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CHARACTERIZATION OF CAMPANATO SPACES ASSOCIATED WITH PARABOLIC SECTIONS*

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Abstract. We study the Campanato spaces $\Lambda^{\kappa}_{q,\mathcal{P}}$ associated with a family $\mathcal{P}$ of parabolic sections which are closely related to the parabolic Monge-Ampère equation. We characterize these spaces in terms of Lipschitz spaces $\mathrm{Lip}^{\alpha}_{\mathcal{P}}$. We also introduce the corresponding Hardy spaces $H^p_{\mathcal{P}}$ and demonstrate the equivalence between the Littlewood-Paley g-functions and atomic decompositions for elements in $H^p_{\mathcal{P}}$. Moreover, we show that Campanato spaces are the duals of Hardy spaces.

Key words. Campanato spaces, Hardy spaces, Lipschitz spaces, Monge-Ampère equations, parabolic sections.

AMS subject classifications. 42B30, 42B35.

1. Introduction. In 1976, Krylov [Kr] introduced the parabolic Monge-Ampère equation

$$-u_t \det D_x^2 u = f, \qquad (x,t) \in \Omega \times (0,T) \subset \mathbb{R}^n \times \mathbb{R}, \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $D_x^2 u$ denotes the Hessian of u in variable x. Since then this equation has been studied extensively. Its connection with maximum principles for parabolic equations was already observed by Krylov, and was developed further by Tso [Ts2] and Nazarov and Ural'tseva [NU]. Equation (1.1) also arose in the work of Tso [Ts1] on the Gauss curvature flow of convex hypersurfaces. The first initial-boundary value problem for (1.1) was studied by R. H. Wang and G. L. Wang [WW1, WW2]. To study Harnack inequality for (1.1), Huang [Hu] introduced parabolic sections and showed that the Besicovitch type covering lemma and Calderón-Zygmund decomposition still holds in this setting. Basing on the theory of parabolic sections, Gutiérrez and Huang [GH] obtained the $W^{2,p}$ estimates for the parabolic Monge-Ampère equation.

In 2003, Caffarelli and Huang [CH] established estimates in BMO and the generalized Campanato-John-Nirenberg spaces BMO_ψ for the second derivatives of solutions to the fully nonlinear elliptic equations $F(D_x^2u, x) = f(x)$, where Ω is a bounded domain in $\mathbb{R}^n$, $x \in \Omega$, $f \in L^n(\Omega)$, $F(M,x)$ is Lipschitz continuous in M, bounded measurable in x, and uniformly elliptic. When $\psi(r) \equiv 1$ or $\psi(r) = r^b$, $0 < b \le 1$, the spaces BMO_ψ is just John-Nirenberg space or Campanato spaces, respectively. In this paper, we will study the Campanato spaces $\Lambda^{\kappa}_{q,\mathcal{P}}$ and Hardy spaces $H^p_{\mathcal{P}}$ associated with a family $\mathcal{P}$ of parabolic sections which is closely related to the parabolic Monge-Ampère equation. Moreover, we show the Campanato spaces are the duals of the corresponding Hardy spaces.

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We first recall the definition of (generalized) parabolic sections. Suppose that $\varphi : [0,\infty) \mapsto [0,\infty)$ is a monotonic increasing function satisfying

$$\varphi(0) = 0, \qquad \lim_{r\to\infty} \varphi(r) = \infty, \qquad \varphi(2r) \le C\varphi(r),$$

where C is a constant depending on φ only. Define the generalized *parabolic sections*, which will be called parabolic sections below for simplicity, by

$$Q_\varphi(z,r) = S(x,r) \times \Big(t - \frac{\varphi(r)}{2}, t + \frac{\varphi(r)}{2}\Big),$$

where $z = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $r > 0$, and S is the (elliptic) sections given in [CG, DL]. Note that this definition reduces to the one given in [Hu] by choosing $\varphi(r) = r$. We will work for a fixed φ satisfying the above description through the paper, and hence use $Q(z,r)$ to express $Q_\varphi(z,r)$ for simplicity. An affine transformation $\widetilde{T}$ on $\mathbb{R}^{n+1}$ is said to *normalize* $Q(z_0,r)$ if

$$K\Big(0, \frac{1}{n}\Big) \subset \widetilde{T}\big(Q(z_0,r)\big) \subset K(0,1),$$

where $K(z,r) = B(x,r) \times \big(t - \frac{r^2}{2}, t + \frac{r^2}{2}\big)$, $\widetilde{T}(x,t) := (Tx, \frac{t-t_0}{\varphi(r)})$, and T is an affine transformation (on $\mathbb{R}^n$) normalizing $S(x_0,r)$; that is,

$$B\Big(0, \frac{1}{n}\Big) \subset T\big(S(x_0,r)\big) \subset B(0,1).$$

Here we use $B(x,r)$ to denote the ball in $\mathbb{R}^n$ centered at x and with radius r. Note that the restriction of $\widetilde{T}$ to t-axis maps $\big(t_0 - \frac{\varphi(r)}{2}, t_0 + \frac{\varphi(r)}{2}\big)$ onto $(-\frac{1}{2}, \frac{1}{2})$. The family $\mathcal{P} = \{Q(z,r) : z = (x,t) \in \mathbb{R}^n \times \mathbb{R},\ r > 0\}$ of parabolic sections satisfies the following properties.

(**A**) There exist positive constants K_1, K_2, K_3 and ε_1, ε_2 such that, given two parabolic sections $Q(z_0,r_0)$, $Q(z,r)$ in $\mathcal{P}$ with $r \le r_0$ and an affine transformation $\widetilde{T}$ that normalizes $Q(z_0,r_0)$, if

$$Q(z_0,r_0)\ \cap\ Q(z,r)\ \ne\ \varnothing,$$

then there exists $z' = (x',t') \in K(0,K_3)$, depending only on both $Q(z_0,r_0)$ and $Q(z,r)$, satisfying

$$\begin{aligned} B\left(x', K_2\Big(\frac{r}{r_0}\Big)^{\varepsilon_2}\right) \times \left(t' - \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}, t' + \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}\right) &\subset \widetilde{T}\big(Q(z,r)\big) \\ \subset B\left(x', K_1\Big(\frac{r}{r_0}\Big)^{\varepsilon_1}\right) \times \left(t' - \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}, t' + \frac{1}{2}\frac{\varphi(r)}{\varphi(r_0)}\right) \end{aligned}$$

and

$$\widetilde{T}(z) = (Tx, t') \in B\left(x', \frac{1}{2}K_2\Big(\frac{r}{r_0}\Big)^{\varepsilon_2}\right) \times \{t'\}.$$

(**B**) There exists $\iota > 0$ such that, for any parabolic section $Q(z_0,r) \in \mathcal{P}$ and $z \notin Q(z_0,r)$, if $\widetilde{T}$ is an affine transformation that normalizes $Q(z_0,r)$, then

$$K\big(\widetilde{T}(z), \epsilon^\iota\big) \cap \widetilde{T}\big(Q(z_0,(1-\epsilon)r)\big) = \varnothing \qquad \text{for } 0 < \epsilon < 1.$$

(**C**) $\bigcap_{r>0} Q(z,r) = \{z\}$ and $\bigcup_{r>0} Q(z,r) = \mathbb{R}^{n+1}$.
In addition, we also assume that a Borel measure ν is given, which is finite on compact sets, $\nu(\mathbb{R}^{n+1}) = \infty$, and satisfies the following *doubling property* with respect to $\mathcal{P}$; that is, there exists a constant A such that

$$\nu\big(Q(z,2r)\big) \le A\nu\big(Q(z,r)\big), \qquad \forall\ Q(z,r) \in \mathcal{P}. \tag{1.2}$$

We start with the definition of Campanato spaces. For $0 \le \kappa < 1$ and $1 \le q \le \infty$, we say that f belongs to $\Lambda^\kappa_{q,\mathcal{P}}$ if $f \in L^q_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and there exists a constant C satisfying

$$\left(\frac{1}{\nu(Q)}\int_Q |f(z) - m_Q(f)|^q d\nu(z)\right)^{1/q} \le C\nu(Q)^\kappa \qquad \text{for all } \ Q \in \mathcal{P}, \tag{1.3}$$

where $m_Q(f) = \frac{1}{\nu(Q)}\int_Q f(z)d\nu(z)$ denotes the mean of f over the parabolic section Q. The left hand side of (1.3) is understood to be $\|f - m_Q(f)\|_{L^\infty(Q,d\nu)}$ in the case of $q = \infty$. We denote by $\|f\|_{\Lambda^\kappa_{q,\mathcal{P}}}$ the infimum of all constants C which make (1.3) valid. Clearly $\|\cdot\|_{\Lambda^\kappa_{q,\mathcal{P}}}$ is only a seminorm and $\|f\|_{\Lambda^\kappa_{q,\mathcal{P}}} = 0$ if and only if f is constant ν-almost everywhere. We will assume the $\Lambda^\kappa_{q,\mathcal{P}}$ spaces to be quotient spaces without further mention.

For $\kappa = 0$ and $1 \le q < \infty$, the space $\Lambda^0_{q,\mathcal{P}}$ is reduced to $BMO^q_{\mathcal{P}}$ which originated in [W]. It was proved in [QW, Theorem 1.2] that $\Lambda^0_{q,\mathcal{P}} = BMO_{\mathcal{P}}$ for all $1 \le q < \infty$, and all seminorms $\|\cdot\|_{\Lambda^0_{q,\mathcal{P}}}$ are equivalent.

Let ρ be the quasi-metric satisfying (2.3) below and f be a continuous function on $\mathbb{R}^{n+1}$. We define the *modulus of continuity* of f by $\omega_f(h) := \sup_{\rho(z,w)\le h} |f(z) - f(w)|$, and f is said to satisfy a *Lipschitz condition of order* α*,* $0 < \alpha \le 1$*, associated with parabolic sections*, denoted by $f \in \mathrm{Lip}^\alpha_{\mathcal{P}}$, if there exists a positive constant C such that $\omega_f(h) \le Ch^\alpha$ for all $h > 0$. The "norm" of f in $\mathrm{Lip}^\alpha_{\mathcal{P}}$ is defined by the lower bound of the constants C. Note that the constant functions have norm zero. We still use $\mathrm{Lip}^\alpha_{\mathcal{P}}$ to denote the above function space modulo the constant functions.

We may characterize Campanato spaces in terms of Lipschitz functions as follows.

Theorem 1.1. *For* $0 < \alpha < \varepsilon$ *and* $1 \le q \le \infty$*, where* ε *is given in* (2.3) *below, the function spaces* $\Lambda^\alpha_{q,\mathcal{P}}$ *and* $\mathrm{Lip}^\alpha_{\mathcal{P}}$ *coincide with equivalent norms.*

As an immediate consequence of Theorem 1.1, we have

Corollary 1.2. *Let* $0 < \kappa < \varepsilon$*, where* ε *is given in* (2.3) *below. All spaces* $\Lambda^\kappa_{q,\mathcal{P}}$*,* $1 \le q \le \infty$*, coincide.*

In 2005, Ding and Lin [DL] introduced the Hardy spaces $H^1_{\mathcal{F}}(\mathbb{R}^n)$ associated to the family $\mathcal{F}$ of (elliptic) sections, and showed that the dual of $H^1_{\mathcal{F}}$ is $BMO_{\mathcal{F}}$ defined in [CG]. Later on, Wu [W] defined the Hardy spaces $H^1_{\mathcal{P}}$ associated to the family $\mathcal{P}$ of generalized parabolic sections and established the duality $H^1_{\mathcal{P}} - BMO_{\mathcal{P}}$. Next we will consider the Hardy spaces $H^p_{\mathcal{P}}$, $1/2 < p \le 1$, and show that the dual spaces of $H^p_{\mathcal{P}}$ are the Campanato spaces.

We define the atomic Hardy space with respect to parabolic sections as follows. Let $1/2 < p \le 1 \le q \le \infty$ with $p < q$. A function $a \in L^q(\mathbb{R}^{n+1}, d\nu)$ is called a (p,q)*-atom* if there exists a parabolic section $Q(z_0, r_0) \in \mathcal{P}$ such that

(i) $\mathrm{supp}(a) \subset Q(z_0, r_0)$;

(ii) $\displaystyle\int_{\mathbb{R}^{n+1}} a(z)d\nu(z) = 0$;

(iii) $\|a\|_{L^q_\nu} \le \nu(Q(z_0, r_0))^{1/q-1/p}$.

The atomic Hardy space $H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1})$ is defined to be

$$H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1}) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^{n+1}) : f(x) = \sum_j \lambda_j a_j(x) \ \text{ in } \mathcal{S}',\ \text{each } a_j \text{ is a } (p,q)\text{-atom and } \sum_j |\lambda_j|^p < \infty \Big\},$$

where $\mathcal{S}(\mathbb{R}^{n+1})$ is the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^{n+1})$ denotes its dual. Define the $H^{p,q}_{\mathcal{P}}$ norm of f by

$$\|f\|_{H^{p,q}_{\mathcal{P}}} = \inf \Big(\sum_j |\lambda_j|^p \Big)^{1/p},$$

where the infimum is taken over all decompositions of $f = \sum_j \lambda_j a_j$ above.

We may imitate literally the proof of [W, Theorem 1.1] to obtain the equivalence of all $H^{p,q}_{\mathcal{P}}$, $1 \le q \le \infty$, and we leave details to the reader.

Theorem 1.3. *Let $1/2 < p \le 1 \le q < \infty$ with $p < q$. Then $H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1}) = H^{p,\infty}_{\mathcal{P}}(\mathbb{R}^{n+1})$ and the norms $\|\cdot\|_{H^{p,q}_{\mathcal{P}}}$ and $\|\cdot\|_{H^{p,\infty}_{\mathcal{P}}}$ are equivalent.*

Since $H^{p,q}_{\mathcal{P}}$ are independent of the choice of q, we define the *Hardy space associated to the family $\mathcal{P}$ of parabolic sections* to be

$$H^{p}_{\mathcal{P}}(\mathbb{R}^{n+1}) := H^{p,\infty}_{\mathcal{P}}(\mathbb{R}^{n+1}) \qquad \text{and} \qquad \|\cdot\|_{H^p_{\mathcal{P}}} := \|\cdot\|_{H^{p,\infty}_{\mathcal{P}}}.$$

Let $\{E_k\}_{k\in\mathbb{Z}}$ denote the approximation to the identity with regular exponent ε and $\big(\dot{\mathcal{M}}^{(\beta,\gamma)}_{\varepsilon}\big)'$ denote the dual space of test function $\dot{\mathcal{M}}^{(\beta,\gamma)}_{\varepsilon}$ (see definitions given in Section 3). Set $D_k = E_k - E_{k-1}$. For $f \in \big(\dot{\mathcal{M}}^{(\beta,\gamma)}_{\varepsilon}\big)'$, the *Littlewood-Paley g-function of f associated to parabolic sections* is defined by

$$\mathfrak{g}(f)(z) := \Big\{ \sum_k |D_k(f)(z)|^2 \Big\}^{1/2}.$$

Using this g-function, we define another Hardy space

$$H^p_{\mathfrak{g}}(\mathbb{R}^{n+1}) := \big\{ f \in \big(\dot{\mathcal{M}}^{(\beta,\gamma)}_{\varepsilon}\big)' : \mathfrak{g}(f) \in L^p(\mathbb{R}^{n+1}, d\nu) \big\}$$

with $\|f\|_{H^p_{\mathfrak{g}}} := \|\mathfrak{g}(f)\|_{L^p_\nu}$. Then we have the g-function characterization of $H^p_{\mathcal{P}}$.

Theorem 1.4. *For $\frac{1}{1+\varepsilon} < p \le 1$, $H^p_{\mathcal{P}} = H^p_{\mathfrak{g}}$ with equivalent norms.*

Descriptions of $(H^p)'$ in terms of Campanato spaces and Lipschitz spaces in various settings, other than $H^p_{\mathcal{P}}$, were obtained by many other authors. The following theorem demonstrates the dual spaces of $H^{p,q}_{\mathcal{P}}$, which generalizes [W, Theorem 1.2]. For $1 \le q \le \infty$, as usual we use q' to denote its conjugate number satisfying $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 1.5. *Let $p = 1 < q \le \infty$ or $1/2 < p < 1 \le q < \infty$. The dual space of $H^{p,q}_{\mathcal{P}}$ is $\Lambda^{1/p-1}_{q',\mathcal{P}}$.*

Remark 1.1. In the classical case, the space BMO can be regarded as the limiting case of Lipschitz spaces. Theorem 1.1 extends this result to the current setting. Also, using Theorem 1.1 together with Theorem 1.5, we get

$$(H^p_{\mathcal{P}})' = (H^{p,q}_{\mathcal{P}})' = \Lambda^{1/p-1}_{q',\mathcal{P}} = \mathrm{Lip}^{1/p-1}_{\mathcal{P}} \qquad \text{for} \quad \frac{1}{1+\varepsilon} < p < 1 \le q < \infty.$$

As an immediate consequence of Theorems 1.3 and 1.5, we have

COROLLARY 1.6. *For* $0 < \kappa < 1$, *all spaces* $\Lambda^{\kappa}_{q,\mathcal{P}}$, $1 < q \le \infty$, *coincide.*

REMARK 1.2. It follows from Corollaries 1.2 and 1.6 that, for $0 < \kappa < \varepsilon$, all spaces $\Lambda^{\kappa}_{q,\mathcal{P}}$, $1 \le q \le \infty$, coincide. When $\varepsilon \le \kappa < 1$, the spaces $\Lambda^{\kappa}_{q,\mathcal{P}}$ coincide for $1 < q \le \infty$ only. By Hölder's inequality, $\Lambda^{\kappa}_{q,\mathcal{P}} \subset \Lambda^{\kappa}_{1,\mathcal{P}}$ for all $1 < q \le \infty$; however, we do not know whether $\Lambda^{\kappa}_{1,\mathcal{P}}$ agrees with the others $\Lambda^{\kappa}_{q,\mathcal{P}}$ when $\kappa \in [\varepsilon, 1)$.

We recall some background material about parabolic sections in the next section. In Section 3 we demonstrate another characterization of $\Lambda^{\kappa}_{q,\mathcal{P}}$ in terms of Lipschitz functions. We prove the g-function characterization of $H^p_{\mathcal{P}}$ in Section 4. The $H^{p,q}_{\mathcal{P}} - \Lambda^{1/p-1}_{q',\mathcal{P}}$ duality is shown in the last section. Throughout the article, the letter C will denote a positive constant that may vary from line to line but remains independent of the main variables. We also write $A \lesssim B$ to indicate that A is majorized by B times a constant independent of A and B, while the notation $A \approx B$ denotes both $A \lesssim B$ and $B \lesssim A$.

2. Elementary properties of parabolic sections. Since the parabolic sections are similar to elliptic cylinders, by properties (**A**) and (**B**) of parabolic sections, it is easy to obtain the following *engulfing property*: There exists a constant $\theta \ge 1$, depending only on ι, K_1, and ε_1, such that for each $z' \in Q(z,r) \in \mathcal{P}$ we have

$$Q(z,r) \subset Q(z',\theta r) \qquad \text{and} \qquad Q(z',r) \subset Q(z,\theta r). \tag{2.1}$$

Define a quasi-metric d on $\mathbb{R}^{n+1}$ with respect to $\mathcal{P}$ by

$$d(z,w) = \inf\{r : z \in Q(w,r) \text{ and } w \in Q(z,r)\},$$

which satisfies the triangle inequality

$$d(z,w) \le \theta\big(d(z,u) + d(u,w)\big) \qquad \text{for any } z,u,w \in \mathbb{R}^{n+1}.$$

Also,

$$Q\Big(z, \frac{r}{2\theta}\Big) \subset B_d(z,r) \subset Q(z,r) \qquad \text{for any } z \in \mathbb{R}^{n+1} \text{ and } r > 0, \tag{2.2}$$

where $B_d(z,r) := \{w \in \mathbb{R}^{n+1} : d(z,w) < r\}$ denotes the d-ball centered at z with radius r. By (1.2) and (2.2), if we choose $k_0 \in \mathbb{N}$ satisfying $2^{k_0-2} \ge \theta$, then

$$\nu(B_d(z,2r)) \le A^{k_0}\nu(B_d(z,r)) \qquad \text{for any } z \in \mathbb{R}^{n+1} \text{ and } r > 0.$$

Hence, $(\mathbb{R}^{n+1}, d, \nu)$ is a space of homogeneous type introduced by Coifman and Weiss [CW].

Macías and Segovia [MS, Theorems 2 and 3] have shown that one can replace d by another quasi-metric ρ such that there exist constants $C > 0$ and $\varepsilon \in (0,1)$ satisfying

$$\begin{cases} \rho(z,w) \approx \inf\{\nu(B_d) : B_d \text{ are } d\text{-balls containing } z \text{ and } w\}; \\ \nu(B_\rho(z,r)) \approx r, \quad \forall\, z \in \mathbb{R}^{n+1},\ r > 0, \text{ where } B_\rho(z,r) := \{w \in \mathbb{R}^{n+1} : \rho(z,w) < r\}; \\ |\rho(z,w) - \rho(z',w)| \le C(\rho(z,z'))^{\varepsilon}[\rho(z,w) + \rho(z',w)]^{1-\varepsilon}, \quad \forall\, z,z',w \in \mathbb{R}^{n+1}. \end{cases} \tag{2.3}$$

Since on spaces of homogeneous type only polynomials of degree zero are considered in the moment condition in the definition of atoms, the range of p for the atom of

$H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1})$ is restricted to $1/2 < p \le 1$ from the viewpoint of spaces of homogeneous type.

Christ [Ch] proved an analogous decomposition of the Euclidean dyadic cubes on spaces of homogeneous type, which was independently obtained by Sawyer and Wheeden [SW] as well.

THEOREM 2.1. *[Ch] Let (X, ρ, ν) be a space of homogeneous type. There exists a collection of open subsets $\{Q^j_k \subset X : j \in \mathbb{Z}, k \in I_j\}$, where I_j is a (finite or infinite) index set depending on j, and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$, C_1 and $C_2 > 0$ such that*

(i) $\nu\big(X \setminus \bigcup_{k\in I_j} Q^j_k\big) = 0$ *for each fixed j;*

(ii) $Q^j_k \cap Q^j_{k'} = \varnothing$ *if* $k \ne k'$;

(iii) for any given Q^j_k and $Q^{j'}_\ell$ with $j > j'$, either $Q^j_k \subset Q^{j'}_\ell$ or $Q^j_k \cap Q^{j'}_\ell = \varnothing$;

(iv) for each (j,k) and any $j' < j$, there is a unique $\ell \in I_{j'}$ such that $Q^j_k \subset Q^{j'}_\ell$;

(v) for each Q^j_k, $\mathrm{diam}(Q^j_k) \le C_1\delta^j$;

(vi) each Q^j_k contains a ball $B(y^j_k, a_0\delta^j)$, where $y^j_k \in Q^j_k$;

(vii) $\nu\{x \in Q^j_k : \rho(x, X \setminus Q^j_k) \le t\delta^j\} \le C_2 t^\eta \nu(Q^j_k) \quad \forall j,k,\ \forall t > 0.$

Properties (i)–(iv) of Theorem 2.1 show that all these subsets have the same properties as dyadic cubes in $\mathbb{R}^{n+1}$. Property (v) implies that all these Q^j_k with the same j may have different measures; however, (v) and (vi) show that they have almost the same measures. That is, for each $j \in \mathbb{Z}$, and $k, \ell \in I_j$, $\nu(Q^j_k) \approx \nu(Q^j_\ell) \approx \delta^j$. We will call all these subsets Q^j_k, $j \in \mathbb{Z}$ and $k \in I_j$, the *dyadic cubes on spaces of homogeneous type.*

Define a function σ on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by

$$\sigma(z,w) = \inf\{r > 0 : w \in Q(z,r)\}.$$

Using the engulfing property (2.1), we can deduce from the properties of elliptic sections (cf. [In]) and obtain that

(**D**) $\sigma(z,w) \le \theta\sigma(w,z)$ for all $z, w \in \mathbb{R}^{n+1}$;

(**E**) $\sigma(z,w) \le \theta^2(\sigma(z,u) + \sigma(u,w))$ for all $z, u, w \in \mathbb{R}^{n+1}$.

Obviously, from the definition of σ, it is easy to see that

(**F**) for a given parabolic section $Q(z,r)$, $w \in Q(z,r)$ if and only if $\sigma(z,w) < r$.

3. Characterizations of Campanato spaces. In this section we demonstrate another characterization of $\Lambda^\kappa_{q,\mathcal{P}}$ in terms of Lipschitz functions.

Let θ be the engulfing constant appearing in (2.1), ρ be the quasi-metric and ε be the regularity exponent given in (2.3). A sequence of operators $\{E_k\}_{k\in\mathbb{Z}}$ is said to be an *approximation to the identity associated to parabolic sections with the regular exponent* ε if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $z, z', w, w' \in \mathbb{R}^{n+1}$, the kernels $E_k(z,w)$ of E_k satisfy the following conditions:

(i) $E_k(z,w) = 0$ if $\rho(z,w) \ge C2^{-k}$ and $|E_k(z,w)| \le C2^k$;

(ii) $$|E_k(z,w) - E_k(z',w)| \le C\Big(\frac{\rho(z,z')}{2^{-k} + \rho(z,w)}\Big)^\varepsilon \frac{2^{-k\varepsilon}}{(2^{-k} + \rho(z,w))^{1+\varepsilon}}$$
$$\text{for } \rho(z,z') \le \frac{1}{2\theta}(2^{-k} + \rho(z,w));$$

(iii) $$|E_k(z,w) - E_k(z,w')| \le C\Big(\frac{\rho(w,w')}{2^{-k} + \rho(z,w)}\Big)^\varepsilon \frac{2^{-k\varepsilon}}{(2^{-k} + \rho(z,w))^{1+\varepsilon}}$$
$$\text{for } \rho(w,w') \le \frac{1}{2\theta}(2^{-k} + \rho(z,w));$$

(iv) $|[E_k(z,w)-E_k(z,w')]-[E_k(z',w)-E_k(z',w')]|$
$$\le C\Big(\frac{\rho(z,z')}{2^{-k}+\rho(z,w)}\Big)^{\varepsilon}\Big(\frac{\rho(w,w')}{2^{-k}+\rho(z,w)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(z,w))^{1+\varepsilon}}$$
$$\text{for } \rho(z,z')\le\frac{1}{2\theta}(2^{-k}+\rho(z,w)) \text{ and } \rho(w,w')\le\frac{1}{2\theta}(2^{-k}+\rho(z,w));$$

(v) $\int_{\mathbb{R}^{n+1}} E_k(z,w)d\nu(w)=1 \qquad \text{for all } k\in\mathbb{Z},\ z\in\mathbb{R}^{n+1};$

(vi) $\int_{\mathbb{R}^{n+1}} E_k(z,w)d\nu(z)=1 \qquad \text{for all } k\in\mathbb{Z},\ w\in\mathbb{R}^{n+1}.$

The existence of such an approximation to the identity follows from Coifman's construction which was first appeared in [DJS].

Fix two exponents $0<\beta\le 1$ and $\gamma>0$. A function f defined on $\mathbb{R}^{n+1}$ is said to be a *test function of type* (β,γ) *centered at* $z_0\in\mathbb{R}^{n+1}$ *with width* $r>0$ if f satisfies

(vii) $|f(z)|\le C\frac{r^\gamma}{(r+\rho(z,z_0))^{1+\gamma}};$

(viii) $|f(z)-f(w)|\le C\Big(\frac{\rho(z,w)}{r+\rho(z,z_0)}\Big)^{\beta}\frac{r^\gamma}{(r+\rho(z,z_0))^{1+\gamma}}$
$$\text{for } \rho(z,w)<\frac{1}{2\theta}(r+\rho(z,z_0));$$

(ix) $\int_{\mathbb{R}^{n+1}} f(z)d\nu(z)=0.$

We write $\mathcal{M}^{(\beta,\gamma)}(z_0,r)$ for the collection of all test functions of type (β,γ) centered at z_0 with width r. If $f\in\mathcal{M}^{(\beta,\gamma)}(z_0,r)$, then the norm of f in $\mathcal{M}^{(\beta,\gamma)}(z_0,r)$ is defined by
$$\|f\|_{\mathcal{M}^{(\beta,\gamma)}(z_0,r)}:=\inf\{C:\text{the above (vii) and (viii) hold}\}.$$

We denote $\mathcal{M}^{(\beta,\gamma)}(0,1)$ simply by $\mathcal{M}^{(\beta,\gamma)}$. Then $\mathcal{M}^{(\beta,\gamma)}$ is a Banach space with the norm $\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$. It is easy to check that for any $z_0\in\mathbb{R}^{n+1}$ and $r>0$, $\mathcal{M}^{(\beta,\gamma)}(z_0,r)=\mathcal{M}^{(\beta,\gamma)}$ with equivalent norms.

Suppose that $\{E_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated to parabolic sections with the regular exponent ε. Set $D_k=E_k-E_{k-1}$. For both $\beta,\gamma\in(0,\varepsilon)$, denote by $\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}$ the closure of $\mathcal{M}^{(\varepsilon,\varepsilon)}$ with respect to the norm $\|\cdot\|_{\mathcal{M}^{(\beta,\gamma)}}$. If $f\in\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}$, we then define $\|f\|_{\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}}=\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$. The dual space $\big(\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}\big)'$ consists of all linear functionals $\mathcal{L}$ from $\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}$ to $\mathbb{C}$ satisfying
$$|\mathcal{L}(f)|\le C\|f\|_{\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}} \qquad \text{for all } \ f\in\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}.$$

The Littlewood-Paley characterization of Lipschitz spaces is presented as follows.

THEOREM 3.1. *For* $0<\alpha<\varepsilon$ *and both* $\beta,\gamma\in(\alpha,\varepsilon)$, *let* $f\in(\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)})'$ *such that*
$$f=\sum_{k\in\mathbb{Z}}D_k\widetilde{D}_k(f) \qquad \Big(or\ \ f=\sum_k\widetilde{\widetilde{D}}_kD_k(f)\Big),$$
where the series converges in $(\dot{\mathcal{M}}_\varepsilon^{(\beta',\gamma')})'$, $\beta<\beta'<\varepsilon$ *and* $\gamma<\gamma'<\varepsilon$. *Then* f *belongs to* $\mathrm{Lip}^\alpha_{\mathcal{P}}$ *if and only if* $\|\widetilde{D}_k(f)\|_\infty\le C2^{-k\alpha}$ (*or* $\|D_k(f)\|_\infty\le C2^{-k\alpha}$) *for some constant* C *and for all* $k\in\mathbb{Z}$. *Moreover,*
$$\|f\|_{\mathrm{Lip}^\alpha_{\mathcal{P}}}\approx\sup_k 2^{k\alpha}\|\widetilde{D}_k(f)\|_\infty \qquad \Big(or\ \ \|f\|_{\mathrm{Lip}^\alpha_{\mathcal{P}}}\approx\sup_k 2^{k\alpha}\|D_k(f)\|_\infty\Big).$$

Here $\widetilde{D}_k(z,w)$, the kernels of $\widetilde{D}_k$, satisfy the following estimates: for $0<\varepsilon'<\varepsilon$, there exists a constant $C>0$ such that

$$|\widetilde{D}_k(z,w)| \le C\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(z,w))^{1+\varepsilon'}},$$

$$|\widetilde{D}_k(z,w)-\widetilde{D}_k(z,w')| \le C\Big(\frac{\rho(w,w')}{(2^{-k}+\rho(z,w))}\Big)^{\varepsilon'}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(z,w))^{1+\varepsilon'}}$$

$$\text{for } |\rho(w,w')| \le \frac{1}{2\theta}(2^{-k}+\rho(z,w)), \text{ where } \theta \text{ is the engulfing constant,}$$

$$\int_{\mathbb{R}^{n+1}} \widetilde{D}_k(z,w)\,dw = 0 \qquad \text{for all } k\in\mathbb{Z} \text{ and } z\in\mathbb{R}^{n+1},$$

$$\int_{\mathbb{R}^{n+1}} \widetilde{D}_k(z,w)\,dz = 0 \qquad \text{for all } k\in\mathbb{Z} \text{ and } w\in\mathbb{R}^{n+1}.$$

The kernels $\widetilde{\widetilde{D}}_k(z,w)$ of $\widetilde{\widetilde{D}}_k$ satisfy the same conditions as $\widetilde{D}_k(z,w)$ but with the roles of z and w interchanged.

Proof. We show the case $f=\sum_{k\in\mathbb{Z}} D_k\widetilde{D}_k(f)$ only. The proof of another case $f=\sum_k \widetilde{\widetilde{D}}_k D_k(f)$ is the same. Suppose $f\in \mathrm{Lip}^{\alpha}_{\mathcal{P}}$, $0<\alpha<\varepsilon$. We may assume that $f(0)=0$. Then

$$|f(z)| = |f(z)-f(0)| \le C(\rho(z,0))^{\alpha}.$$

This shows that f grows slowly at infinity and therefore $f\in(\dot{\mathcal{M}}_{\varepsilon}^{(\beta,\gamma)})'$ for $\beta,\gamma\in(\alpha,\varepsilon)$. Using the condition $\int \widetilde{D}_k(z,w)d\nu(w)=0$, we have

$$\begin{aligned}\widetilde{D}_k(f)(z) &= \int_{\mathbb{R}^{n+1}} \widetilde{D}_k(z,w)f(w)d\nu(w)\\ &= \int_{\mathbb{R}^{n+1}} \widetilde{D}_k(z,w)[f(w)-f(z)]d\nu(w).\end{aligned}$$

Thus,

$$\begin{aligned}\|\widetilde{D}_k(f)\|_{\infty} &\lesssim \int_{\mathbb{R}^{n+1}} |\widetilde{D}_k(z,w)|\rho^{\alpha}(z,w)d\nu(w)\\ &= \int_{\rho(z,w)\le 2^{-k}} |\widetilde{D}_k(z,w)|\rho^{\alpha}(z,w)d\nu(w)\\ &\quad + \sum_{i=0}^{\infty}\int_{2^i2^{-k}<\rho(z,w)\le 2^{i+1}2^{-k}} |\widetilde{D}_k(z,w)|\rho^{\alpha}(z,w)d\nu(w)\\ &\lesssim 2^{-k\alpha}.\end{aligned}$$

To prove the converse implication, by the continuous Calderón reproducing formula,

$$f=\sum_k D_k\widetilde{D}_k(f) \qquad \text{for } f\in(\dot{\mathcal{M}}_{\varepsilon}^{(\beta',\gamma')})' \text{ with } \beta<\beta'<\varepsilon \text{ and } \gamma<\gamma'<\varepsilon,$$

where the series converges in $(\dot{\mathcal{M}}_\varepsilon^{(\beta',\gamma')})'$. Decompose f as

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} D_k \widetilde{D}_k(f)(z) + \sum_{k=-\infty}^{0} \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(0,w)]\widetilde{D}_k(f)(w)d\nu(w) \\ &:= f_1(z) + f_2(z). \end{aligned}$$

The condition $\|\widetilde{D}_k(f)\|_\infty \lesssim 2^{-k\alpha}$ implies that f_1 is continuous and bounded on $\mathbb{R}^{n+1}$. The smoothness condition of $D_k(z,w)$ yields

$$\left| \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(0,w)]\widetilde{D}_k(f)(w)d\nu(w) \right| \lesssim 2^{(\varepsilon-\alpha)k}\rho^\varepsilon(z,0),$$

which implies that f_2 is continuous on any compact subset of $\mathbb{R}^{n+1}$. Hence f is continuous on any compact subset of $\mathbb{R}^{n+1}$. To show $f \in \mathrm{Lip}_{\mathcal{P}}^\alpha$, we write

$$\begin{aligned} f(z) - f(z') &= \sum_k \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(w)d\nu(w) \\ &= \sum_{k\le m} \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(w)d\nu(w) \\ &\quad + \sum_{k>m} \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(w)d\nu(w), \end{aligned}$$

where m is the positive integer satisfying $2^{-m} \le \rho(z,z') < 2^{-m+1}$. For the first sum $\sum_{k\le m}$, we use the smoothness of $D_k(z,w)$ and the size condition on $\widetilde{D}_k(f)$ to get

$$\left| \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(w)d\nu(w) \right| \lesssim \rho^\varepsilon(z,z')2^{(\varepsilon-\alpha)k},$$

which implies

$$\sum_{k\le m} \left| \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(z)d\nu(z) \right| \lesssim \rho^\varepsilon(z,z')2^{(\varepsilon-\alpha)m} \lesssim \rho^\alpha(z,z').$$

In the second sum $\sum_{k>m}$, the size condition of $D_k(z,w)$ and the size condition on $\widetilde{D}_k(f)$ yield

$$\left| \int_{\mathbb{R}^{n+1}} [D_k(z,w) - D_k(z',w)]\widetilde{D}_k(f)(w)d\nu(w) \right| \lesssim 2^{-\alpha k},$$

and hence the second sum is dominated by $C2^{-\alpha m} \le C\rho^\alpha(z,z')$. Therefore, the proof of Theorem 3.1 is concluded. ◻

We now are ready to show Theorem 1.1.

Proof of Theorem 1.1. For $0 < \alpha < \varepsilon$, by Theorem 3.1, it suffices to show

$$\sup_k 2^{k\alpha}\|D_k(f)\|_\infty \lesssim \|f\|_{\Lambda_{q,\mathcal{P}}^\alpha} \lesssim \|f\|_{\mathrm{Lip}_{\mathcal{P}}^\alpha}. \tag{3.1}$$

We consider the case $1 \le q < \infty$ only since the case $q = \infty$ can be similarly handled with minor modification. The first inequality in (3.1) can be verified by

$$\begin{aligned}
|D_k(f)(z)| &= \left| \int_Q D_k(z,w)[f(w) - m_Q(f)]d\nu(w) \right| \\
&\lesssim \frac{1}{\nu(Q)} \int_Q |f(w) - m_Q(f)| d\nu(w) \\
&\le \left(\frac{1}{\nu(Q)} \int_Q |f(w) - m_Q(f)|^q d\nu(w) \right)^{1/q} \lesssim 2^{-k\alpha} \|f\|_{\Lambda^\alpha_{q,\mathcal{P}}},
\end{aligned}$$

where $Q = Q(z, C2^{-k})$ denotes the support of $D_k(z,\cdot)$.

As for the second inequality in (3.1), the estimate

$$|f(w) - m_Q(f)| \le \frac{1}{\nu(Q)} \int_Q |f(w) - f(u)| d\nu(u) \lesssim 2^{-k\alpha} \|f\|_{\mathrm{Lip}^\alpha_{\mathcal{P}}}, \qquad \forall\, w \in Q,$$

implies

$$\left(\frac{1}{\nu(Q)} \int_Q |f(w) - m_Q(f)|^q d\nu(w) \right)^{1/q} \lesssim 2^{-k\alpha} \|f\|_{\mathrm{Lip}^\alpha_{\mathcal{P}}} \qquad \text{for } \nu(Q) \approx 2^{-k},$$

and hence the second inequality follows. □

4. Littlewood-Paley g-function. In this section, we collect from the previous literature some ideas and results that will play a role for showing the equivalence between the characterization of g-function and atomic decomposition for $H^p_{\mathcal{P}}$. More precisely, we define the Littlewood-Paley g-function associated to parabolic sections and another type of Hardy spaces $H^p_{\mathfrak{g}}$ in terms of this g-function. Then we point out that each element in $H^p_{\mathfrak{g}}$ can be written as sum of (p,q)-ρ-atoms that are supported on ρ-balls rather than parabolic sections.

For $f \in \big(\dot{\mathcal{M}}^{(\beta,\gamma)}_\varepsilon\big)'$, the *Littlewood-Paley g-function of f associated to parabolic sections* is defined to be

$$\mathfrak{g}(f)(z) := \Big\{ \sum_k |D_k(f)(z)|^2 \Big\}^{1/2}.$$

Using this g-function, we define another Hardy space by

$$H^p_{\mathfrak{g}}(\mathbb{R}^{n+1}) := \big\{ f \in \big(\dot{\mathcal{M}}^{(\beta,\gamma)}_\varepsilon\big)' : \mathfrak{g}(f) \in L^p(\mathbb{R}^{n+1}, d\nu) \big\}$$

with $\|f\|_{H^p_{\mathfrak{g}}} := \|\mathfrak{g}(f)\|_{L^p_\nu}$. The definition of $H^p_{\mathfrak{g}}(\mathbb{R}^{n+1})$ is independent of the choice of $\{E_k\}_{k\in\mathbb{Z}}$ due to the following Plancherel-Pôlya inequality for $H^p_{\mathfrak{g}}$.

THEOREM 4.1. *[Ha, Theorem 1] Suppose that $\{E_k\}_{k\in\mathbb{Z}}$ and $\{R_k\}_{k\in\mathbb{Z}}$ are approximations to the identity with regularity exponent ε, and $\frac{1}{1+\varepsilon} < p < \infty$. Set $D_k = E_k - E_{k-1}$ and $J_k = R_k - R_{k-1}$. Then, for $f \in \big(\dot{\mathcal{M}}^{(\beta,\gamma)}_\varepsilon\big)'$,*

$$\begin{aligned}
&\left\| \Big\{ \sum_k \sum_\tau \Big(\sup_{z \in Q^{k+N}_\tau} |J_k(f)(z)| \Big)^2 \chi_{Q^{k+N}_\tau} \Big\}^{1/2} \right\|_{L^p_\nu} \\
&\qquad \approx \left\| \Big\{ \sum_k \sum_\tau \Big(\inf_{z \in Q^{k+N}_\tau} |D_k(f)(z)| \Big)^2 \chi_{Q^{k+N}_\tau} \Big\}^{1/2} \right\|_{L^p_\nu},
\end{aligned}$$

where Q_τ^k are the dyadic cubes given in Theorem 2.1.

We say that a function $a \in L^q(\mathbb{R}^{n+1}, d\nu)$ is called a (p,q)-ρ-*atom* if

(i) a is supported on a ρ-ball $B_\rho(z,r)$;

(ii) $\int_{B_\rho(z,r)} a(w)d\nu(w) = 0$;

(iii) $\|a\|_{L^q_\nu} \le \nu(B_\rho(z,r))^{1/q-1/p}$.

THEOREM 4.2. *[Ha, Theorem 4] Suppose $\frac{1}{1+\varepsilon} < p \le 1 < q < \infty$, where ε is given in (2.3). Then $f \in H^p_{\mathfrak{g}}$ if and only if there exist a sequence of (p,q)-ρ-atoms $\{a_i\}$ and a sequence $\{\lambda_i\} \in \ell^p$ such that $f = \sum \lambda_i a_i$ in $\dot{\mathcal{M}}_\varepsilon^{(\beta,\gamma)}$. Moreover,*

$$\|f\|_{H^p_{\mathfrak{g}}} \approx \inf\Big(\sum |\lambda_i|^p\Big)^{1/p},$$

where the infimum is taken over all the above decomposition of f.

REMARK 4.1. If we use $H^{p,q}_\rho$ to express the following atomic Hardy space

$$H^{p,q}_\rho(\mathbb{R}^{n+1}) := \Big\{f \in \mathcal{S}'(\mathbb{R}^{n+1}) : f(z) = \sum_j \lambda_j a_j(z) \ \text{ in } \mathcal{S}', \text{ each } a_j \text{ is a } (p,q)\text{-}\rho\text{-atom and } \sum_j |\lambda_j|^p < \infty\Big\}$$

with norm $\|f\|_{H^{p,q}_\rho} = \inf\big(\sum_j |\lambda_j|^p\big)^{1/p}$, where the infimum is taken over all decompositions of $f = \sum_j \lambda_j a_j$ above, then Theorem 4.2 says $H^p_{\mathfrak{g}} = H^{p,q}_\rho$ with equivalent norms.

We also note that both atomic Hardy spaces $H^{p,q}_\rho$ and $H^{p,q}_{\mathcal{P}}$ coincide with equivalent norms. For any $z \in \mathbb{R}^{n+1}$ and $r > 0$, (2.2) yields

$$\frac{1}{\nu(Q(z,r))} \le \frac{1}{\nu(B_\rho(z,r))} \le \frac{1}{\nu(Q(z,\frac{r}{2\theta}))} \le \frac{A^{1+\log_2\theta}}{\nu(Q(z,r))},$$

where we apply (1.2) to the last inequality. For each (p,q)-ρ-atom a, it is easy to see that $A^{(1+\log_2\theta)(1/q-1/p)}a$ is a (p,q)-atom with respect to $\mathcal{P}$, and hence $H^{p,q}_\rho \subset H^{p,q}_{\mathcal{P}}$ with $\|\cdot\|_{H^{p,q}_{\mathcal{P}}} \le A^{(1+\log_2\theta)(1/p-1/q)}\|\cdot\|_{H^{p,q}_\rho}$. Similarly, we have $H^{p,q}_{\mathcal{P}} \subset H^{p,q}_\rho$ and $\|\cdot\|_{H^{p,q}_\rho} \le \|\cdot\|_{H^{p,q}_{\mathcal{P}}}$.

Summarizing Theorems 1.3 and 4.2 with Remark 4.1, we conclude the proof of Theorem 1.4.

5. Proof of Theorem 1.5. For $p = 1$ and $1 < q \le \infty$, it follows from Theorem 1.3, [W, Theorem 1.2] and [QW, Theorem 1.2].

We now consider $1/2 < p < 1 \le q < \infty$ and let $\kappa = 1/p - 1$. It suffices to show that, if $g \in \Lambda^\kappa_{q',\mathcal{P}}$, then

$$l_g(f) = \int_{\mathbb{R}^{n+1}} f(z)g(z)d\nu(z) \tag{5.1}$$

is a bounded linear functional on $H^{p,q}_{\mathcal{P}}$, and conversely for any bounded linear functional l on $H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1})$, there exists $b \in \Lambda^\kappa_{q',\mathcal{P}}$ such that

$$l(f) = \int_{\mathbb{R}^{n+1}} f(z)b(z)d\nu(z), \qquad \forall\ f \in H^{p,q}_{\mathcal{P}}(\mathbb{R}^{n+1}).$$

We first prove that $\Lambda^{\kappa}_{q',\mathcal{P}} \subset (H^{p,q}_{\mathcal{P}})'$. Write $D = H^{p,q}_{\mathcal{P}} \cap L^q_c(\mathbb{R}^{n+1}, d\nu)$, where $L^q_c(\mathbb{R}^{n+1}, d\nu)$ consists of all functions in $L^q(\mathbb{R}^{n+1}, d\nu)$ with compact supports. Since the set of all the finite linear combinations of (p, q)-atoms is dense in $H^{p,q}_{\mathcal{P}}$, D is a dense subset of $H^{p,q}_{\mathcal{P}}$. Then we will see that, for any $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$, the linear functional l_g defined in (5.1) is bounded on the dense subset D of $H^{p,q}_{\mathcal{P}}$.

For $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$, it is easy to verify that $|g| \in \Lambda^{\kappa}_{q',\mathcal{P}}$. Hence, for $g_1, g_2 \in \Lambda^{\kappa}_{q',\mathcal{P}}$, the functions $\max\{g_1, g_2\}$ and $\min\{g_1, g_2\}$ are both in $\Lambda^{\kappa}_{q',\mathcal{P}}$, with norms majorized by a multiple of $\max\big\{\|g_1\|_{\Lambda^{\kappa}_{q',\mathcal{P}}}, \|g_2\|_{\Lambda^{\kappa}_{q',\mathcal{P}}}\big\}$. The advantage is that now we can approximate a function of $\Lambda^{\kappa}_{q',\mathcal{P}}$ by means of bounded functions with uniformly bounded $\Lambda^{\kappa}_{q',\mathcal{P}}$ norms. For $N \in \mathbb{N}$ and $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$, we set

$$g_N(z) = \begin{cases} N, & \text{if } g(z) \geq N \\ g(z), & \text{if } |g(z)| < N \\ -N, & \text{if } g(z) \leq -N. \end{cases}$$

Then we have $g_N \in \Lambda^{\kappa}_{q',\mathcal{P}}$ and $\|g_N\|_{\Lambda^{\kappa}_{q',\mathcal{P}}} \leq C\|g\|_{\Lambda^{\kappa}_{q',\mathcal{P}}}$.

Set $f = \sum_{k=1}^{\infty} \lambda_k a_k \in D$, where a_k is a (p, q)-atom supported in a parabolic section $Q_k \in \mathcal{P}$. Thus, by the definition of the (p, q)-atom, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{n+1}} f(z) g_N(z) d\nu(z) \right| \\
\leq &\sum_{k=1}^{\infty} |\lambda_k| \left| \int_{\mathbb{R}^{n+1}} a_k(z) g_N(z) d\nu(z) \right| \\
\leq &\sum_{k=1}^{\infty} |\lambda_k| \left| \int_{Q_k} a_k(z) [g_N(x) - m_{Q_k}(g_N)] d\nu(z) \right| \\
\leq &\sum_{k=1}^{\infty} |\lambda_k| \, \|a_k\|_{L^q_\nu} \bigg(\int_{Q_k} |g_N(x) - m_{Q_k}(g_N)|^{q'} d\nu(z) \bigg)^{1/q'} \\
\leq &\sum_{k=1}^{\infty} |\lambda_k| \nu(Q_k)^{1-1/p} \bigg(\frac{1}{\nu(Q_k)} \int_{Q_k} |g_N(z) - m_{Q_k}(g_N)|^{q'} d\nu(z) \bigg)^{1/q'} \\
\leq &C\|f\|_{H^{p,q}_{\mathcal{P}}} \|g\|_{\Lambda^{\kappa}_{q',\mathcal{P}}},
\end{aligned} \tag{5.2}$$

where the last inequality holds by $\sum_k |\lambda_k| \leq \big(\sum_k |\lambda_k|^p\big)^{1/p} \leq C\|f\|_{H^{p,q}_{\mathcal{P}}}$. Since $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$ is a locally q'-th integrable function on $\mathbb{R}^{n+1}$,

$$|f(z) g_N(z)| \leq |f(z) g(z)| \in L^1(\mathbb{R}^{n+1}, d\nu).$$

By the Lebesgue dominated convergence theorem and (5.2),

$$\left| \int_{\mathbb{R}^{n+1}} f(z) g(z) d\nu(z) \right| = \left| \lim_{N\to\infty} \int_{\mathbb{R}^{n+1}} f(z) g_N(z) d\nu(z) \right| \leq C\|f\|_{H^{p,q}_{\mathcal{P}}} \|g\|_{\Lambda^{\kappa}_{q',\mathcal{P}}}.$$

This shows that the linear functional l_g is bounded on D, and $\|l_g\| \leq C\|g\|_{\Lambda^{\kappa}_{q',\mathcal{P}}}$. Consequently, l_g has a unique bounded extension to $H^{p,q}_{\mathcal{P}}$ since D is a dense subset of $H^{p,q}_{\mathcal{P}}$. In this sense we obtain $\Lambda^{\kappa}_{q',\mathcal{P}} \subset (H^{p,q}_{\mathcal{P}})'$.

In order to prove the reverse inclusion $(H^{p,q}_{\mathcal{P}})' \subset \Lambda^{\kappa}_{q',\mathcal{P}}$, we need to show that if l is a bounded linear functional on $H^{p,q}_{\mathcal{P}}$, then there exists $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$ such that, for any

$f \in H^{p,q}_{\mathcal{P}}$,

$$l(f) = \int_{\mathbb{R}^{n+1}} f(z)g(z)d\nu(z).$$

The proof will be divided into the following three steps.

Step 1. Let us first prove $(H^{p,q}_{\mathcal{P}})' \subset (L^q_0(Q, d\nu))'$, where $Q = Q(z, r) \in \mathcal{P}$ is any parabolic section in $\mathbb{R}^{n+1}$ and

$$L^q_0(Q, d\nu) = \Big\{ f \in L^q(\mathbb{R}^{n+1}, d\nu) : \ f = 0 \ \nu\text{-}a.e. \text{ on } Q^c \text{ and } \int_Q f(z)d\nu(z) = 0 \Big\}.$$

Indeed, when $f \in L^q_0(Q, d\nu)$, it is easy to check that $a(z) = f(z)\nu(Q)^{1/q-1/p}\|f\|^{-1}_{L^q_\nu(Q)}$ is a (p, q)-atom. Thus $f(z) = a(z)\nu(Q)^{1/p-1/q}\|f\|_{L^q_\nu(Q)} \in H^{p,q}_{\mathcal{P}}$ and $\|f\|_{H^{p,q}_{\mathcal{P}}} \le \nu(Q)^{1/p-1/q}\|f\|_{L^q_\nu(Q)}$. Therefore, we have

$$|l(f)| \le \|l\|\nu(Q)^{1/p-1/q}\|f\|_{L^q_\nu(Q)}, \tag{5.3}$$

which shows that l is also a bounded linear functional on $L^q_0(Q, d\nu)$. Since $L^q_0(Q, d\nu) \subset L^q(Q, d\nu)$, using the Hahn-Banach extension theorem, we know that l has a unique bounded extension to $L^q(Q, d\nu)$. Since $1 \le q < \infty$, by the Riesz representation theorem, there exists $g \in L^{q'}(Q, d\nu)$ such that

$$l(f) = \int_Q f(z)g(z)d\nu(z), \qquad \forall\ f \in L^q_0(Q, d\nu). \tag{5.4}$$

Furthermore, we have the following fact:

If $\int_Q f(z)b(z)d\nu(z) = 0$ *for all* $f \in L^q_0(Q, d\nu)$*, then* $g(z)$ *is constant for almost every* $z \in Q$.

Indeed, since Q is a bounded convex set, for any $h \in L^q(Q, d\nu)$ we have $h - m_Q(h) \in L^q_0(Q, d\nu)$. Thus

$$0 = \int_Q g(z)\Big(h(z) - m_Q(h)\Big)d\nu(z) = \int_Q h(z)\Big(g(z) - m_Q(g)\Big)d\nu(z), \qquad \forall\ h \in L^q(Q, d\nu).$$

Hence $g(z) = m_Q(g)$ for almost every $z \in Q$.

Step 2. Fix $z_0 \in \mathbb{R}^{n+1}$ and choose a sequence of positive increasing numbers $\{t_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} t_j = \infty$. Then, by property (**C**) of parabolic sections, $\{Q(z_0, r_j)\}_{j=1}^\infty$ is a sequence of parabolic sections with $\bigcup_{j=1}^\infty Q_j = \mathbb{R}^{n+1}$, where $Q_j = Q(z_0, r_j)$. By (5.4), for each Q_j, there exists $g_j \in L^{q'}(Q_j, d\nu)$ satisfying (5.3).

Consider an arbitrary $f \in L^q_0(Q_1, d\nu)$. There exists $g_1 \in L^{q'}(Q_1, d\nu)$ such that

$$l(f) = \int_{Q_1} f(z)g_1(z)d\nu(z). \tag{5.5}$$

By $Q_2 \supset Q_1$, we have $L^q_0(Q_2, d\nu) \supset L^q_0(Q_1, d\nu)$ and $f \in L^q_0(Q_2, d\nu)$. Therefore, there exists $g_2 \in L^{q'}(Q_2, d\nu)$ such that

$$l(f) = \int_{Q_2} f(z)g_2(z)d\nu(z) = \int_{Q_1} f(z)b_2(z)d\nu(z), \tag{5.6}$$

since $\operatorname{supp}(f) \subset Q_1$. From (5.5) and (5.6), we get

$$\int_{Q_1} f(z)\Big(g_1(z) - g_2(z)\Big) d\nu(z) = 0, \qquad \forall\, f \in L^q_0(Q_1, d\nu).$$

Applying the fact shown in Step 1, we have $g_1(z) - g_2(z) = C_1$ for almost every $z \in Q_1$. Now we write

$$g(z) = \begin{cases} g_1(z) & \text{if} \quad z \in Q_1, \\ g_2(z) + C_1 & \text{if} \quad z \in Q_2 \backslash Q_1. \end{cases}$$

Then we obtain

$$l(f) = \int_{Q_j} f(z) g(z) d\nu(z), \qquad \forall\, f \in L^q_0(Q_j, d\nu),\ j = 1, 2.$$

By a method quite similar to the above, we may obtain a function g satisfying

$$l(f) = \int_{Q_j} f(z) g(z) d\nu(z), \qquad \forall\, f \in L^q_0(Q_j, d\nu),\ j = 1, 2, \ldots. \tag{5.7}$$

Step 3. Now we prove that the above $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$ and satisfies

$$l(f) = \int_{\mathbb{R}^{n+1}} f(z) g(z) d\nu(z), \qquad \forall\, f \in H^{p,q}_{\mathcal{P}}. \tag{5.8}$$

We need the following fact about parabolic sections in $\mathbb{R}^{n+1}$.

Assume that $Q_0 = Q(w_0, r') \in \mathcal{P}$ is an arbitrary parabolic section in $\mathbb{R}^{n+1}$. Then there exists j_0 such that $Q_{j_0} \supset Q_0$, where $Q_{j_0} = Q(z_0, r_{j_0})$ is the j_0-th parabolic section of the sequence in Step 2.

Indeed, by $\bigcup_{j=1}^{\infty} Q_j = \mathbb{R}^{n+1}$, there exists a parabolic section $Q_i = Q(z_0, r_i)$ such that $Q(z_0, r_i) \cap Q(w_0, r') \neq \varnothing$ with $r_i \geq r'$. Then there exists $u \in Q(z_0, r_i) \cap Q(w_0, r')$. From (2.1), we have $Q(z_0, r') \subset Q(u, \theta r') \subset Q(u, \theta r_i)$. Since $u \in Q(z_0, r_i) \subset Q(z_0, \theta r_i)$, using (2.1) again, we know $Q(u, \theta r_i) \subset Q(z_0, \theta^2 r_i)$ and therefore $Q(w_0, r') \subset Q(z_0, \theta^2 r_i)$. Now if we take j_0 such that $r_{j_0} \geq \theta^2 r_i$, then $Q(w_0, r') \subset Q(z_0, r_{j_0})$.

Now, let us return to the proof of (5.8). For any $f \in H^{p,q}_{\mathcal{P}}$, we may write $f = \sum_{k=1}^{\infty} \lambda_k a_k$, where a_k is a (p,q)-atom supported in the parabolic section $Q_k \in \mathcal{P}$. By the fact above, for each k there exists j_k such that $Q_k \subset Q_{j_k} = Q(z_0, r_{j_k})$. By the definition of (p,q)-atom, we have $a_k \in L^q_0(Q_{j_k}, d\nu)$. Thus by (5.7),

$$l(a_k) = \int_{Q_{j_k}} a_k(z) g(z) d\nu(z) = \int_{\mathbb{R}^{n+1}} a_k(z) g(z) d\nu(z). \tag{5.9}$$

Since the functional l is linear, by (5.9) we obtain

$$l(f) = \sum_{k=1}^{\infty} \lambda_k l(a_k) = \sum_{k=1}^{\infty} \lambda_k \int_{\mathbb{R}^{n+1}} a_k(z) g(z) d\nu(z) = \int_{\mathbb{R}^{n+1}} f(z) g(z) d\nu(z).$$

Finally, to finish the proof of Step 3, it remains to show that $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$. For any parabolic section $Q \in \mathcal{P}$, let $h \in L^q(Q, d\nu)$ with $\operatorname{supp}(h) \subset Q$ and $\|h\|_{L^q_\nu} \leq 1$. Then

$$a(z) := \frac{1}{2} \nu(Q)^{1/q - 1/p} \Big(h(z) - m_Q(h)\Big) \chi_Q(z)$$

is a (p,q)-atom supported in Q and $\|a\|_{H^{p,q}_{\mathcal{P}}} \le 1$. Thus, (5.9) implies

$$\left|\int_Q a(z)g(z)d\nu(z)\right| = |l(a)| \le \|l\|.$$

Hence

$$\nu(Q)^{1/q-1/p}\left|\int_Q \Big(h(z) - m_Q(h)\Big)g(z)d\nu(z)\right| \le 2\|l\|.$$

That is,

$$\nu(Q)^{1/q-1/p}\left|\int_Q h(z)\Big(g(z) - m_Q(g)\Big)d\nu(z)\right| \le 2\|l\|. \tag{5.10}$$

From (5.10), we have

$$\begin{aligned}&\nu(Q)^{1/q-1/p}\|g - m_Q(g)\|_{L^{q'}_\nu}\\ &= \nu(Q)^{1/q-1/p} \sup_{\|h\|_{L^q_\nu}\le 1}\left|\int_Q h(z)\Big(g(z) - m_Q(g)\Big)d\nu(z)\right| \le 2\|l\|.\end{aligned}$$

Since the parabolic section $Q \in \mathcal{P}$ is arbitrary, we conclude that $g \in \Lambda^{\kappa}_{q',\mathcal{P}}$. This completes the proof of Theorem 1.5.

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