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COHOMOLOGY OF DIGRAPHS AND (UNDIRECTED) GRAPHS*

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Abstract. We construct a cohomology theory on a category of finite digraphs (directed graphs), which is based on the universal calculus on the algebra of functions on the vertices of the digraph. We develop necessary algebraic technique and apply it for investigation of functorial properties of this theory. We introduce categories of digraphs and (undirected) graphs, and using natural isomorphism between the introduced category of graphs and the full subcategory of symmetric digraphs we transfer our cohomology theory to the category of graphs. Then we prove homotopy invariance of the introduced cohomology theory for undirected graphs. Thus we answer the question of Babson, Barcelo, Longueville, and Laubenbacher about existence of homotopy invariant homology theory for graphs. We establish connections with cohomology of simplicial complexes that arise naturally for some special classes of digraphs. For example, the cohomologies of posets coincide with the cohomologies of a simplicial complex associated with the poset. However, in general the digraph cohomology theory can not be reduced to simplicial cohomology. We describe the behavior of digraph cohomology groups for several topological constructions on the digraph level and prove that any given finite sequence of non-negative integers can be realized as the sequence of ranks of digraph cohomology groups. We present also sufficiently many examples that illustrate the theory.

Key words. (co)homology of digraphs, (co)homology of graphs, differential graded algebras, path complex of a digraph, simplicial homology, differential calculi on algebras.

AMS subject classifications. 05C25, 05C38, 16E45, 18G35, 18G60, 55N35, 55U10, 57M15.

1. Introduction. In this paper we consider finite simple digraphs (directed graphs) and (undirected) simple graphs. A simple digraph G is couple (V, E) where V is any set and $E \subset \{V \times V \setminus \text{ diag}\}$. Elements of V are called the vertices and the elements of E – directed edges. Sometimes, to avoid misunderstanding, we shall use the extended notations V_G and E_G instead of V and E, respectively. The fact that $(a, b) \in E$ will be denoted by $a \to b$. A (undirected) graph G is a pair (V, E) (or more precise (V_G, E_G)) where V is a set of vertices and E is a set of unordered pairs (v, w) of vertices. The elements of E are called edges. In this paper we shall consider only simple graphs, which have no edges (v, v) (loops).

A digraph is a particular case of a quiver. A particular example of a digraph is a poset (partially ordered set) when E is just a partial order (that is, $a \to b$ if and only if $a \geq b$). The interest to construction of some type of algebraic topology on the digraphs and graphs is motivated by physical applications of this subject (see, for example, [6], [7], [8]), discrete mathematics [24], [18], [4], and graph theory [1], [2], [18], and [19, Part III].

Dimakis and Müller-Hoissen suggested [7] and [8] a certain approach to construction of cohomologies on digraphs, which is based on the notion of a differential

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calculus on an abstract associative unital algebra $\mathcal{A}$ over a commutative unital ring $\mathbb{K}$. However, this approach remained on intuitive level without a precise definition of the corresponding cochain complex. An explicit and direct construction of chain and cochain complexes on arbitrary finite digraphs was given in [13] (see also [15]). The construction in [13] of n-chains is based on the naturally defined notion of a path of length n on a digraph.

In the present paper we provide an alternative construction of the cochain complex that is equivalent to that of [13] (see Section 4 below). We stay here on the algebraic point of view of [3], [6] to define a functor from the category of digraphs to the category of cochain complexes. We develop necessary algebraic background for this approach, which makes most of the constructions functorial and enables one to use methods of homological algebra [17], [22]. The main construction is based on the universal calculus on the algebra of functions on the vertices of the digraph.

This constructed cohomology theory happens to be closely related to other cohomology theories but is not covered by them. For example, the cohomology groups of a poset coincide with the simplicial cohomology groups of a simplicial complex associated with the poset and with the Hochschild cohomology of corresponding incidence algebra (see [5], [12], and [14]). We would like also to point out, that the digraph cohomology theory gives new geometric connections between the digraphs and cubic lattices of topological spaces (see, [9], [10], and [15]) and new algebraic connections with algebras of quiver and incidence algebras (see [11], [21], [5], [14]).

We introduce categories of digraphs and (undirected) graphs. Using natural isomorphism between introduced category of graphs and full subcategory of symmetric digraphs (see [16, Section 1.1]), we transfer the cohomology theory to the category of graphs. In the papers [1] and [2] the homotopy theory of graphs was constructed, and the question about natural homotopy invariant homology theory of graphs was raised. We prove homotopy invariance of introduced cohomology theory for graphs and give several examples of computations. Note, that the previously known homology theory of digraphs (see, for example, [18, Section 3]) is not homotopy invariant.

We prove functoriality of the cohomology groups for natural maps of digraphs. In particular, for a subcategory of digraphs with the inclusion maps we obtain direct description of relative cohomology groups. We describe behavior of introduced cohomology groups for several transformations of digraphs that are similar to standard topological constructions.

We describe relations between the digraph cohomology and the simplicial cohomology of various simplicial complexes which arise naturally for some special classes of digraphs. Finally, we prove the following cohomology realization theorem:

> *for any finite collection of nonnegative integers $k_0, k_1, \ldots k_n$ with $k_0 \geq 1$, there exists a finite digraph G (that is not a poset) such that the cohomology groups of its differential calculus satisfies the conditions*

$$\dim H^i(\Omega_G) = k_i \quad \text{for all } 0 \leq i \leq n.$$

The paper is organized in the following way. In Section 2 we give a short survey of the classical results on abstract differential calculi on associative algebras [3] in the form that is adapted to further application to digraphs. We provide several technical theorems which are based on the standard algebraic results (see [3], [20], and [22]) which will be helpful in the next sections.

In Section 3, we define the differential calculus on the algebra of functions on a finite set following [7] and [8] and describe its basic properties.

In Section 4, we define the calculus on simple finite digraphs. We use the algebraic machinery developed in previous sections and prove that we have a functor from the category of digraphs to the category of differential calculi with morphisms of the calculi. We describe some cohomology properties of these calculi and prove among others the cited above cohomology realization theorem.

In Section 5, we construct a cohomology theory on the category of undirected graphs that is identified naturally with the full subcategory of symmetric digraphs [16] and prove the homotopy invariance of obtained cohomology theory. Note that our homology theory of graphs is new and its construction realizes the desire of Babson, Barcelo, Longueville, and Laubenbacher " for a homology theory associated to the A-theory of a graph" (see [1, page 32]).

In Section 6 we consider a category of acyclic digraphs and transfer to this case the results of previous sections. We describe a sufficiently wide class of acyclic digraphs for which the cohomology theory admits a geometrical realization in terms of simplicial complexes.

2. Differential calculus on algebras. In this section we give a short survey of classical results on abstract differential calculi on associative algebras in the form that is adapted to further application to digraphs. Starting with a standard construction of a first order calculus from [3], we give two methods for construction of higher order universal differential calculi and prove their equivalence. We provide several technical theorems which are based on the classical algebraic results (see [3], [20], and [22]) which will be helpful in the next sections.

Let $\mathbb{K}$ be a commutative unital ring and $\mathcal{A}$ be an associative unital algebra over $\mathbb{K}$.

DEFINITION 2.1. A *first order differential calculus* on the algebra $\mathcal{A}$ is a pair (Γ, d) where Γ is an $\mathcal{A}$-bimodule, and $d\colon \mathcal{A} \to \Gamma$ is a $\mathbb{K}$-linear map such that

- (i) $d(ab) = (da) \cdot b + a \cdot (db)$ for all $a, b \in \mathcal{A}$ (where $\cdot$ denotes multiplication between the elements of $\mathcal{A}$ and Γ).
- (ii) The minimal left $\mathcal{A}$-module containing $d\mathcal{A}$, coincides with Γ, that is, any element $\gamma \in \Gamma$ can be written in the form

$$\gamma = \sum_i a_i \cdot db_i \tag{2.1}$$

with $a_i, b_i \in \mathcal{A}$, where i run over any finite set of indexes.

By [3, III, §10.2], a mapping d satisfying (i) is called *a derivation of $\mathcal{A}$ into* Γ. The condition (i) implies

$$d1_{\mathcal{A}} = d(1_{\mathcal{A}}1_{\mathcal{A}}) = (d1_{\mathcal{A}})\, 1_{\mathcal{A}} + 1_{\mathcal{A}}\, (d1_{\mathcal{A}}) = 2d1_{\mathcal{A}}$$

and hence $d1_{\mathcal{A}} = 0$. The $\mathbb{K}$-linearity implies then that $d(k1_{\mathcal{A}}) = 0$ for any $k \in \mathbb{K}$.

Let us describe a construction of the first order differential calculus for a general algebra $\mathcal{A}$. The algebra $\mathcal{A}$ can be regarded as a $\mathbb{K}$-module, and the tensor product $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ is also defined as a $\mathbb{K}$-module. In what follows we will always denote $\otimes_{\mathbb{K}}$ simply by $\otimes$.

Note that $\mathcal{A}$ and $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ have also natural structures of $\mathcal{A}$-bimodules. We will denote by $\cdot$ the product of the elements of $\mathcal{A}$ by those of $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$. For all $a, b, c \in \mathcal{A}$, we have

$$c \cdot (a \otimes b) = (ca) \otimes b \quad \text{and} \quad (a \otimes b) \cdot c = a \otimes (bc) \tag{2.2}$$

Define the following operator

$$d\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \qquad da := 1_{\mathcal{A}} \otimes a - a \otimes 1_{\mathcal{A}}, \tag{2.3}$$

and observe that it satisfies the product rule. Hence, d is a derivation from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$. Now we reduce the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathcal{A}$ to obtain a first order differential calculus.

Definition 2.2. Define $\Omega^1_{\mathcal{A}}$ as the minimal left $\mathcal{A}$-submodule of $\mathcal{A} \otimes \mathcal{A}$ containing $d\mathcal{A}$. In other words, $\Omega^1_{\mathcal{A}}$ consists of all finite sums of the elements of $\mathcal{A} \otimes \mathcal{A}$ of the form $a \cdot db$ with $a, b \in \mathcal{A}$ (cf. (2.1)

Proposition 2.3. *$\Omega^1_{\mathcal{A}}$ is a $\mathcal{A}$-bimodule and, hence, $\left(\Omega^1_{\mathcal{A}}, d\right)$ is a first order differential calculus on $\mathcal{A}$.*

Proof. Let $u \in \Omega^1_{\mathcal{A}}$ and $c \in \mathcal{A}$. We need to prove that $c \cdot u$ and $u \cdot c$ belong to $\Omega^1_{\mathcal{A}}$. By definition of $\Omega^1_{\mathcal{A}}$, it suffices to verify this for $u = a \cdot db$ where $a, b \in \mathcal{A}$. Then $c \cdot u = (ca) \cdot db \in \Omega^1_{\mathcal{A}}$ and

$$u \cdot c = (a \cdot db) \cdot c = a \cdot (db \cdot c) = a \cdot (d(bc) - b \cdot dc) = a \cdot d(bc) - (ab) \cdot dc \in \Omega^1_{\mathcal{A}}.$$

Hence, $\Omega^1_{\mathcal{A}}$ satisfies all the requirements of Definition 2.1. □

Let us give an alternative equivalent description of $\Omega^1_{\mathcal{A}}$. Define a $\mathbb{K}$-linear map

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad \mu\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i b_i \tag{2.4}$$

where i runs over a finite index set. By (2.2) the map μ is a homomorphism of $\mathcal{A}$-bimodules. It follows from (2.2), (2.3) and (2.4) that, for all $a, b \in \mathcal{A}$,

$$\mu(a \cdot db) = \mu(a \otimes b) - \mu(ab \otimes 1_{\mathcal{A}}) = ab - ab = 0,$$

so that $a \cdot db \in \ker \mu$ and, hence, $\Omega^1_{\mathcal{A}} \subset \ker \mu$. In fact, the following is true.

Theorem 2.4. *[3, III, §10.10]*

(i) We have the identity $\Omega^1_{\mathcal{A}} = \ker \mu$, where μ is defined by (2.4).

(ii) For every differential calculus of first order (Γ, d') over the algebra $\mathcal{A}$ there exists exactly one epimorphism p of $\mathcal{A}$-bimodules $p\colon \Omega^1_{\mathcal{A}} \to \Gamma$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d} & \Omega^1_{\mathcal{A}} \\ \updownarrow^{\mathrm{id}} & & \downarrow^{p} \\ \mathcal{A} & \xrightarrow{d'} & \Gamma. \end{array} \tag{2.5}$$

Definition 2.5. The pair $\left(\Omega^1_{\mathcal{A}}, d\right)$ is called the *universal* first order differential calculus on $\mathcal{A}$.

EXAMPLE 2.6. Consider the $\mathbb{R}$-algebra $\mathcal{A} = C^m(\mathbb{R})$ and the bimodule $\Gamma = C^{m-1}(\mathbb{R})$ with the usual derivative of functions f from $\mathcal{A}$ that will be denoted by $d'f$. Let us describe explicitly the epimorphism $p : \Omega^1_{\mathcal{A}} \to \Gamma$ from Theorem 2.4(*ii*). Define a mapping $p : \mathcal{A} \otimes \mathcal{A} \to \Gamma$ by

$$p(f \otimes g) = \frac{1}{2}(fg' - f'g)$$

and extend it additively to all elements of $\mathcal{A} \otimes \mathcal{A}$. It is easy exercise to prove that $p|_{\Omega^1_{\mathcal{A}}}$ is a $\mathcal{A}$-bimodule epimorphism, using that $f \otimes g - g \otimes f \in \Omega^1_{\mathcal{A}}$ and

$$p(f \otimes g - g \otimes f) = \frac{1}{2}(fg' - f'g) - \frac{1}{2}(gf' - g'f) = (fg)'. \tag{2.6}$$

Finally, for any $f \in \mathcal{A}$ by (2.6) we have $(p \circ d)f = p(1 \otimes f - f \otimes 1) = f'$ so that $p \circ d$ is the ordinary first order derivative on $\mathcal{A}$.

Let us pass to construction of a higher order differential calculus on $\mathcal{A}$. We start with the following two definitions.

DEFINITION 2.7. A graded unital algebra Λ over a commutative unital ring $\mathbb{K}$ is an associative unital $\mathbb{K}$-algebra that can be written as a direct sum

$$\Lambda = \bigoplus_{p=0,1,\ldots} \Lambda^p$$

of $\mathbb{K}$-modules Λ^p with the following conditions: the unity 1_Λ of Λ belongs to Λ^0 and

$$u \in \Lambda^p, \quad v \in \Lambda^q \quad \Rightarrow \quad u * v \in \Lambda^{p+q},$$

where $*$ denotes multiplication in Λ. If $u \in \Lambda^p$ then p is called the *degree* of u and is denoted by $\deg u$. The operation of multiplication in a graded algebra is called an *exterior* (or a graded) multiplication. A homomorphism $f \colon \Lambda' \to \Lambda''$ of two graded unital $\mathbb{K}$-algebras Λ' and Λ'' is a homomorphism of $\mathbb{K}$-algebras that preserves degree of elements.

DEFINITION 2.8. A differential calculus on an associative unital $\mathbb{K}$-algebra $\mathcal{A}$ is a couple (Λ, d), where Λ is a graded algebra

$$\Lambda = \bigoplus_{p=0,1,\ldots} \Lambda^p$$

over $\mathbb{K}$ such that $\Lambda^0 = \mathcal{A}$, and $d : \Lambda \to \Lambda$ is a $\mathbb{K}$-linear map, such that

(*i*) $d\Lambda^p \subset \Lambda^{p+1}$

(*ii*) $d^2 = 0$

(*iii*) $d(u * v) = (du) * v + (-1)^p u * (dv)$, for all $u \in \Lambda^p$, $v \in \Lambda^q$, where $*$ is the exterior multiplication in Λ;

(*iv*) the minimal left $\mathcal{A}$-submodule of Λ^{p+1} containing $d\Lambda^p$ coincides with Λ^{p+1}, that is, any $w \in \Lambda^{p+1}$ can be represented as a finite sum of the form

$$w = \sum_k a_k * dv_k \tag{2.7}$$

for some $a_k \in \mathcal{A}$ and $v_k \in \Lambda^p$.

The property (iii) in Definition 2.8 is called the *Leibniz rule* or the *product rule*.

A classical example of a differential calculus is the calculus of exterior differential forms on a smooth manifold with the wedge product and with the exterior derivation. This calculus is based on the algebra $\mathcal{A}$ of smooth functions on the manifold.

The following property of a differential calculus will be frequently used.

LEMMA 2.9. *Let (Λ, d) be a differential calculus on $\mathcal{A}$. Then for any $p \geq 0$ any element $w \in \Lambda^p$ can be written as a finite sum*

$$w = \sum_j a_0^j * da_1^j * da_2^j * \cdots * da_p^j, \tag{2.8}$$

where $a_i^j \in \mathcal{A}$ for all $0 \leq i \leq p$ and $*$ is the exterior multiplication in Λ.

Proof. Representation (2.8) for $p = 0$ is true by $\Lambda^0 = \mathcal{A}$. Let us make an inductive step from $p-1$ to p. By part (iv) of Definition 2.8, it suffices to show the existence of the representation (2.8) for $w = a * dv$ with $a \in \mathcal{A}$ and $v \in \Lambda^{p-1}$. By the inductive hypothesis, v admits the representation in the form

$$v = \sum_j a_1^j * da_2^j * da_2^j * \cdots * da_p^j$$

where all $a_i^j \in \mathcal{A}$. Using the associative law, the Leibniz rule and $d^2 = 0$, we obtain

$$dv = \sum_j da_1^j * da_2^j * da_2^j * \cdots * da_p^j,$$

whence (2.8) follows with $a_0^i = a$. □

The first method of construction a differential calculus on $\mathcal{A}$ uses multiple tensor products $\otimes_{\mathbb{K}}$ of $\mathcal{A}$ by itself as in the following definition.

DEFINITION 2.10. Given an arbitrary associative unital $\mathbb{K}$-algebra $\mathcal{A}$, define a graded $\mathbb{K}$-algebra T as follows:

where

$$T = \bigoplus_{p=0,1,\dots} T^p, \quad \text{where} \quad T^p = \begin{cases} \mathcal{A}, & p = 0 \\ \underbrace{\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{p \text{ times } \otimes}, & p \geq 1, \end{cases} \tag{2.9}$$

and the exterior multiplication $T^p \bullet T^q \longrightarrow T^{p+q}$ is defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_p) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_j) := a_0 \otimes a_1 \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q, \tag{2.10}$$

for all $a_i, b_j \in \mathcal{A}$.

It is a trivial exercise to check that the multiplication $\bullet$ is well-defined and that T is indeed a graded associative unital $\mathbb{K}$-algebra with the unity $1_T = 1_{\mathcal{A}}$. The multiplication $\bullet$ by elements of $\mathcal{A} = T^0$ endows each $\mathbb{K}$-module T^p by a structure of $\mathcal{A}$-bimodule.

Note, that the original multiplication in the algebra $\mathcal{A}$ coincides with the exterior multiplication $T^0 \bullet T^0 \to T^0$, and the multiplication $\cdot$ of the elements of $\mathcal{A} = T_0$ and $\mathcal{A} \otimes \mathcal{A} = T_1$ defined in (2.2), coincides with exterior multiplication $T^0 \bullet T^1 \to T^1$.

Define a $\mathbb{K}$-linear map $d\colon T^p \to T^{p+1}$ ($p \geq 0$) by a formula

$$d(a_0 \otimes \cdots \otimes a_p) = \sum_{i=0}^{p+1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1_{\mathcal{A}} \otimes a_i \otimes \cdots \otimes a_p, \tag{2.11}$$

for all $a_i \in \mathcal{A}$. The next result can obtained by straightforward computation.

PROPOSITION 2.11. *For the operator* (2.11) *we have* $d^2 = 0$. *In particular,* d *determines the following cochain complex of* $\mathbb{K}$*-modules*

$$0 \longrightarrow T^0 \xrightarrow{d} T^1 \xrightarrow{d} T^2 \longrightarrow \dots$$

REMARK 2.12. The homomorphism $\varepsilon\colon \mathbb{K} \to \mathcal{A}$ defined by $\varepsilon(k) = k1_{\mathcal{A}}$ evidently satisfies the property $d \circ \varepsilon = 0$. Hence we can equip the cochain complex T^* by the augmentation ε. We shall denote this complex with the augmentation ε by $\widetilde{T}^{\varepsilon}$.

PROPOSITION 2.13. *The map* d *defined in* (2.11) *satisfies the following product rule:*

$$d(u \bullet v) = du \bullet v + (-1)^p u \bullet dv \tag{2.12}$$

for all $u \in T^p$ *and* $v \in T^q$.

Proof. It suffices to prove (2.12) for $u = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in T^p$ and $v = b_0 \otimes b_1 \otimes \cdots \otimes b_q \in T^q$. We have

$$\begin{aligned} d(u \bullet v) &= d(a_0 \otimes a_1 \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q) \\ &= \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes 1_{\mathcal{A}} \otimes a_j \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q \\ &+ (-1)^{p+1} a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p b_0 \otimes 1_{\mathcal{A}} \otimes b_1 \otimes \cdots \otimes b_q \\ &+ \sum_{i=2}^{q+1} (-1)^{p+i} a_0 \otimes \cdots \otimes \otimes a_{p-1} \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1_{\mathcal{A}} \otimes b_i \cdots \otimes b_q \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & du \bullet v + (-1)^p u \bullet dv \\ = & \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes 1_{\mathcal{A}} \otimes a_j \otimes \cdots \otimes a_p) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_q) \\ & + (-1)^{p+1} (a_0 \otimes \cdots \otimes a_p \otimes 1_{\mathcal{A}}) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_q) \qquad [\text{term with } j = p+1] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet (1_{\mathcal{A}} \otimes b_0 \otimes \cdots \otimes b_q) \qquad [\text{term with } i = 0] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet (-1) (b_0 \otimes 1_{\mathcal{A}} \otimes b_1 \otimes \cdots \otimes b_q) \qquad [\text{term with } i = 1] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet \sum_{i=2}^{q+1} (-1)^i b_0 \otimes \cdots \otimes b_{i-1} \otimes 1_{\mathcal{A}} \otimes b_i \otimes \cdots \otimes b_q \,. \end{aligned}$$

Noticing that the terms with $j = p+1$ and $i = 0$ cancel out, we obtain the required identity. □

Now we reduce the graded algebra T introduced above, to obtain a differential calculus in the sense of Definition 2.8.

Definition 2.14. Set $\Omega^0_{\mathcal{A}} = \mathcal{A} = T^0$. For all integers $p \geq 0$, define inductively $\Omega^{p+1}_{\mathcal{A}}$ as the minimal left $\mathcal{A}$-submodule of T^{p+1} containing $d\Omega^p_{\mathcal{A}}$, that is, $\Omega^{p+1}_{\mathcal{A}}$ consists of all the elements of the form (2.7) for some $a_k \in \mathcal{A}$ and $v_k \in \Omega^p_{\mathcal{A}}$.

Clearly, for $p = 1$ Definition 2.14 is consistent with previous Definition 2.2.

Theorem 2.15. *For all* $p, q \geq 0$

$$u \in \Omega^p_{\mathcal{A}},\ v \in \Omega^q_{\mathcal{A}} \ \Rightarrow\ u \bullet v \in \Omega^{p+q}_{\mathcal{A}}. \tag{2.13}$$

Consequently, the direct sum $\Omega_{\mathcal{A}} = \bigoplus_{p=0,1,\dots} \Omega^p_{\mathcal{A}}$, *with the multiplication* $\bullet$ *and with differential* d *is a differential calculus on* $\mathcal{A}$.

Applying (2.13) with $q = 0$, we obtain that $\Omega^p_{\mathcal{A}}$ is also a right $\mathcal{A}$-module, that is, $\Omega^p_{\mathcal{A}}$ is an $\mathcal{A}$-bimodule.

Proof. The proof is by induction on p. For $p = 0$ the statement is trivial, as by definition $\Omega^q_{\mathcal{A}}$ is a left $\mathcal{A}$-module. Let us make an inductive step from $p - 1$ to p. It suffices to prove that $u \bullet v \in \Omega^{p+q}_{\mathcal{A}}$ for $u = a \bullet db$ where $a \in \mathcal{A}$ and $b \in \Omega^{p-1}_{\mathcal{A}}$. We have by the associative law and by the Leibniz rule

$$\begin{aligned} u \bullet v &= (a \bullet db) \bullet v = a \bullet ((db) \bullet v) \\ &= a \bullet [d(b \bullet v) + (-1)^p\, b \bullet dv] \\ &= a \bullet d(b \bullet v) + (-1)^p\, (a \bullet b) \bullet dv. \end{aligned}$$

By the inductive hypothesis we have $b \bullet v \in \Omega^{p+q-1}_{\mathcal{A}}$ whence $d(b \bullet v) \in \Omega^{p+q}_{\mathcal{A}}$ and $a \bullet d(b \bullet v) \in \Omega^{p+q}_{\mathcal{A}}$. Also, we have $a \bullet b \in \Omega^{p-1}_{\mathcal{A}}$ and $dv \in \Omega^{q+1}_{\mathcal{A}}$, whence by the inductive hypothesis $(a \bullet b) \bullet dv \in \Omega^{p+q}_{\mathcal{A}}$. It follows that $u \bullet v \in \Omega^{p+q}_{\mathcal{A}}$.

Finally, $(\Omega_{\mathcal{A}}, d)$ satisfies all the conditions of Definition 2.8 by Propositions 2.11, 2.13, Definition 2.14 and by (2.13). Hence, $(\Omega_{\mathcal{A}}, d)$ is a differential calculus on $\mathcal{A}$. □

Now let us describe a different construction of the differential calculus on $\mathcal{A}$ that is based on the first order differential calculus $\Omega^1_{\mathcal{A}}$ from Definition 2.2. Define for each $p \geq 0$ a $\mathcal{A}$-bimodule $\widetilde{\Omega}^p_{\mathcal{A}}$ by

$$\widetilde{\Omega}^0_{\mathcal{A}} = \mathcal{A} \ \text{ and } \ \widetilde{\Omega}^p_{\mathcal{A}} = \underbrace{\Omega^1_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}}_{p \text{ factors}} \ \text{ for } p \geq 1. \tag{2.14}$$

In particular, $\widetilde{\Omega}^1_{\mathcal{A}} = \Omega^1_{\mathcal{A}}$. Clearly, each $\widetilde{\Omega}^p_{\mathcal{A}}$ is also a $\mathbb{K}$-module. Define the following multiplication $\star$ between the elements $u \in \widetilde{\Omega}^p_{\mathcal{A}}$ and $v \in \widetilde{\Omega}^q_{\mathcal{A}}$:

$$u \star v = \begin{cases} u \cdot v, & \text{if } p = 0 \text{ or } q = 0 \\ u \otimes_{\mathcal{A}} v, & \text{if } p, q \geq 1, \end{cases} \tag{2.15}$$

where $\cdot$ denotes the multiplication in $\widetilde{\Omega}^k_{\mathcal{A}}$ by the elements of $\mathcal{A}$ that comes from the $\mathcal{A}$-bimodule structure of $\widetilde{\Omega}^k_{\mathcal{A}}$. Clearly, multiplication $\star$ is associative, has a unity $1_{\mathcal{A}}$, and makes the direct sum

$$\widetilde{\Omega}_{\mathcal{A}} = \bigoplus_{p=0,1,\dots} \widetilde{\Omega}^p_{\mathcal{A}}$$

into a graded $\mathbb{K}$-algebra. It turns out that the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$ (cf. Definition 2.14) are isomorphic as is stated below.

THEOREM 2.16. *(i) There exists a unique isomorphism $f\colon \widetilde{\Omega}_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ of graded $\mathbb{K}$-algebras given by $\mathcal{A}$-bimodule isomorphisms*

$$f_p\colon \widetilde{\Omega}^p_{\mathcal{A}} \to \Omega^p_{\mathcal{A}}, \quad p \geq 0,$$

where $f_0\colon \mathcal{A} \to \mathcal{A}$ and $f_1\colon \Omega^1_{\mathcal{A}} \to \widetilde{\Omega}^1_{\mathcal{A}}$ are identical maps.

(ii) Define an operator $\widetilde{d}: \widetilde{\Omega}^p_{\mathcal{A}} \to \widetilde{\Omega}^{p+1}_{\mathcal{A}}$ to make the following diagram commutative:

$$\begin{array}{ccc} \widetilde{\Omega}^p_{\mathcal{A}} & \xrightarrow{\widetilde{d}} & \widetilde{\Omega}^{p+1}_{\mathcal{A}} \\ \downarrow^{f_p} & & \downarrow^{f_{p+1}} \\ \Omega^p_{\mathcal{A}} & \xrightarrow{d} & \Omega^{p+1}_{\mathcal{A}} \end{array} \tag{2.16}$$

Then $(\widetilde{\Omega}_{\mathcal{A}}, \widetilde{d})$ is a differential calculus that is isomorphic to $(\Omega_{\mathcal{A}}, d)$.

Clearly, the operators d and $\widetilde{d}$ on $\mathcal{A}$ are the same. As in the proof of Lemma 2.9 we obtain that any element of $\widetilde{\Omega}^p_{\mathcal{A}}$ can be represented as a finite sum of the terms $a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p$, and the following identity holds:

$$\widetilde{d}\left(a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p\right) = \widetilde{d}a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p.$$

Proof. We will use the following property of the tensor product: $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}$ where $\cong$ stands for a $\mathcal{A}$-bimodule isomorphism given by the mapping [3], [20], [22]

$$\varphi : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}, \quad \varphi(a) = a \otimes 1_{\mathcal{A}}. \tag{2.17}$$

In order to construct a necessary mapping f, define first a $\mathcal{A}$-bimodule $\widetilde{T}^p$ by

$$\widetilde{T}^0 = \mathcal{A}, \quad \widetilde{T}^p = \underbrace{(\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} (\mathcal{A} \otimes \mathcal{A})}_{p \text{ factors } \mathcal{A}\otimes\mathcal{A}}, \quad p \geq 1.$$

Since $\Omega^1_{\mathcal{A}}$ is a sub-module of $\mathcal{A} \otimes \mathcal{A}$, it follows that $\widetilde{\Omega}^p_{\mathcal{A}}$ is a sub-module of $\widetilde{T}^p$.

Recall that $\Omega^p_{\mathcal{A}}$ is a sub-module of T^p where T^p was defined by (2.9). Let us show that, for all $p \geq 0$,

$$\widetilde{T}^p \cong T^p. \tag{2.18}$$

For $p = 0$ and $p = 1$ it is obvious as

$$\widetilde{T}^0 = \mathcal{A} = T^0 \quad \text{and} \quad \widetilde{T}^1 = \mathcal{A} \otimes \mathcal{A} = T^1.$$

If (2.18) is already proved for some $p \geq 1$ then the statement follows by the associative law of tensor product and the inductive hypothesis.

Denote by f_p the mapping from $\widetilde{T}^p$ to T^p that provides the isomorphism (2.18). For $p = 0, 1$ the mappings f_p are identity mappings. It follows from (2.17) and properties of the tensor product that for $p \geq 2$ and for

$$u = (a_1 \otimes b_1) \otimes_{\mathcal{A}} (a_2 \otimes b_2) \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} (a_p \otimes b_p) \in \widetilde{T}^p \tag{2.19}$$

where $a_i, b_i \in \mathcal{A}$, we have

$$f_p(u) = a_1 \otimes b_1 a_2 \otimes b_2 a_3 \otimes ... \otimes b_{p-1} a_p \otimes b_p \in T^p. \tag{2.20}$$

Set

$$\widetilde{T} = \bigoplus_{p=0,1,...}^{p} \widetilde{T}^p$$

and define the exterior multiplication $\star$ in $\widetilde{T}$ by (2.15), so that $\widetilde{T}$ becomes a graded $\mathbb{K}$-algebra. Set $f = \oplus_{p\geq 0} f_p$ and show that the mapping $f : \widetilde{T} \to T$ is an isomorphism of the graded algebras $\widetilde{T}$ and T (cf. Definition 2.10). It suffices to verify that

$$f(u \star v) = f(u) \bullet f(v) \tag{2.21}$$

for all $u, v \in \widetilde{T}$. Let $u \in \widetilde{T}^p$ and $v \in \widetilde{T}^q$. If $p = 0$, that is, $u \in \mathcal{A}$, then $u \star v = u \cdot v$ and

$$f(u \star v) = f(u \cdot v) = u \cdot f(v) = f(u) \bullet f(v).$$

The same argument works for $q = 0$. For $p = 1$ it suffices to prove Assume now that $p \geq 1$ and $q \geq 1$. It suffices to verify (2.21) for u as in (2.19) and for

$$v = (\alpha_1 \otimes \beta_1) \otimes_{\mathcal{A}} (\alpha_2 \otimes \beta_2) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (\alpha_q \otimes \beta_q)$$

where $\alpha_j, \beta_j \in \mathcal{A}$. Then by (2.15) and (2.20)

$$f(u \star v) = a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p \alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_q$$

whereas by (2.10)

$$\begin{aligned} f(u) \bullet f(v) &= (a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p) \bullet (\alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_{q-1} \alpha_q \otimes \beta_q) \\ &= a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p \alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_q, \end{aligned}$$

which proves (2.21).

Let us prove that the restriction of f to $\widetilde{\Omega}_{\mathcal{A}}$ provides an isomorphism of the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$, that is,

$$f(\widetilde{\Omega}^p_{\mathcal{A}}) = \Omega^p_{\mathcal{A}}.$$

For $p = 0, 1$ it is clear. Assume $p \geq 2$. By Lemma 2.9 any element of $\Omega^p_{\mathcal{A}}$ can be written as a finite sum of the terms

$$w = v_1 \bullet v_2 \bullet \cdots \bullet v_p$$

where $v_i \in \Omega^1_{\mathcal{A}}$. For the element

$$v := v_1 \star v_2 \star ... \star v_p \in \widetilde{\Omega}^p_{\mathcal{A}}$$

we have by (2.21) and $f|_{\Omega^1_{\mathcal{A}}} = \mathrm{id}$

$$f(v) = f(v_1) \bullet f(v_2) \bullet ... \bullet f(v_p) = v_1 \bullet v_2 \bullet \cdots \bullet v_p = w,$$

which implies the inclusion

$$f(\widetilde{\Omega}^p_{\mathcal{A}}) \supset \Omega^p_{\mathcal{A}}.$$

Let us prove the opposite inclusion. By definition (2.14) of $\widetilde{\Omega}^p_{\mathcal{A}}$, any element of $\widetilde{\Omega}^p_{\mathcal{A}}$ is a finite sum of the terms $v = v_1 \star v_2 \star ... \star v_p$, where $v_i \in \Omega^1_{\mathcal{A}}$. As above we have

$$f(v) = v_1 \bullet v_2 \bullet ... \bullet v_p \tag{2.22}$$

that belongs to $\Omega^p_{\mathcal{A}}$ by Theorem 2.15, whence $f(\widetilde{\Omega}^p_{\mathcal{A}}) \subset \Omega^p_{\mathcal{A}}$. The last argument proves also the uniqueness of the isomorphism of the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$. Indeed, since f_0 and f_1 must be the identical maps, they are uniquely determined, and the uniqueness of f_p follows from (2.22).

Finally, the claim (ii) is a trivial consequence of (i). □

THEOREM 2.17. *The differential calculus* $(\Omega_{\mathcal{A}}, d) \cong (\widetilde{\Omega}_{\mathcal{A}}, \widetilde{d})$ *has the following universal property. For any other differential calculus* (Λ, d') *over* $\mathcal{A}$, *there exists one and only one epimorphism* $p : \Omega_{\mathcal{A}} \to \Lambda$ *of graded* $\mathcal{A}$*-algebras given by*

$$p = \bigoplus_k p_k, \quad p_k : \Omega^k_{\mathcal{A}} \to \Lambda_k$$

with $p_0 = \mathrm{id}$ *and such that, for all* $k \geq 0$, *the following diagram is commutative:*

$$\begin{array}{ccc} \Omega^k_{\mathcal{A}} & \xrightarrow{d} & \Omega^{k+1}_{\mathcal{A}} \\ \downarrow^{p_k} & & \downarrow^{p_{k+1}} \\ \Lambda^k & \xrightarrow{d'} & \Lambda^{k+1} \end{array}$$

Proof. Denote by $*$ the exterior multiplication in Λ. By Lemma 2.9 any element $w \in \Lambda^k$ with $k \geq 1$ can be written as a finite sum

$$w = \sum_j a_0^j * da_1^j * da_2^j * \cdots * da_k^j \tag{2.23}$$

where $a_l^j \in \mathcal{A}$ for all $0 \leq l \leq k$. Consider a graded algebra

$$\widetilde{\Lambda} = \bigoplus_{k=0,1,\ldots} \widetilde{\Lambda}^k,$$

where $\widetilde{\Lambda}^k$ for $k \leq 1$ is defined by

$$\widetilde{\Lambda}^0 = \mathcal{A}, \quad \widetilde{\Lambda}^1 = \Lambda^1, \quad \widetilde{\Lambda}^k = \underbrace{\widetilde{\Lambda}^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \widetilde{\Lambda}^1}_{k \text{ factors}} \text{ for } k \geq 2.$$

The exterior multiplication $\star$ in $\widetilde{\Lambda}$ is defined as in (2.15). The condition (2.23) implies that the maps p_0 and p_1 induce an epimorphism $q : \widetilde{\Lambda} \longrightarrow \Lambda$ of the graded algebras, where $q = \bigoplus_{k=0}^{\infty} q_k$ and $q_k : \widetilde{\Lambda}^k \longrightarrow \Lambda^k$ are defined as follows: q_0 and q_1 are the identity mappings, while for $k \geq 2$ the mapping q_k is defined by

$$q_k(w_1 \star w_2 \star \cdots \star w_k) = q_1(w_1) * q_1(w_2) * \cdots * q_1(w_k) \in \Lambda^k$$

for all $w_i \in \widetilde{\Lambda}^1$. Let $f_0 = \mathrm{Id}$. By Theorems 2.4 and 2.16 we have a unique epimorphism $f_1 = p$ of $\mathcal{A}$-bimodules making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\widetilde{d}} & \widetilde{\Omega}^1_{\mathcal{A}} \\ \downarrow^{f_0} & & \downarrow^{f_1} \\ \mathcal{A} & \xrightarrow{d'} & \widetilde{\Lambda}^1 \end{array} \tag{2.24}$$

The diagram (2.24) induces an epimorphism $f : \widetilde{\Omega}_{\mathcal{A}} \to \widetilde{\Lambda}$ of graded algebras given by $f = \bigoplus_{k=0}^{\infty} f_k$, where for $k \geq 2$ the mapping

$$f_k : \underbrace{\widetilde{\Omega}^1_{\mathcal{A}} \star \cdots \star \widetilde{\Omega}^1_{\mathcal{A}}}_{k \text{ factors}} \longrightarrow \underbrace{\widetilde{\Lambda}^1 \star \cdots \star \widetilde{\Lambda}^1}_{k \text{ factors}}$$

is defined by

$$f_k(w_1 \star \cdots \star w_k) = f_1(w_1) \star \cdots \star f_1(w_k) \in \widetilde{\Lambda}^k$$

for all $w_i \in \widetilde{\Omega}^1_{\mathcal{A}}$.Thus, we obtain an epimorphism $p : \widetilde{\Omega}_{\mathcal{A}} \longrightarrow \Lambda$ of graded algebras defined by

$$p = \bigoplus_{k=0,1,\dots} p_k = \bigoplus_{k=0,1,\dots} q_k \circ f_k :$$

such that $p_0 = \mathrm{Id}$ and $p_1 = p$.

To finish the proof of the theorem we must check the commutativity of the diagram

$$\begin{array}{ccc} \Omega^k_{\mathcal{A}} & \xrightarrow{d} & \Omega^{k+1}_{\mathcal{A}} \\ \downarrow^{p_k} & & \downarrow^{p_{k+1}} \\ \Lambda^k & \xrightarrow{d'} & \Lambda^{k+1} \end{array} \tag{2.25}$$

for all $k \geq 0$. By Theorem 2.16 we can identify in the first line of (2.25) the graded algebra $\Omega_{\mathcal{A}}$ with $\widetilde{\Omega}_{\mathcal{A}}$ and d with $\widetilde{d}$. Let us prove by induction in k that this diagram is commutative. For $k = 0$ this is true by Theorem 2.4. Inductive step from $k-1$ to k assuming $k \geq 1$. It suffices to check the commutativity of (2.25) only on the elements $w \in \Omega^k_{\mathcal{A}}$ of the form $w = a \bullet dv$, where $a \in \mathcal{A}$ and $v \in \Omega^{k-1}_{\mathcal{A}}$. Since p is a homomorphism of graded algebras, the inductive hypothesis and $d'^2 = 0$, we obtain

$$d' p_k(a \bullet dv) = d'(a \star p_k(dv)) = d'(a \star d' p_{k-1}(v)) = d'a \star d' p_{k-1}(v).$$

On the other side, using the Leibniz rule and the inductive hypothesis, we obtain

$$p_{k+1} d(a \bullet dv) = p_{k+1}\,(da \bullet dv) = p_1\,(da) \star p_k\,(dv) = d'a \star d' p_{k-1}(v).$$

The comparison of the above two lines proves that the diagram (2.25) is commutative. $\square$

COROLLARY 2.18. *Under the hypotheses of Theorem 2.17, there exists a two-sided graded ideal*

$$\mathcal{J} = \bigoplus_{l=1,2,\dots} \mathcal{J}^l, \quad \mathcal{J}_l \subset \Omega_{\mathcal{A}}$$

of the graded algebra $\Omega_{\mathcal{A}}$ such that

$$\Lambda^k = \Omega^k_{\mathcal{A}} \Big/ \mathcal{J}^k\,, \quad \Omega_{\mathcal{A}}\mathcal{J}\Omega_{\mathcal{A}} \subseteq \mathcal{J}, \quad d\mathcal{J}^k \subset \mathcal{J}^{k+1} \text{ for all } k \geq 0, \quad \text{and } \mathcal{J}^0 = \{0\}. \tag{2.26}$$

Furthermore, the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & \mathcal{J}^1 & \xrightarrow{d} & \mathcal{J}^2 & \xrightarrow{d} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \xrightarrow{d} & \Omega^1_{\mathcal{A}} & \xrightarrow{d} & \Omega^2_{\mathcal{A}} & \xrightarrow{d} \dots \\
& \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & \\
0 \longrightarrow & \Lambda^0 & \xrightarrow{d'} & \Lambda^1 & \xrightarrow{d'} & \Lambda^2 & \xrightarrow{d'} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots
\end{array} \tag{2.27}$$

where the mappings $\mathcal{J}^k \to \Omega^k_{\mathcal{A}}$ are the identical inclusions. In diagram (2.27) the rows are chain complexes of $\mathbb{K}$-modules, and the columns are exact sequences of $\mathbb{K}$-modules.

Proof. Indeed, define $\mathcal{J}^k = \operatorname{Ker}\{p_k : \Omega^k_{\mathcal{A}} \to \Lambda^k\}$. □

Definition 2.19. The differential calculus $(\Omega_{\mathcal{A}}, d)$ is called *the universal differential calculus* on the algebra $\mathcal{A}$.

Proposition 2.20. *Let $(\Omega_{\mathcal{A}}, d)$ be the universal differential calculus on the algebra $\mathcal{A}$ and $\mathcal{J} \subset \Omega_{\mathcal{A}}$ be a graded ideal, that satisfies the property $d\mathcal{J} \subset \mathcal{J}$. Denote by $d_{\mathcal{J}}$ the map of degree one $\Omega_{\mathcal{A}}/\mathcal{J} \to \Omega_{\mathcal{A}}/\mathcal{J}$ that is induced by d. Then $(\Omega_{\mathcal{A}}/\mathcal{J}, d_{\mathcal{J}})$ is a differential calculus on the algebra $\mathcal{A}$.*

Proof. It is easy to check that $d^2_{\mathcal{J}} = 0$ and $d_{\mathcal{J}}$ satisfies the Leibniz rule. □

Corollary 2.21. *Under assumptions of Corollary 2.18, we have the following cohomology long exact sequence:*

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}) \longrightarrow H^0(\Lambda) \longrightarrow H^1(\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}) \longrightarrow H^1(\Lambda) \longrightarrow \dots$$

Proof. This follows from the commutative diagram (2.27) by means of the standard homology algebra [22]. □

Now we describe properties of quotient calculi that we need for constructing the functorial homology theory of digraphs.

Theorem 2.22. *Let $\mathcal{E}^p \subset \Omega^p_{\mathcal{A}}$ be a $\mathbb{K}$-linear subspace for all $p \geq 1$, such that $\mathcal{E} = \bigoplus_{k=0}^{\infty} \mathcal{E}^p$ is a graded ideal of the exterior algebra $\Omega_{\mathcal{A}}$. Consider a subspace*

$$\mathcal{J} = \bigoplus_{p\geq 1} \mathcal{J}^p \subset \Omega_{\mathcal{A}} = \bigoplus_{p \geq 0} \Omega^p_{\mathcal{A}}, \quad \textit{where} \quad \mathcal{J}^p = \begin{cases} \mathcal{E}^p, & \textit{for } p = 1 \\ \mathcal{E}^p + d\mathcal{E}^{p-1}, & \textit{for } p \geq 2. \end{cases}$$

Then $\mathcal{J} \subset \Omega_{\mathcal{A}}$ is a graded ideal of algebra $\Omega_{\mathcal{A}}$ such that $d\mathcal{J} \subset \mathcal{J}$. In particular, the inclusion $\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}$ is a morphism of cochain complexes.

Proof. Any element $w \in \mathcal{J}^p$ can be represented in the form

$$w = w_1 + w_2 \tag{2.28}$$

where $w_1 \in \mathcal{E}^p$ and $w_2 = d(v),\ v \in \mathcal{E}^{p-1}$. For $x \in \Omega^i_{\mathcal{A}}, y \in \Omega^j_{\mathcal{A}}$ we have

$$xwy = xw_1y + xw_2y = xw_1y + x\,(dv)\,y.$$

The element xw_1y lies in $\mathcal{E}$, since by our assumption $\mathcal{E}$ is an ideal. Now, using the Leibniz rule, we have

$$d(xvy) = (dx)\,vy + (-1)^i xd(vy) = (dx)\,vy + (-1)^i x\,(dv)\,y + (-1)^i(-1)^{p-1}xv\,(dy)\,,$$

and hence

$$\begin{aligned} x(dv)y &= (-1)^i[d(xvy) - (dx)vy + (-1)^{i+p}xv(dy)] \\ &= (-1)^i d(xvy) + (-1)^{i+1}(dx)vy + (-1)^p xv(dy). \end{aligned}$$

In the last sum $(-1)^i d(xvy) \in d\Omega^{i+j+p-1}$ and two others element lie in Ω^{i+j+p}, since $\mathcal{E}$ is an ideal. Thus we proved that $\mathcal{J}$ is an ideal. For an element w with decomposition (2.28) we have

$$dw = dw_1 + dw_2 = dw_1 + d\,(dv) = dw_1 \in d\mathcal{E}^p \in \mathcal{J}^{p+1},$$

which finishes the proof. □

Corollary 2.23. *Under assumptions of Theorem 2.22 we have a commutative diagram of cochain complexes*

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & 0 & \longrightarrow & \mathcal{J}^1 & \xrightarrow{d} & \mathcal{J}^2 & \xrightarrow{d} \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \Omega^0_{\mathcal{A}} & \xrightarrow{d} & \Omega^1_{\mathcal{A}} & \xrightarrow{d} & \Omega^2_{\mathcal{A}} & \xrightarrow{d} \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \Omega^0_{\mathcal{A}} & \xrightarrow{d'} & \Omega^1_{\mathcal{A}}/\mathcal{J}^1 & \xrightarrow{d'} & \Omega^2_{\mathcal{A}}/\mathcal{J}^2 & \xrightarrow{d'} \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \end{array} \tag{2.29}$$

where the columns are exact sequences of $\mathbb{K}$*-modules and the differentials* d' *are induced by* d*. Commutative diagram* (2.29) *induces a cohomology long exact sequence*

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}) \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow H^1(\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}) \longrightarrow H^1(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow \dots$$

Corollary 2.24. *Let for any* $p \geq 1$, $\mathcal{E}^p \subset \mathcal{F}^p$ *be* $\mathbb{K}$*-linear subspaces of* $\Omega^p_{\mathcal{A}}$ *such that* $\mathcal{E} = \overset{\infty}{\underset{p=1}{\oplus}} \mathcal{E}^p$ *and* $\mathcal{F} = \overset{\infty}{\underset{p=1}{\oplus}} \mathcal{F}^p$ *are graded ideals of the exterior algebra* $\Omega_{\mathcal{A}}$*. Define* $\mathcal{J}^p$ *and* $\mathcal{I}^p$ *by*

$$\mathcal{J}^p = \begin{cases} \mathcal{E}^p, & \text{for } p = 1 \\ \mathcal{E}^p + d\left(\mathcal{E}^{p-1}\right), & \text{for } p \geq 2 \end{cases}, \quad \mathcal{I}^p = \begin{cases} \mathcal{F}^p, & \text{for } p = 1 \\ \mathcal{F}^p + d\left(\mathcal{F}^{p-1}\right), & \text{for } p \geq 2, \end{cases},$$

and set $\mathcal{J} = \overset{\infty}{\underset{p=1}{\oplus}} \mathcal{J}^p, \mathcal{I} = \overset{\infty}{\underset{p=1}{\oplus}} \mathcal{I}^p$*. Then* $\mathcal{J}^p \subset \mathcal{I}^p \subset \Omega^p_{\mathcal{A}}$ *, which induces inclusions of cochain complexes*

$$\mathcal{J} \longrightarrow \mathcal{I} \longrightarrow \Omega_{\mathcal{A}}. \tag{2.30}$$

Proof. The inclusions $\mathcal{J}^p \subset \mathcal{I}^p \subset \Omega^p_{\mathcal{A}}$ commute with differentials. □

COROLLARY 2.25. *Under assumptions of Corollary 2.24 we have the following short exact sequence of cochain complexes of* $\mathbb{K}$*-modules*

$$0 \longrightarrow \mathcal{I}/\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}/\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}/\mathcal{I} \longrightarrow 0 \tag{2.31}$$

which can be written in the form of a commutative diagram of $\mathbb{K}$*-modules*

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & \mathcal{I}^1/\mathcal{J}^1 & \longrightarrow & \mathcal{I}^2/\mathcal{J}^2 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \longrightarrow & \Omega^1_{\mathcal{A}}/\mathcal{J}^1 & \longrightarrow & \Omega^2_{\mathcal{A}}/\mathcal{J}^2 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \longrightarrow & \Omega^1_{\mathcal{A}}/\mathcal{I}^1 & \longrightarrow & \Omega^2_{\mathcal{A}}/\mathcal{I}^2 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots
\end{array} \tag{2.32}$$

In (2.32) *all columns are exact and rows are cochain complexes. All the differentials in* (2.32) *are induced by the differential* d*. The diagram* (2.32) *induces a cohomology long exact sequence*

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{I}) \longrightarrow H^1(\mathcal{I}/\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow \dots$$

Proof. The proof is standard, see [20, III, §1] and [22]. □

Now we discuss functorial properties of differential calculi (see, for example, [3], [20], [22]). Consider a category ALG in which objects are associative unital $\mathbb{K}$-algebras and morphisms are homomorphisms of $\mathbb{K}$-algebras.

DEFINITION 2.26. Define a category DC of differential calculi by the following way. An object of DC is a differential calculus $(\Lambda_{\mathcal{A}}, d_{\mathcal{A}})$ on a unital associative algebra $\mathcal{A}$ (see Definition 2.8). A morphism $\lambda\colon (\Lambda_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Lambda_{\mathcal{B}}, d_{\mathcal{B}})$ in the category DC is given by a degree preserving morphism of graded algebras

$$\lambda = \bigoplus_{i=0,1,\dots} \lambda_i\colon \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{B}}, \quad \text{where} \quad \lambda_i\colon \Lambda^i_{\mathcal{A}} \to \Lambda^i_{\mathcal{B}},\ i \geq 0,$$

and $\lambda_0\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a morphism in the category ALG, and the maps λ_i $(i \geq 0)$ are homomorphisms of $\mathbb{K}$-modules which commutes with the differentials.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital associative algebras over a commutative unital ring $\mathbb{K}$ and $g\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism. Now we would like to define an induced by g morphism

$$\lambda = \bigoplus_{0,1,\dots} \lambda_i = \mathcal{U}(g)\colon (\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}, d_{\mathcal{B}})$$

of the universal differential calculus $(\Omega_{\mathcal{A}}, d_{\mathcal{A}})$ to the universal differential calculus $(\Omega_{\mathcal{B}}, d_{\mathcal{B}})$.

Let $T_{\mathcal{A}}, T_{\mathcal{B}}$ be graded algebras defined by algebras $\mathcal{A}$, and $\mathcal{B}$ as in Definition 2.10. Let

$$\phi_k\colon T^k_{\mathcal{A}} \to T^k_{\mathcal{B}}, \qquad k \geq 0,$$

be a homomorphism of $\mathbb{K}$-modules (see [3, II, §3.2]) defined by

$$\phi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k).$$

Denote by

$$\phi = \mathop{\oplus}_{k=0}^{\infty} \phi_k \colon T_{\mathcal{A}} = \mathop{\oplus}_{k=0}^{\infty} T_{\mathcal{A}}^k \longrightarrow T_{\mathcal{B}} = \mathop{\oplus}_{k=0}^{\infty} T_{\mathcal{B}}^k$$

a graded homomorphism of graded $\mathbb{K}$-modules. The map ϕ is a degree preserving homomorphism of graded algebras, since

$$\begin{aligned} &\phi_{k+l}[(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_l)] \\ &= g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k b_0) \otimes g(b_1) \otimes \cdots \otimes g(b_l) \\ &= g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k) g(b_0) \otimes g(b_1) \otimes \cdots \otimes g(b_l) \\ &= \phi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \bullet \phi_l(b_0 \otimes b_1 \otimes \cdots \otimes b_l \ . \end{aligned}$$

The maps ϕ_k commutes with differentials, since $g(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Let λ_i $(i \geq 0)$ be the restriction $\lambda_i = \phi_i|_{\Omega^i_{\mathcal{A}}} : \Omega^i_{\mathcal{A}} \longrightarrow T^i_{\mathcal{B}}$, and set $\lambda = \mathop{\oplus}\limits_{i=0}^{\infty} \lambda_i$.

Proposition 2.27. *The homomorphism of $\mathbb{K}$-modules λ_k is a morphism of differential calculi $(\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}, d_{\mathcal{B}})$.*

Proof. We must check only that $\lambda_k(\Omega^k_{\mathcal{A}}) \subset \Omega^k_{\mathcal{B}}$. This follows from the fact that ϕ_k commutes with the differentials and from the inductive definition of $\Omega^k_{\mathcal{A}}$, $\Omega^k_{\mathcal{B}}$ as in Definition 2.14. □

Theorem 2.28. *We can assign to any associative unital $\mathbb{K}$-algebra $\mathcal{A}$ a universal differential calculus $\mathcal{U}(\mathcal{A}) = (\Omega_{\mathcal{A}}, d_{\mathcal{A}})$ and to homomorphism $g \colon \mathcal{A} \to \mathcal{B}$ of such algebras a morphism $\lambda = \mathcal{U}(g) \colon (\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}, d_{\mathcal{B}})$ of the universal differential calculi. Thus, $\mathcal{U}$ is a functor from the category of associative unital $\mathbb{K}$-algebras to the category of differential calculi.*

Proof. Trivial checking. □

Theorem 2.29. *Let (Ω, d) be a differential calculus on an algebra $\mathcal{A}$ with an exterior multiplication $\bullet$. The multiplication $\bullet$ induces a well-defined associative multiplication*

$$H^p(\Omega) \bullet H^q(\Omega) \longrightarrow H^{p+q}(\Omega).$$

Proof. Let $w, v \in \Omega$ and $dw = 0, dv = 0$. Then $d(wv) = 0$ by Leibniz rule. Now, let $w_1 = w + dx, v_1 = v + dy$, where $dw = 0$ and $dv = 0$. Then we have

$$\begin{aligned} w_1 \bullet v_1 = (w + dx) \bullet (v + dy) &= w \bullet v + w \bullet dy + (dx) \bullet v + (dx) \bullet (dy) = \\ &= w \bullet v + d(\pm w \bullet y) + d(x \bullet v) + d(x \bullet d(y)) \end{aligned}$$

where we have used the Leibniz rule and $d(x \bullet dy) = (dx) \bullet (dy)$. □

Corollary 2.30. *A homomorphism $g \colon \mathcal{A} \to \mathcal{B}$ of $\mathbb{K}$-algebras induces a homomorphism of cohomology rings $H^*(\Omega_{\mathcal{A}}) \to H^*(\Omega_{\mathcal{B}})$. This correspondence is functorial.*

3. Differential calculus on finite sets. From now on let $\mathbb{K}$ be a field. We apply the general constructions of the previous sections to the algebra $\mathcal{A}$ of functions $V \to \mathbb{K}$ defined on a finite set $V = \{0, 1, \dots, n\}$. In construction of differential calculus on $\mathcal{A}$ we follow [7] and [8]. The we describe functorial properties of the calculus in the form which will be helpful in the next sections.

The algebra $\mathcal{A}$ has a $\mathbb{K}$-basis

$$e^i : V \to \mathbb{K}, \quad i = 0, 1, \dots, n, \quad \text{where} \quad e^i(j) = \delta_i^j := \begin{cases} 1_{\mathbb{K}}, & i = j \\ 0, & i \neq j \end{cases}, \quad 0 \leq i, j \leq n,$$

and the following relations are satisfied:

$$e^i e^j = \delta_i^j e^j, \quad \sum_{i=0}^{n} e^i = 1_{\mathcal{A}}. \tag{3.1}$$

Denote by (Ω_V^1, d) the first order differential calculus $(\Omega_{\mathcal{A}}^1, d)$ defined in Section 2 with the exterior multiplication $\bullet$.

THEOREM 3.1. *[8] The $\mathbb{K}$-module Ω_V^1 has a basis $\{e^i \otimes e^j\}$ where $0 \leq i, j \leq n$, $i \neq j$. The differential $d\colon \mathcal{A} \to \Omega_V^1$ on the basic elements e^i of $\mathcal{A}$ is given by the formula*

$$de^i = \sum_{0 \leq j \leq n, \ j \neq i} (e^j \otimes e^i - e^i \otimes e^j). \tag{3.2}$$

Also, the following identity is satisfied:

$$e^i \bullet de^j = \begin{cases} e^i \otimes e^j, & i \neq j \\ -\sum_{k \neq i} e^i \otimes e^k, & i = j. \end{cases} \tag{3.3}$$

Proof. For $0 \leq i, j \leq n$, we have by (2.4)

$$\mu(e^i \otimes e^j) = e^i e^j = \delta_j^i.$$

Hence $e^i \otimes e^j \in \Omega_V^1$ for $i \neq j$ and $e^i \otimes e^i \notin \Omega_V^1$ for $0 \leq i \leq n$. The finite dimensional $\mathbb{K}$-module $\mathcal{A} \otimes \mathcal{A}$ has basis $\{e^i \otimes e^j\}$ for $0 \leq i, j \leq n$ (see [3, II §7.7 Remark]), whence the first statement follows.

The identities (3.2) and (3.3) are proved by direct computation using the definition of d and relations (3.1). $\square$

Let $\Omega_V^k = \Omega_{\mathcal{A}}^k \subset T_{\mathcal{A}}^k$ $(k \geq 0)$, and $\Omega_V = \Omega_{\mathcal{A}}$ be the graded algebra defined in Section 2 with the multiplication $\bullet$. Let us introduce the following notation:

$$e^{i_0 \dots i_k} := e^{i_0} \otimes e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_k}$$

assuming that $i_m \neq i_{m+1}$ for all $0 \leq m \leq k-1$. Clearly, $e^{i_0 \dots i_k}$ are the elements of Ω_V^k.

THEOREM 3.2. *[8] (i) The elements $\{e^{i_0 \dots i_k}\}$ form a $\mathbb{K}$-basis in Ω_V^k.*

(ii) The exterior multiplication $\bullet$ of the basic elements is given by the following formula

$$e^{i_0 \dots i_k} \bullet e^{j_0 \dots j_l} = \begin{cases} 0, & i_k \neq j_0 \\ e^{i_0 \dots i_k j_1 \dots j_l}, & i_k = j_0. \end{cases}$$

(iii) The differential d is given on the basic elements by

$$de^{i_0\dots i_k}=\sum_{m=0}^{k+1}\sum_{j\neq i_{m-1},i_m}(-1)^m\, e^{i_0\dots i_{m-1}ji_m\dots i_k}.$$

Proof. (i) The elements $e^{i_0\dots i_k}$ with $i_m\neq i_{m+1}$ for all $0\leq m\leq k-1$ are linearly independent in the $\mathbb{K}$-module T^k (see [3, II §7.7 Remark]). We must only prove that such elements lie in $\Omega_V^k\subset T^k$. By Theorem 2.16 we have an isomorphism of graded algebras $f\colon\widetilde{\Omega}_{\mathcal{A}}\to\Omega_V=\Omega_{\mathcal{A}}$ with an isomorphism of $\mathbb{K}$-modules

$$f_k\colon\widetilde{\Omega}_{\mathcal{A}}^k\to\Omega_V^k,\quad k\geq 0,$$

which is the identity isomorphism for $k=0,1$. Hence the statement (i) is true for $k=0,1$ by the definition of $\mathcal{A}$ and by Theorem 3.1. For $k\geq 2$, consider an element $w=e^{i_0\dots i_k}\in T^k$ with $i_m\neq i_{m+1}$ for all $0\leq m\leq k-1$. Then the elements $e^{i_0i_1},e^{i_1i_2},\dots,e^{i_{k-1}i_k}$ lie in $\widetilde{\Omega}_{\mathcal{A}}^1$ and hence their $\star$-product

$$\omega=e^{i_0i_1}\star e^{i_1i_2}\star\cdots\star e^{i_{k-1}i_k}$$

is contained in $\widetilde{\Omega}_{\mathcal{A}}^k$, and, hence, $f_k(\omega)\in\Omega_V^k$. By the definition of f_k we have

$$\begin{aligned}f_k\left(e^{i_0i_1}\star e^{i_1i_2}\star\cdots\star e^{i_{k-1}i_k}\right)&=f_1\left(e^{i_0i_1}\right)\bullet f_1\left(e^{i_1i_2}\right)\bullet\cdots\bullet f_1\left(e^{i_{k-1}i_k}\right)\\&=f_1\left(e^{i_0}\otimes e^{i_1}\right)\bullet f_1(e^{i_1}\otimes e^{i_2})\bullet\dots\bullet f_1(e^{i_{k-1}}\otimes e^{i_k})\\&=e^{i_0}\otimes\left(e^{i_1}e^{i_1}\right)\otimes\dots\left(e^{i_{k-1}}e^{i_{k-1}}\right)\otimes e^{i_k}\\&=e^{i_0\dots i_k}\end{aligned}$$

so that $f_k(\omega)\in\Omega_V^k$.

(ii) This follows from the definition of multiplication $\bullet$ in Definition 2.10 and (3.1).

(iii) We prove this by induction on k. For $k=0$ it is proved in Theorem 3.1. For $k=1$, let $i\neq j$. We have using (3.1)

$$\begin{aligned}d\left(e^i\otimes e^j\right)&=1_{\mathcal{A}}\otimes e^i\otimes e^j-e^i\otimes 1_{\mathcal{A}}\otimes e^j+e^i\otimes e^j\otimes 1_{\mathcal{A}}\\&=\sum_k(e^k\otimes e^i\otimes e^j-e^i\otimes e^k\otimes e^j+e^i\otimes e^j\otimes e^k)\\&=(e^i\otimes e^i\otimes e^j-e^i\otimes e^i\otimes e^j-e^i\otimes e^j\otimes e^j+e^i\otimes e^j\otimes e^j)\\&\quad+\sum_{k\neq i}e^k\otimes e^i\otimes e^j-\sum_{k\neq i,k\neq j}e^i\otimes e^k\otimes e^j+\sum_{k\neq j}e^i\otimes e^j\otimes e^k.\end{aligned}$$

The sum in the brackets is equal to zero, and we obtain the result for $k=1$. For $k\geq 2$ we have, using the Leibniz rule,

$$\begin{aligned}&d\left(e^{i_0}\otimes e^{i_1}\otimes\cdots\otimes e^{i_k}\right)\\&=d\left((e^{i_0}\otimes e^{i_1})\bullet(e^{i_1}\otimes e^{i_2}\otimes\cdots\otimes e^{i_k})\right)\\&=(e^{i_0}\otimes e^{i_1})\bullet(e^{i_1}\otimes e^{i_2}\otimes\cdots\otimes e^{i_k})-(e^{i_0}\otimes e^{i_1})\bullet d(e^{i_1}\otimes e^{i_2}\otimes\cdots\otimes e^{i_k}).\end{aligned}$$

The result then follows by the inductive hypotheses and elementary transformations. $\square$

We have the augmentation homomorphism $\varepsilon\colon \mathbb{K} \to \mathcal{A} = \Omega_V^0$ that is induced by

$$\varepsilon(1_{\mathbb{K}}) = 1_{\mathcal{A}} = \sum_{i=0}^{n} e^i \in \mathcal{A}.$$

Since $d\varepsilon\colon \mathbb{K} \to \Omega_V^1$ is trivial, we can consider a cochain complex Ω_V^ε with the augmentation ε

$$0 \longrightarrow \mathbb{K} \xrightarrow{\varepsilon} \Omega_V^0 \longrightarrow \Omega_V^1 \longrightarrow \dots \longrightarrow \Omega_V^n \longrightarrow \dots \tag{3.4}$$

Proposition 3.3. *The cohomology group $H^0(\Omega_V^\varepsilon)$ of the complex* (3.4) *is trivial.*

Proof. Let $w \in \Omega_V^0$ be such that $dw = 0$. The element w can be written in the form $w = \sum_{i=0}^n f_i e^i$ where $f_i \in \mathbb{K}$. By Theorem 3.1 we have

$$dw = \sum_i f_i \left(\sum_{\{k\colon\, k\neq i\}} (e^k \otimes e^i - e^i \otimes e^k) \right) = \sum_{\{k,i\colon\, k\neq i\}} (f_i - f_k) e^k \otimes e^i.$$

Since for $i \neq k$, $e^i \otimes e^k$ are the basic elements, the last sum is trivial if and only if $f_i = f_k$ for all i, k. Then we obtain $w = f_0 \left(\sum_{i=0}^n e^i\right) = f_0\varepsilon(1) = \varepsilon(f_0)$, that is, w belongs to the image of ε. □

Let SET be a category in which objects are finite sets and morphisms are the maps of finite sets. Let V and W be finite sets, and $A(V) = \mathcal{A}_V, A(W) = \mathcal{A}_W$ be algebras of $\mathbb{K}$-valued functions respectively. For any map $F\colon V \to W$ define an induced homomorphism of algebras

$$A(F) = F^*\colon \mathcal{A}_W \to \mathcal{A}_V \ \text{ by } \ F^*(f) = f \circ F,\ f \in \mathcal{A}_W, \ \ f \circ F \in \mathcal{A}_V.$$

We formulate the next Proposition and Corollary, the proofs of which are standard, for conveniences of references.

Proposition 3.4. *The map A is a contravariant functor from the category SET to the category ALG of associative unital algebras.*

Corollary 3.5. *For a finite set V, let $\mathcal{U}(\mathcal{A}_V) = (\Omega_V, d)$ be the universal differential calculus (Ω_V, d) on algebra $\mathcal{A}_V$ of $\mathbb{K}$-valued functions on V. Let us assign to any map $F\colon V \to W$ of finite sets a morphism*

$$\mathcal{U}(A(F)) = \mathcal{U}(F^*)\colon (\Omega_W, d) \to (\Omega_V, d)$$

where F^* is defined above. The composition $\mathcal{U} \circ A$ defines a contravariant functor from the category SET to the category DC of differential calculi.

Now let $F\colon V \to W$ be a identical inclusion of a set $V = \{0, 1, \dots, k\}$ into a set $W = \{0, 1, \dots, n\}$ where $k < n$. As before, let $\mathcal{A}$ and $\mathcal{B}$ be the algebras of $\mathbb{K}$-valued functions on V and W, respectively. Define a $\mathbb{K}$-linear subspace $\mathcal{J}$ of Ω_W by

$$\mathcal{J} = \oplus_{m\geq 0} \mathcal{J}^m \ \text{ where } \ \mathcal{J}^0 = \operatorname{span}\{e^{k+1}, \dots, e^n\} \subset \Omega_W^0 = \mathcal{B}$$

and for $m \geq 1$, a subspace $\mathcal{J}^m$ of Ω_W^m is generated by the elements $e^{i_0 \cdots i_m}$ such that the set $\{i_0, i_1, \dots, i_m\}$ contains at least one number from the set $\{k+1, k+2, \dots, n\}$.

Proposition 3.6. (*i*) *The subspace $\mathcal{J} \subset \Omega_W$ is an graded ideal in the graded algebra Ω_W such that*

$$d\mathcal{J}^m \subset \mathcal{J}^{m+1} \text{ for all } m \geq 0.$$

Thus, the restriction of the differential d to $\mathcal{J}$ induces a cochain complex

$$0 \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \dots$$

of $\mathbb{K}$-modules such that the natural inclusion $\mathcal{J} \longrightarrow \Omega_W$ is a morphism of cochain complexes.

(*ii*) *The factor algebra $\Omega_W/\mathcal{J}$ endowed with the induced differential is a differential calculus which is isomorphic to the differential calculus Ω_V.*

Proof. (*i*) Let $e^{i_0\dots i_p} \in \mathcal{J}^p$, $e^{j_0\dots j_q} \in \Omega_V^q$, $e^{l_0\dots l_r} \in \Omega_W^r$. Then by Theorem 3.2 the product $e^{j_0\dots j_q} \bullet e^{i_0\dots i_p} \bullet e^{l_0\dots l_r}$ lies in $\mathcal{J}^{p+q+r}$. The condition $d\mathcal{J}^m \subset \mathcal{J}^{m+1}$ is satisfied by definition of $\mathcal{J}$ and Theorem 3.2.

(*ii*) Any element $[w] \in \Omega_W^p/\mathcal{J}^p$ has a unique representative $w = \sum w_{i_0\dots i_p} e^{i_0\dots i_p}$ where $w_{i_0\dots i_p} \in \mathbb{K}$ and the sum goes over indices $i_j \in \{1,\dots,k\}$ for $0 \leq j \leq p$. Define a map $s_p\colon \Omega_W^p/\mathcal{J}^p \to \Omega_V^p$ by $s_p[w] = w$ and set

$$s = \bigoplus_p s_p\colon \Omega_W/\mathcal{J} \to \Omega_V.$$

Then the map s is a well-defined homomorphism of graded algebras that commutes with differential. It is easy to see that it an epimorphism with a trivial kernel. Hence it is an isomorphism. □

Remark 3.7. The composition

$$\Omega_W \longrightarrow \Omega_W/\mathcal{J} \xrightarrow{s} \Omega_V,$$

where the first map is a natural projection, coincides with the morphism of $\mathcal{U}(A(F))$ from Corollary 3.5 for the inclusion $F\colon V \to W$.

Corollary 3.8. *Under the hypotheses of Proposition* 3.6 *we have a cohomology long exact sequence*

$$0 \longrightarrow H^0(\mathcal{J}) \longrightarrow H^0(\Omega_W) \longrightarrow H^0(\Omega_V) \longrightarrow H^1(\mathcal{J}) \longrightarrow \dots$$

Theorem 3.9. *For any finite set V the cohomology group $H^p(\Omega_V)$ is trivial for $p \geq 1$.*

Proof. Follows from Theorem 5.4 in [13]. □

Corollary 3.10. *Under assumptions of Proposition 3.6,* $H^p(\mathcal{J}) = 0$ *for* $p \geq 0$.

4. Cohomology of digraphs. In this section we define a cohomology of any finite simple digraph and describe its properties. From intuitive point of view the rank of k-dimensional cohomology group of a digraph G correspond to "the number of $k+1$-dimensional holes in G", but sometimes our intuition is not adequate as we can see from Example 4.8 v) in this section.

Let us briefly recall the construction in [13] of n-chains leading to the notion of a chain complex of a digraph. Let $\mathbb{K}$ be a fixed commutative ring with a unity 1, and G be a digraph with the finite set V of vertices. An elementary p-path on V is any (ordered) sequence $i_0, ..., i_p$ of $p+1$ of vertices that will be denoted by $e_{i_0...i_p}$. Consider the free $\mathbb{K}$-module $\Lambda_p = \Lambda_p(V)$ which is generated by elementary p-paths $e_{i_0...i_p}$, whose elements are called p-paths; and define the boundary operator $\partial\colon \Lambda_{p+1} \to \Lambda_p$ on basic elements by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0...\widehat{i_q}...i_p} \quad (p \geq 1) \quad \text{and} \quad \partial e_i = 0, \tag{4.1}$$

where we set $\Lambda_{-1} = \{0\}$. It follows from the definition that $\partial^2 = 0$. An elementary p-path $e_{i_0...i_p}$ is called regular if $i_k \neq i_{k+1}$ for all k and irregular otherwise. Let I_p be the subspace of Λ_p that is spanned by all irregular p-paths. The operator ∂ is well defined on the quotient space

$$\mathcal{R}_p = \mathcal{R}_p(V) = \Lambda_p / I_p.$$

The module $\mathcal{R}_p$ is linearly isomorphic to the module generated by regular p-paths:

$$\operatorname{span}\left\{e_{i_0...i_p} : i_0...i_p \text{ is regular}\right\}.$$

For simplicity of notation, we will identify $\mathcal{R}_p$ with this space, by setting all irregular p-paths to be equal to 0.

Now the paths on a digraph G are defined in a natural way. A regular elementary p-path $e_{i_0...i_p}$ on the set of vertices V is allowed if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and non-allowed otherwise. Denote by $A_p = A_p(G)$ the subspace of $\mathcal{R}_p$ spanned by the allowed elementary p-paths, and consider the ∂-invariant subspaces $\Omega_p = \Omega_p(G) = \{v \in A_p : \partial v \in A_{p-1}\}$. Thus, for a digraph G, we obtain a chain complex $\Omega_*(G)$ and a dual complex $\Omega^*(G)$, thus leading to the notions of homology reps. cohomology groups of the digraph. Note that the main results of [13] are the formulas for homology groups of the join and product of digraphs.

In this section we construct a cochain complex Ω_G that is naturally isomorphic to $\Omega^*(G)$, investigate its functorial properties, and prove the cohomology realization Theorem 4.22.

All digraphs considered in this section are simple digraphs with a finite set of vertices. Let $H = (V, E)$ be a simple complete digraph consisting of the set of vertices $V = \{0, 1, \ldots, n\}$ and all directed edges $E = \{(i \to j) | \; i \neq j\}$. Let $\mathcal{A}$ be the algebra of $\mathbb{K}$-valued functions on the set V, where $\mathbb{K}$ is a commutative unital ring.

Definition 4.1. The differential calculus on a complete finite simple digraph H is the universal differential calculus (Ω_V, d) on the algebra $\mathcal{A}$ constructed in Section 3 with the multiplication $\bullet$ and the differential d that given in Theorem 3.2.

Now let G be a sub-digraph of the digraph H with the same set of vertices $V = \{0, 1, 2, \ldots, n\}$ and a set $E_G \subset E_H$ of edges. Denote by $g\colon G \to H$ the natural inclusion.

DEFINITION 4.2. (*i*) A basic element $e^{i_0 i_1 \dots i_k} \in \Omega_V^k$ is called *allowed* if $\{i_j, i_{j+1}\} \in E_G$ for all $0 \le j \le k-1$, and non-allowed otherwise.

(*ii*) Let $\mathcal{E}_g^k$ be a $\mathbb{K}$-submodule of Ω_V^k generated by non-allowed elements (in particular, $\mathcal{E}_g^0 = \{0\}$), and set $\mathcal{E}_g = \bigoplus_{k \ge 0} \mathcal{E}_g^k \subset \Omega_V$.

PROPOSITION 4.3. *The set $\mathcal{E}_g$ is a graded ideal of algebra Ω_V.*

Proof. The result follows from Theorem 3.2 (ii). ∎

DEFINITION 4.4. Denote by $\mathcal{J}_g = \bigoplus_{k \ge 0} \mathcal{J}_g^k$, where

$$\mathcal{J}_g^k = \begin{cases} \mathcal{E}_g^k, & \text{for } k = 0, 1, \\ \mathcal{E}_g^k + d\mathcal{E}_g^{k-1}, & \text{for } k \ge 2, \end{cases}$$

a $\mathbb{K}$-submodule of Ω_V^k.

PROPOSITION 4.5. *The set $\mathcal{J}_g$ is a graded ideal of algebra Ω_V, $d\mathcal{J}_g \subset \mathcal{J}_g$, and the inclusion $\mathcal{J}_g \longrightarrow \Omega_V$ is a morphism of cochain complexes. In particular, we have an exact sequence of cochain complexes*

$$0 \longrightarrow \mathcal{J}_g \longrightarrow \Omega_V \longrightarrow \Omega_V / \mathcal{J}_g \longrightarrow 0.$$

Proof. Follows from Proposition 4.3 and Theorem 2.22. ∎

DEFINITION 4.6. Let $g\colon G \to H$ be an inclusion of a digraph G into the simple complete digraph H with the same set V of vertices.

i) The differential calculus on G is the calculus $(\Omega_G, d) = (\Omega_V/\mathcal{J}_g, d)$ on the algebra $\mathcal{A}$ with a differential that is induced from differential d on Ω_V.

ii) The cohomology $H^i(G, \mathbb{K})$ $(i = 0, 1, 2, \dots)$ of the digraph G with coefficients $\mathbb{K}$ is the cohomology of the cochain complex

$$0 \longrightarrow \Omega_G^0 \longrightarrow \Omega_G^1 \longrightarrow \Omega_G^2 \longrightarrow \dots \longrightarrow \Omega_G^n \longrightarrow \dots$$

where $\Omega_G^0 = \Omega_V^0 = \mathcal{A}$.

Note, that there is also a cochain complex with the augmentation ε

$$0 \longrightarrow \mathbb{K} \longrightarrow \Omega_G^0 \longrightarrow \Omega_G^1 \longrightarrow \Omega_G^2 \longrightarrow \dots \longrightarrow \Omega_G^n \longrightarrow \dots$$

PROPOSITION 4.7. *Under assumption above, there is a short exact sequence*

$$0 \longrightarrow \mathbb{K} \longrightarrow H^0(\Omega_G) \longrightarrow H^1(\mathcal{J}_g) \longrightarrow 0$$

and there are isomorphisms $H^i(\Omega_G) \cong H^{i+1}(\mathcal{J}_g)$ for $i \ge 1$.

Proof. Follows directly from Proposition 4.5, Theorem 3.9, and Proposition 3.3. ∎

EXAMPLE 4.8. i) For any digraph G, $H^0(G) = (\mathbb{K})^c$, where c is the number of connected components of the digraph G. The proof follows immediately from definition (see also Section 4.1 of [13]).

ii) Consider the following digraphs:

$I = (V_I, E_I), V_I = \{0, 1\}, E_I = \{0 \to 1\}$;

$II = (V_{II}, E_{II}), V_{II} = \{0, 1\}, E_{II} = \{0 \to 1, 1 \to 0\}$;

$T = (V_T, E_T), V_T = \{0, 1, 2\}, E_T = \{0 \to 1, 1 \to 2, 0 \to 2\}$;
$S = (V_S, E_S), V_S = \{0, 1, 2, 3\}, E_S = \{0 \to 1, 1 \to 2, 0 \to 3, 3 \to 2\}$;
$R = (V_R, E_R), V_R = \{0, 1, 2\}, E_R = \{0 \to 1, 1 \to 2, 0 \to 2, 2 \to 0\}$.

Let G be one of the digraphs from this list. Then $H^0(G, \mathbb{K}) = \mathbb{K}$ and $H^i(G, \mathbb{K}) = 0$ for $i \geq 1$.

iii) Given a digraph G, the underlying graph of G is the graph with the same vertices as G, in which (u, v) is an edge whenever at least one of $(u \to v)$, $(v \to u)$ lies in E_G (cf. page 2 of [16]). Let a digraph G be a tree (that is, the underlying graph is a tree). Then $H^0(G, \mathbb{K}) = \mathbb{K}$ and $H^i(G, \mathbb{K}) = 0$ for $i \geq 1$. This statement follows from ii) and by induction from Theorem 4.17 below (see also [13]).

iv) Define for any $n \geq 5$ the digraph $C_n = (V_{C_n}, E_{C_n})$ as follows: $V_{C_n} = \{0, 1, \ldots, n-1\}$ and E_{C_n} contains exactly one arrow $i \to i+1$ or $i+1 \to i$ for $0 \leq i \leq n-2$ and exactly one arrow $(n-1) \to 0$ or $0 \to (n-1)$.

$C_3 = (V_{C_3}, E_{C_3}), V_{C_3} = \{0, 1, 2\}, E_{C_3} = \{0 \to 1, 1 \to 2, 2 \to 0\}$,
$C_4 = (V_{C_4}, E_{C_4}), V_{C_4} = \{0, 1, 2, 3\}, E_{C_4} = \{0 \to 1, 1 \to 2, 2 \to 3, 3 \to 0\}$,
Consider also a digraph
$D_4 = (V_{D_4}, E_{D_4}), V_{D_4} = \{0, 1, 2, 3\}, E_{D_4} = \{0 \to 1, 1 \to 2, 2 \to 3, 0 \to 3\}$.

Let G be one of the digraphs C_n or D_4 as above. Then $H^0(G, \mathbb{K}) \cong H^1(G, \mathbb{K}) = \mathbb{K}$ and $H^i(G, \mathbb{K}) = 0$ for $i \geq 2$ (see [13]).

v) Let $G = (V_G, E_G)$ be a planar digraph with
$V_G = \{0, 1, 2, 3, 4\}, E_G = \{0 \to 1, 1 \to 2, 2 \to 0, 0 \to 3, 1 \to 3, 2 \to 3, 0 \to 4, 1 \to 4, 2 \to 4\}$.

Then $H^0(G, \mathbb{K}) \cong H^2(G, \mathbb{K}) = \mathbb{K}$, $H^1(G, \mathbb{K}) = 0$, and $H^i(G, \mathbb{K}) = 0$ for $i \geq 3$. The result follows from Theorem 4.21 below (see also [13]).

Consider a commutative diagram

$$\begin{tikzcd} F \arrow[rr, "s"] \arrow[dr, "f"'] & & G \arrow[dl, "g"] \\ & H & \end{tikzcd} \tag{4.2}$$

of inclusions of digraphs F and G into H with the same number of vertices. Let $\mathcal{E}_f$ and $\mathcal{E}_g$ be the subspaces generated by non-allowed elements for the inclusions f and g correspondingly, and $\mathcal{J}_f \subset \Omega_V$, $\mathcal{J}_g \subset \Omega_V$ are the graded ideals defined above.

Theorem 4.9. *We have the inclusions of the chain complexes*

$$\mathcal{J}_g \subset \mathcal{J}_f \subset \Omega_V,$$

which induce a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_f/\mathcal{J}_g \longrightarrow \Omega_V/\mathcal{J}_g \xrightarrow{s^*} \Omega_V/\mathcal{J}_f \longrightarrow 0. \tag{4.3}$$

The cohomology long exact sequence of (4.3) *has the following form*

$$0 \longrightarrow H^0(\Omega_G) \longrightarrow H^0(\Omega_F) \longrightarrow H^1(\mathcal{J}_f/\mathcal{J}_g) \longrightarrow H^1(\Omega_G) \longrightarrow H^1(\Omega_F) \longrightarrow \ldots \tag{4.4}$$

Proof. Any non-allowed element from $\mathcal{E}_g$ is evidently non-allowed in $\mathcal{E}_f$. Now the result follows from Corollaries 2.24 and 2.25. $\square$

Now consider an arbitrary inclusion of digraphs $\sigma\colon F\to G$, where $F=(V_F,E_F)$ and $G=(V_G,E_G)$. Let H_1 and H_2 be complete simple digraphs with the same number of vertices as F and G, respectively. Consider a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & G \\ \downarrow f & & \downarrow g \\ H_1 & \xrightarrow{\sigma_v} & H_2 \end{array} \tag{4.5}$$

where vertical maps are natural inclusions, and $\sigma_v\colon H_1\to H_2$ is the inclusion defined by σ. By Corollary 3.5 and Definition 4.1, the map σ_v induces a morphism $\mathcal{U}(\sigma_v)\colon \Omega_{V_G}\longrightarrow \Omega_{V_F}$. Thus by Proposition 4.5 we can write down the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}_g & \to & \Omega_{V_G} & \to & \Omega_G & \to & 0 \\ & & & & \downarrow \mathcal{U}(\sigma_v) & & & & \\ 0 & \to & \mathcal{J}_f & \to & \Omega_{V_F} & \to & \Omega_F & \to & 0 \end{array} \tag{4.6}$$

where the horizontal rows are exact sequences of cochain complexes.

LEMMA 4.10. *In diagram* (4.6) *we have* $\mathcal{U}(\mathcal{J}_g)\subset \mathcal{J}_f\subset \Omega_{V_F}$ *and hence the induced morphism* $\mathcal{U}(\sigma)\colon \Omega_G\longrightarrow\Omega_F$ *of differential calculus is defined.*

Proof. Let $e^{i_0\cdots i_k}\in\mathcal{E}^k_g\subset\Omega^k_{H_2}$ be a non-allowed element for the digraph G. If all i_j for $j=0,1,\dots,k$ are contained in the image of $\sigma|_{V_F}\colon V_F\to V_G$, then $\mathcal{U}(e^{i_0\cdots i_k})=e^{i_0\cdots i_k}\subset\mathcal{E}_f$ by diagram (4.5) since the map σ is an inclusion. In the opposite case by Proposition 3.6 and Remark 3.7 we obtain $\mathcal{U}(e^{i_0\cdots i_k})=0$. Hence $\mathcal{U}(\mathcal{E}_g)\subset\mathcal{E}_f$. ¿From now the result follows from the definition of $\mathcal{J}$, since vertical maps in diagram (4.6) are morphisms of cochain complexes. □

Denote by *GRI* the category in which objects are simple finite digraphs and the morphisms are inclusions.

THEOREM 4.11. *Let* $\mathcal{U}(G)$ *be a differential calculus* (Ω_G,d) *defined in Definition* 4.6, *and for an inclusion* $\sigma\colon F\to G$ *of graphs let* $\mathcal{U}(\sigma)\colon\Omega_G\longrightarrow\Omega_F$ *be a morphism of differential calculi defined in Lemma* 4.10. *Then* $\mathcal{U}$ *is a contravariant functor from category GRI to the category DC.*

Proof. We must only check that for two inclusions of digraphs

$$\sigma\colon F\to G \ \text{ and } \ \tau\colon G\to M$$

we have $\mathcal{U}(\tau\circ\sigma)=\mathcal{U}(\sigma)\circ\mathcal{U}(\tau)$. By Lemma 4.10 we have a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_m & \to & \mathcal{J}_g & \to & \mathcal{J}_f \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{V_M} & \xrightarrow{\mathcal{U}(\tau_v)} & \Omega_{V_G} & \xrightarrow{\mathcal{U}(\sigma_v)} & \Omega_{V_F} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_M & \xrightarrow{\mathcal{U}(\tau)} & \Omega_G & \xrightarrow{\mathcal{U}(\sigma)} & \Omega_F \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \ . \tag{4.7}$$

By Corollary 3.5 we have $\mathcal{U}(\sigma_v)\circ\mathcal{U}(\tau_v)=\mathcal{U}(\tau_v\sigma_v)$. The commutativity of diagram

$$\begin{tikzcd} \mathcal{J}_m \arrow[rr] \arrow[dr] & & \mathcal{J}_g \arrow[dl] \\ & \mathcal{J}_f & \end{tikzcd}$$

follows from Lemma 4.10. This implies the claim, since the vertical columns in (4.7) are exact sequences. ∎

Remark 4.12. Let $s\colon F\to G$ be an inclusion of digraphs with the same number of vertices. Then $\mathcal{U}(s)$ coincides with the morphism s^* from Theorem 4.9.

Definition 4.13. Let G be a simple digraph with the set of vertices V and the set of edges E_G. Define a simple digraph $\overline{G}$ with the same set of vertices V and with the set of inverse-directed edges

$$E_{\overline{G}}=\{\{i,j\}:\{j,i\}\in E_G\}\,.$$

Note that the mapping $G\longrightarrow\overline{G}$ is an involution on the set of simple digraphs.

Theorem 4.14. *Let G be a simple digraph. Then we have an isomorphism of cochain complexes $\Omega_G\longrightarrow\Omega_{\overline{G}}$ which is given on the basic elements by the following map*

$$e^{i_0i_1\dots i_{p-1}i_p}\longrightarrow(-1)^k e^{i_pi_{p-1}\dots i_1i_0},$$

where $k=1$ for $p=1,2\bmod 4$ and $k=0$ for $p=0,3\bmod 4$.

Proof. Let H be a full simple digraph with the same number of vertices $V=\{0,1,\dots,n\}$ as the digraph G, and $g\colon G\to H,\overline{g}\colon\overline{G}\to H$ be the natural inclusions. Define a $\mathbb{K}$-linear map $\tau:\Omega_V\longrightarrow\Omega_V$ on the basic elements by the following way:

$$\tau\left(e^{i_0i_1\dots i_{p-1}i_p}\right)=e^{i_pi_{p-1}\dots,i_1i_0}.$$

The map τ is an anti-automorphism of the graded algebra Ω_V since

$$\tau(vw)=\tau(w)\tau(v),\quad \tau(v+w)=\tau(v)+\tau(w).$$

We can write down two diagrams

$$\begin{array}{ccc}\Omega_V^{2k+1} & \xrightarrow{d} & \Omega_V^{2k+2}\\ \downarrow\tau & & \downarrow\tau\\ \Omega_V^{2k+1} & \xrightarrow{d} & \Omega_V^{2k+2}\end{array},\qquad \begin{array}{ccc}\Omega_V^{2k} & \xrightarrow{d} & \Omega_V^{2k+1}\\ \downarrow\tau & & \downarrow\tau\\ \Omega_V^{2k} & \xrightarrow{d} & \Omega_V^{2k+1}\end{array}.$$

And it is easy to check, that the first diagram is commutative, that is $\tau d=d\tau$, and the second diagram is anti-commutative that is $\tau d=-d\tau$. Thus, we obtain a commutative diagram of chain complexes

$$\begin{array}{cccccccccc}0\to & \mathbb{K} & \longrightarrow & \Omega_V^0 & \xrightarrow{d} & \Omega_V^1 & \xrightarrow{d} & \Omega_V^2 & \xrightarrow{d} & \Omega_V^3\to\dots\\ & \downarrow= & & \downarrow\tau & & \downarrow-\tau & & \downarrow-\tau & & \downarrow\tau\\ 0\to & \mathbb{K} & \longrightarrow & \Omega_V^0 & \xrightarrow{d} & \Omega_V^1 & \xrightarrow{d} & \Omega_V^2 & \xrightarrow{d} & \Omega_V^3\to\dots\end{array}\tag{4.8}$$

where all the vertical maps are isomorphisms of $\mathbb{K}$-modules.

Since $\tau|_{\mathcal{E}_g}\colon \mathcal{E}_g \to \mathcal{E}_{\overline{g}}$ is clearly bijective, $\mathcal{E}_g = -\mathcal{E}_g$ and $\mathcal{E}_{\overline{g}} = -\mathcal{E}_{\overline{g}}$, the restriction of the vertical maps in (4.8) to $\mathcal{J}_g$ provides isomorphisms $\mathcal{J}_g^p \longrightarrow \mathcal{J}_{\overline{g}}^p$, whence the result follows. ∎

By a digraph with a pointed vertex we mean a couple (G, v) where G is a digraph and v is one of its vertices.

Definition 4.15. Let $\{(G_i, v_i)\}_{i\in A}$ be a finite family of digraphs $G_i = (V_i, E_i)$ with pointed vertices. Assume that all the vertex sets V_i are disjoint. The wedge sum (or bouquet) (G, v) of the digraphs (G_i, v_i) is a digraph with the set V of vertices that is obtained from the disjoint union $U = \bigcup_{i\in A} V_i$ by identification of all pointed vertices v_i, $i \in A$, with one vertex v, and with the following set of edges $E = \bigcup_{i\in A} E_i$ with the same identification of the endpoint.

We shall denote the wedge sum by $G = \bigvee_{i\in A} G_i$.

From now we shall consider a wedge sum of two digraphs $G = G_1 \bigvee G_2$, with pointed vertices $v_1 \in G_1$, $v_2 \in G_2$ and $v \in G$. Denote

$$W_1 = V_1 \setminus \{v_1\}, \quad W_2 = V_2 \setminus \{v_2\}.$$

Let H_1, H_2, and H be complete simple digraphs with the set of vertices V_1, V_2, and V, respectively. Let

$$g_1\colon G_1 \to H_1,\ g_2\colon G_2 \to H_2,\ g\colon G \to H$$

be natural inclusions. The graded ideals

$$\mathcal{E}_{g_1} \subset \Omega_{V_1},\ \mathcal{E}_{g_2} \subset \Omega_{V_2},\ \mathcal{E}_g \subset \Omega_V$$

are defined as above as well as the graded ideals

$$\mathcal{J}_{g_1} \subset \Omega_{V_1},\ \mathcal{J}_{g_2} \subset \Omega_{V_2},\ \mathcal{J}_g \subset \Omega_V.$$

Lemma 4.16. *Let $G = G_1 \bigvee G_2$ as above. Let a basic element $e^{i_0 i_1 \dots i_p} \in \Omega_V^p$ be such that the multiindex $\{i_0, i_1, \dots, i_p\}$ contains at least one vertex $i_k \in W_1$ and at least one vertex $i_m \in W_2$. Then $e^{i_0 i_1 \dots i_p} \in \mathcal{J}_g$.*

Proof. Let $e^{i_0 i_1 \dots i_p}$ be as in the hypotheses. Consider two cases. If the pointed vertex v does not belong to the sequence $\{i_0, i_1, \dots, i_p\}$ then, by definition of the wedge sum, $e^{i_0 i_1 \dots i_p} \in \mathcal{E}_g \subset \mathcal{J}_g$, since it is non-allowed. Now assume that $v \in \{i_0, i_1, \dots, i_p\}$. In this case we have necessarily $p \geq 2$. For $p = 2$ the element $e^{i_0 i_1 \dots i_p}$ can be written as

$$e^{i_1 v i_2} \in \Omega_V^2 \ \text{ where } \ i_1 \in W_1,\ i_2 \in W_2$$

(or $i_1 \in W_2$ and $i_2 \in W_1$). By definition of the wedge sum, $e^{i_1 i_2} \in \mathcal{E}_g$. Hence

$$de^{i_1 i_2} = \left(\left[\sum_i e^{i i_1 i_2} \right] - \left[\sum_{i \neq v} e^{i_1 i i_2} \right] + \left[\sum_i e^{i_1 i_2 i} \right] - e^{i_1 v i_2} \right) \in \mathcal{J}_g.$$

Here first three summands lie in $\mathcal{E}_g$, whence $e^{i_1 v i_2} \in \mathcal{J}_g$. Now consider the case $p \geq 3$. Then there exists $1 \leq l \leq p-1$ such that $v = i_l$. Then either $i_{l-1} \in W_1$ and $i_{l+1} \in W_2$

(or $i_{l-1} \in W_2$ and $i_{l+1} \in W_1$). Using the case $p = 2$ we conclude that $e^{i_{l-1}vi_{l+1}} \in \mathcal{J}_g$, which implies $e^{i_0i_1\dots i_p} \in \mathcal{J}_g$ since $\mathcal{J}_g$ is a two-sided ideal in Ω_V. ☐

THEOREM 4.17. *Let $G = G_1 \bigvee G_2$ where G_i $(i = 1, 2)$ are connected digraphs. Then*

$$H^0(\Omega_G) = \mathbb{K} \quad \textit{and} \quad H^k(\Omega_G) = H^k(\Omega_{G_1}) \oplus H^k(\Omega_{G_2}) \quad \textit{for} \quad k \geq 1.$$

Proof. Let V_i $(i = 1, 2)$ be the set of vertices of the digraphs G_i, and we recall that these sets are disjoint. Let V be the set of vertices of the digraph G. Let H_i $(i = 1, 2)$ be complete simple digraphs with the set of vertices V_i, and H be a complete simple digraph with the set of vertices V. Let

$$f_i \colon H_i \longrightarrow H, \quad i = 1, 2$$

be the natural inclusions of complete simple digraphs. Define for any mapping f_i the graded ideal $\mathcal{J}_i$ of Ω_V as in Proposition 3.6. By definition, $\mathcal{J}_1^k$ is a $\mathbb{K}$-linear subspace of Ω_V^k that is generated by the elements $e^{i_0i_1\dots i_k} \in \Omega_V^k$ such that the set $\{i_0, i_1, \dots, i_k\}$ contains at least one vertex from $V \setminus V_1$. The subspace $\mathcal{J}_2^k \subset \Omega_V^k$ is defined similarly. By Proposition 3.6 the graded ideals $\mathcal{J}_i$ $(i = 1, 2)$ of Ω_V induce short exact sequences of $\mathbb{K}$-modules

$$0 \longrightarrow \mathcal{J}_i \xrightarrow{\hat{f}_i} \Omega_V \xrightarrow{p_i} \Omega_{V_i} \longrightarrow 0, \tag{4.9}$$

where p_i and $\hat{f}_i$ are chain maps. Then $p = (p_1, p_2) \colon \Omega_V \longrightarrow \Omega_{V_1} \oplus \Omega_{V_2}$ is a chain map. We denote by p^k a restriction of p to Ω_V^k. In dimension 0, the map p^0 is a monomorphism with a one-dimensional cokernel generated by $e^{v_1} \oplus 0$ (or $0 \oplus e^{v_2}$), since $p^0(e^v) = e^{v_1} \oplus e^{v_2}$. The map p is an epimorphism in dimensions $k \geq 1$. Indeed, consider an arbitrary element

$$e^{i_0i_1\dots i_k} \oplus e^{j_0j_1\dots j_k} \in \Omega_{V_1}^k \oplus \Omega_{V_2}^k.$$

Set $\alpha = \{i_0i_1 \dots i_k\}$ and define a new multiindex α' by the following rule: if α does not contain the pointed vertex v_1 then $\alpha' = \alpha$, otherwise α' is obtained from α by changing v_1 to v. Similarly, using multiindex $\beta = \{j_0j_1 \dots j_k\}$ we define a multiindex β'. Then we have

$$p_1(e^{\alpha'}) = e^{i_0i_1\dots i_k}, \quad p_2\left(e^{\beta'}\right) = e^{j_0j_1\dots j_k}$$

and it is clear that $p_2(e^{\alpha'}) = 0$, $p_1(e^{\beta'}) = 0$. It follows that

$$p\left(e^{\alpha'} + e^{\beta'}\right) = e^{i_0i_1\dots i_k} \oplus e^{j_0j_1\dots j_k} \in \Omega_{V_1}^k \oplus \Omega_{V_2}^k,$$

which proves that p is an epimorphism in dimensions $k \geq 1$.

Observe that $\mathcal{J}_{12}^k := \mathcal{J}_1^k \cap \mathcal{J}_2^k$ is a graded ideal in Ω_V and $\operatorname{Ker} p = \mathcal{J}_{12}$. Note, that $\mathcal{J}_{12}^0 = \{0\}$ and we have a short exact sequence

$$0 \longrightarrow \Omega_V^0 \xrightarrow{p} \Omega_{V_1}^0 \oplus \Omega_{V_2}^0 \longrightarrow \langle e^{v_1} \oplus 0 \rangle \longrightarrow 0. \tag{4.10}$$

Consider now the case $k \geq 1$. In this case we obtain a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_{12}^k \xrightarrow{\hat{f}} \Omega_V^k \xrightarrow{p} \Omega_{V_1}^k \oplus \Omega_{V_2}^k \longrightarrow 0, \tag{4.11}$$

where $\hat{f} = \hat{f}_1|_{\mathcal{J}_{12}} = \hat{f}_2|_{\mathcal{J}_{12}}$ and p are chain maps. Recall, that the ideal $\mathcal{J}_{12}$ is generated by $e^{i_0 i_1 \dots i_k} \in \Omega_V$ such that the set $\{i_0, i_1, \dots, i_k\}$ contains $i_l \in W_1$ and $i_m \in W_2$.

Let us introduce the following notations:

$$\mathcal{J}'_{12} = \oplus_{k\geq 1}\mathcal{J}^k_{12},\ \ \mathcal{J}'_g = \oplus_{k\geq 1}\mathcal{J}^k_g,\ \ \mathcal{J}'_{g_i} = \oplus_{k\geq 1}\mathcal{J}^k_{g_i},\ \ \Omega'_V = \oplus_{k\geq 1}\Omega^k_V,\ \ \Omega'_{V_i} = \oplus_{k\geq 1}\Omega^k_{V_i}.$$

LEMMA 4.18. *Under the above assumptions ($k \geq 1$) there is a commutative diagram of chain complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}'_{12} & \longrightarrow & \mathcal{J}'_g & \xrightarrow{q} & \mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2} & \longrightarrow & 0 \\ & & || & & \downarrow^{mono} & & \downarrow^{mono} & & \\ 0 & \longrightarrow & \mathcal{J}'_{12} & \longrightarrow & \Omega'_V & \xrightarrow{p} & \Omega'_{V_1} \oplus \Omega'_{V_2} & \to & 0 \end{array} \tag{4.12}$$

where the rows are exact sequences and the two right vertical maps are natural inclusions.

Proof. The bottom exact sequence is the exact sequence (4.11). The inclusion $\mathcal{J}'_{12} \to \mathcal{J}'_g$ follows from Lemma 4.16 and we obtain that the left square is commutative. Set

$$q = p|_{\mathcal{J}'_g} : \mathcal{J}'_g \to \Omega'_{V_1} \oplus \Omega'_{V_2}$$

where we identify $\mathcal{J}'_g$ with a subspace of Ω'_V. It remains to prove that the image of q is $\mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2}$.

Let us first prove that

$$\mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2} \subset q\left(\mathcal{J}'_g\right) \tag{4.13}$$

For $i = 1, 2$ we shall define the grade preserving homomorphisms of $\mathbb{K}$-modules

$$s_i \colon \Omega_{V_i} \longrightarrow \Omega_V.$$

For any basic elements $e^\alpha = e^{i_0 \dots i_k} \in \Omega_{V_1}$, let α' be the multiindex that is equal to α if α does not contain the pointed vertex v_1, and otherwise α' is obtained from α by changing all occurrences of v_1 in α to v. Similarly define $e^{\beta'}$ for $e^\beta = e^{j_0 \dots j_k} \in \Omega_{V_2}$. Then set $s_1(e^\alpha) = e^{\alpha'}$ and $s_2(e^\beta) = e^{\beta'}$ and extend s_1 and s_2 by linearity to all the spaces Ω_{V_1} and Ω_{V_2}, respectively. It follows immediately from this definition, that

$$s_1\left(e^\alpha\right) \in \mathcal{E}_g,\ \text{if } e^\alpha \in \mathcal{E}_{g_1}\ \text{ and }\ s_2\left(e^\beta\right) \in \mathcal{E}_g,\ \text{if } e^\beta \in \mathcal{E}_{g_2}.$$

Note, that the maps s_i do not commutes with the differentials, but they satisfy the following properties:

$$p_i s_j = \begin{cases} \mathrm{Id} \colon \Omega^k_{V_i} \to \Omega^k_{V_i}, & i = j \\ 0 \colon \Omega^k_{V_j} \to \Omega^k_{V_i}, & i \neq j \end{cases} \tag{4.14}$$

and, for $e^\alpha \in \Omega^k_{V_1}$,

$$d_H(s_1(e^\alpha)) = s_1(d_{H_1}(e^\alpha)) + \sum_\gamma f_\gamma e^\gamma,\ \text{where } \sum_\gamma f_\gamma e^\gamma \in \mathcal{J}^k_{12} \tag{4.15}$$

and a similar identity holds for $e^{\beta} \in \Omega^k_{V_2}$. Let

$$u = u_1 \oplus u_2 \in \mathcal{J}^k_{g_1} \oplus \mathcal{J}^k_{g_2} \subset \Omega^k_{V_1} \oplus \Omega^k_{V_2}.$$

By definition of $\mathcal{J}^k_{g_1}$ we have

$$u_1 = w_1 + d_{H_1}(w_1') \in \mathcal{J}^k_{g_1} \subset \Omega^k_{V_1} \quad \text{where} \quad w_1, w_1' \in \mathcal{E}_{g_1}.$$

Note that $s_1(w_1), s_1(w_1') \in \mathcal{E}_g$ whence it follows that

$$s_1(w_1) + d_H(s_1(w_1')) \in \mathcal{J}_g.$$

Using by (4.14) and (4.15) we obtain that

$$p_1(s_1(w_1) + d_H(s_1(w_1'))) = w_1 + p_1(d_H(s_1(w_1'))) = w_1 + p_1\left(s_1(d_{H_1}(w_1')) + \sum_{\gamma} f_{\gamma} e^{\gamma}\right)$$

which equals to $w_1 + d_{H_1}(w_1') = u_1$. Where we have used that

$$\sum_{\gamma} f_{\gamma} e^{\gamma} \in \mathcal{J}_{12} \text{ and } p(\mathcal{J}_{12}) = 0.$$

By the same line of arguments we obtain

$$p_2(s_1(w_1) + d_H(s_1(w_1'))) = w_1 + p_2(d_H(s_1(w_1'))) = 0,$$

and

$$p_2(s_2(w_2) + d_H(s_2(w_2'))) = u_2, \quad p_1(s_2(w_2) + d_H(s_2(w_2'))) = 0.$$

Hence,

$$p((s_1(w_1) + d_H(s_1(w_1')) + s_2(w_2) + d_H(s_2(w_2'))) = u_1 \oplus u_2,$$

which proves the inclusion (4.13).

Let us prove the opposite inclusion. Any element of $\mathcal{J}_g$ has the form $w + dw'$ where

$$w = s_1(u_1) + s_2(u_2) + u_3, \quad w' = s_1(u_1') + s_2(u_2') + u_3'$$

where $u_1, u_1' \in \mathcal{E}_{g_1}$, $u_2, u_2' \in \mathcal{E}_{g_2}$ and $u_3, u_3' \in \mathcal{J}_{12}$. As above we obtain for $i = 1, 2$

$$p_i(w + dw') = u_i + d_{H_i} u_i' \in \mathcal{J}_{g_i}$$

which finishes the proof of Lemma. ∎

By Lemma 4.18 we obtain a commutative diagram of chain complexes in which rows and columns are exact (in dimensions $k \geq 1$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_{12} & \longrightarrow & \mathcal{J}_g & \xrightarrow{q} & J_{g_1} \oplus \mathcal{J}_{g_2} & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_{12} & \longrightarrow & \Omega_V & \xrightarrow{p} & \Omega_{V_1} \oplus \Omega_{V_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Omega_G & \xrightarrow{\cong} & \Omega_{G_1} \oplus \Omega_{G_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.16}$$

In dimension 0 we have isomorphisms

$$\Omega_G^0 = \Omega_V^0, \quad \Omega_{G_1}^0 = \Omega_{V_1}^0, \quad \Omega_{G_2}^0 = \Omega_{V_2}^0$$

and an exact sequence

$$0 \longrightarrow \Omega_V^0 \xrightarrow{p} \Omega_{V_1}^0 \oplus \Omega_{V_2}^0 \longrightarrow \langle e^{v_1} \oplus 0\rangle \longrightarrow 0.$$

Now from (4.16) and the last exact sequence we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega_G^0 & \longrightarrow & \Omega_G^1 & \xrightarrow{d} & \Omega_G^2 & \xrightarrow{d} \dots \\
 & \downarrow & & \downarrow = & & \downarrow = & \\
0 \longrightarrow & \Omega_{G_1}^0 \oplus \Omega_{G_2}^0 & \xrightarrow{d\oplus d} & \Omega_{G_1}^1 \oplus \Omega_{G_2}^1 & \xrightarrow{d\oplus d} & \Omega_{G_1}^2 \oplus \Omega_{G_2}^2 & \xrightarrow{d\oplus d} \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \langle e^{v_1} \oplus 0\rangle & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots
\end{array} \tag{4.17}$$

where the columns are exact sequences, and the rows are chain complexes. Using the obvious identity

$$H^*(\Omega_{G_1} \oplus \Omega_{G_2}) = H^*(\Omega_{G_1}) \oplus H^*(\Omega_{G_2})$$

and the cohomology long exact sequences of (4.17) we finish the proof of the theorem. ∎

COROLLARY 4.19. *Let $G = \bigvee_{i\in A} G_i$ be a finite wedge sum, and all G_i are connected digraphs. Then*

$$H^0(\Omega_G) = \mathbb{K} \ \ \textit{and} \ \ H^m(\Omega_G) = \oplus_{i\in A} H^m(\Omega_{G_i}) \ \ \textit{for} \ \ m \geq 1.$$

Proof. Induction on i. ∎

Let G be a digraph with the set of vertices $V = \{0, 1, \dots, n\}$.

DEFINITION 4.20. [13] (i) The cone CG of the digraph G is obtained by adding a new vertex v to the set of vertices V and new edges $\{i, v\}$ for all $0 \leq i \leq n$.

(ii) The suspension SG of the digraph G is obtained from the digraph G by adding two new vertices v and w and new edges $\{i, v\}$, $\{i, w\}$ for all $0 \leq i \leq n$.

We recall here the following result from [13].

THEOREM 4.21. *[13] For any digraph G we have*

$$H^p(\Omega_{CG}) \cong \begin{cases} \mathbb{K}, & p = 0 \\ 0, & p \geq 1, \end{cases}, \qquad H^{p+1}(\Omega_{SG}) \cong \begin{cases} \mathbb{K}, & p = -1 \\ H^0(\Omega_G^{\varepsilon}), & p = 0 \\ H^p(\Omega_G), & p \geq 1 \end{cases}$$

where Ω_G^{ε} is a cochain complex with the augmentation.

One of the main results of this paper is the following theorem.

THEOREM 4.22. *For any finite collection of nonnegative integers $k_0, k_1, \dots k_n$ such that $k_0 \geq 1$ there exists a digraph G such that the cohomology groups of its differential calculus satisfies the conditions*

$$\dim H^i(\Omega_G) = k_i, \ \text{for all } 0 \leq i \leq n. \tag{4.18}$$

In particular, if $k_0 = 1$ then the digraph G is connected.

Proof. At first we construct a connected digraph G^m $(m \geq 1)$ such that

$$\dim H^p(\Omega_{G^m}) = \begin{cases} 1, & p = 0, m \\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$

For $m = 1$ this is the digraph with the set of vertices $V = \{0, 1, 2\}$ and the set of edges $E = \{\{0, 1\}, \{1, 2\}, \{2, 1\}\}$. Then, by induction we define $G^{m+1} = SG^m$. By Theorem 4.21 it satisfies (4.19).

For any $m \geq 1$, define the digraph F^m as follows. If $k_m = 0$ then F^m consists of a single vertex, and if $k_m \geq 1$ then F^m is equal to the wedge some of k_m copies of G^m. By Theorem 4.21, we have

$$H^p(\Omega_{F^m}) = \begin{cases} 1, & p = 0, \\ k_m, & p = m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

Let F_0 be a digraph, consisting of k_0 vertices and no edges. Now define

$$G = \bigvee_{m=0,1,2,\dots,k_n} F^m.$$

Then (4.18) follows from Theorem 4.21. □

The next result can be helpful for computational purposes.

COROLLARY 4.23. *Under assumption of Theorem 4.22, there exists a digraph G with*

$$k_0 + 2k_1 + 4k_2 + 6k_3 + \cdots + (2n)k_n$$

vertices such that

$$\dim H^i(\Omega_G) = k_i, \ \forall \ 0 \leq i \leq n.$$

Proof. This follows from a direct computation of the number of vertices of the digraphs in the proof of Theorem 4.22. □

The number of vertices of G in Corollary 4.23 can be easily improved. An interesting open question is to find the minimum number of vertices of the digraph satisfying (4.18).

Now let Σ be a finite simplicial complex (see [17], [23]). Consider a digraph $G(\Sigma)$ with the set of vertices $V = \{\sigma \in \Sigma\}$ that coincides with the set of simplexes from Σ and we have an arrow $\sigma \to \tau$ if and only if $(\sigma \supset \tau) \& (\sigma \neq \tau)$. This digraph evidently gives a poset of his vertices $a \geq b$ if and only if there is arrow $a \to b$.

THEOREM 4.24. *(cf. [15]). The dual chain complex to the complex $\Omega_{G(\Sigma)}$ is isomorphic to the simplicial chain complex of the first barycentric subdivision of Σ.*

Proof. The transitivity condition for arrows $(\sigma \to \tau \to \rho) \Longrightarrow (\sigma \to \rho)$ implies for the digraph $G(\Sigma)$ that $d(\mathcal{E}) \in \mathcal{E}$, where $\mathcal{E}$ as usually an ideal generated by non-admissible elements. Hence $\mathcal{J} = \mathcal{E}$, and we can equip $\Omega_{G(\Sigma)} = \Omega_V/\mathcal{J}$ by $\mathbb{K}$-basis of admissible elements $e^{i_0 i_1 \dots i_n}$ and the differential is given by formula in Theorem 3.2 (iii) in which the sum is only by admissible elements. Hence in the dual basis $e_*^{i_0 i_1 \dots i_n}$ differential is given by the formula

$$\delta(e_*^{i_0 i_1 \dots i_n}) = \sum_{p=0}^{n} e_*^{i_0 \dots \breve{i}_p \dots i_n}. \tag{4.21}$$

But every such sequence $i_0 i_1 \dots i_n$ define a unique simplex $i_0 \to i_1 \dots \to i_n$ of the first barycentric subdivision (see, for example, [15] and [12]) with the same as in (4.21) boundary map. □

Thus the Theorem 4.22 gives a non-trivial realization theorem in contrast with the Theorem 4.24, that provides a realization theorem for the digraphs that obtained from simplicial complexes.

Now consider a digraph G with the set of vertices V and the set of edges E_G. Let H be a digraph with the set of vertices W and and the set of edges E_H.

Definition 4.25. A map of sets $F\colon V \to W$ defines a morphism of digraphs

$$f\colon G \to H$$

if for any edge $(i \to j) \in G$ we have $(F(i) \to F(j)) \in H$ is an edge of H, or $F(i) = F(j) \in W$. The last condition means that the edge $(i \to j)$ maps to the vertex $F(i) = F(j)$.

Remark 4.26. It follows from this definition, that if the edge (j, l) is non-admissible in H then for any two vertices $i, k \in V$, for which $F(i) = j, F(k) = l$, the edge (i, k) is non-admissible in G.

The set of digraphs with the morphisms given by maps from Definition 4.25 is a category which we shall denote by GR.

Let

$$\mathcal{E}_H^n \subset \mathcal{J}_H^n \subset \Omega_W^n,\ n \geq 0,\quad \mathcal{J}_H = \bigoplus_{n\geq 0} \mathcal{J}_H^n$$

be as in Definition 4.2 and Definition 4.4.

The factor algebra $\Omega_W/\mathcal{J}_H$ equipped with the induced differential is a differential calculus Ω_H on the digraph H, and by a similar way a differential calculus $\Omega_G = \Omega_V/\mathcal{J}_G$ is defined. By results of Corollary 3.5 we have a contravariant functor from the category SET to the the category DC of differential calculi.

Lemma 4.27. *Let $\mathcal{U}(F^*)\colon (\Omega_W, d) \longrightarrow (\Omega_V, d)$ be the map of chain complexes from Corollary 3.5. Then*

$$(\mathcal{U}(F^*))(\mathcal{J}_H) \subset \mathcal{J}_G$$

and hence an induced morphism of factor complexes

$$\Omega_H = \Omega_W/\mathcal{J}_H \longrightarrow \Omega_G = \Omega_V/\mathcal{J}_G,$$

which we denote by $\mathcal{U}(F^*)$*, is good defined.*

Proof. The morphism $\mathcal{U}(F^*)\colon (\Omega_W, d) \longrightarrow (\Omega_V, d)$ is given on basic elements by the rule

$$\mathcal{U}(F^*)(e^{i_0} \otimes \cdots \otimes e^{i_l}) = F^*(e^{i_0}) \otimes \cdots \otimes F^*(e^{i_l})$$

where $i_0, \ldots, i_l \in W$ and $F^*(e^j) = e^j \circ F$. By Remark 4.26

$$(\mathcal{U}(F^*))(\mathcal{E}_H) \subset \mathcal{E}_G \quad \text{and hence} \quad (\mathcal{U}(F^*))(\mathcal{J}_H) \subset \mathcal{J}_G.$$

□

THEOREM 4.28. *Let a map* $F\colon V \to W$ *be a morphism of digraphs* $G \to H$ *in the sense of Definition 4.25 and*

$$\mathcal{U}(F^*)\colon \Omega_H = \Omega_W/\mathcal{J}_H \longrightarrow \Omega_G = \Omega_V/\mathcal{J}_G$$

the above constructed morphism of differential calculi. Thus we obtain a functor from the category GR *of digraphs to the category of differential calculus* DC*.*

Proof. It is trivial to check that this is a functor. □

COROLLARY 4.29. *A morphism* $F\colon G \to H$ *of digraphs induces a homomorphism of cohomology rings* $H^*(F)\colon H^*(H, \mathbb{K}) \to H^*(G, \mathbb{K})$*. This correspondence is functorial.*

Proof. Follows from Theorems 4.28, 2.28, and 2.29. □

5. Cohomology of undirected graphs. In this section we define a natural equivalence of the subcategory of digraphs to the category of graphs. Thus we transfer the cohomology theory to the category of graphs and prove it is homotopy invariant in relation to the homotopy theory defined in [1] and [2]. Then we present several examples.

DEFINITION 5.1. A simple digraph $G = (V, E)$ is symmetric if $(v \to w) \in E_G$ implies that $(w \to v) \in E_G$.

PROPOSITION 5.2. *The symmetric digraphs with the morphisms defined in Definition 4.25 give a full subcategory* SGR *of the category* GR*.*

Proof. Direct checking. □

DEFINITION 5.3. Let $G = (V_G, E_G), H = (V_H, E_H)$ be (undirected) graphs. A morphism $f\colon G \to H$ is given by a map of vertices $f\colon V_G \to V_H$ such that if $(v, w) \in E_G$ then we have $(f(v), f(w)) \in E_H$ or $f(v) = f(w) \in W$. The last condition means that the edge (v, w) maps to the vertex $f(v) = f(w)$.

PROPOSITION 5.4. *The graphs with the morphisms defined in Definition 5.3 give a category* NGR*.*

Proof. Direct checking. □

Let $G = (V_G, E_G)$ be a graph. Define a symmetric digraph $\mathcal{S}(G) = (V_{\mathcal{S}(G)}, E_{\mathcal{S}(G)})$, where $V_{\mathcal{S}(G)} = V_G$, $(v \to w) \in E_{\mathcal{S}(G)}$ if $(v, w) \in E_G$. Any morphism $f\colon G \to H$ of graphs defines an unique morphism $\mathcal{S}(f)\colon \mathcal{S}(G) \to \mathcal{S}(H)$ of symmetric digraphs. It is easy to check that $\mathcal{S}$ is a functor from the category NGR to the full subcategory SGR (see [16]).

PROPOSITION 5.5. *The functor $\mathcal{S}$ provides an equivalence of categories $NGR \simeq SGR$ with the inverse functor $\mathcal{S}^{-1}$.*

Proof. Direct checking. □

DEFINITION 5.6. The differential calculus on a graph G is the differential calculus $\Omega_{\mathcal{S}(G)}$ on the symmetric digraph $\mathcal{S}(G)$. *The cohomology ring $H^*(G, K)$ of a graph G* is the cohomology ring $H^*(\mathcal{S}(G), \mathbb{K})$.

THEOREM 5.7. *A morphism $f\colon G \to H$ of graphs induces a homomorphism of cohomology rings $H^*(f)\colon H^*(H, \mathbb{K}) \to H^*(G, \mathbb{K})$. This correspondence is functorial.*

Proof. Follows from Corollary 4.29 and Proposition 5.5. □

EXAMPLE 5.8. The statements of the examples below can be trivially checked by direct computing or follows directly from the results of previous section.

i) For any graph G, rank $H^0(G, \mathbb{K})$ coincides with the number of connect components of G.

ii) Let a graph G be a tree. Then $H^i(G, \mathbb{K}) = 0$ for $i \geq 1$.

iii) Let $C_n = (V_{C_n}, E_{C_n})$ be a cyclic graph where

$$V_{C_n} = \{0, 1, 2, \ldots, n-1\} \quad \text{and} \quad E_{C_n} = \{(0,1), (1,2), \ldots, (n-2, n-1), (n-1, 0)\}.$$

Then $H^1(C_n, \mathbb{K}) = 0$ for $n \leq 4$ and $H^1(C_n, \mathbb{K}) = \mathbb{K}$ for $n \geq 5$; and $H^i(C_n, \mathbb{K}) = 0$ for $i \geq 2$ and any n.

iv) Let $S = (V_S, E_S)$ be a graph on Fig. 1. Then $H^2(S, \mathbb{K}) = \mathbb{K}$ and $H^i(S, \mathbb{K}) = 0$ for $i \neq 0, 2$.

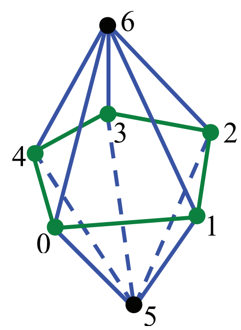

FIG. 1.

v) Let Q be the graph that is given by one-dimensional skeleton of n-dimensional cube. Then $H^i(Q, \mathbb{K}) = 0$ for $i \geq 1$. The similar result is true for a one-dimensional skeleton of any simplex.

Recall a homotopy theory of graphs constructed in [1] and [2].

Let $\mathbf{I}_n (n \geq 0)$ denote a graph with the set of vertices $V_{\mathbf{I}} = \{0, 1, \ldots, n\}$ and the set of edges $(i, i+1)$ for $0 \leq i \leq n-1$. The graph $\mathbf{I}_n$ we shall call *a line graph*. Note that by this definition a one-point graph $\mathbf{I}_0$ is a line graph. Let $\mathbf{I} = \mathbf{I}_1$.

For two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ we define a $\boxdot$- *product* $\Pi = G \boxdot H = (V_\Pi, E_\Pi)$ as a graph with a set of vertices $V_\Pi = V_G \times V_H$ and a set of edges E_Π such that

$$[(x, y), (x', y')] \in E_\Pi \quad \text{for} \quad x, x' \in V_G; y, y' \in V_H$$

if one of the conditions is satisfied

$$x' = x, \ (y, y') \in E_H \quad \text{or} \quad y' = y, \ (x, x') \in E_G.$$

DEFINITION 5.9. [2, Definition 5.1] i) Two maps

$$f_i \colon G \to H, \quad i = 0, 1$$

of a graph G to a graph H are *homotopic* if there exists a line graph $\mathbf{I}_n$ and a morphism

$$F \colon G \boxdot \mathbf{I}_n \to H$$

such that

$$F|_{G \boxdot \{0\}} = f_0 \colon G \boxdot \{0\} \to H, \quad F|_{G \boxdot \{n\}} = f_1 \colon G \boxdot \{n\} \to H.$$

In this case we shall write $f_0 \simeq f_1$.

ii) Two graphs G and H are *homotopy equivalent* if there exist maps

$$f \colon G \to H, \quad g \colon H \to G$$

such that

$$f \circ g \simeq \mathrm{Id}_H, \quad g \circ f \simeq \mathrm{Id}_G \,.$$

In this case we shall write $H \simeq G$. In this case maps f and g are called *homotopy inverses* of each other.

Now we state and prove the theorem that answer on the question from [2, page 32] about construction of natural homotopy invariant homology theory for graphs.

THEOREM 5.10. *i) Let $f \simeq g \colon G \to H$ be two homotopy equivalent maps of graphs. Then these maps induce the equal homomorphisms of cohomology groups.*

ii) Let $G \simeq H$ be two homotopy equivalent graphs. Then they have isomorphic cohomology groups.

Proof. Let $F \colon G \times \mathbf{I} \to H$ be a homotopy between f and g, such that

$$F|_{G \boxdot \{0\}} = f \colon G \to H, \quad F|_{G \boxdot \{1\}} = g \colon G \to H.$$

The morphisms f, g, and F induce morphisms of cochain complexes $f^*, g^* \colon \Omega_{\mathcal{S}(H)} \to \Omega_{\mathcal{S}(G)}$, and

$$F^* \colon \Omega_{\mathcal{S}(H)} \to \Omega_{\mathcal{S}(G \boxdot \mathbf{I})} = \Omega_{\mathcal{S}(G) \boxdot \mathcal{S}(\mathbf{I})}.$$

Let Φ be a composition of morphisms

$$\mathcal{S}(G) \boxdot I_1 \longrightarrow \mathcal{S}(G) \boxdot \mathcal{S}(\mathbf{I}) = \mathcal{S}(G \boxdot \mathbf{I}) \xrightarrow{F} \mathcal{S}(H)$$

where the first morphism is the natural inclusion of digraphs and I_1 is a digraph with $V_{I_1} = \{0, 1\}, E_{I_1} = (0 \to 1)$. Then we have a also a morphism $\Phi^* \colon \Omega_{\mathcal{S}(H)} \to \Omega_{\mathcal{S}(G)\Box I_1}$ of cochain complexes.

Consider the chain complexes $\Omega^*_{\mathcal{S}(G)}, \Omega^*_{\mathcal{S}(H)}, \Omega^*_{\mathcal{S}(G)\Box I_1}$ that are dual to the cochain complexes $\Omega_{\mathcal{S}(G)}, \Omega_{\mathcal{S}(H)}, \Omega_{\mathcal{S}(G)\Box I_1}$, correspondingly. Denote by δ the differentials in the chain complexes, since it is clear from the context what chain complex is under consideration. The morphisms f, g induce morphisms of chain complexes $f_*, g_* \colon \Omega^*_{\mathcal{S}(G)} \to \Omega^*_{\mathcal{S}(H)}$, and the morphism Φ induces a morphism $\Phi_* \colon \Omega^*_{\mathcal{S}(G)\Box I_1} \to \Omega^*_{\mathcal{S}(H)}$. Consider a $\mathbb{K}$-module morphism

$$P \colon \Omega^n_{\mathcal{S}(G)\Box I_1} \to \Omega^{n-1}_{\mathcal{S}(G)}$$

that is dual to the $\mathbb{K}$-module morphism

$$\left[\Omega^*_{\mathcal{S}(G)}\right]_{n-1} \longrightarrow \left[\Omega^*_{\mathcal{S}(G)\Box I_1}\right]_n, \text{ given by } \ x \longrightarrow x \times I_1,$$

that is defined in [13]. Let

$$L_n = (-1)^n P \circ \Phi^* \colon \Omega^n_{\mathcal{S}(H)} \to \Omega^{n-1}_{\mathcal{S}(G)}.$$

Let $w^n \in \Omega^n_{\mathcal{S}(H)}$, $x \in \left[\Omega^*_{\mathcal{S}(G)}\right]_n$. Then, using computation in [13], we have

$$[(\partial L_n + L_{n+1}\partial)(w^n)](x) = (-1)^n \left[\partial P \Phi^*(w^n)\right](x) + (-1)^{n+1}[P\Phi^*\partial(w^n)](x).$$

But δ is dual to ∂ an P is dual to $x \to x \times I_1$, thus the last equals to

$$\begin{aligned} &(-1)^n \left[P\Phi^*(w^n)\right](\delta x) + (-1)^{n+1}[\Phi^*\partial(w^n)](x \times I_1) \\ &= (-1)^n \left[\Phi^*(w^n)\right](\delta x \times I_1) + (-1)^{n+1}[\Phi^*\partial(w^n)](x \times I_1) \end{aligned}$$

(since $\partial\Phi^* = \Phi^*\partial$, $\delta\Phi_* = \Phi_*\delta$, and Φ^* is dual to Φ_*)

$$\begin{aligned} &= (-1)^n w^n[\Phi_*(\delta x \times I_1)] + (-1)^{n+1}[\partial(w^n)][\Phi_*(x \times I_1)] \\ &= (-1)^n w^n[\Phi_*(\delta x \times I_1)] + (-1)^{n+1} w^n[\delta\Phi_*(x \times I_1)] \\ &= (-1)^n w^n[\Phi_*(\delta x \times I_1 - \delta(x \times I_1))] \\ &= (-1)^n w^n \left[\Phi_* \left(\delta(x \times I_1) - \delta x \times I_1\right)\right] \end{aligned}$$

(by [13, Prop 7.3])

$$\begin{aligned} &= (-1)^n w^n \Phi_* \left((-1)^n x \times \delta I_1 + \delta x \times I_1 - \delta x \times I_1\right) \\ &= w^n \left[\Phi_*(x \times \{1\} - x \times \{0\})\right] \\ &= w^n[(g_* - f_*)(x)] = [(f^* - g^*)(w^n)](x). \end{aligned}$$

From now the result follows by [22, Theorem 2.1] for $\mathbf{I}_n = \mathbf{I}_1$. The general case follows by induction. $\square$

6. Cohomology acyclic digraphs.

Definition 6.1. (i) A complete acyclic digraph Γ is a finite simple digraph with a set of vertices $V = \{0, 1, 2, \ldots, n\}$ and the set of directed edges

$$E = \{\{i, j\} : i < j; i, j = 0, 1, 2, \ldots, n\}.$$

(ii) An acyclic digraph G is any subgraph of a complete acyclic digraph Γ.

We have a natural inclusion $\gamma\colon \Gamma \longrightarrow H$, where H is a full finite simple digraph with the set V of vertices defined in Section 4.

By Definition 4.2 and Proposition 4.3 of Section 4 we have a graded ideal

$$\mathcal{E}_\gamma = \bigoplus_{p\geq 0} \mathcal{E}_\gamma^p \subset \bigoplus_{p\geq 0} \Omega_V^p = \Omega_V, \tag{6.1}$$

where $\mathcal{E}_\gamma^0 = \{0\}$ and $\mathcal{E}_\gamma^p$ for $p \geq 1$ is generated by non-allowed elements.

Recall that $\mathcal{A}$ is an algebra of $\mathbb{K}$-valued functions on the set V.

Proposition 6.2. *For $p \geq 0$ we have $d\mathcal{E}_\gamma^p \subset \mathcal{E}_\gamma^{p+1}$, and the differential calculus $(\Omega_\Gamma, d_\Gamma)$ from Definition 4.6 coincides with the calculus $(\Omega_V/\mathcal{E}_\gamma, d)$. In particular, we have an exact sequence of cochain complexes*

$$0 \longrightarrow \mathcal{E}_\gamma \longrightarrow \Omega_V \longrightarrow \Omega_\Gamma \longrightarrow 0.$$

Proof. For $p \geq 1$, an element $e^{i_0 i_1 \dots i_p} \in \mathcal{E}_\gamma^p$ is non-allowed if and only the sequence $\{i_0, i_1, \dots, i_p\}$ is non monotonic increasing. Now the result follows from description of differential on basic elements in Theorem 3.2. □

Corollary 6.3. *(i) The basic elements of the differential calculus $(\Omega_\Gamma, d_\Gamma)$ of the digraph Γ can be represented by classes of elements $e^{i_0 i_1 \dots i_p} \in \Omega_V^p$ such that $0 \leq i_0 < \dots < i_p \leq n$.*

(ii) For $0 \leq k, l \leq n$, the exterior multiplication $\bullet$ of basic elements is given by the following formula

$$(e^{i_0 i_1 \dots i_k}) \bullet (e^{j_0 j_1 \dots j_l}) = \begin{cases} 0, & i_k \neq j_0 \\ e^{i_0 i_1 \dots i_k j_1 \dots j_l}, & i_k = j_0. \end{cases}$$

(iii) The differential d_Γ is given on basic elements by

$$d_\Gamma\left(e^{i_0 i_1 \dots i_k}\right) = \widetilde{\sum_{j \neq i_0}} e^{j i_0 i_1 \dots i_k} - \widetilde{\sum_{j\neq i_0; j \neq i_1}} e^{i_0 j i_1 \dots i_k} + \dots$$

$$+ (-1)^{l+1} \widetilde{\sum_{j \neq i_l; j \neq i_{l+1}}} e^{i_0 i_1 \dots i_l j i_{l+1} \dots i_k} + \dots + (-1)^{k+1} \widetilde{\sum_{j \neq i_k}} e^{i_0 i_1 \dots i_k j},$$

where $\widetilde{\ }$ over the sign $\sum$ means that in summation are presented only the elements with strongly monotonic increasing sets of indices.

Proof. Follows from Theorem 3.2 and the proof of Proposition 6.2. □

We shall omit subscript Γ in the differential, if it is clear from context what cochain complex we consider.

Corollary 6.4. *For $k > n$ we have $\Omega_\Gamma^k = 0$, and the maps $\mathcal{E}_\gamma^k \longrightarrow \Omega_H^k$ are isomorphisms.*

Proof. The space Ω_Γ^k is generated by basic elements $e^{i_0 i_1 \dots i_k}$, where $0 \leq i_0 < i_1 < \dots < i_k \leq n$. Any finite sequence of more than $n + 1$-elements from the set

$V = \{0, 1, 2, \ldots, n\}$ has at least two equal elements. Now the statement follows from Corollary 6.3. $\square$

From now we describe several non-trivial cases in which it is possible to reduce computing the homology of an acyclic digraph to the homology of a simplicial complex. Recall that one of such examples is given by Theorem 4.24. For the definition and basic properties of homology and cohomology groups of a simplicial complex we refer to [17] and [23].

Let $(\Omega_V/\mathcal{E}_\gamma, d) = (\Omega_\Gamma, d)$ be the differential calculus on the complete monotonic digraph Γ with the set of vertices $V = \{0, 1, \ldots, n\}$, and Ω^*_Γ be the dual chain complex with the basis $e^*_{i_0 i_1 \ldots i_p}$ $(0 \le i_0 < i_1 < \cdots < i_p \le n)$ which is dual to the basis described in Corollary 6.3 and with the boundary operator $\delta\colon [\Omega_\Gamma]^*_p \longrightarrow [\Omega_\Gamma]^*_{p-1}$.

Let Δ be simplicial complex consisting of a n-dimensional simplex $\Delta^n = [0, 1, \ldots, n]$ and all its faces $[i_0, i_1, \ldots, i_k]$ that are given by increasing subsequences $i_0 < i_1 < \cdots < i_k$ of $0 < 1 < \cdots < n$, and $C(\Delta^n)$ be a chain complex with k-dimensional modules $C_k(\Delta)$ generated by k-simplexes of Δ^n and the standard boundary map $\partial\colon C_k(\Delta) \to C_{k-1}(\Delta)$.

Theorem 6.5. *The boundary operator $\delta\colon [\Omega_\Gamma]^*_p \longrightarrow [\Omega_\Gamma]^*_{p-1}$ is given on the basic elements by the rule*

$$\delta(e^*_{i_0 i_1 \ldots i_p}) = \sum_{0 \le k \le p} (-1)^k e^*_{i_0 \ldots i_{k-1} \hat{i}_k i_{k+1} \ldots i_p}$$

where $\hat{i}_k$ means omitting the symbol i_k from the multiindex. For $k \ge 0$ the maps $T_k\colon [\Omega_\Gamma]^*_k \to C_k(\Delta^n)$, of $\mathbb{K}$-modules given on basic elements by formulas

$$e^*_{i_0 i_1 \ldots i_p} \longrightarrow [i_0, i_1, \ldots, i_p],\ 0 \le i_0 < i_1 < \cdots < i_p \le n.$$

commute with differentials and define an isomorphism between the chain complexes

$$T = \bigoplus_k T_k\colon \Omega^*_\Gamma \to C(\Delta^n).$$

Proof. Let $e^{j_0 j_1 \cdots j_{p-1}}$ be a basic element of Ω^{p-1}_Γ. Then

$$\begin{aligned}
[\delta(e^*_{i_0 i_1 \ldots i_p})](e^{j_0 j_1 \cdots j_{p-1}}) &= e^*_{i_0 i_1 \ldots i_p}(d e^{j_0 j_1 \cdots j_{p-1}}) \\
&= e^*_{i_0 i_1, \ldots i_p} \left[\sum_{q=0}^{p} \widetilde{\sum_k} (-1)^q e^{j_0 j_1 \ldots, j_{q-1} k j_q \ldots j_{p-1}} \right] \\
&= \sum_{q=0}^{p} \widetilde{\sum_k} (-1)^q e^*_{i_0 i_1 \ldots i_p} \left[e^{j_0 j_1 \ldots, j_{q-1} k j_q \ldots j_{p-1}} \right]
\end{aligned}$$

where $\widetilde{}$ means that only elements with monotonic multiindices are used in the summation. We have

$$e^*_{i_0 i_1 \ldots i_p} \left[e^{j_0 j_1 \ldots j_{q-1} k j_q \ldots j_{p-1}} \right] = 1 \quad \text{only if} \quad \{i_0 i_1 \ldots i_p\} = \{j_0 j_1 \ldots j_{q-1} k j_q \ldots j_{p-1}\}$$

for some place q. This means that the sequence $\{j_0, j_1, \ldots, j_{p-1}\}$ is obtained from the sequence $\{i_0, i_1, \ldots, i_p\}$ by deleting a term $i_q = k$. For such basic elements we have

$$[\delta(e^*_{i_0 i_1 \ldots i_p})](e^{i_0 i_1 \ldots i_{q-1} i_{q+1} \ldots i_p}) = (-1)^q,\ 0 \le q \le p.$$

Hence, we obtain

$$\delta(e^*_{i_0 i_1 \dots i_p}) = (-1)^q \sum_{0\leq q\leq p} e^*_{i_0 i_1 \dots i_{q-1}\hat{i}_q i_{q+1}\dots i_p}$$

which finishes the proof of the first statement of the theorem. The second statement follows from this one since T gives one-to-one correspondence between basic elements commuting with differentials. ∎

Corollary 6.6. *Under the assumptions of Theorem 6.5 we have*

$$H^p(\Omega_\Gamma) = \begin{cases} \mathbb{K}, & p = 0 \\ 0, & p \geq 1. \end{cases}$$

Now let G be an acyclic digraph with a set of vertices $V = \{0, 1, 2, \dots, n\}$, Γ be the complete acyclic graph with the same set of vertices V and H be the complete simple digraph with the set of vertices V.

We have a commutative diagram of inclusions of digraph as (4.2)

$$\begin{array}{ccc} G & \xrightarrow{s} & \Gamma \\ & \searrow^{g} \quad {}^{\gamma}\swarrow & \\ & H & \end{array} \tag{6.2}$$

The exact sequence of chain complexes (4.3) has the following form

$$0 \longrightarrow \mathcal{J}_g/\mathcal{E}_\gamma \longrightarrow \Omega_\Gamma \longrightarrow \Omega_G \longrightarrow 0. \tag{6.3}$$

Theorem 6.7. *For $p \geq 1$, we have an isomorphism $H^p(\Omega_G) \cong H^{p+1}(\mathcal{J}_g/\mathcal{E}_\gamma)$ and an exact sequence*

$$0 \longrightarrow \mathbb{K} \longrightarrow H^0(\Omega_G) \longrightarrow H^1(\mathcal{J}_g/\mathcal{E}_\gamma) \longrightarrow 0.$$

Proof. Follows from Corollary 6.6 and exact sequence (6.3). ∎

Definition 6.8. Let $\mathcal{E}^0_s = 0$ and $\mathcal{E}^p_s$, $p \geq 1$ be a subspace of Ω^p_Γ generated by those $e^{i_0 i_1 \dots i_p} \in \Omega^p_\Gamma$ that are non-allowed elements for the digraph G. Let

$$\mathcal{J}^p_s = \mathcal{E}^p_s + d_\Gamma \mathcal{E}^{p-1}_s \subset \Omega^p_\Gamma$$

where d_Γ is the differential in Ω_Γ described in Corollary 6.3. Denote

$$\mathcal{E}_s = \bigoplus_{0\leq i\leq n} \mathcal{E}^i_s,\ \mathcal{J}_s = \bigoplus_{0\leq i\leq n} \mathcal{J}^i_s.$$

Proposition 6.9. *The submodule $\mathcal{J}_s \subset \Omega_\Gamma$ is a graded ideal such that the inclusion is a morphism of chain complexes and the exact sequence*

$$0 \longrightarrow \mathcal{J}_s \longrightarrow \Omega_\Gamma \longrightarrow \Omega_\Gamma/\mathcal{J}_s \longrightarrow 0 \tag{6.4}$$

is isomorphic to the exact sequence in (6.3), *and hence $\Omega_\Gamma/\mathcal{J}_s \cong \Omega_G$.*

Proof. The subspace $\mathcal{E}_s \in \Omega_\Gamma$ is a graded ideal. As in the proof of Theorem 2.22, we can see that $\mathcal{J}_s \subset \Omega_\Gamma$ is a graded ideal and the inclusion is a morphism of chain complexes. By definition, we have a graded isomorphism $\Omega_V/\mathcal{E}_\gamma \longrightarrow \Omega_\Gamma$. It follows from Definition 6.8 and Corollary 6.3 that a restriction of this map to $\mathcal{J}_g/\mathcal{E}_\gamma \subset \Omega_V/\mathcal{E}_\gamma$ correctly defines a graded isomorphism $\mathcal{J}_g/\mathcal{E}_\gamma \longrightarrow \mathcal{J}_s$ which agrees with the differentials. The Proposition is proved. □

Let Γ be a complete acyclic digraph with the set $V = \{0, 1, 2, \ldots, n\}$ of vertices and the set of edges

$$E = \{\{i, j\} : i < j; i, j = 0, 1, 2, \ldots, n\}.$$

Let F and G be acyclic digraphs with the same number of vertices such that we have a commutative diagram of inclusions

$$\begin{tikzcd} F \arrow[rr, "r"] \arrow[dr, "t"'] & & G \arrow[dl, "s"] \\ & \Gamma & \end{tikzcd} \tag{6.5}$$

where the horizontal map is an inclusion $r \colon F \longrightarrow G$ and swallow maps are inclusions into Γ.

THEOREM 6.10. *Let $\mathcal{E}_s$ and $\mathcal{E}_t$ be the subspaces generated by non-admissible elements in Ω_Γ for the inclusions s and t respectively,, and $\mathcal{J}_s \subset \Omega_\Gamma$, $\mathcal{J}_t \subset \Omega_\Gamma$ are the ideals defined in Definition* 6.8. *Then we have the inclusions of chain complexes*

$$\mathcal{J}_s \subset \mathcal{J}_t \subset \Omega_\Gamma,$$

which induce a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_t/\mathcal{J}_s \longrightarrow \Omega_\Gamma/\mathcal{J}_s \longrightarrow \Omega_\Gamma/\mathcal{J}_t \longrightarrow 0 \tag{6.6}$$

where $(\Omega_\Gamma/\mathcal{J}_s, d) = (\Omega_G, d)$ is a differential calculus of the digraph G and $(\Omega_\Gamma/\mathcal{J}_t, d) = (\Omega_F, d)$ is a differential calculus of the digraph F. The cohomology long exact sequence of (6.6) has the following form

$$0 \longrightarrow H^0(\Omega_G) \longrightarrow H^0(\Omega_F) \longrightarrow H^1(\mathcal{J}_t/\mathcal{J}_s) \longrightarrow H^1(\Omega_G) \longrightarrow \ldots \tag{6.7}$$

Proof. Similarly to the proof of Theorem 4.9. □

REMARK 6.11. In the case of an inclusion $r \colon F \to G$ from (6.5) we can write down a commutative diagram similarly to (4.2)

$$\begin{tikzcd} F \arrow[rr, "r"] \arrow[dr, "f"'] & & G \arrow[dl, "g"] \\ & H & \end{tikzcd}$$

where H is a simple complete digraph. Theorem 4.9 is applicable in this situation, as well. The advantage of Theorem 6.10 is in simplification of all computations, since we can work with very small number of basic elements directly described in Corollary 6.3.

Remark 6.12. The acyclic digraphs with the morphisms defined in Definition 4.25 give a full subcategory MGR of the category GR. The realization Theorem 4.22 is true in the category of acyclic digraphs. It is easy to see, that we can define suspension and a wedge sum in the category of acyclic digraphs. Now let D_2 be an acyclic digraph that has two vertices and no edges. Then

$$H^k(D_2,\mathbb{K}) \cong \begin{cases} \mathbb{K}^2, & k=0 \\ 0, & k\neq 0 \end{cases}, \qquad H^k(SD_2,\mathbb{K}) \cong \begin{cases} \mathbb{K}, & k=0,1 \\ 0, & k\geq 2. \end{cases}$$

Now we can repeat all the constructions from the proof of Theorem 4.22 in the category of acyclic digraphs.

Example 6.13. For $n \geq 2$ and any digraph G define n-suspension inductively by $S^n(G) = S(S^{n-1}G)$. Let D_2 be the digraph from Remark 6.12. Then for $n \geq 1$ we have

$$H^0(S^nD_2,\mathbb{K}) = H^n(S^nD_2,\mathbb{K}) = \mathbb{K}$$

and $H^i(S^nD_2,\mathbb{K}) = 0$ for $i \neq 0, n$.

Now we describe sufficiently wide classes of acyclic digraphs for which there exists a geometric realization of cohomology theory by cohomology theory of simplicial complexes.

Let Γ be a complete acyclic digraph with the set $V = \{0,1,\dots,n\}$ and Θ be a complete acyclic digraph with the set $W = \{0,1,\dots,k\}$ ($k \leq n$) of vertices. We have a natural inclusion $\sigma\colon \Theta \to \Gamma$. By Lemma 4.10 we have a morphism of cochain complexes $\mathcal{U}(\sigma)\colon \Omega_\Gamma \to \Omega_\Theta$ which induces a morphism of chain complexes $\mathcal{U}(\sigma)^*\colon \Omega_\Theta^* \to \Omega_\Gamma{*}$ by the standard rule

$$\mathcal{U}(\sigma)^*(f) = f\circ \mathcal{U}(\sigma), \quad f\colon \Omega_\Theta \to \mathbb{K}.$$

Proposition 6.14. *There exists a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega_\Theta^* & \xrightarrow{\mathcal{U}(\sigma)^*} & \Omega_\Gamma{*} \\ \downarrow T & & \downarrow T \\ C(\Delta^k) & \xrightarrow{\tau_*} & C(\Delta) \end{array}$$

where τ_* is induced by natural inclusion $\tau\colon \Delta^k \to \Delta^n$ on the first k-face.

Proof. Follows from Proposition 3.6, Corollary 6.3, and Theorem 6.5. ∎

Let Γ be a complete acyclic digraph with the set $V = \{0,1,\dots,n\}$ of vertices and the set E_Γ of edges, and let $s\colon G_k \to \Gamma$ be the natural inclusion of the sub-digraph G_k with the same set of vertices and the set of edges $E_k = E_{G_k} = E_\Gamma \setminus \{\{k,k+1\}\}$ where k is a number $0 \leq k \leq n-1$. That is the digraph G_k is obtained from Γ by deleting exactly one edge $\{k,k+1\}$.

Proposition 6.15. *(i) Let $\mathcal{E}_s$, $\mathcal{J}_s$ be subspaces of Ω_Γ from Definition 6.8. Then $\mathcal{J}_s^p = \mathcal{E}_s^p$ for $p \geq 0$.*

(ii) The basic elements of the differential calculus $(\Omega_{G_k}, d_{G_k}) = (\Omega_\Gamma/\mathcal{J}_s, d)$ of the digraph G_k can be represented by classes $[e^{i_0i_1\dots i_p}] \in \Omega^p_{G_k}$ of elements $e^{i_0i_1\dots i_p} \in \Omega^p_\Gamma$ such that $0 \leq i_0 < i_1 < \dots < i_p \leq n$ and $\{k,k+1\}$ is not a subset of $\{i_0,i_1,\dots,i_p\}$.

The exterior multiplication and differential are described by Corollary 6.3, where in summing for differential we must delete summands $\pm e^{i_0,\dots i_m}$ *in which* $\{k, k+1\}$ *is a subset of* $\{i_0, i_1, \dots, i_m\}$.

Proof. For $p = 0, 1$ the statement follows from Definition 6.8. In the case $p \geq 2$ we have the inclusion $\mathcal{E}_s^p \subset \mathcal{J}_s^p$. Any basic element of $\mathcal{E}_s^{p-1}$ has the form $w = e^{i_0 i_1 \dots [k][k+1] \dots i_{p-1}}$. To finish the proof we note that dw is described in Corollary 6.3 (iii), where the summing contains only elements with strongly monotonic increasing sets of indices. Since we can not put integer number between k and $k+1$ to obtain monotonic increasing sequence, the all elements e^{α} in the sum satisfy condition $\{k, k+1\} \subset \{\alpha\}$. Hence $dw \in \mathcal{E}_s^p$. From now the Proposition follows. □

Let Δ be a simplicial complex given by the simplex $\Delta^n = [0, 1, \dots, n]$ and all its faces. Denote by Δ_k the k-th face $\Delta_k = [0, 1, \dots \check{k}, \dots n]$, and let

$$\Delta_{k,k+1} = \Delta_k \cup \Delta_{k+1}$$

be a simplicial complex that geometrically corresponds to the union of two $(n-1)$-faces of Δ^n and we have a natural inclusion $\tau\colon \Delta_{k,k+1} \to \Delta$.

THEOREM 6.16. *We have a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega^*_{G_k} & \overset{\mathcal{U}(s)^*}{\longrightarrow} & \Omega_{\Gamma}* \\ \downarrow T' & & \downarrow T \\ C\left(\Delta_{k,k+1}\right) & \overset{\tau_*}{\longrightarrow} & C(\Delta) \end{array} \tag{6.8}$$

where T' *and* T *are isomorphisms, and* τ_* *is induced by the natural inclusion* τ.

Proof. In diagram (6.8) the right vertical isomorphism and the horizontal morphisms are already defined. We must define T' and check the commutativity.

Consider a basic element $[e_{i_0 \dots i_p}]^* \in [\Omega_{G_k}]_p^*$ that is dual to $[e^{i_0 \cdots i_p}] \in \Omega^p_{G_k}$ described in Proposition 6.15. Define T' on the basic elements by

$$T'([e_{i_0 \dots i_p}]^*) = [i_0, \dots, i_p] \subset C^p(\Delta).$$

and extend to $[\Omega_{G_k}]_p^*$ by linearity.

The map T' is evidently a one-to-one correspondence between basic elements. Checking that it commutes with differentials and commutativity of diagram 6.8 is routine. □

Denote by $\Gamma = (V, E)$ a complete acyclic digraph with a set of vertices $V = \{0, 1, 2, \dots, n\}$. Let $K \subset V$ be a subset such that $n \notin K$. Consider an acyclic sub-digraph $s\colon G_K \to \Gamma$ with the same number of vertices and the set of edges

$$E_K = E_{G_K} = E_{\Gamma} \setminus \{\{i \to i+1\} |\ i \in K\}.$$

THEOREM 6.17. *There exists a simplicial complex* S_K *with an inclusion* $\tau\colon S_K \to \Delta$ *such that the following diagram is commutative*

$$\begin{array}{ccc} \Omega^*_{G_K} & \overset{\mathcal{U}(s)^*}{\longrightarrow} & \Omega_{\Gamma}* \\ \downarrow T' & & \downarrow T \\ C\left(S_K\right) & \overset{\tau_*}{\longrightarrow} & C(\Delta) \end{array} \tag{6.9}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion τ.

Proof. The proof is based on the same line of arguments as the proof of Theorem 6.16, that we briefly repeat. We can check directly that $\mathcal{J}_s = \mathcal{E}_s$ in this case. The basic elements of the differential calculus $(\Omega_{G_K}, d_K) = (\Omega_\Gamma/\mathcal{J}_s, d)$ of the digraph G_K can be represented by classes $[e^{i_0 i_1 \ldots i_p}] \in \Omega^p_{G_K}$ of elements $e^{i_0 i_1 \ldots i_p} \in \Omega^p_\Gamma$ such that $0 \leq i_0 < i_1 < \cdots < i_p \leq n$ and, for any $k \in K$, $\{k, k+1\}$ is not a subset of $\{i_0, i_1, \ldots, i_p\}$. The differential d_K is given on the basic elements by

$$
\begin{aligned}
d_K \left[e^{i_0 i_1 \ldots i_p}\right] = \widehat{\sum_{j \neq i_0}} [e^{j i_0 i_1 \ldots i_p}] - \widehat{\sum_{j \neq i_0; j \neq i_1}} [e^{i_0 j i_1 \ldots i_p}] + \ldots \\
+(-1)^{l+1} \widehat{\sum_{j \neq i_l; j \neq i_{l+1}}} [e^{i_0 i_1 \ldots i_l j i_{l+1} \ldots i_p}] + \cdots + (-1)^{k+1} \widehat{\sum_{j \neq i_p}} [e^{i_0 i_1 \ldots i_p j}]
\end{aligned}
$$

where $\widehat{\ }$ means that every element $e^{i_0, \ldots i_m}$ of the sum satisfies the conditions $0 \leq i_0 < \cdots < i_m \leq n$ and, for any $k \in K$, the pair $\{k, k+1\}$ is not a subset of $\{i_0, i_1, \ldots, i_m\}$. Let for a number $\Delta_{k,k+1}$ $(k \in K)$ be a simplicial complex, defined above, that geometrically corresponds to union of two $(n-1)$-faces of Δ^n. It is given by the union of all simplexes from Δ^n that does not contain the edge $\{k, k+1\}$. Set

$$
S_K = \bigcap_{k \in K} \Delta_{k,k+1} .
$$

Equivalently, S_K can be described as the union of all the simplexes from Δ^n that do not contain an edge in the form $\{k, k+1\}$ for $k \in K$. In the diagram (6.9) we must define a chain map T' and check commutativity. Define T' on the basic elements as in the proof of Theorem 6.16 and extend to $[\Omega_{G_K}]^*_p$ by linearity. We obtain an isomorphism $T' \colon [\Omega_{G_K}]^* \longrightarrow C(S_K)$ of $\mathbb{K}$-modules which commutes with differentials and it is easy to check that diagram (6.9) is commutative. □

Now let Γ be a complete acyclic digraph with the set $V = \{0, 1, \ldots, n\}$ of vertices $(n \geq 2)$ and the set E_Γ of edges. Let $s \colon F_k \to \Gamma$ be the natural inclusion of the sub-digraph F_k with the same set of vertices and the set of edges

$$
E_k = E_{F_k} = E_\Gamma \setminus \{(k \to k+2)\} \quad \text{where } \ 0 \leq k \leq n-2.
$$

THEOREM 6.18. *There exists a simplicial complex $\Delta_{k,k+2}$ with an inclusion $\tau \colon \Delta_{k,k+2} \to \Delta$ such that the following diagram is commutative*

$$
\begin{array}{ccc}
\Omega^*_{F_K} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_\Gamma * \\
\downarrow T' & & \downarrow T \\
C\left(\Delta_{k,k+2}\right) & \xrightarrow{\tau_*} & C(\Delta)
\end{array}
\tag{6.10}
$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion τ.

Proof. It is easy to check that in this case

$$
\mathcal{J}^0_s = \{0\}, \quad \mathcal{J}^p_s = \langle e^{i_0 \ldots i_p} | \{k, k+2\} \subset \{i_0, i_1, \ldots, i_p\}, p \geq 1 \rangle.
$$

The basic elements of $[\Omega_{F_k}]^*_m$ are given by the classes $[e^*_{i_0 \ldots i_m}]$, where $\{i_0, \ldots, i_m\}$ does not contain the pair $\{i_l, i_{l+1}\} = \{k, k+2\}$. Then the proof is finished similarly to that of Theorems 6.16 and 6.17. □

Now let $\Gamma = (V, E)$ be a complete acyclic digraph as above, and $K \subset V$ be a subset such that $n \notin K$, and $L \subset V$ such that $n \notin L$ and $(n-1) \notin L$. Consider an acyclic sub-digraph $s\colon G_{K,L} \to \Gamma$ with the same number of vertices as Γ and the set of edges

$$E_K = E_{G_K} = E_\Gamma \setminus [\{(i \to i+1)|i \in K\} \cup \{(j \to j+2)|j \in L\}]$$

THEOREM 6.19. *There exists a simplicial complex $S_{K,L}$ with an inclusion $\tau\colon S_{K,L} \to \Delta^n$ such that there exists a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega^*_{G_{K,L}} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_\Gamma * \\ \downarrow T' & & \downarrow T \\ C\left(S_{K,L}\right) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array} \tag{6.11}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion τ.

Proof. The simplicial complex $S_{K,L}$ is defined as

$$S_{K,L} = \left(\bigcap_{k \in K} \Delta_{k,k+1}\right) \bigcap \left(\bigcap_{l \in L} \Delta_{l,l+2}\right).$$

The same line of arguments as in the proof of Theorems 6.16, 6.17, and 6.18 finishes the proof. □

COROLLARY 6.20. *Let G be a digraph G_k or F_k from Theorems* 6.16*,* 6.18*. Then $H^0(G, \mathbb{K}) = \mathbb{K}$ and $H^i(G, \mathbb{K}) = 0$ for $i \geq 1$.*

Theorem 6.19 obviously reduces computation of cohomology for a wide class of digraphs to that of simplicial complexes.

EXAMPLE 6.21. Let Γ be a complete acyclic digraph with the set of vertices $V = \{0, 1, 2, 3, 4\}$ and let G be the digraph that is obtained from Γ by removing the edges $(1 \to 2), (2 \to 3), (1 \to 3)$. Then G satisfies the hypotheses of Theorem 6.19 and, hence, can be realized as a simplicial complex. Let G_0 be the digraph that is obtained from G by further removing the edge $(0 \to 4)$. Digraph G_0 does not satisfy the hypothesis of Theorem 6.19, and one can show that it does not admit a geometric realization as a simplicial complex. The chain complex of the digraph G_0 was explicitly described in [15].

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