \in \{13, 17, 26, 35\}$ and not formal for $i \in \{14, 18\}$. The assumptions about the Betti numbers are needed to ensure that the Lie algebra cohomology is isomorphic to the solvmanifold's cohomology.

7.3. Contact structures. Some of the connected and simply-connected five-dimensional solvable Lie groups $G_{5.i}$ which admit a lattice Γ possess a left-invariant contact form. Obviously, it also defines a contact form on the corresponding solvmanifold. By this way, we showed that the manifolds $G_{5.i}/\Gamma$ for $i \in \{4, 5, 6\}$ and quotients of almost nilpotent groups with non-abelian nilradical (i.e. $i \geq 19$) by lattices are contact.

But $\mathbb{R}^5$, $U_3(\mathbb{R}) \times \mathbb{R}^2$, $G_{4.1} \times \mathbb{R}$ and $G_{5.i}$ do not have a left-invariant contact form for $i \in \{1, 2, 3, 7, \ldots, 18\}$, see e.g. [14]. For some of the nilmanifolds, we can provide a contact structure by another approach.

Theorem 7.3.1. *Let $G \in \{\mathbb{R}^5, U_3(\mathbb{R}) \times \mathbb{R}^2, G_{4.1} \times \mathbb{R}, G_{5.1}, G_{5.3}\}$ and Γ a lattice G. Then G/Γ admits a contact structure.*

Proof. For G chosen as in the theorem, the dimension of the center is greater than or equal to two. Therefore, we can find a two-dimensional closed normal subgroup that lies in the center such that its intersection with Γ is a lattice in it. By Theorem

TABLE 7.1: 5-*dimensional indecomposable non-nilmanifolds*

	b_1	b_2	formal	Comment	Example
$G_{5.7}^{p,q,r}$	1	0	yes	$-1 < r < p < q < 1$, $pqr \neq 0$, $p+q+r=-1$	7.2.1 (i)
$G_{5.7}^{p,q,-1}$	1	2	yes	$p = -q \in]0,1[$	7.2.1 (ii)
$G_{5.7}^{1,-1,-1}$	1	4	yes		7.2.1 (iii)
$G_{5.8}^{-1}$	2	3	no		7.2.2
$G_{5.13}^{-1-2q,q,r}$	≥ 1	≥ 0	?	$q \in [-1,0] \setminus \{\frac{1}{2}\}$, $r \neq 0$	7.2.5
$G_{5.13}^{-1,0,r}$	≥ 1	≥ 2	?	$r \neq 0$	7.2.6
$G_{5.14}^{0}$	≥ 2	≥ 3	?		7.2.7
$G_{5.15}^{-1}$	1	2	no		7.2.9
$G_{5.17}^{p,-p,r}$	≥ 1	≥ 0	?	$p \neq 0,\ r \notin \{0, \pm 1\}$	7.2.11
$G_{5.17}^{p,-p,\pm 1}$	≥ 1	≥ 2	?	$p \neq 0$	7.2.12
$G_{5.17}^{0,0,r}$	≥ 1	≥ 2	?	$r \notin \{0, \pm 1\}$	7.2.13
$G_{5.17}^{0,0,\pm 1}$	≥ 1	≥ 4	?		7.2.14
$G_{5.18}^{0}$	≥ 1	≥ 2	?		7.2.15
$G_{5.20}^{-1}$	2	1	yes		7.2.17
$G_{5.26}^{0,\pm 1}$	≥ 2	≥ 1	?		7.2.18
$G_{5.33}^{-1,-1}$	2	1	yes		no
$G_{5.35}^{-2,0}$	≥ 2	≥ 1	?		7.2.21

3.5, G/Γ has the structure of a principal T^2-bundle over a three dimensional closed orientable manifold. Then the following result of Lutz implies the claim. ◻

THEOREM 7.3.2 ([36]). *The total space of a principal T^2-bundle over a closed orientable 3-manifold admits a contact form.* ◻

Unfortunately, we did not find a contact structure on the manifold of Proposition 7.2.9. If such exists, this yields a five-dimensional non-formal contact solvmanifold with $b_1 = 1$.

8. Six-dimensional solvmanifolds. There are 164 types of connected and simply-connected indecomposable solvable Lie groups in dimension six, most of them depending on parameters. For classifying six-dimensional solvmanifolds, we restrict ourselves to the following types:

(1) nilmanifolds
(2) symplectic solvmanifolds that are quotients of indecomposable non-nilpotent groups
(3) products of lower-dimensional solvmanifolds

Although we have to make some restrictions to get a manageable number of cases, one certainly has to consider types (1) and (3). The further restriction in (2) is justified by the large number of indecomposable non-nilpotent solvable Lie algebras in dimension six: There are 140 types of it. The author has decided to consider the most interesting among them. Since we are not able to refute a symplectic form's existence in the non-completely solvable case, we shall partly make even more restrictions.

8.1. Nilmanifolds. There are 34 isomorphism classes of nilpotent Lie algebras in dimension six. Each of them possesses a basis with rational structure constants and therefore determines a nilmanifold. They are listed on page 228 in Table 8.1 which is taken from [50]. The corresponding Lie algebras are listed in Appendix A. Among the 34 classes of nilmanifolds, there are 26 which admit a symplectic form.

TABLE 8.1: *6-dimensional nilmanifolds*

$b_1(G/\Gamma)$	$b_2(G/\Gamma)$	Comment	$\mathfrak{g}$
6	15	Torus, symplectic	$6\mathfrak{g}_1$
5	11	symplectic	$\mathfrak{g}_{3.1} \oplus 3\mathfrak{g}_1$
5	9	not symplectic	$\mathfrak{g}_{5.4} \oplus \mathfrak{g}_1$
4	9	symplectic	$\mathfrak{g}_{5.1} \oplus \mathfrak{g}_1$
4	8	symplectic	$2\mathfrak{g}_{3.1}$
4	8	symplectic	$\mathfrak{g}_{6.N4}$
4	8	symplectic	$\mathfrak{g}_{6.N5}$
4	7	symplectic	$\mathfrak{g}_{5.5} \oplus \mathfrak{g}_1$
4	7	symplectic	$\mathfrak{g}_{4.1} \oplus 2\mathfrak{g}_1$
4	6	not symplectic	$\mathfrak{g}_{6.N12}$
3	8	symplectic	$\mathfrak{g}_{6.N3}$
3	6	symplectic	$\mathfrak{g}_{6.N1}$
3	6	symplectic	$\mathfrak{g}_{6.N6}$
3	6	symplectic	$\mathfrak{g}_{6.N7}$
3	5	symplectic	$\mathfrak{g}_{5.2} \oplus \mathfrak{g}_1$
3	5	not symplectic	$\mathfrak{g}_{5.3} \oplus \mathfrak{g}_1$
3	5	symplectic	$\mathfrak{g}_{5.6} \oplus \mathfrak{g}_1$
3	5	symplectic	$\mathfrak{g}_{6.N8}$
3	5	symplectic	$\mathfrak{g}_{6.N9}$
3	5	symplectic	$\mathfrak{g}_{6.N10}$
3	5	not symplectic	$\mathfrak{g}_{6.N13}$
3	5	not symplectic	$\mathfrak{g}^{1}_{6.N14}$
3	5	not symplectic	$\mathfrak{g}^{-1}_{6.N14}$
3	5	symplectic	$\mathfrak{g}_{6.N15}$
3	5	symplectic	$\mathfrak{g}_{6.N17}$
3	4	symplectic	$\mathfrak{g}_{6.N16}$
2	4	symplectic	$\mathfrak{g}_{6.N11}$
2	4	symplectic	$\mathfrak{g}^{1}_{6.N18}$
2	4	symplectic	$\mathfrak{g}^{-1}_{6.N18}$
2	3	symplectic	$\mathfrak{g}_{6.N2}$
2	3	symplectic	$\mathfrak{g}_{6.N19}$
2	3	symplectic	$\mathfrak{g}_{6.N20}$
2	2	not symplectic	$\mathfrak{g}_{6.N21}$
2	2	not symplectic	$\mathfrak{g}_{6.N22}$

Recall that a nilmanifold is formal or Kählerian if and only if the corresponding Lie algebra is abelian.

8.2. Candidates for the existence of lattices. Among the 61 types of indecomposable unimodular almost nilpotent Lie algebras in dimension six that are listed in Tables A.9 – A.16, there are some that cannot be the Lie algebra of a connected and simply-connected Lie group which admits a lattice.

Instead of the small German letters for the Lie algebras in the mentioned tables, we use again capital Latin letters with the same subscripts for the corresponding connected and simply-connected Lie groups. If any, we chose the same designation for the parameters $a, b, c, h, s, \varepsilon$ of $G_{6.i}$ as for their Lie algebras.

PROPOSITION 8.2.1. *Let* $i \in \{13, \ldots, 38\}$, *i.e.* $\mathrm{Nil}(G_{6.i}) = U_3(\mathbb{R}) \times \mathbb{R}^2$. *Then it is necessary for* $G_{6.i}$ *to contain a lattice that one of the following conditions holds:*

$i = 15$,	$i = 18 \wedge a = -1$,	$i = 21 \wedge a = 0$,	$i = 23 \wedge a = 0$,
$i = 25 \wedge b = 0$,	$i = 26$,	$i = 29 \wedge b = 0$,	$i = 32 \wedge a = 0$,
$i = 33 \wedge a = 0$,	$i = 34 \wedge a = 0$,	$i = 35 \wedge a = -b$,	$i = 36 \wedge a = 0$,
$i = 37 \wedge a = 0$,	$i = 38$.		

Proof. This can be seen analogous as in the proof of Proposition 7.2.16. Denote $\{X_1, \ldots, X_6\}$ the basis used for the description of the Lie algebra in Appendix A. Then the existence of a lattice implies that $[X_6, X_1]$ has no component in X_1-direction and this yields the claim. □

PROPOSITION 8.2.2. *Let* $i \in \{39, \ldots, 47\}$, *i.e. the nilradical of* $G_{6.i}$ *is* $G_{4.1} \times \mathbb{R}$. *If* $G_{6.i}$ *admits a lattice, then holds* $i = 39 \wedge h = -3$ *or* $i = 40$.

Proof. Use the designation $X_1, \ldots, X_6$ as above. Then $\langle X_1, X_2 \rangle$ is the commutator of the nilradical of $\mathfrak{g}_{6.i}$. Analogous as in the proof of Proposition 7.2.19, one shows that $\mathrm{ad}(X_6)|_{\langle X_1, X_2 \rangle}$ has trace equal to zero. This is only satisfied for $i = 39 \wedge h = -3$ or $i = 40$. □

PROPOSITION 8.2.3.

(i) Let $i \in \{54, \ldots, 70\}$, *i.e. the nilradical of* $G_{6.i}$ *is* $G_{5.1}$. *If* $G_{6.i}$ *admits a lattice, then holds* $i = 54 \wedge l = -1$, $i = 63$, $i = 65 \wedge l = 0$ *or* $i = 70 \wedge p = 0$.

(ii) No connected and simply-connected almost nilpotent Lie group with nilradical $G_{5.2}$ *or* $G_{5.5}$ *admits a lattice.*

Proof. This follows in the same manner as the last proposition. The trace of $\mathrm{ad}(X_6)$ restricted to the commutator of the nilradical must be zero. □

8.3. Symplectic solvmanifolds whose first Betti number equals one. If we are looking for solvmanifolds with $b_1 = 1$, it is necessary that the corresponding Lie algebra is unimodular, almost nilpotent and has $b_1 = 1$ itself. Note that the latter forces the algebra to be indecomposable. In Tables A.20 – **??** on pages 257 – **??** we have listed all possible values that can arise as b_1 for the classes of unimodular non-nilpotent solvable indecomposable Lie algebras in dimension six.

Since we are mainly interested in symplectic 6-manifolds, we now investigate which Lie algebras contained in Tables **??** – A.16 that satisfy $b_1 = 1$ are *cohomologically symplectic*, i.e. there is a closed element $\omega \in \bigwedge^2 \mathfrak{g}^*$ such that ω^3 is not exact.

Note, if a unimodular Lie algebra is cohomologically symplectic, then each compact quotient of the corresponding Lie group by a lattice is symplectic. If the Lie algebra is completely solvable, this is even necessary for the quotient to be symplectic.

Proposition 8.3.1. *Let $\mathfrak{g}_{6.i}$ be a unimodular almost-nilpotent Lie algebra with $b_1(\mathfrak{g}_{6.i}) = 1$. Then we have:*
$\mathfrak{g}_{6.i}$ *is cohomologically symplectic if and only if* $i \in \{15, 38, 78\}$.

Proof. For $i \in \{15, 38, 78\}$ one computes all symplectic forms up to exact summands as
$i = 15: \ \omega = (\lambda + \mu)\, x_{16} + \lambda\, x_{25} - \mu\, x_{34}, \quad \lambda, \mu \in \mathbb{R} \setminus \{0\}, \lambda \neq -\mu,$
$i = 38: \ \omega = \lambda\, x_{16} + \mu\, x_{24} + \frac{\lambda}{2}\, x_{25} - \frac{\lambda}{2}\, x_{34} + \mu\, x_{35}, \quad \lambda, \mu \in \mathbb{R}, \lambda \neq 0, -\frac{3}{2}\lambda^3 \neq 2\lambda\mu^2,$
$i = 78: \ \omega = \lambda\, x_{14} + \lambda\, x_{26} + \lambda\, x_{35}, \quad \lambda \in \mathbb{R} \setminus \{0\}.$

If $i \notin \{15, 38, 78\}$, then the conditions on the parameters of $\mathfrak{g}_{6.i}$ to ensure its unimodularity and $b_1(\mathfrak{g}_{6.i}) = 1$ imply that there are no closed elements of $\bigwedge^2 \mathfrak{g}_{6.i}^*$ without exact summands which contain one of the elements $x_{16}, x_{26}, x_{36}, x_{46}$ or x_{56}. Therefore, $\mathfrak{g}_{6.i}$ cannot be cohomologically symplectic. □

We now examine the three Lie groups that have cohomologically symplectic Lie algebras.

The next theorem was announced in [6]. It provides an example of a symplectic non-formal 6-manifold with $b_1 = 1$. Since it is a solvmanifold, this manifold is symplectically aspherical. Hence, we found an example for which Kędra, Rudyak and Tralle looked in [34, Remark 6.5].

Theorem 8.3.2.
(i) The completely solvable Lie group $G_{6.15}^{-1}$ contains a lattice.
(ii) If Γ is any lattice in $G := G_{6.15}^{-1}$, then $M := G/\Gamma$ is a symplectic and non-formal manifold with $b_1(M) = 1$ and $b_2(M) = 2$.

Proof. ad (i): Let $N = U_3(\mathbb{R}) \times \mathbb{R}^2$ denote the nilradical of G. We can identify N with $\mathbb{R}^5$ as a manifold and the multiplication given by

$$(a, b, c, r, s) \cdot (x, y, z, v, w) = (a + x + bz, b + y, c + z, r + v, s + w),$$

i.e. $[N, N] = \{(x, 0, 0, 0, 0) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ and $\overline{N} := N/[N, N] \cong \mathbb{R}^4$.

By definition of G, we have $G = \mathbb{R} \ltimes_\mu N$, where

$$\forall_{t\in\mathbb{R}} \mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}\left(t \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}\right) \circ \log^N, \tag{8}$$

and since $\exp^{\mathbb{R}^4} = \mathrm{id}$, the induced maps $\overline{\mu}(t)\colon \overline{N} \to \overline{N}$ are given by

$$\begin{aligned} \overline{\mu}(t)\big((y, z, v, w)\big) &= \exp^{GL(4,\mathbb{R})}\left(t \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\right) \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 \\ -te^{-t} & 0 & e^{-t} & 0 \\ 0 & -te^{t} & 0 & e^{t} \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}. \end{aligned}$$

One calculates that $\widetilde{\mu}\colon \mathbb{R} \to \mathrm{A}(N)$ given by

$$\forall_{t\in\mathbb{R}} \forall_{(x,y,z,v,w)\in N} \widetilde{\mu}(t)\big((x, y, z, v, w)\big) = \big(x, \overline{\mu}(t)\big((y, z, v, w)\big)\big) \tag{9}$$

is a one-parameter group, and since the derivations of (8) and (9) in zero are equal, we have $\mu \equiv \widetilde{\mu}$.

Let $t_1 = \ln(\frac{3+\sqrt{5}}{2})$, then $\overline{\mu}(t_1)$ is conjugate to $A := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. The transformation matrix $T \in \mathrm{GL}(4,\mathbb{R})$ with $TAT^{-1} = \overline{\mu}(t_1)$ is

$$T = \begin{pmatrix} 1 & -\frac{2(2+\sqrt{5})}{3+\sqrt{5}} & 0 & 0 \\ 1 & \frac{1+\sqrt{5}}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & \frac{2(2+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & -\frac{(1+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} \end{pmatrix}.$$

Denote by $\{b_1, \ldots, b_4\}$ the basis of $\mathbb{R}^4$ for which $\overline{\mu}(t_1)$ is represented by A, i.e. b_i is the i-th column of T. One calculates

$$\begin{aligned} b_{11}b_{22} - b_{12}b_{21} &= \sqrt{5}, \\ b_{i1}b_{j2} - b_{i2}b_{j1} &= 0 \quad \text{for } i < j, (i,j) \neq (1,2). \end{aligned}$$

This implies that we have for $\gamma_0 := (\sqrt{5}, 0_{\mathbb{R}^4})$, $\gamma_i := (b_{i0}, b_i) \in N$ with arbitrary $b_{i0} \in \mathbb{R}$, $i = 1, \ldots, 4$,

$$[\gamma_1, \gamma_2] = \gamma_0,\ [\gamma_1, \gamma_3] = [\gamma_1, \gamma_4] = [\gamma_2, \gamma_3] = [\gamma_2, \gamma_4] = [\gamma_3, \gamma_4] = e_N.$$

We can choose the b_{i0} such that the following equations hold:

$$(10) \qquad \begin{array}{rcl} \mu(t_1)(\gamma_0) & = & \gamma_0, \\ \mu(t_1)(\gamma_1) & = & \gamma_1^2 \;\; \gamma_2 \;\; \gamma_3^2 \;\; \gamma_4, \\ \mu(t_1)(\gamma_2) & = & \gamma_1 \;\; \gamma_2 \;\; \gamma_3 \;\; \gamma_4, \\ \mu(t_1)(\gamma_3) & = & \gamma_3^2 \;\; \gamma_4, \\ \mu(t_1)(\gamma_4) & = & \gamma_3 \;\; \gamma_4. \end{array}$$

Note that (10) leads to the equation $(\mathrm{id} - {}^{\tau}A)\begin{pmatrix} b_{10} \\ b_{20} \\ b_{30} \\ b_{40} \end{pmatrix} = \begin{pmatrix} 1 + \frac{2(1+\sqrt{5})}{3+\sqrt{5}} \\ \frac{1+\sqrt{5}}{3+\sqrt{5}} \\ 0 \\ 0 \end{pmatrix}$ which has the (unique) solution $b_{10} = -\frac{1+\sqrt{5}}{3+\sqrt{5}}$, $b_{20} = -\frac{11+5\sqrt{5}}{7+3\sqrt{5}}$ and $b_{30} = b_{40} = 0$.

We claim that $t_1\mathbb{Z} \ltimes_\mu \langle \exp^N \left(\mathrm{Span}_\mathbb{Z} \log^N(\{\gamma_0, \ldots, \gamma_4\})\right)\rangle$ defines a lattice in G:

It suffices to show that $\langle \exp^N \left(\mathrm{Span}_\mathbb{Z} \log^N(\{\gamma_0, \ldots, \gamma_4\})\right)\rangle$ defines a lattice in N, so let us prove this assertion. There exist uniquely $Y_0, \ldots, Y_4 \in \mathfrak{n}$ with $\exp^N(Y_i) = \gamma_i$ for $i \in \{0, \ldots, 4\}$. If we prove that $\mathfrak{Y} := \{Y_0, \ldots, Y_4\}$ is a basis of $\mathfrak{n}$ with rational structure constants, then Theorem 2.1 (i) implies that $\langle \exp^N(\mathrm{Span}_\mathbb{Z}\mathfrak{Y})\rangle$ is a lattice in N.

We identify $\mathfrak{n}$ with $\mathbb{R}^5$ and brackets given by the Campbell-Hausdorff formula, see e.g. [57, Chapter 2.15]. Since $\mathfrak{n}$ is 2-step nilpotent (and $\exp^N$ is a diffeomorphism), the formula yields for all $V, W \in \mathfrak{n}$

$$\log^N\left(\exp^N(V)\exp^N(W)\right) = V + W + \frac{1}{2}[V, W].$$

Since $U_3(\mathbb{R})$ can be considered as a group of matrices, one can easily calculate its exponential map. Then, its knowledge implies that the exponential map resp. the logarithm of N is given by

$$\exp^N\big((x,\,y,\,z,\,v,\,w)\big) = (x+\frac{1}{2}yz,\,y,\,z,\,v,\,w),$$
$$\log^N\big((x,\,y,\,z,\,v,\,w)\big) = (x-\frac{1}{2}yz,\,y,\,z,\,v,\,w),$$

and we obtain $Y_0 = (\sqrt{5}, 0_{\mathbb{R}^4})$, $Y_1 = (b_{10} - \frac{1}{2}, b_1)$, $Y_2 = (b_{20} + \frac{(2+\sqrt{5})(1+\sqrt{5})}{(3+\sqrt{5})^2}, b_2)$, $Y_3 = (0, b_3)$, $Y_4 = (0, b_4)$, $[Y_1, Y_2] = Y_0$. The other brackets vanish.

ad (ii): Let Γ be an arbitrary lattice in G. By completely solvability and Theorem 3.10 (ii), we get the minimal model of $M = G/\Gamma$ as the minimal model $\mathcal{M}$ of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. The latter has the closed generator x_6 and the non-closed generators satisfy

$$\delta x_1 = -x_{23},\ \delta x_2 = -x_{26},\ \delta x_3 = x_{36},\ \delta x_4 = -x_{26} - x_{46},\ \delta x_5 = -x_{36} + x_{56},$$

which implies $b_1(M) = 1$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{array}{lll} \delta x_{12} = x_{126}, & \delta x_{13} = -x_{136}, & \delta x_{14} = x_{126} + x_{146} - x_{234}, \\ \delta x_{15} = x_{136} - x_{156} - x_{235}, & \delta x_{16} = x_{236}, & \delta x_{24} = 2x_{246}, \\ \delta x_{25} = x_{236}, & \delta x_{34} = -x_{236}, & \delta x_{35} = -2x_{356}, \\ \delta x_{45} = x_{256} - x_{346}, & & \end{array}$$

i.e. $b_2(M) = 2$.

The minimal model $\rho\colon (\bigwedge V, d) \to (\bigwedge \mathfrak{g}^*, \delta)$ must contain three closed generators y, z_1, z_2 which map to $x_6, x_{16} + x_{25}$ and $x_{16} - x_{34}$. $\rho(yz_1) = x_{256}$ and $\rho(yz_2) = -x_{346}$ are closed and not exact. But in the generation of y, z_1 and z_2 is one (and up to a scalar only one) element that maps onto an exact form, namely $\rho(y(z_1 + z_2)) = \delta x_{45}$. The minimal model's construction in the proof of Theorem 1.1.2 implies that there is another generator u of degree two such that $\rho(u) = x_{45}$ and $du = y(z_1 + z_2)$. Since $\rho(yu) = x_{456}$ is closed and non-exact, there are no further generators of degree less than or equal to two in V. But this implies for each closed element c of degree two that $y\,(u+c)$ is closed and non-exact in $\mathcal{M}$. Using the notation of Theorem 1.1.5, we have $u \in N^2, y \in V^1$ and $\mathcal{M}$ is not formal.

Finally, the existence of a symplectic form on G/Γ follows from Proposition 8.3.1. □

Proposition 8.3.3.

(i) Each quotient of the Lie group $G^0_{6.38}$ by a lattice is symplectic. $G^0_{6.38}$ contains a lattice Γ with $b_1(G^0_{6.38}/\Gamma) = 1$.

(ii) If the Lie group $G^0_{6.38}$ contains a lattice Γ such that $M := G^0_{6.38}/\Gamma$ satisfies $b_1(M) = 1$ and $b_2(M) = 2$, then M is a symplectic and non-formal manifold.

Proof. The proof is similar to that of the last theorem. Therefore, we just give a sketch of the proof.

ad (i): The existence of a symplectic form on each quotient of $G := G^0_{6.38}$ by a lattice follows from Proposition 8.3.1.

The nilradical N of G is the same as in the proof of Theorem 8.3.2, so we have $[N,N]=\mathbb{R}$ and $\overline{N}=N/[N,N]=\mathbb{R}^4$. If $\overline{\mu}(t)\colon \overline{N}\to\overline{N}$ is defined by

$$\overline{\mu}(t)\big((y,z,v,w)\big)=\exp^{GL(4,\mathbb{R})}(t\begin{pmatrix}0&1&0&0\\-1&0&0&0\\-1&0&0&1\\0&-1&-1&0\end{pmatrix})\begin{pmatrix}y\\z\\v\\w\end{pmatrix}$$
$$=\begin{pmatrix}\cos(t)&\sin(t)&0&0\\-\sin(t)&cos(t)&0&0\\-t\cos(t)&-t\sin(t)&\cos(t)&sin(t)\\t\sin(t)&-t\cos(t)&-\sin(t)&\cos(t)\end{pmatrix}\begin{pmatrix}y\\z\\v\\w\end{pmatrix},$$

one calculates that $\mu\colon\ \mathbb{R}\to \mathrm{A}(N)$ given by

$$\mu(t)\big((x,y,z,v,w)\big)=\big(x-\sin^2(t)yz+\frac{\sin(t)\cos(t)}{2}(z^2-y^2)+t\frac{\sqrt{3}}{8}(y-z),\\ \overline{\mu}(t)\big((y,z,v,w)\big)\big)$$

is a one-parameter group with $d_e\big(\mu(t)\big)=\exp^{A(\mathfrak{n})}(t\underbrace{\begin{pmatrix}0&0&0&0&0\\0&-1&0&0&0\\0&0&1&0&0\\0&-1&0&-1&0\\0&0&-1&0&1\end{pmatrix}}_{=\mathrm{ad}(X_6)})$, i.e. $G=N\rtimes_\mu\mathbb{R}$. (Here X_6 is chosen as in the last line of Table **??** on page **??**.) For $t_1:=\frac{\pi}{3}$ we have

$$\mu(t_1)\big((x,y,z,v,w)\big)=\big(x-\frac{3}{4}yz+\frac{\sqrt{3}}{8}(z^2-y^2)+\frac{\pi}{8\sqrt{3}}(y-z),\,\overline{\mu}(t)\big((y,z,v,w)\big)\big),$$

and in order to construct a lattice in G, it is enough to construct a lattice in N that is preserved by $\mu(t_1)$. $\overline{\mu}(t_1)$ is conjugate to $A:=\begin{pmatrix}-1&-3&0&0\\1&2&0&0\\-2&-3&-1&-3\\1&1&1&2\end{pmatrix}$ and the transformation matrix $T\in\mathrm{GL}(4,\mathbb{R})$ with $TAT^{-1}=\overline{\mu}(t_1)$ is

$$T=\begin{pmatrix}\frac{\sqrt{3}}{\pi}&0&0&0\\-\frac{3}{\pi}&-\frac{6}{\pi}&0&0\\0&0&-\frac{2}{\sqrt{3}}&-\sqrt{3}\\0&0&0&1\end{pmatrix}.$$

Denote by $\{b_1,\dots,b_4\}$ the basis of $\mathbb{R}^4$ for which $\overline{\mu}(t_1)$ is represented by A, i.e. b_i is the i-th column of T. One calculates

$$b_{11}b_{22}-b_{12}b_{21}=\frac{-6\sqrt{3}}{\pi^2},$$
$$b_{i1}b_{j2}-b_{i2}b_{j1}=0\quad\text{for } i<j,(i,j)\neq(1,2).$$

This implies that we have for $\gamma_0:=(b_{11}b_{22}-b_{12}b_{21},0_{\mathbb{R}^4})$, $\gamma_i:=(b_{i0},b_i)\in N$ with arbitrary $b_{i0}\in\mathbb{R}$, $i=1,\dots,4$,

$$[\gamma_1,\gamma_2]=\gamma_0,\ [\gamma_1,\gamma_3]=[\gamma_1,\gamma_4]=[\gamma_2,\gamma_3]=[\gamma_2,\gamma_4]=[\gamma_3,\gamma_4]=e_N.$$

If we set $b_{10} = \frac{1488\sqrt{3}+72\sqrt{3}\pi-19\sqrt{3}\pi^2+4\pi^3}{128\pi^2}$, $b_{20} = \frac{2736\sqrt{3}+216\sqrt{3}\pi-25\sqrt{3}\pi^2+12\pi^3}{128\pi^2}$ and $b_{30} = b_{40} = 0$, we obtain

$$\begin{array}{rcl} \mu(t_1)(\gamma_0) & = & \gamma_0, \\ \mu(t_1)(\gamma_1) & = & \gamma_1^{-1}\ \gamma_2\ \gamma_3^{-2}\ \gamma_4, \\ \mu(t_1)(\gamma_2) & = & \gamma_1^{-3}\ \gamma_2^{2}\ \gamma_3^{-3}\ \gamma_4, \\ \mu(t_1)(\gamma_3) & = & \gamma_3^{-1}\ \gamma_4, \\ \mu(t_1)(\gamma_4) & = & \gamma_3^{-3}\ \gamma_4^{2}. \end{array}$$

Then $\langle \exp^N \left(\mathrm{Span}_{\mathbb{Z}} \log^N(\{\gamma_0, \ldots, \gamma_4\})\right)\rangle$ is a lattice in N. This can be seen by a similar computation as in the proof of the last theorem. Finally, one checks that the abelianisation of this lattice is isomorphic to $\mathbb{Z}$, hence the corresponding solvmanifold has $b_1 = 1$.

ad (ii): Let Γ be a lattice in G such that $b_1(G/\Gamma) = 1$ and $b_2(G/\Gamma) = 2$.

The Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$ has the closed generator x_6 and δ is given on the non-closed generators by

$$\delta x_1 = -x_{23},\ \delta x_2 = x_{36},\ \delta x_3 = -x_{26},\ \delta x_4 = -x_{26} + x_{56},\ \delta x_5 = -x_{36} - x_{46},$$

which implies $H^1(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_6]\rangle$.

One computes the differential of the non-exact generators of degree two in the Chevalley-Eilenberg complex as

$$\begin{array}{ll} \delta x_{12} = -x_{136}, & \delta x_{13} = x_{126}, \\ \delta x_{14} = x_{126} - x_{156} - x_{234}, & \delta x_{15} = x_{136} + x_{146} - x_{235}, \\ \delta x_{16} = -x_{236}, & \delta x_{24} = -x_{256} - x_{346}, \\ \delta x_{25} = x_{236} + x_{246} - x_{356}, & \delta x_{34} = -x_{236} + x_{246} - x_{356}, \\ \delta x_{35} = x_{256} + x_{346}, & \delta x_{45} = x_{256} - x_{346}, \end{array}$$

i.e. $H^2(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_{16} + \frac{1}{2}x_{25} - \frac{1}{2}x_{34}], [x_{24} + x_{35}]\rangle$.

This implies that G/Γ and $(\bigwedge \mathfrak{g}^*, \delta)$ have the same Betti numbers and therefore, by Theorem 3.10, they share their minimal model.

Now, the proof of non-formality is similar to the proof in the last theorem. □

Theorem 8.3.4.

(i) The completely solvable Lie group $G := G_{6.78}$ possesses a lattice.

(ii) For each lattice the corresponding quotient is a symplectic and formal manifold with $b_1 = b_2 = 1$.

Proof. ad (i): By definition, we have $G = \mathbb{R} \ltimes_\mu N$ with $N = G_{5.3}$ and $\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t\,\mathrm{ad}(X_6)) \circ \log^N$, where $\{X_1, \ldots, X_6\}$ denotes a basis of $\mathfrak{g}$ as in the second row of Table A.14. Note that $\{X_1, \ldots, X_5\}$ is a basis for the nilradical $\mathfrak{n}$. One computes

$$(11)\qquad \mu(t)_* := d_e\big(\mu(t)\big) = \exp^{A(\mathfrak{n})}(t\,\mathrm{ad}(X_6)) = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-t} & -te^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^t \end{pmatrix}.$$

Using $\mathfrak{n} = \langle X_5\rangle \ltimes_{ad(X_5)} \big(\langle X_1\rangle \oplus \langle X_2, X_3, X_4 \,|\, [X_2, X_4] = X_3\rangle\big)$, we can determine the Lie group N.

As a smooth manifold N equals $\mathbb{R}^5$, and the multiplication is given by

$$(a,b,c,r,s)\cdot(x,y,z,v,w)$$
$$= (a+x+bw+\frac{rw^2}{2}\,,\, b+y+rw\,,\, c+z+bv+\frac{r^2w}{2}+rvw\,,\, r+v\,,\, s+w).$$

Now, Theorem 4.2 enables us to compute the exponential map of N as

$$\exp^N(xX_1+yX_2+zX_3+vX_4+wX_5)$$
$$= \big(x+\frac{yw}{2}+\frac{vw^2}{6}\,,\, y+\frac{vw}{2}\,,\, z+\frac{yv}{2}+\frac{v^2w}{3}\,,\, v\,,\, w\big),$$

and therefore, we also obtain the logarithm of N

$$\log^N\big((x,y,z,v,w)\big)$$
$$= (x-\frac{yw}{2}+\frac{vw^2}{12})X_1+(y-\frac{vw}{2})X_2+(z-\frac{yv}{2}-\frac{v^2w}{12})X_3+vX_4+wX_5.$$

Finally, a short computation shows that (11) implies

$$\mu(t)\big((x,y,z,v,w)\big) = (e^t x, y, e^{-t}(z-tw), e^{-t}v, e^t w).$$

Let $t_1 := \ln(\frac{3+\sqrt{5}}{2})$, $b_0 := -\frac{2t_1}{1+\sqrt{5}}$ and consider for $t\in\mathbb{R}$ the automorphisms $I(t)\colon\, N\to N$ given by

$$I(t)\big((x,y,z,v,w)\big)$$
$$= (0,tb_0,0,0,0)(x,y,z,v,w)(0,tb_0,0,0,0)^{-1} = (x+tb_0w, y, z+tb_0v, v, w),$$

and $\nu(t) := \mu(t)\circ I(t)\colon\, N\to N$. It is easy to see that $\nu\colon\,\mathbb{R}\to \mathrm{A}(N)$ is a one-parameter group in N.

We shall show that there exists a lattice Γ_N in N preserved by $\nu(t_1)$, and this then implies the existence of a lattice in $G_{6.78}$, namely $t_1\mathbb{Z}\ltimes_\nu \Gamma_N$.

For the remainder of the proof, we identify $\mathfrak{n}\equiv\mathbb{R}^5$ with respect to the basis $\{X_1,\ldots,X_5\}$ of $\mathfrak{n}$. Under this identification, consider the basis $\{Y_1,\ldots,Y_5\}$ of $\mathfrak{n}$, Y_i being the i-th column of $T=(T_{ij})\in\mathrm{GL}(5,\mathbb{R})$, where T has the following entries:
$T_{11} = \frac{10(161+72\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})^2}{1165+521\sqrt{5}}$, $\quad T_{13} = \frac{5(2+\sqrt{5})(161+72\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})^2}{1525+682\sqrt{5}}$,
$T_{14} = \frac{328380+146856\sqrt{5}-(159975+71543\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})^2}{202950+90762\sqrt{5}}$, $\quad T_{22}=T_{33} = -\frac{(5+3\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}}$,
$T_{24} = -\frac{(158114965+70711162\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{141422324+63245986\sqrt{5}}$, $\quad T_{25} = \frac{5(3940598+1762585\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{17622890+7881196\sqrt{5}}$,
$T_{31} = \frac{1}{2}(5+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})$, $\quad T_{33}=T_{22}$, $\quad T_{35} = -\frac{597+267\sqrt{5}+(3808+1703\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{369+165\sqrt{5}}$,
$T_{45} = -\frac{2(2+\sqrt{5})}{3+\sqrt{5}}$, $\quad T_{54} = \ln(\frac{2}{3+\sqrt{5}})$, $\quad T_{55} = -\frac{2\ln(\frac{3+\sqrt{5}}{2})}{1+\sqrt{5}}$, $\quad T_{15}=T_{34}=T_{44}=1$,
$T_{12}=T_{21}=T_{23}=T_{32}=T_{41}=T_{42}=T_{43}=T_{51}=T_{52}=T_{53}=0$.

Let $\gamma_i := \exp^N(Y_i)$ for $i\in\{1,\ldots 5\}$ and
$S_1 = \frac{92880525355200+41537433696024\sqrt{5}}{57403321562460+25671545829588\sqrt{5}} - \frac{(3591421616495+1606132574069\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})^2}{57403321562460+25671545829588\sqrt{5}}$,
$S_2 = -\frac{(228826127+102334155\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{141422324+63245986\sqrt{5}}$, $\quad S_3 = 1-\frac{(757189543+338625458\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{848533944+379475916\sqrt{5}}$,
$S_4 = \frac{724734510+324111126\sqrt{5}-(325041375+145362922\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})^2}{724734510+324111126\sqrt{5}}$,

$S_5 = \frac{(120789085+54018521\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{74651760+33385282\sqrt{5}}$

$S_6 = -\frac{466724522940+208725552012\sqrt{5}}{24(12018817440+5374978561\sqrt{5})} + \frac{(3393446021605+1517595196457\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{24(12018817440+5374978561\sqrt{5})}$.

One computes $\gamma_1 = (T_{11}, 0, T_{31}, 0, 0)$, $\gamma_2 = (0, T_{22}, 0, 0, 0)$, $\gamma_3 = (T_{13}, 0, T_{33}, 0, 0)$, $\gamma_4 = (S_1, S_2, S_3, T_{44}, T_{54})$ and $\gamma_5 = (S_4, S_5, S_6, T_{45}, T_{55})$.

Moreover, if A denotes the matrix $\begin{pmatrix} 1 & 0 & 1 & \frac{13}{6} & \frac{11}{6} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 1 & 0 & 2 & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, we can calculate $TAT^{-1} = \nu(t_1)_* := d_e(\nu(t))$. Since $\nu(t_1) = \exp^N \circ \nu(t_1)_* \circ \log^N$, this yields $\nu(t_1)(\gamma_1) = \gamma_1\,\gamma_3$, $\nu(t_1)(\gamma_2) = \gamma_2$, $\nu(t_1)(\gamma_3) = \gamma_1\,\gamma_3^2$, $\nu(t_1)(\gamma_4) = \gamma_1^2\,\gamma_2^{-2}\,\gamma_4^2\,\gamma_5$ and $\nu(t_1)(\gamma_5) = \gamma_1^2\,\gamma_2^{-1}\,\gamma_4\,\gamma_5$.

Therefore, we have shown that $\nu(t_1)$ preserves the subgroup Γ_N of N which is generated by $\gamma_1, \ldots, \gamma_5$. In order to complete the proof of (i), it suffices to show that Γ_N is a lattice in N.

Since $\mathfrak{n}$ is 3-step nilpotent, the Baker-Campbell-Hausdorff formula (see e.g. [57, Chapter 2.15]) yields for all $V, W \in \mathfrak{n}$

$$\log^N\left(\exp^N(V)\exp^N(W)\right) = V + W + \frac{1}{2}[V,W] + \frac{1}{12}([[V,W],W] - [[V,W],V]).$$

Therefore, we obtain by a short calculation $[Y_2, Y_4] = Y_3$, $[Y_2, Y_5] = Y_1$ and $[Y_4, Y_5] = \frac{1}{2}Y_1 + Y_2 + \frac{1}{2}Y_3$, i.e. the basis $\{Y_1, \ldots, Y_5\}$ has rational structure constants. Theorem 2.1 then implies that Γ_N is a lattice in N.

ad (ii): Let Γ be a lattice in $G := G_{6.78}$. By completely solvability and Theorem 3.10 (ii), the minimal model of $M = G/\Gamma$ is the same as the minimal model $\mathcal{M}$ of the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, \delta)$. In view of Theorem 1.1.6, it suffices to prove that the latter is 2-formal. On the non-closed generators of $(\bigwedge \mathfrak{g}^*, \delta)$ the differential is given by

$$\delta x_1 = x_{16} - x_{25},\ \delta x_2 = -x_{45},\ \delta x_3 = -x_{24} - x_{36} - x_{46},\ \delta x_4 = -x_{46},\ \delta x_5 = x_{56},$$

i.e. $H^1(\bigwedge \mathfrak{g}^*, \delta) = \langle [x_6] \rangle$. Further, one calculates $H^2(\bigwedge \mathfrak{g}^*, \delta) = \langle x_{14} + x_{26} + x_{35} \rangle$. The minimal model $\rho\colon (\bigwedge V, d) \to (\bigwedge \mathfrak{g}^*, \delta)$ then must contain two closed generators y, z which map to x_6 and $x_{14} + x_{26} + x_{35}$. Since $\rho(yz) = x_{146} + x_{356}$ is closed and non-exact, there are no other generators of degree two in $(\bigwedge V, d)$, hence up to degree two, all generators are closed. This implies the minimal model's 2-formality.

Moreover, $x_{14} + x_{26} + x_{35}$ defines a symplectic form. □

8.4. Symplectic solvmanifolds whose first Betti number is greater than one. In this section, we examine which Lie groups G can give rise to a six-dimensional solvmanifold G/Γ with $b_1(G/\Gamma) > 1$. Again, we just consider indecomposable connected and simply-connected solvable Lie groups. The nilradical of such a group has not dimension equal to three, see e.g. [42]. Proposition 3.4 then tells us that indecomposable solvable Lie groups have nilradicals of dimension greater than three. Moreover, the nilpotent ones were considered in Section 8.1, hence we can assume that G is non-nilpotent, i.e. $\dim \mathrm{Nil}(G) \in \{4, 5\}$. The corresponding Lie algebras are listed in Tables A.9 – A.19.

In Section 8.2, we have excluded some groups G since they cannot admit lattices. Clearly, we omit them in the considerations below.

By Theorem 3.10(ii), we have in the completely solvable case an isomorphism from Lie algebra cohomology to the solvmanifold's cohomology, i.e. the Lie algebra $\mathfrak{g}$ must satisfy $b_1(\mathfrak{g}) > 1$, too. In the last section, we saw that $\mathfrak{g}^0_{6.38}$ is the only non-completely solvable but cohomologically symplectic Lie algebra with $b_1(\mathfrak{g}) = 1$. Therefore, for each lattice Γ in $G^0_{6.38}$ with $b_1(G^0_{6.38}/\Gamma) > 1$, the quotient is symplectic. We now turn to Lie algebras with $b_1(\mathfrak{g}) > 1$. The possible values of b_1 can be read of in Tables A.20 – **??**.

The remaining algebras to examine are $\mathfrak{g}_{6.i}$ as in (12), see below.

As above, we just consider such Lie algebras that are cohomologically symplectic, although this condition is only in the completely solvable case necessary for the existence of a symplectic form on G/Γ.

$$\begin{array}{lll}
i=2,\ a=0; & i=3,\ d=-1; & i=6,\ a=-\tfrac{1}{2},\ b=0;\\
i=9,\ b=0; & i=10,\ a=0; & i=21,\ a=0;\\
i=23,\ a=0; & i=25,\ b=0; & i=26;\\
i=29,\ b=0; & i=33,\ a=0; & i=34,\ a=0;\\
i=36,\ a=0; & i=54,\ l=-1; & i=63;\\
i=65,\ l=0; & i=70,\ p=0; & i=83,\ l=0;\\
i=84; & i=88,\ \mu_0=\nu_0=0; & i=89,\ \nu_0 s=0;\\
i=90,\ \nu_0=0; & i=92,\ \nu_0\mu_0=0; & i=93,\ \nu_0=0;\\
i=102; & i=105; & i=107;\\
i=113; & i=114; & i=115;\\
i=116; & i=118; & i=120;\\
i=125; & i=129; & i=135.
\end{array} \tag{12}$$

Proposition 8.4.1. *Let $\mathfrak{g}_{6.i}$ be one of the Lie algebras listed in (12).*

Then $\mathfrak{g}_{6.i}$ is cohomologically symplectic if and only if it is contained in the following list:

$$\begin{array}{llll}
b_1=2: & i=3\ \wedge\ d=-1, & i=10\ \wedge\ a=0, & i=21\ \wedge\ a=0,\\
 & i=36\ \wedge\ a=0, & i=54\ \wedge\ l=-1, & \\
 & i=70\ \wedge\ p=0, & i=118\ \wedge\ b=\pm1. & \\
b_1=3: & i=23\ \wedge\ a=0\ \wedge\ \varepsilon\neq0, & i=29\ \wedge\ b=0. &
\end{array}$$

Proof. This is done by a case by case analysis as described in the proof of Proposition 8.3.1. We list the symplectic forms for the Lie algebras that are cohomologically symplectic. In the cases with $b_1 = 2$, the symplectic forms are given by

$$\begin{array}{lll}
i=3,\ d=-1 : & \omega=\lambda\, x_{16}+\mu\, x_{23}+\nu\, x_{45}, & \lambda\mu\nu\neq0,\\
i=10,\ a=0 : & \omega=\lambda\, x_{16}+\mu\, x_{23}+\nu\, x_{45}, & \lambda\mu\nu\neq0,\\
i=21,\ a=0 : & \omega=\lambda\, x_{12}+\mu\, x_{36}+\nu\, x_{45}, & \lambda\mu\nu\neq0,\\
i=36,\ a=0 : & \omega=\lambda\, x_{12}+\mu\, x_{36}+\nu\, x_{45}, & \lambda\mu\nu\neq0,\\
i=54,\ l=-1 : & \omega=\lambda\,(x_{12}+x_{23})+\mu\, x_{34}+\nu\, x_{56}, & \lambda\nu\neq0,\\
i=70,\ p=0 : & \omega=\lambda\,(x_{13}+x_{24})+\mu\, x_{34}+\nu\, x_{56}, & \lambda\nu\neq0,\\
i=118,\ b=\pm1: & \omega=\lambda\,(x_{13}\pm x_{24})+\mu\,(x_{14}-x_{23})+\nu\, x_{56}, & (\lambda^2+\mu^2)\nu\neq0.
\end{array}$$

In the cases with $b_1 = 3$, we have the symplectic forms

$$\omega=\lambda\,(x_{12}+\varepsilon\, x_{35})+\mu\,(x_{16}+x_{24})+\nu\,(x_{23}-\varepsilon\, x_{56})+\rho\, x_{25}+\sigma\, x_{46}$$

with $\lambda\mu\nu \neq 0$ for $i = 23,\ a = 0,\ \varepsilon \neq 0$,

$$\omega = \lambda\,(x_{13} + \varepsilon\, x_{45}) + \mu\,(x_{16} + x_{24}) + \nu\,(x_{23} - \varepsilon\, x_{56}) + \rho\, x_{26} + \sigma\, x_{34}$$

with $\lambda \neq 0, \rho \neq \frac{(\lambda+\varepsilon)\mu\nu}{\lambda}$ for $i = 29,\ b = 0,\ \varepsilon \neq 0$ and

$$\omega = \lambda\, x_{12} + \mu\, x_{13} + \nu\,(x_{16} + x_{24}) + \rho\, x_{26} + \sigma\, x_{34} + \tau\, x_{56}$$

with $\nu(\nu\sigma + \mu\tau) \neq 0$ for $i = 29,\ b = 0,\ \varepsilon \neq 0$. □

Provided there is a lattice in one of the ten Lie groups $G_{6.i}$ in the last proposition whose Lie algebras are cohomologically symplectic, we can ensure that the corresponding solvmanifold is symplectic. In the completely solvable case, i.e. $i \in \{3, 21, 23, 29, 54\}$, we can determine cohomological properties of the potential solvmanifolds.

Proposition 8.4.2.
(i) There is a lattice in the completely solvable Lie group $G_{6.3}^{0,-1}$.
(ii) For each lattice the corresponding solvmanifold is symplectic, not formal and satisfies $b_1 = 2$ *as well as* $b_2 = 3$.

Proof. ad (i) : We have $G := G_{6.3}^{0,-1} = \mathbb{R} \ltimes_\mu \mathbb{R}^4$ with $\mu(t) = \exp^{GL(4,\mathbb{R})}(t\,\mathrm{ad}(X_6))$, where $X_6 \in \mathfrak{g}_{6.3}^{0,-1}$ is chosen as in Table **??**, i.e.

$$\mu(t) = \begin{pmatrix} 1 & -t & \frac{t^2}{2} & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & e^{t} \end{pmatrix}.$$

Set $t_1 := \ln(\frac{3+\sqrt{5}}{2})$. Then $\mu(t_1)$ is conjugate to $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$. (This can be seen by the use of Mathematica-software.) Hence G admits a lattice.

ad (ii): By completely solvability and Theorem 3.10 (ii), the solvmanifold's minimal model is the same as the minimal model of the Chevalley-Eilenberg complex. In view of Theorem 1.1.6, it suffices to prove that the latter is not 2-formal.

Using the knowledge of the Chevalley-Eilenberg complex, one can compute the minimal model up to generators of degree two. This implies the statement about the Betti numbers. Moreover, it is easy to see that the minimal model is not 1-formal. □

Proposition 8.4.3.
(i) There is a lattice in the completely solvable Lie group $G_{6.21}^{0}$.
(ii) For each lattice the corresponding solvmanifold is symplectic, not formal and satisfies $b_1 = 2$ *as well as* $b_2 = 3$.

Proof. The proof of (ii) is analogous to that of (ii) in the last proposition.

ad (i): In order to prove the existence of a lattice, we use the same argumentation as in the proof of Theorem 8.3.2 (i). (Note that $G_{6.15}^{-1}$ and $G := G_{6.21}^{0}$ share their nilradical N.) But of course, we now have a different initial data: $G = \mathbb{R} \ltimes_\mu N$ with

$$\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}) \circ \log^N \text{ and}$$

$$\overline{\mu}(t)\big((y,z,v,w)\big) = \exp^{GL(4,\mathbb{R})}(t \begin{pmatrix} 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}) \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix}.$$

Arguing analogous as in (9), one obtains

$$\mu(t)\big((x,y,z,v,w)\big) = \big(x - \frac{t}{2}y^2, \overline{\mu}(t)\big((x,y,z,v,w)\big)\big).$$

Let $t_1 = \ln(\frac{3+\sqrt{5}}{2})$, $A := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -\frac{1}{t_1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} & 1 \\ 0 & 0 & \frac{2}{3+\sqrt{5}} & 1 \end{pmatrix}$.
Then we have $TAT^{-1} = \overline{\mu}(t_1)$. Denote the i-th column of T by b_i. Analogous calculations as in loc. cit. imply the existence of a lattice generated by $\gamma_0 := (\frac{1}{t_1}, 0_{\mathbb{R}^4})$ and $\gamma_i := (b_{i0}, b_i)$, $i \in \{1, \dots, 4\}$, where $b_{20} \in \mathbb{R}$ arbitrary and $b_{10} = -\frac{1}{2t_1}$ as well as $b_{30} = b_{40} = 0$. □

Proposition 8.4.4.

(i) Let $\varepsilon = \pm 1$. There is a lattice in the completely solvable Lie group $G^{0,0,\varepsilon}_{6.23}$.

(ii) If there is a lattice in $G^{0,0,\varepsilon}_{6.23}$, $\varepsilon \neq 0$, then the corresponding solvmanifold is symplectic, non-formal and satisfies $b_1 = 3$ as well as $b_2 = 5$.

Proof. ad (i): $G^{0,0,\varepsilon}_{6.23}$ has the same nilradical N as $G^{-1}_{6.15}$ and the latter is described at the beginning of the proof of Theorem 8.3.2.

By definition, $G^{0,0,\varepsilon}_{6.23} = \mathbb{R} \ltimes_\mu N$ with

$$\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t \begin{pmatrix} 0 & 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}) \circ \log^N.$$

The functions $\exp^N, \log^N$ also can be found in the proof of Theorem 8.3.2. Using their knowledge, we calculate

$$\mu(t)\big((x,y,z,v,w)\big) = \big(x - \frac{t}{2}y^2 - t\varepsilon\,,\, y\,,\, z - ty\,,\, \frac{t^2}{2}y - tz + v\,,\, w\big).$$

If $\varepsilon = \pm 1$, then the map $\mu(2)$ preserves the lattice

$$\{(x,y,z,v,w) \in N \,|\, x,y,z,v,w \in \mathbb{Z}\} \subset N.$$

Therefore, $G^{0,0,\varepsilon}_{6.23}$ admits a lattice.

ad (ii): By completely solvability, the Betti numbers of the Chevalley-Eilenberg complex coincide with the solvmanifold's Betti numbers. A short calculation yields the first Betti numbers of the former as $b_1 = 3$ and $b_2 = 5$.

As above, the knowledge of the Chevalley-Eilenberg complex enables us to compute the first stage of the minimal model. It is easy to see that it is not 1-formal. □

Proposition 8.4.5.

(i) Let $\varepsilon \in \{0, \pm 1\}$. There is a lattice in the completely solvable Lie group $G^{0,0,\varepsilon}_{6.29}$.

(ii) If there is a lattice in $G^{0,0,\varepsilon}_{6.29}$, $\varepsilon \in \mathbb{R}$, then the corresponding solvmanifold is symplectic, non-formal and has $b_1 = 3$ as well as $b_2 = \left\{ \begin{array}{l} 5,\ \text{if } \varepsilon \neq 0 \\ 6,\ \text{if } \varepsilon = 0 \end{array} \right\}$.

Proof. The argumentation is analogous to the last proof, but this time we have

$$\mu(t)\big((x, y, z, v, w)\big) = \big(x - \frac{\varepsilon}{6}t^3 z + \frac{\varepsilon}{2}t^2 v - \varepsilon t w\, ,\, y\, ,\, z\, ,\, -tz + v\, ,\, \frac{1}{2}t^2 z - tv + w\big).$$

For $\varepsilon \in \{0, \pm 1\}$, $\mu(6)$ preserves the integer lattice mentioned in the last proof. This implies (i).

In order to prove (ii), we consider the minimal model up to generators of degree one and can deduce the non-formality. □

The following result is due to Fernández, de Léon and Saralegui. Its proof can be found in [19, Section 3]. Note that the cohomological results are independent of the choice of the lattice, since the Lie group in the proposition is completely solvable.

Proposition 8.4.6. *The completely solvable Lie group $G^{0,-1}_{6.54}$ admits a lattice. For each such, the corresponding solvmanifold is symplectic, non-formal and satisfies $b_1 = 2$ as well as $b_2 = 5$.* □

Summing up the results concerning completely solvable Lie groups that admit symplectic quotients, we obtain:

Theorem 8.4.7. *All six-dimensional symplectic solvmanifolds that can be written as quotient of a non-nilpotent completely solvable indecomposable Lie group are contained in one of the last five propositions, Theorem 8.3.2 or Theorem 8.3.4.* □

To end this section, we consider the four cohomologically symplectic Lie algebras $\mathfrak{g}_{6.i}$ of Proposition 8.4.1 that are not completely solvable, this means $i = 10 \wedge a = 0$, $i = 36 \wedge a = 0$, $i = 70 \wedge p = 0$ or $i = 118 \wedge b = \pm 1$. C Clearly, the existence of a lattice implies that the corresponding solvmanifold is symplectic. But in order to make a statement about cohomological properties, one needs an assumption about the first two Betti numbers to ensure the knowledge of the cohomology algebra.

Proposition 8.4.8.

(i) Each quotient of the Lie group $G := G^{0,0}_{6.10}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.

(ii) If there is a lattice in G such that the corresponding solvmanifold satisfies $b_1 = 2$ and $b_2 = 3$, then it is symplectic and not formal.

Proof. We have $G = \mathbb{R} \ltimes_\mu \mathbb{R}^4$ with $\mu(t) = \exp^{GL(4,\mathbb{R})}(t\, \mathrm{ad}(X_6))$ and $X_6 \in \mathfrak{g}^{0,0}_{6.10}$

chosen as in Table **??**, i.e.

$$\mu(t) = \begin{pmatrix} 1 & -t & \frac{t^2}{2} & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}.$$

$\mu(\pi)$ is conjugate to $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. (This can be seen by the use of Mathematica-software.) Hence G admits a lattice Γ.

A short calculation yields that the abelianisation of this lattice is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_2{}^2$, i.e. $b_1(G/\Gamma) = 2$.

Using the assumptions of (ii), one calculates the minimal model up to generators of degree one and proves the non-formality. □

Proposition 8.4.9.

(i) Each quotient of the Lie group $G := G^{0,0}_{6.36}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.

(ii) If there is a lattice in the Lie group G such that the corresponding solvmanifold satisfies $b_1 = 2$ and $b_2 = 3$, then it is symplectic and not formal.

Proof. The proof of (ii) is analogous to the last one.

ad (i): Using another initial data, we can argue as in the proof of Proposition 8.4.3. □

(i) Each quotient of the Lie group $G := G^{0,0}_{6.70}$ by a lattice is symplectic and G admits a lattice Γ with $b_1(G/\Gamma) = 2$.

(ii) If there is a lattice Γ in G such that $b_1(G/\Gamma) = 2$ and $b_2(G/\Gamma) = 3$, then G/Γ is formal.

Proof. ad (i): By definition, we have $G = \mathbb{R} \ltimes_\mu N$ with $N = G_{5.1}$ and $\mu(t) = \exp^N \circ \exp^{A(\mathfrak{n})}(t\,\mathrm{ad}(X_6)) \circ \log^N$, where $\{X_1, \ldots, X_6\}$ denotes a basis of $\mathfrak{g}$ as in the second row of Table **??**. Note that $\{X_1, \ldots, X_5\}$ is a basis of the nilradical $\mathfrak{n}$. One computes

$$\begin{aligned} \mu(t)_* &:= d_e\big(\mu(t)\big) = \exp^{A(\mathfrak{n})}(t\,\mathrm{ad}(X_6)) \\ &= \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) & 0 \\ 0 & 0 & -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Using $\mathfrak{n} = \langle X_5 \rangle \ltimes_{ad} \langle X_1, \ldots, X_4 \,|\, \rangle$, $\mathrm{ad}(X_5)(X_3) = -X_1$, we can determine the Lie group N.

As a smooth manifold N equals $\mathbb{R}^5$, and the multiplication is given by

$$(a, b, c, r, s) \cdot (x, y, z, v, w) = \big(a + x + cw\,,\, b + y + rw\,,\, c + z\,,\, r + v\,,\, s + w\big).$$

By Theorem 4.2, we can obtain the exponential map of N as

$$\exp^N(xX_1 + yX_2 + zX_3 + vX_4 + wX_5) = (x + \frac{zw}{2}\,,\, y + \frac{vw}{2}\,,\, z\,,\, v\,,\, w),$$

and obviously, this implies

$$\log^N\big((x,y,z,v,w)\big) = (x - \frac{wz}{2})X_1 + (y - \frac{vw}{2})X_2 + zX_3 + vX_4 + wX_5.$$

¿From $\mu(t) = \exp^N \circ \mu(t)_* \circ \log^N$ we get

$$\mu(t)\big((x,y,z,v,w)\big) = (\cos(t)\,x + \sin(t)\,y\,,\,-\sin(t)\,x + \cos(t)\,y\,,\\ \cos(t)\,z + \sin(t)\,v\,,\,-\sin(t)\,z + \cos(t)\,v\,,\,w),$$

and $\mu(\pi)$ preserves the lattice $\{(x,y,z,v,w) \in N \,|\, x,y,z,v,w \in \mathbb{Z}\}$.

The corresponding solvmanifold has $b_1 = 2$ since the abelianisation of this lattice is isomorphic to $\mathbb{Z}^2 \oplus {\mathbb{Z}_2}^4$.

ad (ii): A calculation of the minimal model up to generators of degree two shows that the minimal model is 2-formal. By Theorem 1.1.6, the solvmanifold is formal. □

PROPOSITION 8.4.10.
(i) $G := G^{0,\pm1,-1}_{6.118}$ *admits a lattice such that the first Betti number of the corresponding solvmanifold equals two and the second Betti number equals five.*
(ii) *If there is a lattice* Γ *in* G *such that* $b_1(G/\Gamma) = 2$ *and* $b_2(G/\Gamma) = 3$*, then* G/Γ *is symplectic and formal.*

Proof. The construction of the lattices mentioned in (i) can be found in [60]. In loc. cit. $G^{0,1,-1}_{6.118}$ is denoted by G_3 and $G^{0,-1,-1}_{6.118}$ by G_1, respectively. The Betti numbers of the quotient of $G^{0,-1,-1}_{118}$ are determined explicitly. In the case of $G^{0,1,-1}_{118}$, one can make an analogous computation.

Assume there is a lattice that satisfies the condition of (ii). A calculation of the solvmanifolds's minimal model up to generators of degree two shows its 2-formality. Theorem 1.1.6 then implies formality. □

9. Relations with the Lefschetz property. We have seen in Section 1.2 that a compact Kähler manifold is formal, Hard Lefschetz and its odd-degree Betti numbers are even. Even if a manifold has a complex structure, these conditions are not sufficient as the following theorem which is mentioned in [30] shows. Recall, we have seen above that $G^{1,-1,-1}_{5.7}$ admits a lattice.

THEOREM 9.1. *Let* Γ *be an arbitrary lattice in* $G^{1,-1,-1}_{5.7}$*. Then the solvmanifold* $M := G^{1,-1,-1}_{5.7}/\Gamma \times S^1$ *is formal, Hard Lefschetz and has even odd-degree Betti numbers. Moreover,* M *possesses a complex structure but it cannot be Kählerian.*

Proof. From Proposition 7.2.1 follows that the Lie group $G := G^{1,-1,-1}_{5.7} \times \mathbb{R}$ possesses a lattice Γ. The Chevalley-Eilenberg complex of its Lie algebra

$$\langle\, X_1, \ldots, X_6 \,|\, [X_1, X_5] = X_1,\ [X_2, X_5] = X_2,\ [X_3, X_5] = -X_3,\ [X_4, X_5] = -X_4 \,\rangle$$

is given by

$$\delta x_1 = -x_{15},\ \delta x_2 = -x_{25},\ \delta x_3 = x_{35},\ \delta x_4 = x_{45},\ \delta x_5 = \delta x_6 = 0,$$

where $\{x_1, \ldots, x_6\}$ is a basis of the left-invariant one-forms on G. Since G is completely solvable, Theorem 3.10 (ii) enables us to compute the cohomology of M as

$$\begin{aligned} &H^1(M,\mathbb{R}) = \langle [x_5], [x_6] \rangle,\\ &H^2(M,\mathbb{R}) = \langle [x_{13}], [x_{14}], [x_{23}], [x_{24}], [x_{56}] \rangle,\\ &H^3(M,\mathbb{R}) = \langle [x_{135}], [x_{136}], [x_{145}], [x_{146}], [x_{235}], [x_{236}], [x_{245}], [x_{246}] \rangle,\\ &H^4(M,\mathbb{R}) = \langle [x_{1234}], [x_{1356}], [x_{1456}], [x_{2356}], [x_{2456}] \rangle,\\ &H^5(M,\mathbb{R}) = \langle [x_{12345}], [x_{12346}] \rangle. \end{aligned} \tag{13}$$

Let $[\omega] \in H^2(M,\mathbb{R})$ represent a symplectic form on M. A short calculation shows that there are $a,b,c,d,e \in \mathbb{R}$ with $e(bc-ad) \neq 0$ and

$$[\omega] = a[x_{13}] + b[x_{14}] + c[x_{23}] + d[x_{24}] + e[x_{56}].$$

Since $[x_5] \cup [\omega]^2 = 2(bc-de)[x_{12345}] \neq 0$ and $[x_6] \cup [\omega]^2 = 2(bc-de)[x_{12346}] \neq 0$, the homomorphism $L^2\colon\ H^1(M,\mathbb{R}) \to H^5(M,\mathbb{R})$ is an isomorphism.

In the basis (13), the homomorphism $L^1\colon\ H^2(M,\mathbb{R}) \to H^4(M,\mathbb{R})$ is represented by the matrix $\begin{pmatrix} -d & c & -b & -a & 0 \\ e & 0 & 0 & 0 & a \\ 0 & e & 0 & 0 & b \\ 0 & 0 & e & 0 & c \\ 0 & 0 & 0 & e & d \end{pmatrix}$ which has $2e^3(ad-bc) \neq 0$ as determinant, hence M is Hard Lefschetz.

We define an almost complex structure J on G by

$$JX_1 = X_2,\ JX_2 = -X_1,\ JX_3 = X_4,\ JX_4 = -X_3,\ JX_5 = X_6,\ JX_6 = -X_5,$$

which induces an almost complex structure on M. It is easy to see that the Nijenhuis tensor vanishes, hence M is a complex manifold.

M is a non-toral solvmanifold which is a quotient of a completely solvable Lie group. Therefore, M cannot be Kählerian by Theorem 3.12. □

The authors of [33] considered the relations between the above three properties for closed symplectic manifolds. We want to try to complete [33, Theorem 3.1 Table 1] in the case of symplectic solvmanifolds. Actually, the mentioned table deals with symplectically aspherical closed manifolds, but note that symplectic solvmanifolds are symplectically aspherical.

We start our investigations by the examination of the Lefschetz property in dimension four.

Theorem 9.2. *A four-dimensional symplectic solvmanifold is not (Hard) Lefschetz if and only if it is a non-toral nilmanifold. Especially, the (Hard) Lefschetz property is independent of the choice of the symplectic form.*

Proof. By Theorem 6.2, there are five classes of four-dimensional symplectic solvmanifolds. Three of them are nilmanifolds and satisfy the claim by Corollary 2.9.

There remain two non-nilmanifolds to consider. We start with a quotient of the Lie group which has $\mathfrak{g}_{3.1}^{-1} \oplus \mathfrak{g}_1$ as Lie algebra, see Table A.1. The Lie group is completely solvable, hence the Lie algebra cohomology is isomorphic to the solvmanifold's cohomology. If $x_1,\dots,x_4$ denote the left-invariant one-forms which are dual to the basis given in Table A.1, one computes

$$H^1 = \langle [x_3],[x_4]\rangle, H^2 = \langle [x_{12}],[x_{34}]\rangle, H^3 = \langle [x_{123}],[x_{124}]\rangle. \tag{14}$$

The class representing a symplectic form must be of the form $[a\,x_{12} + b\,x_{34}]$ with $a,b \neq 0$ and obviously, the Lefschetz map with respect to this class is an isomorphism.

Now, consider a solvmanifold G/Γ such that the Lie algebra of G is $\mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_1$ and $b_1(G/\Gamma) = 2$. A short computation yields that the Lie algebra cohomology of $\mathfrak{g}_{3.5} \oplus \mathfrak{g}_1$ is the same as in (14). Since G/Γ is compact and parallelisable, we see further $b_i(G/\Gamma) = 2$ for $i \in \{1,2,3\}$, and Theorem 3.10 (i) implies that (14) also gives the cohomology of G/Γ. We have yet seen that a symplectic four-manifold with this cohomology is Hard Lefschetz. □

Denote KT "the" four-dimensional symplectic nilmanifold with $b_1(KT) = 3$. We have seen that KT is not formal and not Lefschetz. Its square has the following properties:

THEOREM 9.3 ([33]). *$KT \times KT$ is not formal, not Lefschetz and has even odd-degree Betti numbers.* □

Next, we are looking for an example of a formal manifold that is not Lefschetz and has even odd degree Betti numbers resp. an odd odd degree Betti number.

THEOREM 9.4. *The Lie group $G_{6.78}$ admits a lattice Γ, see above. $M := G_{6.78}/\Gamma$ is a formal solvmanifold with $b_1(M) = 1$ that admits a symplectic form ω such that (M, ω) is not Hard Lefschetz. Moreover, $(M \times M, \omega \times \omega)$ is a formal symplectic manifold with even odd degree Betti numbers that is not Hard Lefschetz.*

Proof. By Theorem 8.3.4, M is a formal symplectic manifold with Betti numbers $b_1(M) = b_2(M) = 1$. Note that this implies that $M \times M$ is symplectic and formal (the latter property by Proposition 1.1.7).

Corollary 1.2.3 forces M to be not Lefschetz and since [20, Proposition 4.2] says that a product is Lefschetz if and only if both factors are Lefschetz, $M \times M$ is not Lefschetz.

M is a six-dimensional solvmanifold and so it is parallelisable. Hence the fact $b_0(M) = b_1(M) = b_2(M) = 1$ implies $b_3(M) = 2$. This and Poincaré Duality imply $b_1(M \times M) = b_{11}(M \times M) = 2$, $b_3(M \times M) = b_9(M \times M) = 6$ and $b_5(M \times M) = b_7(M \times M) = 4$. □

In 1990, Benson and Gordon [4, Example 3] constructed an eight-dimensional non-exact symplectic and completely solvable Lie algebra that does not satisfy the Hard Lefschetz property, but they did not know whether the corresponding connected and simply-connected Lie group G^{BG} admits a lattice.

Fernández, de León and Saralegui computed in [19, Proposition 3.2] the minimal model of the complex of the left-invariant differential forms on G^{BG}. It is formal and its cohomology of odd degree is even-dimensional. If G^{BG} admits a lattice, by completely solvability, the corresponding solvmanifold would be a symplectic and formal manifold with even odd degree Betti numbers that violates the Hard Lefschetz property.

In 2000, Tralle [55] claimed that a lattice does not exist but Sawai and Yamada noted 2005 Tralle's proof would contain calculatory errors and constructed a lattice [51, Theorem 1]. This proves the next theorem.

THEOREM 9.5. *There exists an eight-dimensional symplectic and formal solvmanifold M^{BG} with even odd degree Betti numbers that is not Hard Lefschetz.* □

We sum up the above results in Table 9.1 on page 245. It is an enlargement of [33, Theorem 3.1 Table 1].

Unfortunately, the missing example does not arise among the six-dimensional solvmanifolds that possess the same cohomology as the corresponding Lie algebra. We omit the discussion here.

TABLE 9.1: *Relations of the Kähler properties*

Formality	Hard Lefschetz	$b_{2i+1} \equiv 0(2)$	Example
yes	yes	yes	Kähler
yes	yes	no	impossible
yes	no	yes	M^{BG}, $G_{6.78}/\Gamma \times G_{6.78}/\Gamma$
yes	no	no	$G_{6.78}/\Gamma$
no	yes	yes	?
no	yes	no	impossible
no	no	yes	$KT \times KT$
no	no	no	KT

Appendix A. Lists of Lie Algebras.

In Table A.1, we give the isomorphism classes of Lie algebras of the simply-connected solvable Lie groups up to dimension four that possesses lattices. The designation $\mathfrak{g}_{i,j}$ means the j-th indecomposable solvable Lie algebra of dimension i. The choice of the integer j bases on the notation of [40]. The superscripts, if any, give the values of the continuous parameters on which the algebra depends.

TABLE A.1: *Solvmanifolds up to dimension four*

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_1$		abelian
$2\mathfrak{g}_1$		abelian
$3\mathfrak{g}_1$		abelian
$\mathfrak{g}_{3.1}$	$[X_2, X_3] = X_1$	nilpotent
$\mathfrak{g}_{3.4}^{-1}$	$[X_1, X_3] = X_1,\ [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^{0}$	$[X_1, X_3] = -X_2,\ [X_2, X_3] = X_1$	no
$4\mathfrak{g}_1$		abelian
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1$	nilpotent
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_1$	$[X_1, X_3] = X_1,\ [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_1$	$[X_1, X_3] = -X_2,\ [X_2, X_3] = X_1$	no
$\mathfrak{g}_{4.1}$	$[X_2, X_4] = X_1,\ [X_3, X_4] = X_2$	nilpotent
$\mathfrak{g}_{4.5}^{p,-p-1}$	$[X_1, X_4] = X_1,\ [X_2, X_4] = pX_2,$ $[X_3, X_4] = (-p-1)X_3,\ \ -\frac{1}{2} \le p < 0$	yes
$\mathfrak{g}_{4.6}^{-2p,p}$	$[X_1, X_4] = -2pX_1,\ [X_2, X_4] = pX_2 - X_3,$ $[X_3, X_4] = X_2 + pX_3,\ p > 0$	no
$\mathfrak{g}_{4.8}^{-1}$	$[X_2, X_3] = X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = -X_3$	yes
$\mathfrak{g}_{4.9}^{0}$	$[X_2, X_3] = X_1,\ [X_2, X_4] = -X_3,\ [X_3, X_4] = X_2$	no

We do not claim that the corresponding Lie groups admit a lattice for all parameters. We just know that there exist such for certain parameters!

The set of non-isomorphic five dimensional nilpotent Lie algebras is exhausted by three types of decomposable algebras and six indecomposables which are listed in the next table. The designation is taken from [41].

Table A.2: *5-dimensional nilpotent algebras*

	$[X_i, X_j]$
$5\mathfrak{g}_1$	abelian
$\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$	$[X_2, X_3] = X_1$
$\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$	$[X_2, X_4] = X_1,\ [X_3, X_4] = X_2$
$\mathfrak{g}_{5.1}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2$
$\mathfrak{g}_{5.2}$	$[X_2, X_5] = X_1,\ [X_3, X_5] = X_2,\ [X_4, X_5] = X_3$
$\mathfrak{g}_{5.3}$	$[X_2, X_4] = X_3,\ [X_2, X_5] = X_1,\ [X_4, X_5] = X_2$
$\mathfrak{g}_{5.4}$	$[X_2, X_4] = X_1,\ [X_3, X_5] = X_1$
$\mathfrak{g}_{5.5}$	$[X_3, X_4] = X_1,\ [X_2, X_5] = X_1,\ [X_3, X_5] = X_2$
$\mathfrak{g}_{5.6}$	$[X_3, X_4] = X_1,\ [X_2, X_5] = X_1,\ [X_3, X_5] = X_2,\ [X_4, X_5] = X_3$

There are 24 classes of solvable and non-nilpotent decomposable Lie algebras in dimension five. The unimodular among them are the ones in Table A.3.

Table A.3: *5-dimensional decomposable unimodular non-nilpotent algebras*

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{3.4}^{-1} \oplus 2\mathfrak{g}_1$	$[X_1, X_3] = X_1,\ [X_2, X_3] = -X_2$	yes
$\mathfrak{g}_{3.5}^{0} \oplus 2\mathfrak{g}_1$	$[X_1, X_3] = -X_2,\ [X_2, X_3] = X_1$	no
$\mathfrak{g}_{4.2}^{-2} \oplus \mathfrak{g}_1$	$[X_1, X_4] = -2X_1,\ [X_2, X_4] = X_2,$ $[X_3, X_4] = X_2 + X_3$	yes
$\mathfrak{g}_{4.5}^{p,-p-1} \oplus \mathfrak{g}_1$	$[X_1, X_4] = X_1,\ [X_2, X_4] = pX_2,$ $[X_3, X_4] = (-p-1)X_3,\ -\frac{1}{2} \leq p < 0$	yes
$\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathfrak{g}_1$	$[X_1, X_4] = -2pX_1,\ [X_2, X_4] = pX_2 - X_3,$ $[X_3, X_4] = X_2 + pX_3,\ p > 0$	no
$\mathfrak{g}_{4.8}^{-1} \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = -X_3$	yes
$\mathfrak{g}_{4.9}^{0} \oplus \mathfrak{g}_1$	$[X_2, X_3] = X_1,\ [X_2, X_4] = -X_3,\ [X_3, X_4] = X_2$	no

Except for $\mathfrak{g}_{4.2} \oplus \mathfrak{g}_1$, to each class of algebras there is a connected and simply-connected solvable Lie group admitting a lattice and has a Lie algebra belonging to the class.

Mubarakzjanov's list in [41] contains 33 classes of five-dimensional indecomposable non-nilpotent solvable Lie algebras, namely $\mathfrak{g}_{5.7}, \ldots, \mathfrak{g}_{5.39}$. We list the unimodular among them in Tables A.4 to A.7.

Note that there is a minor misprint in [41] which has been corrected in the list below.

Table A.4: *5-dimensional indecomposable unimodular almost abelian algebras*

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.7}^{p,q,r}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = pX_2,$ $[X_3, X_5] = qX_3,\ [X_4, X_5] = rX_4,$ $-1 \leq r \leq q \leq p \leq 1,\ pqr \neq 0,\ p + q + r = -1$	yes

(*Continued on next page*)

TABLE A.4: *5-dimensional indecomposable unimodular almost abelian algebras*

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.8}^{-1}$	$[X_2, X_5] = X_1,\ [X_3, X_5] = X_3,\ [X_4, X_5] = -X_4,$	yes
$\mathfrak{g}_{5.9}^{p,-2-p}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = X_1 + X_2,\ [X_3, X_5] = pX_3,$ $[X_4, X_5] = (-2 - p)X_4,\ p \geq -1$	yes
$\mathfrak{g}_{5.11}^{-3}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = X_2 + X_3,\ [X_4, X_5] = -3X_4,$	yes
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = (-1 - 2q)X_2,$ $[X_3, X_5] = qX_3 - rX_4,\ [X_4, X_5] = rX_3 + qX_4,$ $-1 \leq q \leq 0,\ q \neq -\frac{1}{2},\ r \neq 0$	no
$\mathfrak{g}_{5.14}^{0}$	$[X_2, X_5] = X_1,\ [X_3, X_5] = -X_4,\ [X_4, X_5] = X_3$	no
$\mathfrak{g}_{5.15}^{-1}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = -X_3,\ [X_4, X_5] = X_3 - X_4$	yes
$\mathfrak{g}_{5.16}^{-1,q}$	$[X_1, X_5] = X_1,\ [X_2, X_5] = X_1 + X_2,$ $[X_3, X_5] = -X_3 - qX_4,\ [X_4, X_5] = qX_3 - X_4$ $q \neq 0$	no
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[X_1, X_5] = pX_1 - X_2,\ [X_2, X_5] = X_1 + pX_2,$ $[X_3, X_5] = -pX_3 - rX_4,\ [X_4, X_5] = rX_3 - pX_4$ $r \neq 0$	no
$\mathfrak{g}_{5.18}^{0}$	$[X_1, X_5] = -X_2,\ [X_2, X_5] = X_1,$ $[X_3, X_5] = X_1 - X_4,\ [X_4, X_5] = X_2 + X_3$	no

TABLE A.5: *5-dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.19}^{p,-2p-2}$	$[X_2, X_3] = X_1,\ [X_1, X_5] = (1 + p)X_1,\ [X_2, X_5] = X_2,$ $[X_3, X_5] = pX_3,\ [X_4, X_5] = (-2p - 2)X_4,\ p \neq -1$	yes
$\mathfrak{g}_{5.20}^{-1}$	$[X_2, X_3] = X_1,\ [X_2, X_5] = X_2,\ [X_3, X_5] = -X_3,$ $[X_4, X_5] = X_1$	yes
$\mathfrak{g}_{5.23}^{-4}$	$[X_2, X_3] = X_1,\ [X_1, X_5] = 2X_1,\ [X_2, X_5] = X_2 + X_3,$ $[X_3, X_5] = X_3,\ [X_4, X_5] = -4X_4$	yes
$\mathfrak{g}_{5.25}^{p,4p}$	$[X_2, X_3] = X_1,\ [X_1, X_5] = 2pX_1,\ [X_2, X_5] = pX_2 + X_3,$ $[X_3, X_5] = -X_2 + pX_3,\ [X_4, X_5] = -4pX_4,\ p \neq 0$	no
$\mathfrak{g}_{5.26}^{0,\varepsilon}$	$[X_2, X_3] = X_1,\ [X_2, X_5] = X_3,\ [X_3, X_5] = -X_2,$ $[X_4, X_5] = \varepsilon X_1,\ \varepsilon = \pm 1$	no
$\mathfrak{g}_{5.28}^{-\frac{3}{2}}$	$[X_2, X_3] = X_1,\ [X_1, X_5] = -\frac{1}{2}X_1,\ [X_2, X_5] = -\frac{3}{2}X_2,$ $[X_3, X_5] = X_3 + X_4,\ [X_4, X_5] = X_4$	yes

There are ten classes of decomposable nilpotent Lie algebras in dimension six: $6\mathfrak{g}_1$, $\mathfrak{g}_{3.1} \oplus 3\mathfrak{g}_1$, $2\mathfrak{g}_{3.1}$, $\mathfrak{g}_{4.1} \oplus 2\mathfrak{g}_1$ and $\mathfrak{g}_{5.i} \oplus \mathfrak{g}_1$ for $i \in \{1, \ldots 6\}$.

Table A.8 contains the six-dimensional indecomposable nilpotent real Lie algebras. They base on Morozov's classification in [38], where nilpotent algebras over a field of characteristic zero are determined. Note that over $\mathbb{R}$, there is only one isomorphism class of Morozov's indecomposable type 5 resp. type 10 and type 14 resp. 18 splits into two non-isomorphic ones.

Table A.6: 5-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{4.1}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.30}^{-\frac{4}{3}}$	$[X_2, X_4] = X_1,\ [X_3, X_4] = X_2,\ [X_1, X_5] = \frac{2}{3}X_1,$ $[X_2, X_5] = -\frac{1}{3}X_2,\ [X_3, X_5] = -\frac{4}{3}X_3,\ [X_4, X_5] = X_4,$	yes

Table A.7: 5-*dimensional indecomposable unimodular algebras with nilradical* $3\mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{5.33}^{-1,-1}$	$[X_1, X_4] = X_1,\ [X_3, X_4] = -X_3,$ $[X_2, X_5] = X_2,\ [X_3, X_5] = -X_3$	yes
$\mathfrak{g}_{5.35}^{-2,0}$	$[X_1, X_4] = -2X_1,\ [X_2, X_4] = X_2,\ [X_3, X_4] = X_3$ $[X_2, X_5] = -X_3,\ [X_3, X_5] = X_2$	no

Table A.8: 6-*dimensional indecomposable nilpotent algebras*

	$[X_i, X_j]$
$\mathfrak{g}_{6.N1}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_5] = X_6$
$\mathfrak{g}_{6.N2}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_1, X_5] = X_6$
$\mathfrak{g}_{6.N3}$	$[X_1, X_2] = X_6,\ [X_1, X_3] = X_4,\ [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N4}$	$[X_1, X_2] = X_5,\ [X_1, X_3] = X_6,\ [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N5}$	$[X_1, X_3] = X_5,\ [X_1, X_4] = X_6,\ [X_2, X_3] = -X_6,\ [X_2, X_4] = X_5$
$\mathfrak{g}_{6.N6}$	$[X_1, X_2] = X_6,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N7}$	$[X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N8}$	$[X_1, X_2] = X_3 + X_5,\ [X_1, X_3] = X_4,\ [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N9}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_5] = X_6,\ [X_2, X_3] = X_5$
$\mathfrak{g}_{6.N10}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,\ [X_1, X_4] = X_6,$ $[X_2, X_3] = -X_6,\ [X_2, X_4] = X_5$
$\mathfrak{g}_{6.N11}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N12}$	$[X_1, X_3] = X_4,\ [X_1, X_4] = X_6,\ [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N13}$	$[X_1, X_2] = X_5,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_6,\ [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N14}^{\pm 1}$	$[X_1, X_3] = X_4,\ [X_1, X_4] = X_6,\ [X_2, X_3] = X_5,\ [X_2, X_5] = \pm X_6$
$\mathfrak{g}_{6.N15}$	$[X_1, X_2] = X_3 + X_5,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_6,\ [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N16}$	$[X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_1, X_5] = X_6,$ $[X_2, X_3] = X_5,\ [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N17}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_6,\ [X_2, X_5] = X_6$
$\mathfrak{g}_{6.N18}^{\pm 1}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_6,$ $[X_2, X_3] = X_5,\ [X_2, X_5] = \pm X_6$
$\mathfrak{g}_{6.N19}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,$ $[X_1, X_5] = X_6,\ [X_2, X_3] = X_6$
$\mathfrak{g}_{6.N20}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,$ $[X_1, X_5] = X_6,\ [X_2, X_3] = X_5,\ [X_2, X_4] = X_6$
$\mathfrak{g}_{6.N21}$	$[X_1, X_2] = X_3,\ [X_1, X_5] = X_6,\ [X_2, X_3] = X_4,$ $[X_2, X_4] = X_5,\ [X_3, X_4] = X_6$

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TABLE A.8: 6-*dimensional indecomposable nilpotent algebras*

	$[X_i, X_j]$
$\mathfrak{g}_{6.N22}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,\ [X_1, X_5] = X_6,$ $[X_2, X_3] = X_4,\ [X_2, X_4] = X_5,\ [X_3, X_4] = X_6$

Mubarakzjanov's list in [42] contains 99 classes of six-dimensional indecomposable almost nilpotent Lie algebras, namely $\mathfrak{g}_{6.1}, \ldots, \mathfrak{g}_{6.99}$.

As first remarked by Turkowski, there is one algebra missing. The complete (and partly corrected) list can be found in the article [10] of Campoamor-Stursberg[13], where the missing algebra is denoted by $\mathfrak{g}^*_{6.92}$.

We list the unimodular among this 100 algebras in Tables A.9 to A.16 (where some minor misprints have been corrected). Note that there is no table with Lie algebras with nilradical $\mathfrak{g}_{5.6}$ since the only such algebra is not unimodular.

TABLE A.9: 6-*dimensional indecomposable unimodular almost abelian algebras*

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.1}^{a,b,c,d}$	$[X_1, X_6] = X_1,\ [X_2, X_6] = aX_2,\ [X_3, X_6] = bX_3,$ $[X_4, X_6] = cX_4,\ [X_5, X_6] = dX_5$ $0 < \lvert d\rvert \le \lvert c\rvert \le \lvert b\rvert \le \lvert a\rvert \le 1,\ a+b+c+d = -1$	yes
$\mathfrak{g}_{6.2}^{a,c,d}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = X_1 + aX_2,\ [X_3, X_6] = X_3,$ $[X_4, X_6] = cX_4,\ [X_5, X_6] = dX_5$ $0 < \lvert d\rvert \le \lvert c\rvert \le 1,\ 2a+c+d = -1$	yes
$\mathfrak{g}_{6.3}^{-\frac{d+1}{3},d}$	$[X_1, X_6] = -\frac{d+1}{3}X_1,\ [X_2, X_6] = X_1 - \frac{d+1}{3}X_2,$ $[X_3, X_6] = X_2 - \frac{d+1}{3}X_3,\ [X_4, X_6] = X_4,\ [X_5, X_6] = dX_5$ $0 < \lvert d\rvert \le 1,$	yes
$\mathfrak{g}_{6.4}^{-\frac{1}{4}}$	$[X_1, X_6] = -\frac{1}{4}X_1,\ [X_2, X_6] = X_1 - \frac{1}{4}X_2,$ $[X_3, X_6] = X_2 - \frac{1}{4}X_3,\ [X_4, X_6] = X_3 - \frac{1}{4}X_4,\ [X_5, X_6] = X_5$	yes
$\mathfrak{g}_{6.6}^{a,b}$	$[X_1, X_6] = X_1,\ [X_2, X_6] = aX_2,\ [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = bX_4,\ [X_5, X_6] = X_4 + bX_5,\ a \le b,\ a+b = -\frac{1}{2}$	yes
$\mathfrak{g}_{6.7}^{a,-\frac{2}{3}a}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = X_1 + aX_2,\ [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = -\frac{3}{2}aX_4,\ [X_5, X_6] = X_4 - \frac{3}{2}aX_5,\ a \neq 0$	yes
$\mathfrak{g}_{6.8}^{a,b,c,p}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = bX_2,\ [X_3, X_6] = cX_3,$ $[X_4, X_6] = pX_4 - X_5,\ [X_5, X_6] = X_4 + pX_5$ $0 < \lvert c\rvert \le \lvert b\rvert \le \lvert a\rvert,\ a+b+c+2p = 0$	no
$\mathfrak{g}_{6.9}^{a,b,p}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = bX_2,\ [X_3, X_6] = X_2 + bX_3,$ $[X_4, X_6] = pX_4 - X_5,\ [X_5, X_6] = X_4 + pX_5,$ $a \neq 0,\ a+2b+2p = 0$	no
$\mathfrak{g}_{6.10}^{a,-\frac{3}{2}a}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = X_1 + aX_2,\ [X_3, X_6] = X_2 + aX_3,$ $[X_4, X_6] = -\frac{3}{2}aX_4 - X_5,\ [X_5, X_6] = X_4 - \frac{3}{2}aX_5$	no
$\mathfrak{g}_{6.11}^{a,p,q,s}$	$[X_1, X_6] = aX_1,\ [X_2, X_6] = pX_2 - X_3,\ [X_3, X_6] = X_2 + pX_3,$ $[X_4, X_6] = qX_4 - sX_5,\ [X_5, X_6] = sX_4 + qX_5,$	no

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[13]The author wishes to express his gratitude to R. Campoamor-Stursberg for providing him with copies of [10] and [42].

Table A.9: *6-dimensional indecomposable unimodular almost abelian algebras*

	$[X_i, X_j]$	cpl. solv.
	$as \neq 0, a + 2p + 2q = 0$	
$\mathfrak{g}_{6.12}^{-4p,p}$	$[X_1, X_6] = -4pX_1,\ [X_2, X_6] = pX_2 - X_3,$ $[X_3, X_6] = X_2 + pX_3,\ [X_4, X_6] = X_2 + pX_4 - X_5,$ $[X_5, X_6] = X_3 + X_4 + pX_5,\ p \neq 0$	no

Table A.10: *6-dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.13}^{a,b,h}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = (a + b)X_1,\ [X_2, X_6] = aX_2,$ $[X_3, X_6] = bX_3,\ [X_4, X_6] = X_4,\ [X_5, X_6] = hX_5,$ $a \neq 0,\ 2a + 2b + h = -1$	yes
$\mathfrak{g}_{6.14}^{a,b}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = (a + b)X_1,\ [X_2, X_6] = aX_2,$ $[X_3, X_6] = bX_3,\ [X_4, X_6] = X_4,\ [X_5, X_6] = X_1 + (a + b)X_5,$ $a \neq 0,\ a + b = -\frac{1}{3}$	yes
$\mathfrak{g}_{6.15}^{-1}$	$[X_2, X_3] = X_1,\ [X_2, X_6] = X_2 + X_4,$ $[X_3, X_6] = -X_3 + X_5,\ [X_4, X_6] = X_4,\ [X_5, X_6] = -X_5,$	yes
$\mathfrak{g}_{6.17}^{-\frac{1}{2},0}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = -\frac{1}{2}X_1,\ [X_2, X_6] = -\frac{1}{2}X_2,$ $[X_3, X_6] = X_4,\ [X_5, X_6] = X_5,$	yes
$\mathfrak{g}_{6.18}^{a,-2a-3}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = (1 + a)X_1,\ [X_2, X_6] = aX_2,$ $[X_3, X_6] = X_3 + X_4,\ [X_4, X_6] = X_4,$ $[X_5, X_6] = -(2a + 3)X_5,\ a \neq -\frac{3}{2}$	yes
$\mathfrak{g}_{6.19}^{-\frac{4}{3}}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = -\frac{1}{3}X_1,\ [X_2, X_6] = -\frac{4}{3}X_2,$ $[X_3, X_6] = X_3 + X_4,\ [X_4, X_6] = X_4,\ [X_5, X_6] = X_1 - \frac{1}{3}X_5$	yes
$\mathfrak{g}_{6.20}^{-3}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = X_1,\ [X_3, X_6] = X_3 + X_4,$ $[X_4, X_6] = X_1 + X_4,\ [X_5, X_6] = -3X_5$	yes
$\mathfrak{g}_{6.21}^{a}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = 2aX_1,\ [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = aX_3,\ [X_4, X_6] = X_4,\ [X_5, X_6] = -(4a + 1)X_5$ $a \neq -\frac{1}{4}$	yes
$\mathfrak{g}_{6.22}^{-\frac{1}{6}}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = -\frac{1}{3}X_1,\ [X_2, X_6] = -\frac{1}{6}X_2 + X_3,$ $[X_3, X_6] = -\frac{1}{6}X_3,\ [X_4, X_6] = X_4,\ [X_5, X_6] = X_1 - \frac{1}{3}X_5$	yes
$\mathfrak{g}_{6.23}^{a,-7a,\varepsilon}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = 2aX_1,\ [X_2, X_6] = aX_2 + X_3,$ $[X_3, X_6] = aX_3 + X_4,\ [X_4, X_6] = aX_4,$ $[X_5, X_6] = \varepsilon X_1 - 5aX_5,\ \varepsilon a = 0$	yes
$\mathfrak{g}_{6.25}^{b,-1-b}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = -bX_1,$ $[X_2, X_6] = X_2,\ [X_3, X_6] = -(1 + b)X_3,$ $[X_4, X_6] = bX_4 + X_5,\ [X_5, X_6] = bX_5$	yes
$\mathfrak{g}_{6.26}^{-1}$	$[X_2, X_3] = X_1,\ [X_2, X_6] = X_2,\ [X_3, X_6] = -X_3$ $[X_4, X_6] = X_5,\ [X_5, X_6] = X_1$	yes
$\mathfrak{g}_{6.27}^{-2b,b}$	$[X_2, X_3] = X_1,\ [X_1, X_6] = -bX_1,\ [X_2, X_6] = -2bX_2,$ $[X_3, X_6] = bX_3 + X_4,\ [X_4, X_6] = bX_4 + X_5,$ $[X_5, X_6] = bX_5,\ b \neq 0$	yes

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TABLE A.10: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.28}^{-2}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2X_1,\ [X_2,X_6]=X_2+X_3,$ $[X_3,X_6]=X_3,\ [X_4,X_6]=-2X_4+X_5,\ [X_5,X_6]=-2X_5$	yes
$\mathfrak{g}_{6.29}^{-2b,b,\varepsilon}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=-bX_1,\ [X_2,X_6]=-2bX_2,$ $[X_3,X_6]=bX_3+X_4,\ [X_4,X_6]=bX_4+X_5,$ $[X_5,X_6]=\varepsilon X_1+bX_5,\ \varepsilon b=0$ (?)	yes
$\mathfrak{g}_{6.32}^{a,-6a-h,h,\varepsilon}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2aX_1,\ [X_2,X_6]=aX_2+X_3,$ $[X_3,X_6]=-X_2+aX_3,\ [X_4,X_6]=\varepsilon X_1+(2a+h)X_4,$ $[X_5,X_6]=-(6a+h)X_5,\ a>-\frac{1}{4}h,\ \varepsilon h=0$	no
$\mathfrak{g}_{6.33}^{a,-6a}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2aX_1,\ [X_2,X_6]=aX_2+X_3,$ $[X_3,X_6]=-X_2+aX_3,\ [X_4,X_6]=-6aX_4,$ $[X_5,X_6]=X_1+2aX_5,\ a\geq 0$	no
$\mathfrak{g}_{6.34}^{a,-4a,\varepsilon}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2aX_1,\ [X_2,X_6]=aX_2+X_3,$ $[X_3,X_6]=-X_2+aX_3,\ [X_4,X_6]=-2aX_4,$ $[X_5,X_6]=\varepsilon X_1-2aX_5,\ \varepsilon a=0$	no
$\mathfrak{g}_{6.35}^{a,b,c}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=(a+b)X_1,\ [X_2,X_6]=aX_2,$ $[X_3,X_6]=bX_3,\ [X_4,X_6]=cX_4+X_5,$ $[X_5,X_6]=-X_4+cX_5,\ a+b+c=0,\ a^2+b^2\neq 0$	no
$\mathfrak{g}_{6.36}^{a,-2a}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2aX_1,\ [X_2,X_6]=aX_2+X_3,$ $[X_3,X_6]=aX_3,\ [X_4,X_6]=-2aX_4+X_5,$ $[X_5,X_6]=-X_4-2aX_5$	no
$\mathfrak{g}_{6.37}^{-a,-2a,s}$	$[X_2,X_3]=X_1,\ [X_1,X_6]=2aX_1,\ [X_2,X_6]=aX_2+X_3,$ $[X_3,X_6]=-X_2+aX_3,\ [X_4,X_6]=-2aX_4+sX_5,$ $[X_5,X_6]=-sX_4-2aX_5,\ s\neq 0$	no
$\mathfrak{g}_{6.38}^{0}$	$[X_2,X_3]=X_1,\ [X_2,X_6]=X_3+X_4,$ $[X_3,X_6]=-X_2+X_5,\ [X_4,X_6]=X_5,\ [X_5,X_6]=-X_4$	no

TABLE A.11: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.39}^{-4-3h,h}$	$[X_1,X_5]=X_2,\ [X_4,X_5]=X_1,\ [X_1,X_6]=(1+h)X_1,$ $[X_2,X_6]=(2+h)X_2,\ [X_3,X_6]=-(4+3h)X_3,$ $[X_4,X_6]=hX_4,\ [X_5,X_6]=X_5,\ h\neq-\frac{4}{3}$	yes
$\mathfrak{g}_{6.40}^{-\frac{3}{2}}$	$[X_1,X_5]=X_2,\ [X_4,X_5]=X_1$ $[X_1,X_6]=-\frac{1}{2}X_1,\ [X_2,X_6]=\frac{1}{2}X_2,$ $[X_3,X_6]=X_2+\frac{1}{2}X_3,\ [X_4,X_6]=-\frac{3}{2}X_4,\ [X_5,X_6]=X_5$	yes
$\mathfrak{g}_{6.41}^{-1}$	$[X_1,X_5]=X_2,\ [X_4,X_5]=X_1$ $[X_2,X_6]=X_2,\ [X_3,X_6]=-X_3,$ $[X_4,X_6]=X_3-X_4,\ [X_5,X_6]=X_5$	yes
$\mathfrak{g}_{6.42}^{-\frac{5}{3}}$	$[X_1,X_5]=X_2,\ [X_4,X_5]=X_1,$ $[X_1,X_6]=-\frac{2}{3}X_1,\ [X_2,X_6]=\frac{1}{3}X_2,\ [X_3,X_6]=X_3$ $[X_4,X_6]=-\frac{5}{3}X_4,\ [X_5,X_6]=X_3+X_5$	yes
$\mathfrak{g}_{6.44}^{-7}$	$[X_1,X_5]=X_2,\ [X_4,X_5]=X_1,$	yes

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TABLE A.11: *6-dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{4.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
	$[X_1, X_6] = 2X_1,\ [X_2, X_6] = 3X_2,\ [X_3, X_6] = -7X_3$ $[X_4, X_6] = X_4,\ [X_5, X_6] = X_4 + X_5$	
$\mathfrak{g}_{6.47}^{-3,\varepsilon}$	$[X_1, X_5] = X_2,\ [X_4, X_5] = X_1,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = X_2,\ [X_3, X_6] = -3X_3$ $[X_4, X_6] = \varepsilon X_2 + X_4,\ \varepsilon \in \{0, \pm 1\}$	yes

TABLE A.12: *6-dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{5.1}$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.54}^{2(1+l),l}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = lX_2,\ [X_3, X_6] = (-1 - 2l)X_3$ $[X_4, X_6] = (-2 - l)X_4,\ [X_5, X_6] = 2(1 + l)X_5$	yes
$\mathfrak{g}_{6.55}^{-4}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = -3X_2,\ [X_3, X_6] = 4X_3$ $[X_4, X_6] = X_1 + X_4,\ [X_5, X_6] = -3X_5$	yes
$\mathfrak{g}_{6.56}^{\frac{4}{3}}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = -\frac{1}{3}X_2,\ [X_3, X_6] = X_2 - \frac{1}{3}X_3$ $[X_4, X_6] = -\frac{5}{3}X_4,\ [X_5, X_6] = \frac{4}{3}X_5$	yes
$\mathfrak{g}_{6.57}^{-\frac{2}{3}}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = -\frac{4}{3}X_2,\ [X_3, X_6] = \frac{5}{3}X_3$ $[X_4, X_6] = -\frac{2}{3}X_4,\ [X_5, X_6] = X_4 - \frac{2}{3}X_5$	yes
$\mathfrak{g}_{6.61}^{-\frac{3}{4}}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = 2X_1,\ [X_2, X_6] = -\frac{3}{2}X_2,\ [X_3, X_6] = X_3$ $[X_4, X_6] = -\frac{5}{2}X_4,\ [X_5, X_6] = X_3 + X_5$	yes
$\mathfrak{g}_{6.63}^{-1}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = X_1,\ [X_2, X_6] = -X_2,\ [X_3, X_6] = X_3$ $[X_4, X_6] = X_2 - X_4$	yes
$\mathfrak{g}_{6.65}^{4l,l}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = lX_1 + X_2,\ [X_2, X_6] = lX_2,\ [X_3, X_6] = -3lX_3 + X_4$ $[X_4, X_6] = -3lX_4,\ [X_5, X_6] = 4lX_5$	yes
$\mathfrak{g}_{6.70}^{4p,p}$	$[X_3, X_5] = X_1,\ [X_4, X_5] = X_2,$ $[X_1, X_6] = pX_1 + X_2,\ [X_2, X_6] = -X_1 + pX_2,$ $[X_3, X_6] = -3pX_3 + X_4,\ [X_4, X_6] = -X_3 - 3pX_4,$ $[X_5, X_6] = 4pX_5$	no

TABLE A.13: *6-dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{5.2}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.71}^{-\frac{7}{4}}$	$[X_2, X_5] = X_1,\ [X_3, X_5] = X_2,\ [X_4, X_5] = X_3,$ $[X_1, X_6] = \frac{5}{4}X_1,\ [X_2, X_6] = \frac{1}{4}X_2,\ [X_3, X_6] = -\frac{3}{4}X_3$ $[X_4, X_6] = -\frac{7}{4}X_4,\ [X_5, X_6] = X_5$	yes

TABLE A.14: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{5.3}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.76}^{-1}$	$[X_2,X_4]=X_3,\ [X_2,X_5]=X_1,\ [X_4,X_5]=X_2$ $[X_1,X_6]=-X_1,\ [X_3,X_6]=X_3,$ $[X_4,X_6]=X_4,\ [X_5,X_6]=-X_5$	yes
$\mathfrak{g}_{6.78}$	$[X_2,X_4]=X_3,\ [X_2,X_5]=X_1,\ [X_4,X_5]=X_2$ $[X_1,X_6]=-X_1,\ [X_3,X_6]=X_3,$ $[X_4,X_6]=X_3+X_4,\ [X_5,X_6]=-X_5$	yes

TABLE A.15: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{5.4}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.83}^{0,l}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=lX_2,\ [X_3,X_6]=lX_3,$ $[X_4,X_6]=-lX_4,\ [X_5,X_6]=-X_4-lX_5$	yes
$\mathfrak{g}_{6.84}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=X_2,\ [X_4,X_6]=-X_4,\ [X_5,X_6]=X_3$	yes
$\mathfrak{g}_{6.88}^{0,\mu_0,\nu_0}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=\mu_0X_2+\nu_0X_3,\ [X_3,X_6]=-\nu_0X_2+\mu_0X_3,$ $[X_4,X_6]=-\mu_0X_4+\nu_0X_5,\ [X_5,X_6]=-\nu_0X_4-\mu_0X_5$	cpl. solv. $\Updownarrow$ $\nu_0=0$
$\mathfrak{g}_{6.89}^{0,\nu_0,s}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=sX_2,\ [X_3,X_6]=\nu_0X_5,$ $[X_4,X_6]=-sX_4,\ [X_5,X_6]=-\nu_0X_3$	cpl. solv. $\Updownarrow$ $\nu_0=0$
$\mathfrak{g}_{6.90}^{0,\nu_0}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=X_4,\ [X_3,X_6]=\nu_0X_5,$ $[X_4,X_6]=X_2,\ [X_5,X_6]=-\nu_0X_3,\ \nu_0\neq 1$	cpl. solv. $\Updownarrow$ $\nu_0=0$
$\mathfrak{g}_{6.91}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=X_4,\ [X_3,X_6]=X_5,$ $[X_4,X_6]=X_2,\ [X_5,X_6]=-X_3$	no
$\mathfrak{g}_{6.92}^{0,\mu_0,\nu_0}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=\nu_0X_3,\ [X_3,X_6]=-\mu_0X_2,$ $[X_4,X_6]=\mu_0X_5,\ [X_5,X_6]=-\nu_0X_4$	no
$\mathfrak{g}_{6.92^*}^{0}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=X_4,\ [X_3,X_6]=X_5,$ $[X_4,X_6]=-X_2,\ [X_5,X_6]=-X_3$	no
$\mathfrak{g}_{6.93}^{0,\nu_0}$	$[X_2,X_4]=X_1,\ [X_3,X_5]=X_1,$ $[X_2,X_6]=X_4+\nu_0X_5,\ [X_3,X_6]=\nu_0X_4,$ $[X_4,X_6]=X_2-\nu_0X_3,\ [X_5,X_6]=-\nu_0X_2$	cpl. solv. $\Updownarrow$ $\lvert\nu_0\rvert\leq\frac{1}{2}$

TABLE A.16: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{5.5}$

	$[X_i, X_j]$	cpl. solv.
$\mathfrak{g}_{6.94}^{-2}$	$[X_3,X_4]=X_1,\ [X_2,X_5]=X_1,\ [X_3,X_5]=X_2$ $[X_2,X_6]=-X_2,\ [X_3,X_6]=-2X_3,$ $[X_4,X_6]=2X_4,\ [X_5,X_6]=X_5$	yes

The six-dimensional solvable Lie algebras with four-dimensional nilradical were classified by Turkowski in [56]. We list the unimodular among them in Tables A.17 – A.19. Note that there is no table with Lie algebras with nilradical $\mathfrak{g}_{4.1}$ since the only such algebra is not unimodular.

The equations for the twenty-fifth algebra in Turkowoski's list contain a minor misprint that we have corrected here.

Table A.17: 6-*dimensional indecomposable unimodular algebras with nilradical* $4\mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.101}^{a,b,c,d}$	$[X_5,X_1]=aX_1,\ [X_5,X_2]=cX_2,\ [X_5,X_4]=X_4,$ $[X_6,X_1]=bX_1,\ [X_6,X_2]=dX_2,\ [X_6,X_3]=X_3,$ $a+c=-1,\ b+d=-1,\ ab\neq 0,\ c^2+d^2\neq 0$	yes
$\mathfrak{g}_{6.102}^{-1,b,-2-b}$	$[X_5,X_1]=-X_1,\ [X_5,X_2]=X_2,\ [X_5,X_3]=X_4,$ $[X_6,X_1]=bX_1,\ [X_6,X_2]=(-2-b)X_2,$ $[X_6,X_3]=X_3,\ [X_6,X_4]=X_4$	yes
$\mathfrak{g}_{6.105}^{-2,-1}$	$[X_5,X_1]=-2X_1,\ [X_5,X_3]=X_3+X_4,$ $[X_5,X_4]=X_4,\ [X_6,X_1]=-X_1,\ [X_6,X_2]=X_2$	yes
$\mathfrak{g}_{6.107}^{-1,b,0}$	$[X_5,X_1]=-X_1,\ [X_5,X_2]=-X_2,\ [X_5,X_3]=X_3+X_4,$ $[X_5,X_4]=X_4,\ [X_6,X_1]=X_2,\ [X_6,X_2]=-X_1$	no
$\mathfrak{g}_{6.113}^{a,b,-a,d}$	$[X_5,X_1]=aX_1,\ [X_5,X_2]=-aX_2,\ [X_5,X_3]=X_4,$ $[X_6,X_1]=bX_1,\ [X_6,X_2]=dX_2,\ [X_6,X_3]=X_3$ $[X_6,X_4]=X_4,\ a^2+b^2\neq 0,\ a^2+d^2\neq 0,\ b+d=-2$	no
$\mathfrak{g}_{6.114}^{a,-1,-\frac{a}{2}}$	$[X_5,X_1]=aX_1,\ [X_5,X_3]=-\frac{a}{2}X_3+X_4,$ $[X_5,X_4]=-X_3+\frac{a}{2}X_4,\ [X_6,X_1]=-X_1,$ $[X_6,X_2]=X_2,\ a\neq 0$	no
$\mathfrak{g}_{6.115}^{-1,b,c,-c}$	$[X_5,X_1]=X_1,\ [X_5,X_2]=X_2,$ $[X_5,X_3]=-X_3+bX_4,\ [X_5,X_4]=-bX_3-X_4,$ $[X_6,X_1]=cX_1+X_2,\ [X_6,X_2]=-X_1+cX_2$ $[X_6,X_3]=-cX_3,\ [X_6,X_4]=-cX_4,\ b\neq 0$	no
$\mathfrak{g}_{6.116}^{0,-1}$	$[X_5,X_1]=X_2,\ [X_5,X_3]=X_4,\ [X_5,X_4]=-X_3$ $[X_6,X_1]=X_1,\ [X_6,X_2]=X_2,$ $[X_6,X_3]=-X_3,\ [X_6,X_4]=-X_4$	no
$\mathfrak{g}_{6.118}^{0,b,-1}$	$[X_5,X_1]=X_2,\ [X_5,X_2]=-X_1,\ [X_5,X_3]=bX_4,$ $[X_5,X_4]=-bX_3,\ [X_6,X_1]=X_1,\ [X_6,X_2]=X_2$ $[X_6,X_3]=-X_3,\ [X_6,X_4]=-X_4,\ b\neq 0$	no
$\mathfrak{g}_{6.120}^{-1,-1}$	$[X_5,X_2]=-X_2,\ [X_5,X_4]=X_4,\ [X_5,X_6]=X_1,$ $[X_6,X_2]=-X_1,\ [X_6,X_3]=X_3$	yes
$\mathfrak{g}_{6.125}^{0,-2}$	$[X_5,X_3]=X_4,\ [X_5,X_4]=-X_3,\ [X_5,X_6]=X_1,$ $[X_6,X_2]=-2X_2,\ [X_6,X_3]=X_3,\ [X_6,X_4]=X_4$	no

In the introduction of [42], Mubarakzjanov quotes his own result that a six-dimensional solvable Lie algebra with three-dimensional nilradical is decomposable. Therefore, by Proposition 3.4, we have listed all unimodular indecomposable solvable Lie algebras of dimension six.

The first Betti numbers of the six-dimensional unimodular indecomposable Lie algebras are listed in Table A.20. The word "always" means that the certain value

TABLE A.18: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.129}^{-2,-2}$	$[X_2, X_3] = X_1,\ [X_5, X_1] = X_1,\ [X_5, X_2] = X_2,$ $[X_5, X_4] = -2X_4,\ [X_6, X_1] = X_1,$ $[X_6, X_3] = X_3,\ [X_6, X_4] = -2X_4$	yes

TABLE A.19: 6-*dimensional indecomposable unimodular algebras with nilradical* $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$ *(continued)*

	$[X_i, X_j]$	c. s.
$\mathfrak{g}_{6.135}^{0,-4}$	$[X_2, X_3] = X_1,\ [X_5, X_2] = X_3,\ [X_5, X_3] = -X_1,$ $[X_6, X_1] = 2X_1,\ [X_6, X_2] = X_2,$ $[X_6, X_3] = X_3,\ [X_6, X_4] = -4X_4$	no

arises independent of the parameters on which the Lie algebra depends, but we suppose that the parameters are chosen such that Lie algebra is unimodular. The word "otherwise" in the tables means that this value arises for all parameters such that the Lie algebra is unimodular and the parameters are not mentioned in another column of the Lie algebra's row.

TABLE A.20: $b_1(\mathfrak{g}_{6.i})$ *for* $\mathfrak{g}_{6.i}$ *unimodular*

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$	$b_1 = 4$	$b_1 = 5$
1	always	-	-	-	-
2	$a \neq 0$	$a = 0$	-	-	-
3	$d \neq -1$	$d = -1$	-	-	-
4	always	-	-	-	-
6	$a, b \neq 0$	$a = -\frac{1}{2} \wedge b = 0$	-	-	-
7	always	-	-	-	-
8	always	-	-	-	-
9	$b \neq 0$	$b = 0$	-	-	-
10	$a \neq 0$	$a = 0$	-	-	-
11	always	-	-	-	-
12	always	-	-	-	-
13	$b \neq 0 \wedge h \neq 0$	otherwise	$a = -\frac{1}{2} \wedge b = h = 0$	-	-
14	otherwise	$a = -\frac{1}{3} \wedge b = 0$	-	-	-
15	always	-	-	-	-
17	-	always	-	-	-
18	$a \neq 0$	$a = 0$	-	-	-
19	always	-	-	-	-
20	-	always	-	-	-
21	$a \neq 0$	$a = 0$	-	-	-
22	always	-	-	-	-

(*Continued on next page*)

TABLE A.20: $b_1(\mathfrak{g}_{6.i})$ *for* $\mathfrak{g}_{6.i}$ *unimodular*

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$	$b_1 = 4$	$b_1 = 5$
23	$a \neq 0$	-	$a = 0$	-	-
25	$b \notin \{-1, 0\}$	$b \in \{-1, 0\}$	-	-	-
26	-	always	-	-	-
27	always	-	-	-	-
28	always	-	-	-	-
29	$b \neq 0$	-	$b = 0$	-	-
32	$h \notin \{-2a, -6a\}$	otherwise	-	-	-
33	$a \neq 0$	-	$a = 0$	-	-
34	$a \neq 0$	-	$a = 0$	-	-
35	$a, b \neq 0$	otherwise	-	-	-
36	$a \neq 0$	$a = 0$	-	-	-
37	always	-	-	-	-
38	always	-	-	-	-
39	$h \neq 0$	$h = 0$	-	-	-
40	always	-	-	-	-
41	always	-	-	-	-
42	always	-	-	-	-
44	always	-	-	-	-
47	-	always	-	-	-
54	$l \notin \{-2, -1, -\frac{1}{2}\}$	$l \in \{-2, -1, -\frac{1}{2}\}$	-	-	-
55	always	-	-	-	-
56	always	-	-	-	-
57	always	-	-	-	-
61	always	-	-	-	-
63	-	always	-	-	-
65	$l \neq 0$	-	$l = 0$	-	-
70	$p \neq 0$	$p = 0$	-	-	-
71	always	-	-	-	-
76	always	-	-	-	-
78	always	-	-	-	-
83	$l \neq 0$	-	-	$l = 0$	-
84	-	always	-	-	-
88	$\mu_0 \neq 0 \vee \nu_0 \neq 0$	-	-	-	$\mu_0 = \nu_0 = 0$
89	$\nu_0 \neq 0 \wedge s \neq 0$	-	otherwise	-	$\nu_0 = s = 0$
90	$\nu_0 \neq 0$	-	$\nu_0 = 0$	-	-
91	always	-	-	-	-
92	$\mu_0 \neq 0 \wedge \nu_0 \neq 0$	-	otherwise	-	$\mu_0 = \nu_0 = 0$
92*	always	-	-	-	-
93	$\nu_0 \neq 0$	-	$\nu_0 = 0$	-	-
94	always	-	-	-	-
101	-	always	-	-	-
102	-	always	-	-	-
105	-	always	-	-	-

(*Continued on next page*)

TABLE A.20: $b_1(\mathfrak{g}_{6.i})$ *for* $\mathfrak{g}_{6.i}$ *unimodular*

i	$b_1 = 1$	$b_1 = 2$	$b_1 = 3$	$b_1 = 4$	$b_1 = 5$
107	-	always	-	-	-
113	-	always	-	-	-
114	-	always	-	-	-
115	-	always	-	-	-
116	-	always	-	-	-
118	-	always	-	-	-
120	-	always	-	-	-
125	-	always	-	-	-
129	-	always	-	-	-
135	-	always	-	-	-

Appendix B. Integer Polynomials. In this article, we often try to use necessary conditions for a matrix to be conjugated to an integer matrix. We state briefly the used results. Vice versa, we sometimes want to find integer matrices with given minimal polynomial. We also present a few constructions.

Let be $n \in \mathbb{N}_+$, $\mathbb{K}$ a field and $A \in \mathrm{M}(n, n; \mathbb{K})$. The *characteristic polynomial* of A is the monic polynomial

$$P_A(X) := \det(X\,\mathrm{id} - A) \in \mathbb{K}[X],$$

and the *minimal polynomial* $m_A(X)$ is the unique monic divisor of lowest degree of $P_A(X)$ in $\mathbb{K}[X]$ such that $m_A(A) = 0$. (Note, by the theorem of Cayley-Hamilton, one has $P_A(A) = 0$.)

If two matrices are conjugated, then they have the same characteristic resp. minimal polynomials.

$\lambda \in \overline{\mathbb{K}}$ is called *root* of A if λ is a root of the characteristic polynomial, considered as polynomial in $\overline{\mathbb{K}}[X]$, where $\overline{\mathbb{K}}$ denotes the algebraic closure of $\mathbb{K}$.

The next proposition follows directly from [35, Corollaries XIV.2.2, XIV.2.3].

PROPOSITION B.1. *Let* $n \in \mathbb{N}_+$. *If* $A \in \mathrm{M}(n, n; \mathbb{C})$ *and* $B \in \mathrm{M}(n, n; \mathbb{Q})$ *are conjugated via an element of* $\mathrm{GL}(n, \mathbb{C})$*, then holds* $P_A(X) = P_B(X) \in \mathbb{Q}[X]$, $m_A(X) = m_B(X) \in \mathbb{Q}[X]$ *and* $m_A(X)$ *divides* $P_A(X)$ *in* $\mathbb{Q}[X]$. ∎

PROPOSITION B.2. *If* $P(X) \in \mathbb{Z}[X]$, $m(X) \in \mathbb{Q}[X]$ *are monic polynomials and* $m(X)$ *divides* $P(X)$ *in* $\mathbb{Q}[X]$*, then holds* $m(X) \in \mathbb{Z}[X]$.

Proof. Let $P(X), m(X)$ be as in the proposition and $f(X) \in \mathbb{Q}[X]$ non-constant with $P(X) = f(X)\, m(X)$. There exist $k, l \in \mathbb{Z} \setminus \{0\}$ such that

$$k\, f(X) = \sum_i a_i X^i,\ \ l\, m(X) = \sum_j b_j X^j \in \mathbb{Z}[X]$$

are primitive. (An integer polynomial is called *primitive* if its coefficients are relatively prime.) We have

$$kl\, P(X) = (\sum_i a_i X^i)(\sum_j b_j X^j)$$

and claim $kl = \pm 1$.

Otherwise, there is a prime $p \in \mathbb{N}$ that divides kl. Since the coefficients of $k\,f(X)$ resp. $l\,m(X)$ are relatively prime, there are minimal $i_0, j_0 \in \mathbb{N}$ such that p does not divide a_{i_0} resp. b_{j_0}.

The coefficient of $X^{i_0+j_0}$ of $kl\,f(X)\,m(X)$ is

$$a_{i_0}b_{j_0} + a_{i_0-1}b_{j_0+1} + a_{i_0+1}b_{j_0-1} + \dots$$

and p divides each summand except the first. But since $p\,|\,kl$, p divides the whole sum. This is a contradiction. □

Theorem B.3. *Let* $n \in \mathbb{N}_+$ *and* $A \in \mathrm{M}(n,n;\mathbb{C})$ *be conjugated to an integer matrix. Then holds* $P_A(X), m_A(X) \in \mathbb{Z}[X]$.

Proof. This follows from the preceding two propositions. □

Lemma B.4 ([29, Lemma 2.2]). *Let* $P(X) = X^3 - kX^2 + lX - 1 \in \mathbb{Z}[X]$.

Then P *has a double root* $X_0 \in \mathbb{R}$ *if and only if* $X_0 = 1$ *or* $X_0 = -1$ *for which* $P(X) = X^3 - 3X^2 + 3X - 1$ *or* $P(X) = X^3 + X^2 - X - 1$ *respectively.* □

Proposition B.5 ([27, Proposition 5]). *Let* $\lambda_i \in \mathbb{R}_+$ *with* $\lambda_i + \frac{1}{\lambda_i} = m_i \in \mathbb{N}_+$ *and* $m_i > 2$ *for* $i \in \{1,2\}$.

Then there exists no element in $\mathrm{SL}(3,\mathbb{Z})$ *with roots* $\lambda_1, \lambda_2, \frac{1}{\lambda_1\lambda_2}$. □

Proposition B.6. *Let* $P(X) = X^4 - mX^3 + pX^2 - nX + 1 \in \mathbb{Z}[X]$.

Then P *has a root with multiplicity* > 1 *if and only if the zero set of* P *equals* $\{1,1,a,a^{-1}\}$, $\{-1,-1,a,a^{-1}\}$, $\{a,a^{-1},a,a^{-1}\}$ *or* $\{a,-a^{-1},a,-a^{-1}\}$ *for fixed* $a \in \mathbb{C}$.

Proof. The most part of the proof was done by Harshavardhan in the proof of [27, Propositon 2].

We set $S := m^2 + n^2$ and $T := mn$ and get the discriminant D of $P(X)$ as

$$D = 16p^4 - 4Sp^3 + (T^2 - 80T - 128)p^2 + 18S(T+8)p + 256 - 192T + 48T^2 - 4T^3 - 27S^2. \tag{15}$$

Note that $P(X)$ has a root of multiplicity > 1 if and only if $D = 0$. Solving $D = 0$ for S, we see

$$S = -\frac{2}{27}p^3 + \frac{1}{3}pT + \frac{8}{3}p \pm \frac{2}{27}\sqrt{(p^2 - 3T + 12)^3},$$

and since S and T are integers, there is $q \in \mathbb{Z}$ with

$$p^2 - 3T + 12 = q^2,$$

which implies

$$\begin{aligned} S &= 4p + \frac{1}{27}(p^3 - 3pq^2 \pm 2q^3) \\ T &= \frac{1}{3}(p^2 - q^2 + 12). \end{aligned} \tag{16}$$

We first consider the plus sign in equation (16). Then one has

$$\begin{aligned} (m+n)^2 &= S + 2T = \frac{1}{27}(p + 2q + 6)(p - q + 6)^2, \\ (m-n)^2 &= S - 2T = \frac{1}{27}(p + 2q - 6)(p - q - 6)^2, \end{aligned}$$

and this implies the existence of $k_i, l_i \in \mathbb{N}$, $i = 1, 2$, such that

$$3k_1^2 = (p + 2q + 6)k_2^2,$$
$$3l_1^2 = (p + 2q - 6)l_2^2.$$

We shall show: $|m| = |n|$

[If $l_2 = 0$, the claim is proved. Therefore, we can assume $l_2 \neq 0$.

Case 1: $k_2 = 0$

Then holds $k_1 = 0$ and this means $S + 2T = 0$, i.e. $(m + n)^2 = 0$, so we have $m = -n$.

Case 2: $k_2 \neq 0$

We write $k := \frac{k_1}{k_2}$ and $l := \frac{l_1}{l_2}$. Then holds

$$3k^2 = p + 2q + 6 \in \mathbb{Z},$$
$$3l^2 = p + 2q - 6 \in \mathbb{Z},$$

and $3(k^2 - l^2) = 12$. Therefore, we have $k^2 - l^2 = 4$, so $k^2 = 4$, $l^2 = 0$, i.e. $l_1 = 0$, $S - 2T = 0$ and $m = n$.]

Now, consider the minus sign in equation (16). Then one has

$$(m + n)^2 = S + 2T = \frac{1}{27}(p - 2q + 6)(p + q + 6)^2,$$
$$(m - n)^2 = S - 2T = \frac{1}{27}(p - 2q - 6)(p + q - 6)^2,$$

and shows analogously as above $|m| = |n|$.

We have shown: If $P(X)$ has a multiple root, then holds $m = \pm n$.

If $m = n$, then one calculates the solutions of $D = 0$ in (15) as the following

(i) $p = -2 + 2m$,
(ii) $p = -2 - 2m$,
(iii) $p = 2 + \frac{m^2}{4}$,

and if $m = -n$, then the real solution of $D = 0$ in (15) is

(iv) $p = -2 + \frac{m^2}{4}$.

Moreover, a short computation yields the zero set of $P(X)$ in the cases (i) – (iv) as $\{1, 1, a, a^{-1}\}$, $\{-1, -1, a, a^{-1}\}$, $\{a, a^{-1}, a, a^{-1}\}$,$\{a, -a^{-1}, a, -a^{-1}\}$, respectively. □

Proposition B.7 ([1, Proposition 4.4.14]). *Let $\mathbb{K}$ be a field and*

$$m(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1X^1 + a_0 \in \mathbb{K}[X]$$

a monic polynomial. Then $\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 0 & 1 & -a_{n-1} \end{pmatrix}$ *has minimal polynomial* $m(X)$. □

If one is willing to construct an integer matrix with given characteristic and minimal polynomial, one always can chose any matrix M which has the desired polynomials and try to find an invertible matrix T such that $T^{-1}MT$ has integer entries.

Of course, this can be difficult. In the case of 4×4 - matrices we have the following easy construction.

Proposition B.8 ([27, Section 2.3.1]).

(i) *Let integers $m, n, p \in \mathbb{Z}$ be given.*
Choose $m_1, \ldots, m_4 \in \mathbb{Z}$ such that $\sum_{i=1}^{4} m_i = m$ and set

$$\begin{aligned} a &:= -m_1^2 p + m_1^3 m_2 + m_1^3 m_3 + m_1^3 m_4 + m_1 n - 1, \\ b &:= (-m_2 - m_1)p + m_1 m_2^2 + m_1 m_2 m_3 + m_1 m_2 m_4 + m_2^2 m_3 + m_2^2 m_4 \\ &\quad + m_1^2 m_2 + m_1^2 m_3 + m_1^2 m_4 + n, \\ c &:= m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4 - p. \end{aligned}$$

Then the matrix $\begin{pmatrix} m_1 & 0 & 0 & a \\ 1 & m_2 & 0 & b \\ 0 & 1 & m_3 & c \\ 0 & 0 & 1 & m_4 \end{pmatrix}$ *has* $X^4 - mX^3 + pX^2 - nX + 1$ *as characteristic polynomial.*

(ii) *Let $m \in 2\mathbb{Z}$ be an even integer. Then the matrix* $\begin{pmatrix} \frac{m}{2} & 0 & -1 & 0 \\ 0 & \frac{m}{2} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ *has the characteristic polynomial $(X^2 - \frac{m}{2}X + 1)^2$, and $(X^2 - \frac{m}{2}X + 1)$ as minimal polynomial.* □

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REFERENCES

[1] W. A. Adkins and St. H. Weintraub, *Algebra*, Springer (1992).
[2] L. Auslander, *An exposition of the structure of solvmanifolds – part I*, Bull. Amer. Math. Soc., 79:2 (1973), pp. 227–261.
[3] L. Auslander, L. Green, and F. Hahn, *Flows on Homogeneous Spaces*, Princeton University Press (1963).
[4] Ch. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology, 27:4 (1988), pp. 513–518.
[5] Ch. Benson and C. S. Gordon, *Kähler structures on compact solvmanifolds*, Proc. Amer. Math. Soc., 108:4 (1990), pp. 971–980.
[6] Ch. Bock, *Geography of non-formal symplectic and contact manifolds*, arXiv:0812.1447.
[7] S Bosch, *Lineare Algebra*, Springer (2006).
[8] K. S. Brown, *Cohomology of Groups*, Springer (1982).
[9] R. Campoamor-Stursberg, *Symplectic forms on six dimensional real solvable Lie algebras*, arXiv:math.DG/0507499.
[10] R. Campoamor-Stursberg, *Some remarks concerning the invariants of rank one solvable real Lie algebras*, Algebra Colloq., 12:3 (2005), pp. 497–518.
[11] B.-Y. Chu, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc., 197 (1974), pp. 145–159.
[12] L. Corwin and F. P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications*, Cambridge University Press (1990).
[13] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math., 29:3 (1975), pp. 245–274.
[14] A. Diatta, *Left invariant contact structures on Lie groups*, Differential Geom. Appl., 26:5 (2008), pp. 544–552.

[15] A. Diatta and B. Foreman, *Lattices in contact Lie groups and 5-dimensional contact solvmanifolds*, arXiv:0904.3113.
[16] J. Dixmier, *Sur les algèbres dérivées d'une algèbre de Lie*, Proc. Cambride Philos. Soc., 51 (1955), pp. 541–544.
[17] J. Dixmier, *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Soc. Math. France, 85 (1957), pp. 113–121.
[18] M. Fernández and A. Gray, *Compact symplectic solvmanifolds not admitting complex structures*, Geom. Dedicata, 34:3 (1990), pp. 295–299.
[19] M. Fernández, M. de León and M. Saralegui, *A six-dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math., 33:1 (1996), pp. 19–35.
[20] M. Fernández and V. Muñoz, *Formality of Donaldson submanifolds*, Math. Z., 250:1 (2005), pp. 149–175.
[21] M. Fernández and V. Muñoz, *The geography of non-formal manifolds*, in O. Kowalski, E. Musso, D. Perrone: *Complex, Contact and Symmetric Manifolds*, pp. 121–129, Prog. Math. vol. 234, Birkhäuser (2005).
[22] S. Fukuhara and K. Sakamoto, *Classification of T^2-bundles over T^2*, Tokyo J. Math., 6:2 (1983), pp. 311–327.
[23] H. Geiges, *Symplectic structures on T^2-bundles over T^2*, Duke Math. J., 67:3 (1992), pp. 539–555.
[24] H. Geiges, *Symplectic manifolds with disconnected boundary of contact type*, Internat. Math. Res. Notices (1994), no. 1, pp. 23–30.
[25] V. V. Gorbatsevich, *Symplectic structures and cohomologies on some solvmanifolds*, Sibiran Math. J., 44:2 (2003), pp. 260–274.
[26] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1994).
[27] R. Harshavardhan, *Geometric structures of Lie type on 5-manifolds*, Ph.D. Thesis, Cambridge University (1996).
[28] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc., 106:1 (1989), pp. 65–71.
[29] K. Hasegawa, *Four-dimensional compact solvmanifolds with and without complex analytic structures*, arXiv:math.CV/0401413.
[30] K. Hasegawa, *Complex and Kähler structures on compact solvmanifolds*, J. Symplectic Geom., 3:4 (2005), pp. 749–767.
[31] K. Hasegawa, *A note on compact solvmanifolds with Kähler structure*, Osaka J. Math., 43:1 (2006), pp. 131–135.
[32] A. Hattori, *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I, 8 (1960), pp. 289–331.
[33] R. Ibáñez, Y. Rudiak, A. Tralle, and L. Ugarte, *On certain geometric and homotopy properties of closed symplectic manifolds*, Topology Appl., 127 (2003), no. 1–2, pp. 33–45.
[34] J. Kędra, Y. Rudyak, and A. Tralle, *Symplectically aspherical manifolds*, J. Fixed Point Theory Appl., 3:1 (2008), pp. 1–21.
[35] S. Lang, *Algebra*, 3. ed., Addison-Wesley (1993).
[36] R. Lutz, *Sur la géométrie des structures de contact invariantes*, Ann. Inst. Fourier (Grenoble), 29:1 (1979), xvii, pp. 283–306.
[37] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in Math., 21:3 (1976), pp. 293–329.
[38] V. V. Morozov, *Classifikaziya nilpotentnych algebrach Lie shestovo poryadka*, Izv. Vysš. Učebn. Zaved. Mathematika, 5:4 (1958), pp. 161–171.
[39] G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2), 60 (1954), pp. 1–27.
[40] G. M. Mubarakzjanov, *O rasreshimych algebrach Lie*, Izv. Vysš. Učebn. Zaved. Mathematika, 32:1 (1963), pp. 114–123.
[41] G. M. Mubarakzjanov, *Classifkaziya vehestvennych structur algebrach Lie pyatovo poryadko*, Izv. Vysš. Učebn. Zaved. Mathematika, 34:3 (1963), pp. 99–106.
[42] G. M. Mubarakzjanov, *Classifkaziya rasreshimych structur algebrach Lie shestovo poryadka a odnim nenilpoentnym bazisnym elementom*, Izv. Vysš. Učebn. Zaved. Mathematika, 35:4 (1963), pp. 104–116.
[43] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2), 65 (1957), pp. 391–404.
[44] K. Nomizu, *On the cohomology of homogeneous spaces of nilpotent Lie Groups*, Ann. of Math. (2), 59 (1954), pp. 531–538.
[45] T. Nôno, *On the singularity of general linear groups*, J. Sci. Hiroshima Univ. Ser. A, 20 (1956/1957), pp. 115–123.
[46] J. Oprea and A. Tralle, *Symplectic Manifolds with no Kähler Structure*, Lecture Notes in Math., 1661, Springer (1997).

[47] A. L. Onishchik and E. B. Vinberg, *Lie Groups and Lie Algebras III*, Springer (1994).
[48] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer (1972).
[49] M. Saitô, *Sur certaines groupes de Lie résolubles – parties I et II*, Sci. Papers Coll. Gen. Ed. Univ. Tokyo, 7 (1957), 1–11, pp. 157–168.
[50] S. M. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra, 157 (2001), no. 2–3, pp. 311–333.
[51] H. Sawai and T. Yamada, *Lattices on Benson-Gordon type solvable Lie groups*, Topology Appl., 149 (2005), no. 1–3, pp. 85–95.
[52] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press (1951).
[53] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math., 47 (1977), pp. 269–331.
[54] W. P. Thurston, *Some simple examples of compact symplectic manifolds*, Proc. Amer. Math. Soc., 55:2 (1976), pp. 467–468.
[55] A. Tralle, *On solvable Lie groups without lattices*, in M. Fernández, J. A. Wolf: *Global Differential Geometry*, pp. 437–441, Contemp. Math., 288, Amer. Math. Soc. (2001).
[56] P. Turkowski, *Solvable Lie algebras of dimension six*, J. Math. Phys., 31:6 (1990), pp. 1344–1350.
[57] V. S. Vararadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer (1984).
[58] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Glenview (1971).
[59] Ch. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press (1997).
[60] T. Yamada, *A pseudo-Kähler structure on a nontoral compact complex parallelizable solvmanifold*, Geom. Dedicata, 112 (2005), pp. 115–122.