Asian J. Math.
Vol. 19, No. 5, pp. 933–950, November 2015

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STRUCTURE AT INFINITY OF EXPANDING GRADIENT RICCI SOLITON*

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Abstract. We study the geometry at infinity of expanding gradient Ricci solitons $(M^n, g, \nabla f)$, $n \geq 3$, with finite asymptotic curvature ratio without curvature sign assumptions. We mainly prove that they have a noncollapsed cone structure at infinity. Certain topological informations still can be obtained under conditions only involving asymptotic Ricci curvature ratio. Furthermore, we derive a quantitative relationship between (small) asymptotic curvature ratio and asymptotic volume ratio.

Key words. Ricci flow, expanding gradient Ricci soliton, asymptotic cone.

AMS subject classifications. 53, 58.

1. Introduction. The central theme of this paper is the study of the geometry at infinity of noncompact Riemannian manifolds. We will focus on the notion of asymptotic cone whose definition is recalled below.

Definition 1.1. *Let (M^n, g) be a complete noncompact Riemannian manifold and let $p \in M^n$. An* ***asymptotic cone*** *of (M^n, g) at p is a pointed Gromov-Hausdorff limit, provided it exists, of the sequence $(M^n, t_k^{-2}g, p)_k$ where $t_k \to +\infty$.*

Usually, the existence of an asymptotic cone is guaranteed by an assumption of nonnegative curvature. More precisely, if (M^n, g) satisfies the nonnegativity assumption $\mathrm{Ric} \geq 0$ then the existence of a limit is guaranteed by Bishop-Gromov theorem and Gromov's precompactness theorem: see [Pet06]. We mention two striking results in this direction.

In case of nonnegative sectional curvature, any asymptotic cone of (M^n, g) exists and is unique: it is the metric cone over its ideal boundary $M(\infty)$. Moreover, $M(\infty)$ is an Alexandrov space of curvature bounded below by 1: see [GK95].

In case of nonnegative Ricci curvature and positive asymptotic volume ratio, i.e. $\lim_{r\to+\infty} \mathrm{Vol}\, B(p, r)/r^n > 0$, Cheeger and Colding proved that any asymptotic cone is a metric cone $C(X)$ over a length space X of diameter not greater than π: see [CC96]. Nonetheless, even in this case, uniqueness is not ensured: see Perelman [Per97].

In this paper, we consider the existence of asymptotic cone on *expanding gradient Ricci solitons* where no curvature sign assumption is made. Instead, we require the finiteness of the *asymptotic curvature ratio.* Such situation has already been investigated by [Che12] in the case of expanding gradient Ricci soliton with vanishing asymptotic curvature ratio.

Recall that the **asymptotic curvature ratio** of a complete noncompact Riemannian manifold (M^n, g) is defined by

$$\mathrm{A}(g) := \limsup_{r_p(x)\to+\infty} r_p(x)^2 |\,\mathrm{Rm}(g)(x)|.$$

Note that it is well-defined since it does not depend on the reference point $p \in M^n$. Moreover, it is invariant under scalings. This geometric invariant has generated a lot

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of interest: see for example [BKN89], [PT01], [LS00], [Lot03], [ESS89] for a static study of the asymptotic curvature ratio and [CLN06], [Ham95], [Per02], [CL04], [Che12] linking this invariant with the Ricci flow. Note also that Gromov [Gro82] and Lott-Shen [LS00] have shown that any paracompact manifold can support a complete metric g with finite $\mathrm{A}(g)$. Therefore, the only geometric constraint is the Ricci soliton structure.

Now, we recall that an **expanding gradient Ricci soliton** is a triple $(M^n, g, \nabla f)$ where (M^n, g) is a Riemannian manifold and f is a smooth function on M^n such that

$$\mathrm{Ric} + \frac{g}{2} = \mathrm{Hess}(f). \tag{1}$$

It is said **complete** if (M^n, g) is complete and if the vector field ∇f is complete. By [Zha08], the completeness of (M^n, g) suffices to ensure the completeness of ∇f. In case of completeness, the associated Ricci flow is defined on $(-1, +\infty)$ by

$$g(\tau) = (1+\tau)\phi_\tau^* g,$$

where $(\phi_\tau)_\tau$ is the 1-parameter family of diffeomorphisms generated by $\nabla(-f)/(1+\tau)$. A canonical example is the Gaussian soliton $(\mathbb{R}^n, \mathrm{eucl}, |x|^2/4)$. Along the paper, it is essential to keep this example in mind to get a geometric feeling of the proofs: see [Cao97] for other examples of expanding gradient Ricci solitons.

The main result of this paper is the following theorem.

Theorem 1.2. *[Cone structure at infinity] Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete expanding gradient Ricci soliton with finite $\mathrm{A}(g)$.*

For $p \in M^n$, $(M^n, t^{-2}g, p)_t$ Gromov-Hausdorff converges to a metric cone $(C(S_\infty), d_\infty, x_\infty)$ over a compact length space S_∞.

Moreover,

1) $C(S_\infty) \setminus \{x_\infty\}$ is a smooth manifold with a $C^{1,\alpha}$ metric g_∞ compatible with d_∞ and the convergence is $C^{1,\alpha}$ outside the apex x_∞.

2) (S_∞, g_{S_∞}) where g_{S_∞} is the metric induced by g_∞ on S_∞, is the $C^{1,\alpha}$ limit of the rescaled levels of the potential function f,
$(f^{-1}(t^2/4), t^{-2}g_{t^2/4})$ where $g_{t^2/4}$ is the metric induced by g on $f^{-1}(t^2/4)$.

Finally, we can ensure that

$$|K_{g_{S_\infty}} - 1| \leq \mathrm{A}(g), \quad \textit{in the Alexandrov sense}, \tag{2}$$

$$\frac{\mathrm{Vol}(S_\infty, g_{S_\infty})}{n} = \lim_{r\to+\infty} \frac{\mathrm{Vol}\, B(q,r)}{r^n}, \quad \forall q \in M^n. \tag{3}$$

As a direct consequence of Theorem 1.2, in case of vanishing asymptotic curvature ratio, we get the following :

Corollary 1.3 (Asymptotically flatness). *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete expanding gradient Ricci soliton. Assume*

$$\mathrm{A}(g) = 0.$$

Then, with the notations of Theorem 1.2,

$$(S_\infty, g_{S_\infty}) = \amalg_{i\in I}(\mathbb{S}^{n-1}/\Gamma_i, g_{std}) \quad \textit{and} \quad (C(S_\infty), d_\infty, x_\infty) = (C(S_\infty), \mathrm{eucl}, 0)$$

where Γ_i are finite groups of Euclidean isometries acting freely on $\mathbb{S}^{n-1}$ and $|I|$ is the (finite) number of ends of M^n.

Moreover, for $p \in M^n$,

$$\sum_{i\in I} \frac{\omega_n}{|\Gamma_i|} = \lim_{r\to+\infty} \frac{\operatorname{Vol} B(p,r)}{r^n},$$

where ω_n is the volume of the unit Euclidean ball.

Remark 1.4. *Theorem 1.2 ensures existence AND uniqueness of the asymptotic cone of an expanding gradient Ricci soliton with finite asymptotic curvature ratio.*

Remark 1.5. *Another consequence of Theorem 1.2 is to provide examples of expanding gradient Ricci soliton coming out from metric cones. Indeed, under assumptions and notations of Theorem 1.2, since f is proper (lemma 2.2), we may take any p such that $\nabla f(p) = 0$. Then, the pointed sequence $(M^n, t^{-2}g, p)_t$ is isometric to the pointed family $(M^n, g(t^{-2}-1), p)$ since the 1-parameter family of diffeomorphisms generated by $-\nabla f/(1+\tau)$, for $\tau \in (-1,+\infty)$, fixes p. Therefore, by Theorem 1.2, such an expanding gradient Ricci soliton comes out from a metric cone. A similar situation has already been encountered in the case of Riemannian manifolds with nonnegative curvature operator and positive asymptotic volume ratio : see [SS10].*

Remark 1.6. *The major difficulty to prove Theorem 1.2 is to ensure the existence of an asymptotic cone because no assumption of curvature sign is assumed. For this purpose, we have to control the growth of the metric balls of such an expanding gradient Ricci soliton : see Theorem 3.4.*

This approach was initiated in [Che12]. Here we derive more precise quantitative estimates and clarify the geometric role of the potential function by systematically analysing the geometry of its levels : the spatial flow generated by the vector field $\nabla f/|\nabla f|^2$ acts as a powerful substitute for the Ricci flow. We also generalize a topological result in [Che12] by using a weaker assumption: finite asymptotic Ricci curvature ratio. See Theorem 2.10.

In view of lemmas 2.4 and 2.5, the asymptotic volume ratio $\operatorname{AVR}(g) := \lim_{r\to+\infty} \operatorname{Vol} B(p,r)/r^n$ is well-defined and positive in case of finite asymptotic curvature ratio $\mathrm{A}(g)$. How can we link these two invariants in a global inequality ? This is the purpose of section 4. For example, we get

Proposition 1.7. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete expanding gradient Ricci soliton.*

Assume $\mathrm{A}(g) < \epsilon$, where ϵ is a universal constant small enough (less than $3/5$). Then,

$$\sum_{i\in I} \frac{\omega_n}{|\Gamma_i|} \frac{1}{(1+\mathrm{A}(g))^{\frac{n-1}{2}}} \leq \operatorname{AVR}(g) \leq \sum_{i\in I} \frac{\omega_n}{|\Gamma_i|} \frac{1}{(1-\mathrm{A}(g))^{\frac{n-1}{2}}}, \tag{4}$$

where $|I|$ is the (finite) number of ends of M^n and $|\Gamma_i|$ is the order of the fundamental group of the i-th end of M^n.

Finally, one can ask if some geometric information can lead to some restriction on the number of ends. In this direction, we would like to mention a recent sharp result due to Munteanu and Wang (Theorem 1.4 of [MW12]):

Theorem 1.8. *Let $(M, g, \nabla f)$ be an expanding gradient Ricci soliton. Assume that $R \geq -\frac{n-1}{2}$. Then either M is connected at infinity or M is isometric to the product $\mathbb{R} \times N$ where N is a compact Einstein manifold and $\mathbb{R}$ the Gaussian expanding Ricci soliton.*

Organization. In section 2, we study the geometry of the levels of the potential function. From this, we get some information about the topology at infinity of expanding gradient Ricci soliton. In section 3, we first estimate the volume growth of geodesic balls (Theorem 3.4), then we finish the proof of Theorem 1.2. In section 4, we establish geometric inequalities involving the asymptotic curvature ratio $\mathrm{A}(g)$ and the asymptotic volume ratio $\mathrm{AVR}(g)$ of an expanding gradient Ricci soliton. The last section do not depend on Theorem 1.2. It can be read independently of section 3.

Acknowledgements. The authors would like to thank Gilles Carron for the proof of Theorem 3.4. The first author appreciates his advisor Gérard Besson for his constant encouragement. The second author would like to thank his advisor Laurent Bessières for his constant support and his precious remarks. Finally, the authors are grateful to Ovidiu Munteanu and Jiaping Wang for sending us their results on Ricci solitons.

2. Geometry and topology of the level sets of f. In this section, we consider a complete expanding gradient Ricci soliton (EGS) $(M^n, g, \nabla f)$ satisfying one of the two following basic assumptions:

Assumption 1 ($A1$):

$$\mathrm{Ric} \geq -\frac{C}{1+r_p^2} g,$$

for some nonnegative constant C and some $p \in M^n$ where r_p denotes the distance function from the point p.

Assumption 2 ($A2$):

$$|\,\mathrm{Ric}\,| \leq \frac{C}{1+r_p^2},$$

for some nonnegative constant C and some $p \in M^n$.

Of course, (A2) implies (A1) and finite asymptotic curvature ratio implies (A2).

We recall the basic differential equations satisfied by an expanding gradient Ricci soliton [CLN06].

Lemma 2.1. *Let $(M^n, g, \nabla f)$ be a complete EGS. Then:*

$$\Delta f = R + \frac{n}{2}, \tag{5}$$

$$\nabla R + 2\,\mathrm{Ric}(\nabla f) = 0, \tag{6}$$

$$|\nabla f|^2 + R = f + Cst. \tag{7}$$

In the following, we will assume w.l.o.g. $Cst = 0$.

Lemma 2.2. *[Growth of the potential function]*
Let $(M^n, g, \nabla f)$ be a complete EGS and $p \in M^n$.

1) If $R \geq -C$ where $C \geq 0$, then $f(x)+C \leq (r_p(x)/2+\sqrt{f(p)+C})^2$.

2) Under $(A1)$ then

$$f(x) \geq \frac{r_p(x)^2}{4} - \frac{\pi}{2}Cr_p(x) + f(p) - |\nabla f(p)|.$$

In particular, under $(A1)$, f behaves at infinity like $\frac{r_p(x)^2}{4}$.

3) If $\mathrm{Ric} \geq 0$ then

$$\min_{M^n} f + \frac{r_p(x)^2}{4} \leq f(x) \leq \left(\frac{r_p(x)}{2} + \sqrt{\min_{M^n} f}\right)^2.$$

Proof. Let $p \in M^n$. Let $x \in M^n$ and $\gamma : [0, r_p(x)] \to M^n$ be a geodesic from p to x. If $R \geq -C$ then (7) gives $(2\sqrt{f(\gamma(t))+C})' \leq 1$. Therefore, after integration, we get $2\sqrt{f(x)+C} - 2\sqrt{f(p)+C} \leq r_p(x)$. The result follows easily.
To get a lower bound for f, we apply the Taylor-Young integral formula to $f \circ \gamma$:

$$f(x) = f(p) + d_pf(\gamma'(0)) + \int_0^{r_p(x)} (r_p(x)-t)\,\mathrm{Hess}\, f(\gamma'(t), \gamma'(t))dt.$$

Using (1) and $(A1)$, we get

$$f(x) \geq f(p) - |\nabla f(p)| + \frac{r_p(x)^2}{4} - C\int_0^{r_p(x)} \frac{r_p(x)-t}{1+t^2}dt.$$

Hence the desired inequality. To prove the last statement, note that under 3), f is a strictly convex function for $\mathrm{Hess}\, f \geq \frac{g}{2}$. Moreover, by $(A1)$, f is a proper function $(C=0)$. Therefore f attains its minimum at a unique point $p_0 \in M^n$. Now, it suffices to apply the previous results to f and p_0. □

Remark 2.3. *Note that the bound on the Ricci curvature is not optimal to get a growth of type $r_p(x)^2/4$.*

Under assumptions of lemma 2.2 and using (7), the levels of f, $M_t := f^{-1}(t)$, are well-defined compact hypersurfaces for $t > 0$ large enough, in particular, they have a finite number of connected components. We will also denote the sublevels (resp. superlevels) of f by $M_{\leq t} := f^{-1}(]-\infty, t])$ (resp. $M_{\geq t} := f^{-1}([t, +\infty[)$). g_t will stand for the metric induced on M_t by the ambient metric g.

The next lemma is concerned with the volume of the sublevels of f. In case of nonnegative scalar curvature, this lemma has already been proved by several authors: see [CN10], [Zha11] for instance. See also [Che12].

Lemma 2.4. *[Volume of sub-levels of f] Let $(M^n, g, \nabla f)$ be a complete EGS satisfying $(A1)$.*

Then, for $1 << t_0 \leq t$,

$$\frac{\mathrm{Vol}\, M_{\leq t}}{\mathrm{Vol}\, M_{\leq t_0}} \geq \left(\frac{t+nC}{t_0+nC}\right)^{(n/2-nC)}. \tag{8}$$

Moreover, if $R \geq 0$ then $t \to \dfrac{\mathrm{Vol}\, M_{\leq t}}{t^{n/2}}$ is a nondecreasing function for t large enough.

Proof. By assumption and (5), $-nC \leq R = \Delta f - n/2$. After integrating these inequalities over $M_{\leq t}$, using (7) and integration by part, one has

$$-nC \operatorname{Vol} M_{\leq t} \leq \int_{M_t} |\nabla f| - \frac{n}{2} \operatorname{Vol} M_{\leq t} \leq \sqrt{t - \inf_{M_t} R} \operatorname{Vol} M_t - \frac{n}{2} \operatorname{Vol} M_{\leq t}.$$

Now,

$$\frac{d}{dt} \operatorname{Vol} M_{\leq t} = \int_{M_t} \frac{d\mu_t}{|\nabla f|} \geq \frac{\operatorname{Vol} M_t}{\sqrt{t - \inf_{M_t} R}}.$$

Hence,

$$\left(\frac{n}{2} - nC\right) \operatorname{Vol} M_{\leq t} \leq (t - \inf_{M_t} R) \frac{d}{dt} \operatorname{Vol} M_{\leq t} \leq (t + nC) \frac{d}{dt} \operatorname{Vol} M_{\leq t}.$$

The first inequality follows by integrating this differential inequality.

The last statement is obtained by letting $C = 0$ in the previous estimates. The lower Ricci bound is only used to ensure the existence of the levels of f. □

We pursue by estimating the volume of the hypersurfaces M_t.

LEMMA 2.5. *[Volume of levels I] Let $(M^n, g, \nabla f)$ be a complete EGS satisfying (A1).*

Then, for $1 << t_0 \leq t$, there exists a function $a \in L^1(+\infty)$ such that

$$\frac{\operatorname{Vol} M_t}{\operatorname{Vol} M_{t_0}} \geq \exp\left(\int_{t_0}^t a(s) ds\right) \left(\frac{t}{t_0}\right)^{\frac{n-1}{2}}. \tag{9}$$

Moreover, if $\operatorname{Ric} \geq 0$ *then* $t \to \dfrac{\operatorname{Vol} M_t}{t^{(n-1)/2}}$ *is a nondecreasing function for* $t > \min_{M^n} f$.

Proof. The second fundamental form of M_t is $h_t = \dfrac{\operatorname{Hess} f}{|\nabla f|}$. Now,

$$\frac{d}{dt} \operatorname{Vol} M_t = \int_{M_t} \frac{H_t}{|\nabla f|} d\mu_t,$$

where H_t is the mean curvature of the hypersurface M_t. Hence,

$$\frac{d}{dt} \operatorname{Vol} M_t = \int_{M_t} \frac{R - \operatorname{Ric}(\mathbf{n}, \mathbf{n}) + \frac{n-1}{2}}{t - R} d\mu_t \geq \frac{(n-1)\inf_{M_t} \operatorname{Ric} + \frac{n-1}{2}}{t - \inf_{M_t} R} \operatorname{Vol} M_t,$$

where $\mathbf{n} = \nabla f / |\nabla f|$. Therefore,

$$\begin{aligned} \ln \frac{\operatorname{Vol} M_t}{\operatorname{Vol} M_{t_0}} &\geq \int_{t_0}^t \frac{(n-1)(\inf_{M_s} \operatorname{Ric} + \inf_{M_s} R/2s)}{s - \inf_{M_s} R} ds + \frac{n-1}{2} \ln\left(\frac{t}{t_0}\right) \\ &= \int_{t_0}^t a(s) ds + \frac{n-1}{2} \ln\left(\frac{t}{t_0}\right), \end{aligned}$$

where $a(s) := \dfrac{(n-1)(\inf_{M_s} \operatorname{Ric} + \inf_{M_s} R/2s)}{s - \inf_{M_s} R}$.

In view of the lower Ricci bound and lemma 2.2, one has $a \in L^1(+\infty)$. The desired inequality and the case $C = 0$ now follow easily. □

Lemma 2.6. *[Volume of the levels II] Let $(M^n, g, \nabla f)$ be a complete EGS satisfying (A2).*

Then, for $1 << t_0 \leq t$, there exists a function $b \in L^1(+\infty)$ such that

$$\frac{\operatorname{Vol} M_t}{\operatorname{Vol} M_{t_0}} \leq \exp\left(\int_{t_0}^t b(s)ds\right)\left(\frac{t}{t_0}\right)^{\frac{n-1}{2}}. \tag{10}$$

Proof. The proof goes along the same line of the previous one. Here, the lower Ricci bound is only used to ensure the existence of the hypersurfaces M_t for t large enough by lemma 2.2. □

Lemma 2.7. *[Diameter growth] Let $(M^n, g, \nabla f)$ be a complete EGS satisfying (A2).*

Then, for $1 << t_0 \leq t$, there exists a function $c \in L^1(+\infty)$ such that

$$\frac{\operatorname{diam}(g_t)}{\sqrt{t}} \leq \exp\left(\int_{t_0}^t c(s)ds\right)\frac{\operatorname{diam}(g_{t_0})}{\sqrt{t_0}}. \tag{11}$$

Assume only $\operatorname{Ric} \geq 0$ *then* $t \to \dfrac{\operatorname{diam}(g_t)}{\sqrt{t}}$ *is a nondecreasing function for* $t > \min_{M^n} f$.

Proof. Let ϕ_t be the flow associated to the vector field $\nabla f/|\nabla f|^2$. If $v \in TM_{t_0}$, define $V(t) := d\phi_{t-t_0}(v) \in TM_t$ for $1 << t_0 \leq t$. Now,

$$\frac{d}{dt}g(V(t),V(t)) = 2\frac{\operatorname{Hess} f(V(t),V(t))}{|\nabla f|^2} = 2\frac{\operatorname{Ric}(V(t),V(t)) + g(V(t),V(t))/2}{t-R}.$$

Hence,

$$\begin{aligned}\frac{d}{dt}g(V(t),V(t)) &\leq \frac{2\sup_{M_t}|\operatorname{Ric}|+1}{t-\sup_{M_t} R}g(V(t),V(t))\\ &= \left(c(t)+\frac{1}{t}\right)g(V(t),V(t)),\end{aligned}$$

where $c(t) := \dfrac{2\sup_{M_t}|\operatorname{Ric}| + \sup_{M_t} R/t}{t-\sup_{M_t} R}$.

Therefore,

$$\ln\left(\frac{g(V(t),V(t))}{g(V(t_0),V(t_0))}\right) \leq \int_{t_0}^t c(s)ds + \ln\left(\frac{t}{t_0}\right).$$

In view of the upper Ricci bound, one can see that $c \in L^1(+\infty)$. The desired estimate is now immediate.

The proof of the last assertion uses the same arguments. □

According to the growth of f in case of finite asymptotic ratio ($\mathrm{A}(g) < +\infty$), the hypersurface $M_{t^2/4}$ "looks like" the geodesic sphere S_t of radius t. Therefore, the next lemma deals with curvature bounds of the levels $M_{t^2/4}$ for t large.

Lemma 2.8. *Let $(M^n, g, \nabla f)$ be a complete EGS with finite* $\mathrm{A}(g)$*.*
Then,

$$\limsup_{t\to+\infty} |\operatorname{Rm}(t^{-2}g_{t^2/4}) - \mathrm{Id}_{\Lambda^2}| \leq \mathrm{A}(g), \tag{12}$$

$$1 - \mathrm{A}(g) \leq \liminf_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq \limsup_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq 1 + \mathrm{A}(g), \tag{13}$$

In case of nonnegative sectional curvature, one has

$$1 \leq \liminf_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq \limsup_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq 1 + \mathrm{A}(g). \tag{14}$$

Proof. Take a look at Gauss equation applied to $M_{t^2/4}$.

$$\begin{aligned}\operatorname{Rm}(g_{t^2/4})(X,Y) &= \operatorname{Rm}(g)(X,Y) + \det h_{t^2/4}(X,Y)\\ &= \operatorname{Rm}(g)(X,Y) + \frac{\det(\operatorname{Ric} + g/2)}{|\nabla f|^2}\Big|_{M_{t^2/4}},\end{aligned}$$

where X and Y are tangent to $M_{t^2/4}$. After rescaling the metric $g_{t^2/4}$ by t^{-2}, and using the fact that $t^2/|\nabla f|^2\big|_{M_{t^2/4}} \to 4$ as $t \to +\infty$, we get all the desired inequalities. □

Using the classification result of [BW08] together with inequality (12) of lemma 2.8 , we get the following topological information:

Corollary 2.9 (Small asymptotic ratio). *Let $(M^n, g, \nabla f)$, $n \geq 3$ be a complete EGS such that* $\mathrm{A}(g) < 1$.

Then, outside a compact set K, $M^n \setminus K$ is a disjoint union of a finite number of ends, each end being diffeomorphic to $\mathbb{S}^{n-1}/\Gamma \times (0, +\infty)$ where $\mathbb{S}^{n-1}/\Gamma$ is a spherical space form.

This result can be linked with previous results concerning the topology at infinity of Riemannian manifolds with cone structure at infinity and finite (vanishing) asymptotic curvature ratio: [Pet06], [LS00], [GPZ94], [Che12].

It is natural to ask if we can obtain an analogue topological information at infinity when the assumption on $A(g)$ is replaced by $(A2)$ or, equivalently, $\limsup_{r_p(x)\to+\infty} r_p(x)^2|\operatorname{Ric}(g)(x)| < C$. The following theorem confirms this under an additional condition that $|\operatorname{Ric}(g)| \leq \alpha R_g$ and $|\nabla \operatorname{Ric}| \leq \beta|\nabla R|$ on $(M^n, g, \nabla f)$ for some nonnegative constants α and β. Recall that the traced second Bianchi identity reads as $2\operatorname{div}(\operatorname{Ric}) = \nabla R$. See [Che11] for more discussions on this inequality.

Theorem 2.10. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete EGS satisfying*

$$\limsup_{r_p(x)\to+\infty} r_p(x)^2|\operatorname{Ric}(g)(x)| < \eta, \quad |\operatorname{Ric}(g)| \leq \alpha R_g, \quad |\nabla \operatorname{Ric}(g)| \leq \beta|\nabla R_g|,$$

where η, α and β are nonnegative. Then outside a compact set K, $M^n\setminus K$ is a disjoint union of a finite number of ends, each end being diffeomorphic to $N^{n-1}\times(0,\infty)$, where N^{n-1} is an η-almost Einstein manifold with positive normalised Ricci curvature, i.e. $|Ric - (n-2)g| < \eta$ on N.

Proof. As in the proof of lemma 2.8, we first write down the Gauss equation for the Ricci curvature of $M_{t^2/4}$:

$$\operatorname{Ric}(g_{t^2/4})(X,X) = \operatorname{Ric}(g)(X,X) - \operatorname{Rm}(g)(X,\mathbf{n},\mathbf{n},X) \tag{15}$$

$$+ \sum_i \det h_{t^2/4}(X,E_i), \tag{16}$$

where X is tangent to $M_{t^2/4}$ and $(E_i)_i$ is an orthonormal basis of $TM_{t^2/4}$. The only term we have to understand is the radial curvature $\mathrm{Rm}(g)(\cdot,\mathbf{n},\mathbf{n},\cdot)$. As such an EGS is likely to behave like a metric cone at infinity, we expect the radial curvatures decay faster than the spherical ones. This is exactly the content of the next

Claim 1. $\mathrm{Rm}(g)(\cdot,\boldsymbol{n},\boldsymbol{n},\cdot)|_{M_{t^2/4}}=O(t^{-3})$.

Proof of Claim 1. First recall a well-known identity on gradient Ricci solitons [PW10] :

$$(\nabla_X \mathrm{Ric})(Y,Z)-(\nabla_Y \mathrm{Ric})(X,Z)=\mathrm{Rm}(g)(X,Y,\nabla f,Z),$$

for any vector field X,Y,Z. Therefore, for X tangent to $M_{t^2/4}$,

$$\mathrm{Rm}(g)(X,\mathbf{n},\mathbf{n},X)=\frac{1}{|\nabla f|}\left((\nabla_X \mathrm{Ric})(\mathbf{n},X)-(\nabla_{\mathbf{n}} \mathrm{Ric})(X,X)\right).$$

In particular, using the assumption on $\nabla\mathrm{Ric}$, we get the estimate

$$|\mathrm{Rm}(g)(\cdot,\mathbf{n},\mathbf{n},\cdot)|\leq\frac{2|\nabla\mathrm{Ric}|}{|\nabla f|}\leq\frac{2\beta|\nabla R|}{|\nabla f|}. \tag{17}$$

The next lemma asserts that ∇R can be controlled by R, more precisely

Lemma 2.11. *Let $(M^n,g,\nabla f)$, $n\geq 3$, be a complete EGS satisfying*

$$|\mathrm{Ric}(g)|\leq\alpha R_g,\quad |\nabla\mathrm{Ric}(g)|\leq\beta|\nabla R_g|,$$

where α and β are nonnegative. Then, for $p\in M^n$ and $r>0$,

$$\sup_{B(p,r/2)}|\nabla R_g|\leq c\left(n,\alpha,\beta,\sup_{A(p,r/2,r)}\frac{|\nabla f|}{r}\right)\sup_{B(p,r)}R_g.$$

Proof of Lemma 2.11. The proof follows closely the proof of Shi's estimates [Chap. 6, [CLN06]]. Therefore, we will be sketchy. By assumption, the scalar curvature satisfies the following differential inequality

$$\begin{aligned}\Delta_{\nabla f}R^2:=\Delta R^2+\langle\nabla f,\nabla R^2\rangle&=2|\nabla R|^2-2R(R+2|\mathrm{Ric}|^2)\\&\geq 2|\nabla R|^2-2(1+2\alpha^2R)\cdot R^2.\end{aligned}$$

By a similar calculation, $|\nabla R|^2$ satisfies

$$\Delta_{\nabla f}|\nabla R|^2\geq 2|\nabla^2R|^2-C(\alpha,\beta)(R+1)\cdot|\nabla R|^2,$$

where $C(\alpha,\beta)$ is a positive constant depending on α and β. In the spirit of Shi, one considers the function $F:=(R^2+a)|\nabla R|^2$ where a is universally proportional to $\sup_{B(p,r)}R^2$. Then, by the previous differential inequalities, we have

$$\Delta_{\nabla f}F\geq|\nabla R|^4-C(\alpha,\beta)(R+1)\cdot F.$$

Heuristically, $|\nabla R|^4$ is locally controlled by $(R+1)\cdot F$, i.e., by definition of F, $|\nabla R|^2$ is locally controlled by R^2. This can be done rigorously by studying the maximum of $\phi\cdot F$ where ϕ is a nice test function with compact support in $B(p,r)$. $\square$

Now, by lemma 2.11, by the asymptotic behaviour of the norm of the gradient (7), and by the estimate (17), one can conclude the claim is true. ☐

Finally, by claim 1 and (16), one has

$$\limsup_{t\to+\infty} |\operatorname{Ric}(t^{-2}g_{t^2/4}) - (n-2)t^{-2}g_{t^2/4}| = \limsup_{t\to+\infty} |t^2 \operatorname{Ric}(g)| < \eta,$$

since

$$\begin{aligned}\sum_i \det h_{t^2/4}(\cdot, E_i) &= \sum_i \det\left(\frac{2\operatorname{Ric}+g}{2|\nabla f|}\right)(\cdot, E_i)\\ &= (n-2)t^{-2}g_{t^2/4}(\cdot,\cdot) + o(1).\end{aligned}$$

☐

REMARK 2.12. *The constraint* $|\operatorname{Ric}(g)| \leq \alpha R_g$ *is satisfied for instance if* $\operatorname{Ric}(g) \geq 0$ *with* $\alpha = 1$. *Note that this constraint is essentially used to prove that radial curvatures decay faster-than-quadratic. If this assumption is removed, we still get a control by the Ricci curvature ratio on how far from being Einstein (with a non determined constant) the levels of the potential function are.*

We end by the following lemma which establishes some links between the volume growth of metric balls, sublevels and levels:

LEMMA 2.13. *Let* $(M^n, g, \nabla f)$ *be a complete EGS satisfying* $(A1)$.
Then for any $q \in M$,

$$\liminf_{r\to+\infty} \frac{\operatorname{Vol} B(q,r)}{r^n} = \liminf_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} \geq \limsup_{t\to+\infty} \frac{\operatorname{Vol} M_{t^2/4}}{nt^{n-1}} > 0. \tag{18}$$

Moreover, if we assume $(A2)$ *then,*

$$\lim_{r\to+\infty} \frac{\operatorname{Vol} B(q,r)}{r^n} = \lim_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} = \lim_{t\to+\infty} \frac{\operatorname{Vol} M_{t^2/4}}{nt^{n-1}} < +\infty. \tag{19}$$

Proof. Under $(A1)$, lemma 2.2 tells us that the potential function f is equivalent at infinity to $r_p^2/4$ for $p \in M$. So the first equality is clear. Next, for $1 << t_0 \leq t$, by lemma 2.5

$$\begin{aligned}\frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} &\geq \frac{1}{t^n}\int_{t_0^2/4}^{t^2/4} \frac{\operatorname{Vol} M_s}{\sqrt{s+nC}}ds\\ &\geq \frac{2^{n-1}}{t^n}\exp\left(\int_{t_0^2/4}^{t^2/4} a(s)ds\right)\frac{\operatorname{Vol} M_{t_0^2/4}}{t_0^{n-1}}\int_{t_0^2/4}^{t^2/4}\frac{s^{\frac{n-1}{2}}}{\sqrt{s+nC}}ds.\end{aligned}$$

So, letting $t \to +\infty$ in the previous inequality gives for any large t_0,

$$\liminf_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} \geq \exp\left(\int_{t_0^2/4}^{+\infty} a(s)ds\right)\frac{\operatorname{Vol} M_{t_0^2/4}}{nt_0^{n-1}}.$$

Hence inequality (18) by making $t_0 \to +\infty$.

In case of an upper Ricci bound, using lemma 2.6, we get

$$\limsup_{r\to+\infty} \frac{\operatorname{Vol} B(q,r)}{r^n} = \limsup_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} \leq \liminf_{t\to+\infty} \frac{\operatorname{Vol} M_{t^2/4}}{nt^{n-1}} < +\infty.$$

Therefore, all the limits exist and are equal. Moreover, they are positive and finite. ☐

3. Proof of the main Theorem 1.2.

3.1. Gromov-Hausdorff convergence. We still denote the 1-parameter group generated by the vector field $\nabla f/|\nabla f|^2$ by $(\phi_t)_t$.

PROPOSITION 3.1. *Let $(M^n, g, \nabla f)$ be a complete EGS satisfying $(A2)$. Then, for $1 << s \leq t$,*

$$e^{\left(\int_s^t a(u)du\right)} d_{s^{-1}g_s}(p,q) \leq d_{t^{-1}g_t}(\phi_{t-s}(p), \phi_{t-s}(q)) \leq e^{\left(\int_s^t b(u)du\right)} d_{s^{-1}g_s}(p,q),$$

for any $p, q \in M_s$, with $a, b \in L^1(+\infty)$.

Proof. Let v be a tangent vector to M_s. As in the proof of lemma 2.7, we define $V(t) := d\phi_{t-s}(v)$. Using $(A2)$, the same calculation shows that

$$\left(a(t) + \frac{1}{t}\right) g(V(t), V(t)) \leq \frac{d}{dt} g(V(t), V(t)) \leq \left(b(t) + \frac{1}{t}\right) g(V(t), V(t)),$$

where a and b are two functions in $L^1(+\infty)$. Integrating these differential inequalities, we get for any v tangent to M_s,

$$\begin{aligned} \exp\left(\int_s^t a(u)du\right) s^{-1} g_s(v,v) &\leq t^{-1} g_t(d\phi_{t-s}(v), d\phi_{t-s}(v)) \\ &\leq \exp\left(\int_s^t b(u)du\right) s^{-1} g_s(v,v). \end{aligned}$$

The desired inequalities follow easily by applying the previous inequality to a curve γ of M_s with $v := \dot\gamma$. □

As a direct consequence, we get the following important

COROLLARY 3.2. *Let $(M^n, g, \nabla f)$ be a complete EGS satisfying $(A2)$. Then, for $1 << s \leq t$, $\phi_{t-s} : (M_s, s^{-1}g_s) \to (M_t, t^{-1}g_t)$ is an $o(1)$-Gromov-Hausdorff approximation, as s tends to $+\infty$.*

Proof. $\phi_{t-s} : M_s \to M_t$ is a diffeomorphism by construction. Moreover, by proposition 3.1 and lemma 2.7, one has

$$|d_{t^{-1}g_t}(\phi_{t-s}(p), \phi_{t-s}(q)) - d_{s^{-1}g_s}(p,q)| \leq c(t_0)\left(\exp\left(\int_s^{+\infty} a(u)du\right) - 1\right),$$

where $a \in L^1(+\infty)$ and, with the notations of lemma 2.7,

$$c(t_0) := e^{\left(\int_{t_0}^{+\infty} c(s)ds\right)} \frac{\mathrm{diam}(g_{t_0})}{\sqrt{t_0}},$$

for $s \geq t_0 >> 1$. □

REMARK 3.3. *If $(M_{t_k}, d_{t_k^{-1}g_{t_k}})_k$ Gromov-Hausdorff (sub)converges to a metric space (S_∞, d_{S_∞}), corollary 3.2 shows that the 1-parameter family $(M_t, d_{t^{-1}g_t})_t$ Gromov-Hausdorff converges to the same metric space (S_∞, d_{S_∞}).*

Now, we are in a position to begin the proof of Theorem 1.2.

Let $(M^n, g, \nabla f)$ be a complete expanding gradient Ricci soliton with finite asymptotic curvature ratio. Let $p \in M^n$ and $(t_k)_k$ any sequence tending to $+\infty$.

Claim 2. *The sequence $(M^n, t_k^{-2}g, p)_k$ of pointed Riemannian manifolds contains a convergent subsequence in the pointed Gromov-Hausdorff sense.*

According to Gromov's precompactness theorem (see Chap.10 of [Pet06] for instance), the claim follows from the following theorem, which is essentially due to Gilles Carron [Che12].

Theorem 3.4. *Let $(M^n, g, \nabla f)$ be a complete EGS with finite asymptotic curvature ratio. Then there exist positive constants c, c' such that for any $x \in M^n$ and any radius $r > 0$,*

$$cr^n \leq \operatorname{Vol} B(x,r) \leq c'r^n.$$

The arguments are essentially the same as in [Che12]. We give the proof for completeness.

Proof. Along the proof, c, c' will denote constants independent of t which can vary from line to line.

Step1. We begin by the

Lemma 3.5. *There exists $R_0 > 0$ and $c > 0$ such that for $r_p(x) \geq R_0$,*

$$c(r_p(x)/2)^n \leq \operatorname{Vol} B(x, r_p(x)/2).$$

Proof. Indeed, by lemmas 2.5, 2.7 and 2.8, we know that there exist a positive constant i_0 such that $\operatorname{inj}(M_{t^2/4}, t^{-2}g_{t^2/4}) \geq i_0$ for $t \geq t_0 >> 1$. Therefore, for $t \geq t_0 >> 1$ and $x \in M_{t^2/4}$,

$$\operatorname{Vol}_{g_{t^2/4}} B_{g_{t^2/4}}(x, i_0 t/2) \geq ct^{n-1}, \tag{20}$$

for some positive constant c.

Claim 3.

$$\phi_v(B_{g_{t^2/4}}(x, i_0t/2)) \subset B_g(x, r_p(x)/2),$$

for $v \in [0, \alpha t^2/4]$, where α is a positive constant independent of t.

Proof of Claim 3. Let $y \in B_{g_{t^2/4}}(x, i_0t/2)$. By the triangular inequality,

$$d_g(x, \phi_v(y)) \leq d_g(x,y) + d_g(y, \phi_v(y)) \leq i_0t/2 + d_g(y, \phi_v(y)).$$

Thus, it suffices to control the growth of the function $\psi(v) := d_g(y, \phi_v(y))$ for $v \geq 0$. Now, for $t \geq t_0 >> 1$,

$$\begin{aligned}\psi(v) &\leq \int_0^v |\psi'(s)|ds \leq \int_0^v \frac{ds}{|\nabla f|(\phi_s(y))} \\ &= \int_0^v \frac{ds}{\sqrt{s + t^2/4 - R(\phi_s(y))}} \\ &\leq \int_0^v \frac{ds}{\sqrt{s}} = 2\sqrt{v}.\end{aligned}$$

Therefore,

$$d_g(x, \phi_v(y)) \leq i_0 t/2 + 2\sqrt{v} \leq (i_0/2 + \sqrt{\alpha})t,$$

for $v \in [0, \alpha t^2/4]$ and any $\alpha > 0$. The claim now follows by using the growth of the potential function given by lemma 2.2 and choosing a suitable α sufficiently small. □

By Claim 3 and the coarea formula, we have

$$\operatorname{Vol} B_g(x, r_p(x)/2) \geq \operatorname{Vol}\{(\phi_v(B_{g_{t^2/4}}(x, i_0 t/2)); v \in [0, \alpha t^2/4])\} \tag{21}$$

$$= \int_0^{\alpha t^2/4} \int_{B_{g_{t^2/4}}(x, i_0 t/2)} \frac{|\operatorname{Jac}(\phi_v)|}{|\nabla f|} dA_{t^2/4} dv. \tag{22}$$

Now, for $t \geq t_0 >> 1$ and $y \in M_{t^2/4}$, the maps $s \to \phi_s(y)$ for $s \geq 0$ are expanding since, as in the proof of lemma 2.7, for $v \in TM_{t^2/4}$,

$$\frac{d}{ds} g(d\phi_s(v), d\phi_s(v)) = 2\frac{\operatorname{Ric}(d\phi_s(v), d\phi_s(v)) + g(d\phi_s(v), d\phi_s(v))/2}{|\nabla f|^2} \geq 0.$$

Combining this fact with inequalities (20) and (22), we get

$$\begin{aligned}\operatorname{Vol} B_g(x, r_p(x)/2) &\geq \frac{\alpha t^2/4}{\max_{M_{t^2/4}} |\nabla f|} \operatorname{Vol}_{g_{t^2/4}} B_{g_{t^2/4}}(x, i_0 t/2)\\ &\geq c t^n.\end{aligned}$$

This ends the proof of lemma 3.5. □

Step 2. For $r_p(x) \geq R_0$, we know that $\operatorname{Ric} \geq -C^2/r_p(x)^2$ on the ball $B(x, r_p(x)/2)$ for C independent of x since $\mathrm{A}(g) < +\infty$. Therefore, by Bishop-Gromov theorem, for $r \leq r_p(x)/2$,

$$\operatorname{Vol} B(x, r) \geq \frac{\operatorname{Vol}(n, -(C/r_p(x))^2, r)}{\operatorname{Vol}(n, -(C/r_p(x))^2, r_p(x)/2)} \operatorname{Vol} B(x, r_p(x)/2),$$

where $\operatorname{Vol}(n, -k^2, r)$ denotes the volume of a ball of radius r in the n-dimensional hyperbolic space of constant curvature $-k^2$.

Now,

$$\begin{aligned}\operatorname{Vol}(n, -k^2, r) &= \operatorname{Vol}(\mathbb{S}^{n-1}) \int_0^r \left(\frac{\sinh(kt)}{k}\right)^{n-1} dt\\ &\geq \frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{n} r^n,\end{aligned}$$

and

$$\operatorname{Vol}(n, -(C/r_p(x))^2, r_p(x)/2) = \left(\frac{\operatorname{Vol}(\mathbb{S}^{n-1})}{C^n} \int_0^{C/2} \sinh(u)^{n-1} du\right) r_p(x)^n.$$

To sum it up, we get by lemma 3.5,

$$\operatorname{Vol} B(x, r) \geq c r^n, \quad \text{for } r \leq r_p(x)/2 \text{ and } r_p(x) \geq R_0.$$

Next, inside the compact ball $B(p, R_0)$, we will get a similar lower bound for $r \leq R_0$ because of the continuity of the function $(x, r) \to \operatorname{Vol} B(x, r)/r^n$.

Now, for any $r > 0$, choose $x \in M^n$ such that $r_p(x) = r/2$. Then $B(x, r/4) \subset B(p, r)$ and the lower bound follows for any ball centered at p.

Finally, for any $x \in M^n$ and radius r satisfying $r \geq 2r_p(x)$, $B(p, r/2) \subset B(x, r)$. Hence the lower bound for any balls $B(x, r)$ for $r \leq r_p(x)/2$ and $r \geq 2r_p(x)$. Since for $r \in [r_p(x)/2, 2r_p(x)]$, $\operatorname{Vol} B(x, r) \geq \operatorname{Vol} B(x, r_p(x)/2) \geq c(r_p(x)/2)^n \geq c(r/4)^n$, the proof of the lower bound is finished.

Step 3. To get an upper bound, once again, by the Bishop theorem,

$$\operatorname{Vol} B(x, r) \leq \operatorname{Vol}(n, -(C/r_p(x))^2, r) \leq c' r^n, \quad \text{for } r \leq r_p(x)/2,$$

where $c' := \operatorname{Vol}(\mathbb{S}^{n-1}) \max_{u \in [0, C/2]} u^{-n} \int_0^u (\sinh(s))^{n-1} ds$.

For $r \geq r_p(x)/2$, $B(x, r) \subset B(p, 3r)$ and $\operatorname{Vol} B(p, r) \leq c' r^n$ for r large enough (say $3R_0/2$) by lemma 2.13. So, $\operatorname{Vol} B(x, r) \leq \operatorname{Vol} B(p, 3r) \leq c' 3^n r^n$ for $r_p(x) \geq R_0$ and $r \geq r_p(x)/2$.

Invoking again the continuity of the volume ratio on $B(p, R) \times [0, R_0]$, we end the proof of Theorem 3.4. □

3.2. $C^{1,\alpha}$ **convergence.** We finish the proof of Theorem 1.2.

Step 1: $(f^{-1}(t^2/4), t^{-2} g_{t^2/4})_t$ converges in the $C^{1,\alpha}$-topology to a compact smooth manifold S_∞ with a $C^{1,\alpha}$ metric g_∞.

Indeed, according to the lemmas 2.5, 2.7 and 2.8, we are in a position to apply the $C^{1,\alpha}$-compactness theorem [GW88], [Pet87], [Kas89], to a sequence $(M_{t_k^2/4}, t_k^{-2} g_{t_k^2/4})_k$ where $(t_k)_k$ is a sequence tending to $+\infty$. Thanks to corollary 3.2, this shows the second part of Theorem 1.2 : the limit does not depend on the sequence $(t_k)_k$. Moreover, by inequality (13) of lemma 2.8, we immediately get the estimate (2). Equally, by equality (19) of lemma 2.13, equality (3) follows.

Step 2: For $0 < a < b$, consider the annuli $(M_{at^2/4 \leq s \leq bt^2/4}, t^{-2} g) =: (M_{a,b}(t), t^{-2} g)$ for positive t . Because of the finiteness of $\mathrm{A}(g)$, it follows that

$$\lim_{t \to +\infty} \sup_{M_{a,b}(t)} |\operatorname{Rm}(t^{-2} g)| \leq \mathrm{A}(g).$$

Moreover, by a local version of Cheeger's injectivity radius estimate [CGT], lemma 3.5 and by the finiteness of $\mathrm{A}(g)$, there exists a positive constant ι_0 such that for any $x \in M^n$,

$$\operatorname{inj}(x, g) \geq \iota_0 r_p(x).$$

Now, consider the sequence of pointed complete Riemannian manifolds $(M^n, t_k^{-2} g, p)_k$. By Theorem 3.4, $(M^n, t_k^{-2} g, p)$ Gromov-Hausdorff subconverges to a metric space $(X_\infty, d_\infty, x_\infty)$. By a local form of the $C^{1,\alpha}$-compactness theorem [BKN89] and by the previous annuli estimates, one can deduce that $X_\infty/\{x_\infty\}$ is a smooth manifold with a $C^{1,\alpha}$ metric compatible with d_∞ and that the convergence is $C^{1,\alpha}$ outside the apex x_∞. Moreover, according to the previous step, this limit is the metric cone over (S_∞, g_∞), in particular, this shows the uniqueness of the asymptotic cone. □

Finally, we prove corollary 1.3.

Proof of Corollary 1.3. It is not straightforward since the convergence is only $C^{1,\alpha}$. Still, one can apply the results of the proof of Theorem 78 of the book [Pet06]. We sum up the major steps. On the one hand, one shows that the limit metric g_{S_∞} is weakly Einstein, hence smooth by elliptic regularity. On the other hand, one sees, in polar coordinates, that the metrics $t^{-2}g_{t^2/4}$ $C^{1,\alpha}$-converges to a constant curvature metric. These facts with Theorem 1.2 suffice to prove corollary 1.3. □

4. Volume monotonicity and geometric inequalities.

4.1. Volume monotonicity. We begin by stating volume monotonicity results:

Combining the lemma 2.13 with the monotonicity results of lemma 2.4 and 2.5, we get the

Corollary 4.1. *Let $(M^n, g, \nabla f)$ be a complete EGS.*
1) Assume $(A2)$ and $R \geq 0$.

Then, $t \to \operatorname{Vol} M_{\leq t^2/4}/t^n$ is nondecreasing and

$$0 < \operatorname{AVR}(g) := \lim_{r\to+\infty} \frac{\operatorname{Vol} B(p,r)}{r^n} = \lim_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} < +\infty. \tag{23}$$

2) Only assume $\operatorname{Ric} \geq 0$ *then*

$$0 < \lim_{t\to+\infty} \frac{\operatorname{Vol} M_{t^2/4}}{n t^{n-1}} \leq \operatorname{AVR}(g) = \lim_{t\to+\infty} \frac{\operatorname{Vol} M_{\leq t^2/4}}{t^n} \tag{24}$$

with equality if, for instance, the scalar curvature is bounded from above.

The following corollary was already known in a more general context by [BKN89].

Corollary 4.2. *Let $(M^n, g, \nabla f)$ be a complete EGS. Assume*

$$\operatorname{Ric} \geq 0 \quad \textit{and} \quad \mathrm{A}(g) = 0.$$

Then $(M^n, g, \nabla f)$ is isometric to the Gaussian expanding soliton.

Proof. As in [CN10] and in the proof of lemma 2.2, in case of $\operatorname{Ric} \geq 0$, f is a proper strictly convex function, hence M^n is diffeomorphic to $\mathbb{R}^n$. Therefore, by corollary 1.3, corollary 4.1,

$$\omega_n = \lim_{r\to+\infty} \frac{\operatorname{Vol} B(p,r)}{r^n} = \operatorname{AVR}(g).$$

The result now follows by the rigidity part of Bishop-Gromov theorem. □

4.2. Geometric Inequalities. Here, we link $\mathrm{A}(g)$ and $\operatorname{AVR}(g)$ in a global inequality. An easy way is to use the Gauss-Bonnet theorem (see [Ber03]) which is only valid for a global odd dimension.

Proposition 4.3. *Let $(M^n, g, \nabla f)$ be a complete EGS with n odd. Assume*

$$\operatorname{Ric} \geq 0 \quad \textit{and} \quad \mathrm{A}(g) < +\infty. \tag{25}$$

Then

$$\frac{\omega_n}{(1+\mathrm{A}(g))^{\frac{n-1}{2}}} \leq \operatorname{AVR}(g). \tag{26}$$

Proof. As we have already seen, $M_{t^2/4}$ is diffeomorphic to a $(n-1)$-sphere for $t > \min_{M^n}$, since $\mathrm{Ric} \geq 0$. Therefore, apply the Gauss-Bonnet formula to the $(n-1)$-sphere $M_{t^2/4}$:

$$2 = \chi(M_{t^2/4}) = \frac{2}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{M_{t^2/4}} \mathbf{K},$$

where

$$\mathbf{K} = \frac{1}{(n-1)!} \sum_{i_1 < \ldots < i_{n-1}} \epsilon_{i_1,\ldots,i_{n-1}} \mathrm{Rm}_{i_1,i_2} \wedge \ldots \wedge \mathrm{Rm}_{i_{n-2},i_{n-1}},$$

and Rm is the curvature form of the metric $t^{-2}g_{t^2/4}$ and $\epsilon_{i_1,\ldots i_{n-1}}$ is the signature of the permutation $(i_1, \ldots, i_{n-1})$.

As $\mathrm{Vol}(\mathbb{S}^{n-1}) = n\omega_n$ and making $t_k \to +\infty$ where $(t_k)_k$ is as in Theorem 1.2, one has by Theorem 1.2 and corollary 4.1,

$$n\omega_n \leq (1 + \mathrm{A}(g))^{\frac{n-1}{2}} \mathrm{Vol}(S_\infty, g_{S_\infty}) \leq (1 + \mathrm{A}(g))^{\frac{n-1}{2}} (n \, \mathrm{AVR}(g)).$$

□

In case n is not necessarily odd, we still get such an inequality for a small asymptotic curvature ratio:

Proof of Proposition 1.7. Along the proof, we will assume that M^n has only one end. In case of more than one end, the following arguments can be applied to each end and the proposition is established by summing over the ends.

Therefore, consider the connected compact hypersurfaces $(M_{t^2/4}, t^{-2}g_{t^2/4})$ for t large enough. By lemma 2.8,

$$1 - \mathrm{A}(g) \leq \liminf_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq \limsup_{t\to+\infty} K_{t^{-2}g_{t^2/4}} \leq 1 + \mathrm{A}(g).$$

If $\mathrm{A}(g)$ is less than 1, then, by Myers' theorem, $\Gamma = \pi_1(M_{t^2/4})$ is finite and by the Bishop theorem, we get the second inequality.

If we consider the Riemannian finite universal coverings $(\widetilde{M_{t^2/4}}, \widetilde{t^{-2}g_{t^2/4}})$ of these hypersurfaces, the previous curvature inequalities will be preserved and if $\mathrm{A}(g)$ is small enough (less than 3/5) then

$$\frac{1}{4} < \frac{1 - \mathrm{A}(g)}{1 + \mathrm{A}(g)}.$$

Therefore, by Klingenberg's result, [BS09] for a survey on sphere theorems, the injectivity radius of $(\widetilde{M_{t^2/4}}, \widetilde{t^{-2}g_{t^2/4}})$ will be asymptotically greater than $\pi/\sqrt{1 + \mathrm{A}(g)}$. Thus,

$$\begin{aligned}
\frac{n\omega_n}{(1+\mathrm{A}(g))^{\frac{n-1}{2}}} = \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{(1+\mathrm{A}(g))^{\frac{n-1}{2}}} &\leq \lim_{t\to+\infty} \mathrm{Vol}(\widetilde{M_{t^2/4}}, \widetilde{t^{-2}g_{t^2/4}}) \\
&= |\Gamma| \lim_{t\to+\infty} \mathrm{Vol}(M_{t^2/4}, t^{-2}g_{t^2/4}) \\
&\leq |\Gamma| n \, \mathrm{AVR}(g).
\end{aligned}$$

□

As a direct consequence, we get

Corollary 4.4. *Let* $(M^n, g, \nabla f)$ *be a complete EGS with* $n \geq 3$. *Assume* $\mathrm{A}(g) = 0$.

Then, with the notations of proposition 1.7,

$$\mathrm{AVR}(g) = \sum_{i \in I} \frac{\omega_n}{|\Gamma_i|}.$$

Of course, this corollary is weaker than corollary 1.3 but it can be proved directly as above.

Remark 4.5. *Note that when* n *is odd,* $|\Gamma| = 1$, *since the hypersurfaces are orientable. Moreover, in this case, one can assume only* $\mathrm{A}(g) < 1$ *to get the same result by using a "light" version of the Klingenberg's theorem (which is also due to him) which asserts that "any orientable compact even-dimensional Riemannian manifold* (N, h) *with sectional curvature in* $(0, 1]$ *has* $\mathrm{inj}(N, h) \geq \pi$*".*

Remark 4.6. *The assumption* $n \geq 3$ *is sharp in the following sense: there exists a complete two-dimensional expanding gradient soliton with nonnegative scalar curvature, asymptotically flat (i.e.* $\mathrm{A}(g) = 0$*) such that* $\mathrm{AVR}(g) < \omega_2$, *see section* 5, *chap.*4 *of [CL04].*

Remark 4.7. *Note that these inequalities do not depend on the geometry of* f. *Thus, are these inequalities more universal ? For instance, do they hold for Riemannian manifold with nonnegative Ricci curvature, positive* AVR, *and finite A? This will be the subject of forthcoming papers.*

At this stage, one can ask if there are some rigidity results concerning the asymptotic curvature ratio $\mathrm{A}(g)$ of a nonnegatively curved expanding gradient Ricci soliton. In fact, Huai-dong Cao has built a 1-parameter family of expanding gradient Ricci solitons with nonnegative sectional curvature: see [Cao97]. These examples are rotationnally symmetric, they behave at infinity like metric cones and their asymptotic curvature ratios take any values in $(0, +\infty)$.

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