Asian J. Math.
Vol. 19, No. 5, pp. 871–886, November 2015

PERTURBATION OF BAUM-BOTT RESIDUES*

FILIPPO BRACCI† AND TATSUO SUWA‡

In memory of Marco Brunella

Abstract. We prove that Baum-Bott residues vary continuously in an appropriate sense under smooth deformations of holomorphic foliations. This provides an effective way of computing residues.

Key words. Holomorphic foliations, Baum-Bott residues, deformations.

AMS subject classifications. 37F75, 32S65, 32A27.

1. Introduction. A holomorphic foliation $\mathcal{F}$ on a complex manifold M is known to produce a "holomorphic action", as discovered by P. Baum and R. Bott in [4], on the virtual bundle $TM/\mathcal{F}$. Such a partial holomorphic action provides a holomorphic connection for the bundle $TM/\mathcal{F}$ along $\mathcal{F}$ outside the singularities of $\mathcal{F}$ and thus produces localization of sufficiently high degree classes of $TM/\mathcal{F}$ around the singularities of $\mathcal{F}$. Such localizations give rise to the "Baum-Bott residues" (see [4, Thm. 2], [11, Ch.VI, Thm. 3.7]). When the singularity is isolated the Baum-Bott residue can be expressed in terms of a Grothendieck residue (see [4, (0.6)]). When the singular set is non-isolated in some cases some formulas are available (see [4, Thm. 3] and [5]) but, in general, explicit computation of the residues is rather difficult.

The aim of the present paper is to study the behavior of the Baum-Bott residues under smooth deformations. This provides an effective tool for computing residues explicitly.

More in details, we consider a smooth deformation of a complex manifold. This is essentially a smooth fibration over a smooth manifold, whose fibers are complex manifolds (see Section 2). On each such a fiber we consider a holomorphic foliation which varies smoothly (see Section 3). We prove that the Baum-Bott residues (when taken together suitably) vary continuously under smooth deformations.

We state here a simple consequence of our main Theorem 5.4 for the case of classes of top degree, referring the reader to Section 5 for the general case. Thus, let P be a real manifold, the "parameter space". Let $\widetilde{M} := \{M_t\}_{t\in P}$, be a deformation of complex manifolds of dimension n. Let $\widetilde{\mathcal{F}} := \{\mathcal{F}_t\}$ be a deformation of holomorphic foliations on M_t. Then $\widetilde{\mathcal{F}}$ defines naturally a smooth foliation on $\widetilde{M}$ (see Section 3).

Suppose the singular set S_{t_0} of $\mathcal{F}_{t_0}$ in M_{t_0} is compact and connected. The analytic set S_{t_0} is contained in a connected component in $\widetilde{M}$ of the singular set of the smooth foliation $\widetilde{\mathcal{F}}$, and we denote by S_t the intersection of such component with M_t. The set S_t is contained in the singular set of $\mathcal{F}_t$ but in general may not be connected. Thus, we let $S_t = \cup S_t^\lambda$ be the connected components decomposition of S_t. Under some assumption on $T\widetilde{M}/\widetilde{\mathcal{F}}$, which is always satisfied for instance if $\widetilde{\mathcal{F}}$ is locally generated by a single vector field, we have:

*Received December 28, 2012; accepted for publication April 25, 2014. Research partially supported by the ERC grant "HEVO - Holomorphic Evolution Equations" n. 277691 and a grant of JSPS.

†Dipartimento Di Matematica, Università di Roma "Tor Vergata", Via Della Ricerca Scientifica 1, 00133, Roma, Italy (fbracci@mat.uniroma2.it).

‡Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan (tsuwa@sci.hokudai.ac.jp).

Theorem 1.1. *Suppose that S_t is compact for all $t \in P$. Let φ be a homogeneous symmetric polynomial of degree n and denote by $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda)$ the Baum-Bott residue of $\mathcal{F}_t$ at S_t^λ. Then*

$$\lim_{t\to t_0} \sum_\lambda \mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda) = \mathrm{BB}_\varphi(\mathcal{F}_{t_0}; S_{t_0}).$$

A general version of the previous theorem is Theorem 5.4, whose proof is contained in Sections 4 and 5. The rough idea of the proof is to construct a special connection on the regular part of the virtual bundle $T\widetilde{M}/\widetilde{\mathcal{F}}$ such that on each fiber M_t it induces the special connection given by the Baum-Bott action and to see the residues as the integral of a smooth form on $\widetilde{M}$ along the fibers.

In Section 6 we give explicit examples of the previous result. In particular, aside from explicit computation, the examples show that if the residues in the same connected component of $\widetilde{M}$ are not taken together, continuity is lost.

A part of this work was done while the first named author was visiting the University of Tokyo. We would like to thank Prof. J. Noguchi for providing us inspiring environment for research.

2. Deformation of manifolds. The theory of deformation of complex structures was first systematically developed by K. Kodaira and D. C. Spencer [7], here we recall the basic material relevant for our needs.

Definition 2.1. A *deformation of manifolds* is a triple $(\widetilde{M}, P, \pi)$, where P is a C^∞ manifold of real dimension s, called the *parameter space*, $\widetilde{M}$ is a C^∞ manifold of real dimension $2n+s$, called the *ambient manifold*, and $\pi : \widetilde{M} \to P$ is a surjective C^∞ map such that there exists a covering $\{U_\alpha\}$ (called an *adapted deformation coordinates covering*) of $\widetilde{M}$ with the following properties:

1. for each α, the open set U_α is diffeomorphic to $D\times V$, where D is an open set of $\mathbb{C}^n$ and V is an open set of $\mathbb{R}^s$, with coordinates $(z_1^\alpha, \dots, z_n^\alpha, t_1^\alpha, \dots, t_s^\alpha)$,
2. $\pi(U_\alpha)$ is diffeomorphic to V and π is compatible with the projection $D\times V \to V$,
3. on $U_\alpha \cap U_\beta \neq \emptyset$ we may express as

$$\begin{cases} z_i^\beta = z_i^\beta(z^\alpha, t^\alpha) & i = 1, \dots, n \\ t_j^\beta = t_j^\beta(t^\alpha) & j = 1, \dots, s \end{cases} \tag{2.1}$$

and, for each fixed t^α, the map $z^\alpha \mapsto z^\beta(z^\alpha, t^\alpha)$ is holomorphic.

For $t \in P$ we let $M_t := \pi^{-1}(t)$ be the fiber over t. By definition the fibers M_t, for $t \in P$, are complex manifolds. In particular we can define the sheaf $\widetilde{\mathcal{O}}_{\widetilde{M}}$ of C^∞ functions holomorphic along the fibers on $\widetilde{M}$ so that $f \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U)$ if for all $x \in U$, $f|_{U_t} \in \mathcal{O}_{M_t}(U_t)$, where $t = \pi(x)$, $U_t = U \cap M_t$ and $\mathcal{O}_{M_t}$ denotes the sheaf of holomorphic functions on M_t.

Remark 2.2. Let $U_\alpha \subset \widetilde{M}$ be a coordinate chart of an adapted coordinate covering for $\widetilde{M}$. A function f belongs to $\widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha)$ if and only if $f(z_\alpha, t_\alpha)$ is a C^∞ function such that $f(\cdot, t_\alpha)$ is holomorphic (note that this is well defined by (2.1)).

Definition 2.3. Let E be a C^∞ complex vector bundle of rank r over $\widetilde{M}$. We say that E is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$*-(vector) bundle* if there exists a trivializing atlas $\{U_\alpha\}$ for E, with frames $\{e_1^\alpha, \ldots, e_r^\alpha\}$ for $E|_{U_\alpha}$, such that the transition matrices with respect to those frames have entries which are local sections of $\widetilde{\mathcal{O}}_{\widetilde{M}}$. Such frames $\{e_1^\alpha, \ldots, e_r^\alpha\}$ are called *$\widetilde{\mathcal{O}}_{\widetilde{M}}$-frames*.

Given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E, we denote by $\widetilde{\mathcal{O}}_{\widetilde{M}}(E)$ the $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module of $\widetilde{\mathcal{O}}_{\widetilde{M}}$ sections of E. Namely, $s \in \widetilde{\mathcal{O}}_{\widetilde{M}}(E)(U)$ is a C^∞ section of E over the open set $U \subset \widetilde{M}$ such that in any $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame $\{e_1^\alpha, \ldots, e_r^\alpha\}$ over U_α with $U_\alpha \cap U \neq \emptyset$ the section s is given by

$$s(z^\alpha, t^\alpha) = \sum_{j=1}^{r} f_j^\alpha(z^\alpha, t^\alpha) e_j^\alpha, \quad f_j^\alpha \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha \cap U).$$

Let $T_{\mathbb{R}}\pi := \operatorname{Ker} \pi_*$. Since the fibers of the fibration $\pi : \widetilde{M} \to P$ are holomorphic, we can define the complex vector bundles

$$T\pi := \bigcup_{x \in \widetilde{M}} T_x M_{\pi(x)}, \quad \overline{T}\pi := \bigcup_{x \in \widetilde{M}} \overline{T}_x M_{\pi(x)}.$$

Local frames for $T\pi$ and $\overline{T}\pi$ in an adapted deformation coordinates covering are given respectively by $\{\frac{\partial}{\partial z_i^\alpha}\}_{i=1,\ldots,n}$ and $\{\frac{\partial}{\partial \overline{z}_i^\alpha}\}_{i=1,\ldots,n}$ and

$$T_{\mathbb{R}}\pi \otimes \mathbb{C} = T\pi \oplus \overline{T}\pi.$$

Using an adapted deformation coordinates covering, by (2.1), it is easy to see that $T\pi$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-vector bundle over $\widetilde{M}$. Moreover, it has a natural structure of *$\widetilde{\mathcal{O}}_{\widetilde{M}}$-Lie algebra*, namely, using local coordinates, one can easily see that if $v, w \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U)$ then

$$[v, w] \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U).$$

3. Deformation of foliations. Deformations of holomorphic foliations, especially from the viewpoint of moduli spaces, have been studied by a number of authors (*e.g.*, [6], [9], [10]). Here we consider C^∞ families of singular holomorphic foliations.

Let $\mathcal{S}$ be an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module. We say that $\mathcal{S}$ is *coherent* if, for each point $x \in \widetilde{M}$ there exists an open neighborhood $U \subset \widetilde{M}$ of x and two integers $p, q \geq 0$ such that

$$\widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{S}|_U \to 0, \tag{3.1}$$

is an exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$-modules, where φ is a suitable $\widetilde{\mathcal{O}}_{\widetilde{M}}$-morphism.

Definition 3.1. Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds. A *deformation of foliations* on $(\widetilde{M}, P, \pi)$ is a coherent $\widetilde{\mathcal{O}}_{\widetilde{M}}$-submodule $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ such that $[\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}] \subset \widetilde{\mathcal{F}}$.

Given a deformation of foliations $\widetilde{\mathcal{F}}$ on a deformation of manifolds $(\widetilde{M}, P, \pi)$, we denote by $\mathcal{C}_P^\infty$ the sheaf of germs of complex valued smooth functions on P, and for each $t \in P$, by $\mathcal{I}_t := \{f \in \mathcal{C}_P^\infty : f(t) = 0\}$ the ideal sheaf of smooth functions vanishing at t. The set $\mathcal{R} := \pi^*\mathcal{C}_P^\infty$ is the sheaf of smooth functions on $\widetilde{M}$ that are constant along the fibers, and it is naturally a subsheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}$. Noting that $\mathcal{R}/\pi^*\mathcal{I}_t$ is supported on $M_t = \pi^{-1}(t)$, we define

$$\mathcal{F}_t := \widetilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t.$$

Note that $\widetilde{\mathcal{O}}_{\widetilde{M}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t = \mathcal{O}_{M_t}$, the sheaf of holomorphic functions on M_t. Hence, if $\mathcal{E}$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module over $\widetilde{M}$, then $\mathcal{E} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$ is an $\mathcal{O}_{M_t}$-module over M_t.

In particular, the sheaf $\mathcal{F}_t$ is an $\mathcal{O}_{M_t}$-module. In adapted deformation coordinates, if $X_1, \ldots, X_r$ are local generators of $\widetilde{\mathcal{F}}$, given by

$$X_j(z^\alpha, t^\alpha) = \sum f_{ij}(z^\alpha, t^\alpha) \frac{\partial}{\partial z_i^\alpha},$$

then $\mathcal{F}_{t_0}$ is locally generated by the $X_j(z^\alpha, t_0^\alpha)$'s. Namely it is generated by the vector fields

$$X_j(z^\alpha, t_0^\alpha) = \sum f_{ij}(z^\alpha, t_0^\alpha) \frac{\partial}{\partial z_i^\alpha} \tag{3.2}$$

obtained by evaluating $f_{ij}(z^\alpha, t^\alpha)$ at $t = t_0$. From this remark, it follows easily:

LEMMA 3.2. *For all $t \in P$, the sheaf $\mathcal{F}_t$ defines a holomorphic foliation on M_t.*

The normal sheaf $\mathcal{N}_{\widetilde{\mathcal{F}}}$ of $\widetilde{\mathcal{F}}$ is defined by the following exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}$-modules on $\widetilde{M}$:

$$0 \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}} \longrightarrow 0. \tag{3.3}$$

The *singular set* of $\widetilde{\mathcal{F}}$ is by definition

$$S(\widetilde{\mathcal{F}}) := \{x \in \widetilde{M} : \mathcal{N}_{\widetilde{\mathcal{F}},x} \text{ is not } \mathcal{O}_{\widetilde{M},x} - \text{free}\}.$$

REMARK 3.3. As in the case of usual singular holomorphic foliations, even if $\widetilde{\mathcal{F}}$ is locally free, it is possible that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is not locally free. On the other hand, if $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is locally free, so is $\widetilde{\mathcal{F}}$, as $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is locally free.

The *rank* of $\widetilde{\mathcal{F}}$ is defined to be the rank of the locally free part of $\widetilde{\mathcal{F}}$.

LEMMA 3.4. *For each point $x \in \widetilde{M}$ there exists an open neighborhood $U \subset \widetilde{M}$ of x and two integers $p, q \geq 0$ such that*

$$\widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}}|_U \to 0, \tag{3.4}$$

is an exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$-modules. Moreover,

$$S(\widetilde{\mathcal{F}})|_U = \{x \in U : \operatorname{rank} \varphi_x \textit{ is not maximal}\}.$$

Proof. Since $\widetilde{\mathcal{F}}$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-coherent and $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-locally free, from (3.3) it follows that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-coherent as well, so that (3.4) holds. The final statement follows from (3.4) and standard commutative algebra. □

LEMMA 3.5. *For each $t \in P$ such that $M_t \not\subset S(\widetilde{\mathcal{F}})$ the following sequence of $\mathcal{O}_{M_t}$-modules over M_t is exact:*

$$0 \to \widetilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \xrightarrow{\iota} \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to 0. \tag{3.5}$$

Proof. Since taking tensor products is right exact, it suffices to prove that ι is injective.

It is true on the stalk over each $x \in M_t$ such that $x \notin S(\widetilde{\mathcal{F}})$, since $\mathcal{N}_{\mathcal{F},x}$ is $\widetilde{\mathcal{O}}_{\widetilde{M},x}$-free. We note that according to Lemma 3.4, $S(\widetilde{\mathcal{F}})|_{U\cap M_t} = \{x \in U \cap M_t : \operatorname{rank}\varphi_x \text{ is not maximal}\}$. Hence, for t fixed, these equations give rise to an analytic subset $S(\widetilde{\mathcal{F}}) \cap M_t$ of M_t, provided $M_t \not\subset S(\widetilde{\mathcal{F}})$. As a consequence, $S(\widetilde{\mathcal{F}}) \cap M_t$ is thin in M_t. This shows that, since $\widetilde{\mathcal{F}}$ is a subsheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$, ι is also injective on the stalk over $x \in S(\widetilde{\mathcal{F}}) \cap M_t$. ☐

For each $t \in P$ we have the following exact sequence of $\mathcal{O}_{M_t}$-modules:

$$0 \longrightarrow \mathcal{F}_t \longrightarrow \mathcal{O}_{M_t}(TM_t) \longrightarrow \mathcal{N}_{\mathcal{F}_t} \longrightarrow 0. \tag{3.6}$$

Definition 3.6. Let $t \in P$. If $M_t \subset S(\widetilde{\mathcal{F}})$, we let $S(\mathcal{F}_t) := M_t$. Otherwise we let

$$S(\mathcal{F}_t) := \{x \in M_t : \mathcal{N}_{\mathcal{F}_t,x} \text{ is not } \mathcal{O}_{M_t} - \text{free}\}.$$

Proposition 3.7. *For all $t \in P$ it holds*

$$S(\mathcal{F}_t) = S(\widetilde{\mathcal{F}}) \cap M_t.$$

Proof. If $M_t \subset S(\widetilde{\mathcal{F}})$ there is nothing to prove.

Thus, assume $M_t \not\subset S(\widetilde{\mathcal{F}})$. Since $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t = \mathcal{O}_{M_t}(TM_t)$, comparing (3.5) and (3.6) we see that

$$\mathcal{N}_{\mathcal{F}_t} = \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t, \tag{3.7}$$

from which the statement follows at once. ☐

4. Relative Bott vanishing for a deformation of foliations. In this section we discuss a Bott type vanishing theorem for deformations of foliations. Thus, we let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. In this section we assume

$$S(\widetilde{\mathcal{F}}) = \emptyset.$$

This means that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ and hence $\widetilde{\mathcal{F}}$ is locally free so that there exists an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-subbundle $\tilde{F}$ of $T\pi$ such that $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}_{\widetilde{M}}(\tilde{F})$.

We refer to [4] for the notion of partial connections (see also [1], [2], [11]). As an example, given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E over $\widetilde{M}$, we can define a "relative $\overline{\partial}$-connection" for E along $\overline{T}\pi$ as follows. We define

$$\overline{\partial}_E : C^\infty_{\widetilde{M}}(E) \to C^\infty_{\widetilde{M}}(\overline{T}^*\pi \otimes E),$$

imposing that, given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame $\{\sigma_1^\alpha, \dots, \sigma_r^\alpha\}$, and a C^∞ section of E, $\sigma^\alpha := \sum f_j^\alpha \sigma_j^\alpha$, it holds

$$\overline{\partial}_E(\sigma^\alpha) = \sum_{j=1}^{r} \sum_{i=1}^{n} \frac{\partial f_j^\alpha}{\partial \overline{z}_i^\alpha} d\overline{z}_i^\alpha \otimes \sigma_j^\alpha.$$

Since the transition matrices for E with respect to $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frames contains only entries in $\widetilde{\mathcal{O}}_{\widetilde{M}}$, it is easy to see that such a definition is well posed and it is a partial connection for E along $\overline{T}\pi$.

Definition 4.1. Let E be an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle over $\widetilde{M}$ and let $\mathcal{E}$ be the sheaf of its $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections. A *partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection* for E along $\widetilde{\mathcal{F}}$ is a $\mathbb{C}$-linear map

$$\delta : \mathcal{E} \to \widetilde{\mathcal{F}}^* \otimes \mathcal{E}$$

with the properties that for all $X \in \widetilde{\mathcal{F}}$, $f, g \in \widetilde{\mathcal{O}}_{\widetilde{M}}$ and $\sigma \in \mathcal{E}$

$$\delta_{(fX)}(g\sigma) = f\,(g\delta_X(\sigma) + dg(X)\sigma)\,.$$

Moreover, it is said to be *flat* if

$$\delta_X \circ \delta_Y - \delta_Y \circ \delta_X - \delta_{[X,Y]} = 0, \quad \forall X, Y \in \widetilde{\mathcal{F}}.$$

If δ is as above, it induces a (C^∞) partial connection

$$\delta : C^\infty_{\widetilde{M}}(E) \to C^\infty_{\widetilde{M}}(\tilde{F}^* \otimes E)$$

such that, for $X \in \widetilde{\mathcal{F}}$ and $\sigma \in \mathcal{E}$, we have $\delta_X(\sigma) \in \mathcal{E}$. Thus

$$\delta \oplus \bar{\partial}_E : C^\infty_{\widetilde{M}}(E) \to C^\infty_{\widetilde{M}}((\tilde{F}^* \oplus \overline{T}^*\pi) \otimes E)$$

is a partial connection. We say that a connection $\nabla : C^\infty_{\widetilde{M}}(E) \to C^\infty_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E)$ extends $\delta \oplus \bar{\partial}_E$ if $\nabla_X = (\delta \oplus \bar{\partial}_E)_X$ for all sections X of $F \oplus \overline{T}\pi$. Such a connection ∇ always exists (cf. [4]).

We have the following "relative Bott vanishing" theorem for actions of deformations of foliations:

Theorem 4.2. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$ of rank p. Assume that $S(\mathcal{F}) = \emptyset$. Let $\mathcal{E}$ be the sheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections of an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E over $\widetilde{M}$. Assume there exists a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection δ for $\mathcal{E}$ along $\widetilde{\mathcal{F}}$. Then, for any connection ∇ for E extending $\delta \oplus \bar{\partial}_E$, denoting by $\iota_t : M_t \hookrightarrow \widetilde{M}$ the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all $t \in P$ and all symmetric homogeneous polynomials φ of degree $d > n - p$.

Proof. Let $\tilde{F}$ be the $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle whose associated sheaf of sections is $\widetilde{\mathcal{F}}$. Write

$$T\widetilde{M} \otimes \mathbb{C} = \tilde{F} \oplus F_1 \oplus \overline{T}\pi \oplus \pi^*(TP \otimes \mathbb{C}),$$

where F_1 is any C^∞ complement of $\tilde{F}$ in $T\pi$.

Let K be the curvature of ∇. Let $\{s_1, \ldots, s_p\}$ be a local $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame for $\tilde{F}$, and $\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\}$ the natural frame for $\overline{T}\pi$ in adapted deformation coordinates. Since $\tilde{F}$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-subbundle of $T\pi$, we can write $s_j = \sum_{k=1}^n a_k(z,t)\frac{\partial}{\partial z_k}$ for $j = 1, \ldots, p$ and $a_k \in \widetilde{\mathcal{O}}_{\widetilde{M}}$. Hence, $[s_j, \frac{\partial}{\partial \bar{z}_k}] = 0$ for $j = 1, \ldots, p$ and $k = 1, \ldots, n$.

Arguing similarly as in the proof of [4, Prop. 3.27] (see also [2, Thm. 6.1]) since $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections of E generate as $C^{\infty}_{\widetilde{M}}$-module the sheaf of C^{∞}-sections of E, one can see that

$$K(s_j, s_k) = K(s_j, \frac{\partial}{\partial \overline{z}_h}) = K(\frac{\partial}{\partial \overline{z}_h}, \frac{\partial}{\partial \overline{z}_l}) = 0$$

for all $j, k = 1, \ldots, p$ and $h, l = 1, \ldots, n$. In fact, for the second term, given σ an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section of E, we have

$$K(s_j, \frac{\partial}{\partial \overline{z}_h})(\sigma) = \nabla_{s_j}(\nabla_{\frac{\partial}{\partial \overline{z}_h}}\sigma) - \nabla_{\frac{\partial}{\partial \overline{z}_h}}(\nabla_{s_j}\sigma) - \nabla_{[s_j, \frac{\partial}{\partial \overline{z}_h}]}\sigma = 0,$$

because $\nabla_{\frac{\partial}{\partial \overline{z}_h}}\sigma = (\overline{\partial}_E)_{\frac{\partial}{\partial \overline{z}_h}}\sigma = 0$ by definition, since σ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section; $\nabla_{s_j}\sigma$ is another $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section of E, hence $\nabla_{\frac{\partial}{\partial \overline{z}_h}}(\nabla_{s_j}\sigma) = (\overline{\partial}_E)_{\frac{\partial}{\partial \overline{z}_h}}(\nabla_{s_j}\sigma) = 0$; and $[s_j, \frac{\partial}{\partial \overline{z}_h}] = 0$. The first and third terms vanish as δ and $\bar{\partial}_E$ are flat.

As a consequence, the entries of the matrix representing K are 2-forms belonging to the ideal generated by a dual basis of F_1 (which has dimension $n - p$) and by $dt_1, \ldots, dt_s$, where these latter are a basis of $\pi^*(T^*P)$. Therefore, if φ has degree d greater than $n - p$, it follows that

$$\varphi(\nabla) = \sum \omega_j \wedge dt_j,$$

for some $(2d-1)$-forms ω_j, hence, $\iota^*(\varphi(\nabla)) = 0$. □

We recall that if M is a complex manifold and $\mathcal{F}$ is a non-singular holomorphic foliation on M then there exists a natural holomorphic partial connection δ for the normal bundle of the foliation $\mathcal{N}_{\mathcal{F}}$ along $\mathcal{F}$ given by the so called *Baum-Bott action* (see [4], [11]). Such a partial connection is *flat*, in the sense similar to the one in Definition 4.1. It is defined as follows:

$$\delta_X(\sigma) := \rho([X, \tilde{\sigma}]) \tag{4.1}$$

where $\sigma \in \mathcal{N}_{\mathcal{F}}$ is a holomorphic section of the normal bundle to the foliation, $\tilde{\sigma} \in \mathcal{O}_M(TM)$ is a holomorphic section of the tangent bundle to M such that $\rho(\tilde{\sigma}) = \sigma$, where $\rho : \mathcal{O}_M(TM) \to \mathcal{N}_{\mathcal{F}}$ is the natural projection, and $X \in \mathcal{F}$.

We are going to show that a deformation of foliations gives rise to a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$ such that its "restriction" to each fiber M_t is the holomorphic flat partial connection for the normal bundle to $\mathcal{F}_t$ given by the Baum-Bott action:

Proposition 4.3. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. Assume that $S(\widetilde{\mathcal{F}}) = \emptyset$. Then there exists a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection $\tilde{\delta}$ for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$. Moreover, if $\iota_t : M_t \hookrightarrow \widetilde{M}$ is the natural embedding, then $\iota_t^*(\tilde{\delta})$ is the holomorphic flat partial connection for $\mathcal{N}_{\mathcal{F}}$ along $\mathcal{F}_t$ given by the Baum-Bott action.*

Proof. Let $\tilde{\rho} : \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \to \mathcal{N}_{\widetilde{\mathcal{F}}}$ be the natural projection. For $X \in \widetilde{\mathcal{F}}$ and $\sigma \in \mathcal{N}_{\widetilde{\mathcal{F}}}$ we define

$$\tilde{\delta}_X(\sigma) := \tilde{\rho}([X, \tilde{\sigma}]), \tag{4.2}$$

where $\tilde{\sigma} \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is such that $\tilde{\rho}(\tilde{\sigma}) = \sigma$. Involutivity of $\widetilde{\mathcal{F}}$ shows that $\tilde{\delta}$ is well-defined and flatness follows from the Jacobi identity, so that $\tilde{\delta}$ is a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$.

Comparing (4.2) with (4.1), it is easy to see that $\iota_t^*(\tilde{\delta})$ is the flat partial $\mathcal{O}_{M_t}$-connection for $\mathcal{N}_{\mathcal{F}_t}$ along $\mathcal{F}_t$ given by the Baum-Bott action. □

In particular, Theorem 4.2 and Proposition 4.3 imply the following:

COROLLARY 4.4. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. Assume that $S(\widetilde{\mathcal{F}}) = \emptyset$. Then there exists a connection ∇ for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ such that, denoting by $\iota_t : M_t \hookrightarrow \widetilde{M}$ the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all $t \in P$ and all symmetric homogeneous polynomials φ of degree $d > n - p$.

5. Residues of Baum-Bott type on deformations of manifolds. In this section we assume $(\widetilde{M}, P, \pi)$ is a deformations of manifolds and $\widetilde{\mathcal{F}}$ is a deformation of foliations on $\widetilde{M}$. We also assume that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ admits a C^∞ *locally free resolution*, namely, there exists an exact sequence of $\mathcal{C}^\infty_{\widetilde{M}}$-modules:

$$0 \to \mathcal{E}_q \to \cdots \to \mathcal{E}_0 \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}} \to 0, \tag{5.1}$$

such that each $\mathcal{E}_j$ is locally $\mathcal{C}^\infty_{\widetilde{M}}$-free.

REMARK 5.1. Every coherent $\mathcal{O}_M$-module on a complex manifold M admits a real analytic locally free resolution (see [3]). This fact is used in the original construction of the Baum-Bott residues in [4]. What we need is a relative version of this. In practice, a resolution as above often arises naturally with a given $\widetilde{\mathcal{F}}$. The simplest is the case where $\widetilde{\mathcal{F}}$ is locally $\widetilde{\mathcal{O}}_{\widetilde{M}}$-free; we may let $q = 1$ and $\mathcal{E}_1 = \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}$, $\mathcal{E}_0 = \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}$. This applies in particular to the case where $\widetilde{\mathcal{F}}$ is generated locally by a single vector field.

Let E_j be the vector bundle over $\widetilde{M}$ whose sheaf of C^∞ sections is $\mathcal{E}_j$. Then $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is a virtual bundle in the K-group $K(\widetilde{M})$ and its total Chern class is defined as

$$c(\mathcal{N}_{\widetilde{\mathcal{F}}}) = \prod_{i=0}^{q} c(E_i)^{(-1)^i}.$$

We briefly sketch here the theory we need for localizing characteristic classes and obtaining the associated residues in the framework of the Chern-Weil theory adapted to the Čech-de Rham cohomology, and refer the reader to [4, Section 4], [8] and [11, Ch.II, 8] for details.

Let $\tilde{U}_1$ be an open neighborhood of $S(\widetilde{\mathcal{F}})$ and let $\tilde{U}_0 := \widetilde{M} \setminus S(\widetilde{\mathcal{F}})$. We denote by $(\nabla_0^\bullet, \nabla_1^\bullet)$ the family of $q+1$ connections compatible with (5.1) and adapted to the covering $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ of $\widetilde{M}$. Namely, $\nabla_l^\bullet = (\nabla_l^q, \dots, \nabla_l^0)$, $l = 0, 1$ is a family such that ∇_l^j is a connection for $E_j|_{\tilde{U}_l}$, $j = 0, \dots, q$, $l = 0, 1$ and the following diagram is commutative for $i = 1, \dots, q$ and $l = 0, 1$:

$$(5.2)\qquad \begin{array}{ccc} E_i|_{\tilde{U}_l} & \xrightarrow{\nabla_l^i} & C^\infty_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E_i|_{\tilde{U}_l}) \\ \downarrow & & \downarrow \\ E_{i-1}|_{\tilde{U}_l} & \xrightarrow{\nabla_l^{i-1}} & C^\infty_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E_{i-1}|_{\tilde{U}_l}). \end{array}$$

Moreover, let $N_{\tilde{F}}$ be the vector bundle on $\tilde{U}_0$ whose sheaf of sections is $\mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}|_{\tilde{U}_0}$. Let ∇ be an extension of the flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection $\tilde{\delta}$ for $\mathcal{N}_{\widetilde{\mathcal{F}}}|_{\tilde{U}_0}$ along $\widetilde{\mathcal{F}}$ given by Proposition 4.3. It is then possible to choose $\nabla_0^\bullet$ to be compatible with ∇ (in the sense explained before).

Now, we let φ be a homogeneous symmetric polynomial of degree $d > n - p$. One can define the class $\varphi(\mathcal{N}_{\widetilde{\mathcal{F}}})$ in the Čech-de Rham cohomology $\check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}})$ which is represented by

$$\varphi(\nabla_*^\bullet) := (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)),$$

where, by the compatibility condition, $\varphi(\nabla_0^\bullet) = \varphi(\nabla)$ is a $2d$-form on $\tilde{U}_0$, $\varphi(\nabla_1^\bullet)$ is the $2d$-form on $\tilde{U}_1$ associated to the family $\nabla_1^\bullet$ and $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ is a $(2d-1)$-form on $\tilde{U}_0 \cap \tilde{U}_1$ such that $d\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet)$. The Čech-de Rham cohomology $\check{\mathrm{H}}^*(\tilde{\mathcal{U}})$ is naturally isomorphic to the de Rham cohomology $H^*_{\mathrm{dR}}(\widetilde{M}, \mathbb{C})$.

If $M_t \not\subset S(\widetilde{\mathcal{F}})$, tensorizing (5.1) with $\mathcal{R}/\pi^*\mathcal{I}_t$ we obtain the following exact sequence of $\mathcal{C}^\infty_{M_t}$-modules (cf. the proof of Lemma 3.5):

$$(5.3)\quad 0 \to \mathcal{E}_q \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \cdots \to \mathcal{E}_0 \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to 0,$$

where $\mathcal{E}_j \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$ is the sheaf of C^∞ sections of the restriction of the bundle E_j to M_t. By (3.7), it is then easy to see the following:

LEMMA 5.2. *Let $t \in P$ and let $\iota_t : M_t \to \widetilde{M}$ be the natural embedding. If $M_t \not\subset S(\widetilde{\mathcal{F}})$ then $(\iota_t^*(\nabla_0^\bullet), \iota_t^*(\nabla_1^\bullet))$ is a family of connections for the virtual bundle $\mathcal{N}_{\mathcal{F}_t}$ compatible with* (5.3).

By Corollary 4.4 and by the compatibility condition, it follows that for all homogeneous symmetric polynomials φ of degree $d > n-p$, the class $\varphi(\mathcal{N}_{\mathcal{F}_t})$ is represented in the Čech-de Rham cohomology associated to the covering $\tilde{\mathcal{U}} \cap M_t$ of M_t by the cocyle

$$\begin{aligned} \varphi(\iota_t^*\nabla_*^\bullet) &= (\iota_t^*\varphi(\nabla_0^\bullet), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) = (\iota_t^*\varphi(\nabla), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) \\ &= (0, \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)). \end{aligned}$$

Suppose that $M_t \not\subset S(\widetilde{\mathcal{F}})$ and that $S(\mathcal{F}_t)$, which is $S(\widetilde{\mathcal{F}}) \cap M_t$ by Proposition 3.7, is compact. Since $\tilde{U}_0 \cap M_t = M_t \setminus S(\mathcal{F}_t)$, the above cocycle $\varphi(\iota_t^*\nabla_*^\bullet)$ defines a localization of $\varphi(\mathcal{N}_{\mathcal{F}_t})$, call it $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$, in the relative Čech-de Rham cohomology $\check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t))$. The *Baum-Bott residue* is the image of $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$ by the Alexander homomorphism

$$A : \check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t)) \to H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^*.$$

If $S(\mathcal{F}_t)$ is made of k connected components and $\tilde{U}_1$ is small enough, then $H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^*$ is a direct sum of k addends, and we can consider the Baum-Bott residue at each connected component of $S(\mathcal{F}_t)$. Note that if $\tilde{U}_1 \cap M_t$ is a regular neighborhood of $S(\mathcal{F}_t)$, we have $H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^* \xrightarrow{\sim} H_{2n-2d}(S(\mathcal{F}_t), \mathbb{C})$ and the above Alexander homomorphism is an isomorphism. Thus in this case the above residue, as well as the ones corresponding to the connected components of $S(\mathcal{F}_t)$, does not depend on $\tilde{U}_1$.

Now, let $S'(\widetilde{\mathcal{F}}) \subseteq S(\widetilde{\mathcal{F}})$ be a connected component. We assume that

$$S_t := S'(\widetilde{\mathcal{F}}) \cap M_t \text{ is compact} \quad \forall t \in P.$$

Note that S_t may not be connected. Let $\tilde{U}'_1$ be a neighborhood of $S'(\widetilde{\mathcal{F}})$, small enough so that it does not intersect with any other components of $S(\widetilde{\mathcal{F}})$, and $\tilde{R}$ a real manifold of dimension $2n+s$ with boundary in $\tilde{U}'_1$ such that $S'(\widetilde{\mathcal{F}})$ is contained in the interior of $\tilde{R}$ and that $\partial\tilde{R}$ is transverse to M_t for all $t \in P$. Moreover, we can take $\tilde{R}$ in such a way that $R_t := \tilde{R} \cap M_t$ is compact for all $t \in P$.

We let $U_t := \tilde{U}'_1 \cap M_t$. By the previous construction, we can express the *Baum-Bott residue* $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) \in H^{2n-2d}_{\mathrm{dR}}(U_t)^*$ as follows:

$$\text{(5.4)} \quad \mathrm{BB}_\varphi(\mathcal{F}_t; S_t) : H^{2n-2d}_{\mathrm{dR}}(U_t) \ni [\tau] \mapsto \int_{R_t} \iota_t^* \varphi(\nabla_1^\bullet) \wedge \tau - \int_{\partial R_t} \iota_t^* \varphi(\nabla_0^\bullet, \nabla_1^\bullet) \wedge \tau.$$

REMARK 5.3. 1. If $d = n$, the Baum-Bott residue is a complex number given by

$$\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) = \int_{R_t} \iota_t^* \varphi(\nabla_1^\bullet) - \int_{\partial R_t} \iota_t^* \varphi(\nabla_0^\bullet, \nabla_1^\bullet).$$

2. As mentioned above, if U_t is a regular neighborhood of S_t, $H^{2n-2d}_{\mathrm{dR}}(U_t)^* \xrightarrow{\sim} H_{2n-2d}(S_t, \mathbb{C})$ and one can remove the dependence on $\tilde{U}'_1$ or $\tilde{R}$ in this construction.

Now we are in good shape to prove our main result:

THEOREM 5.4. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$ of rank p. Suppose that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ admits a C^∞ locally free resolution. Let $S'(\widetilde{\mathcal{F}}) \subseteq S(\widetilde{\mathcal{F}})$ be a connected component of the singular set of $\widetilde{\mathcal{F}}$ and let $S_t := S'(\widetilde{\mathcal{F}}) \cap M_t$. Assume that for all $t \in P$ the set S_t is compact and $S_t \neq M_t$. Let φ be a homogeneous symmetric polynomial of degree $d > n - p$. Under these assumptions, the Baum-Bott residue $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t)$ is continuous in $t \in P$. Namely, for any C^∞ $(2n-2d)$-form $\tilde{\tau}$ on $\widetilde{M}$ such that $\iota_t^*(\tilde{\tau})$ is closed for all $t \in P$,*

$$\lim_{t \to t_0} \mathrm{BB}_\varphi(\mathcal{F}_t; S_t)\left(\iota_t^*(\tilde{\tau})\right) = \mathrm{BB}_\varphi(\mathcal{F}_{t_0}; S_{t_0})\left(\iota_{t_0}^*(\tilde{\tau})\right).$$

Proof. From the previous construction and (5.4) it follows that the Baum-Bott residues on M_t are expressed by means of smooth forms on $\widetilde{M}$. Hence, they vary continuously. ∎

Note that, if S_t is not connected and $S_t = \cup_\lambda S_t^\lambda$ is its connected components decomposition, then

$$\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) = \sum_\lambda \mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda).$$

6. Examples. Let $\mathbb{P}^3$ denote the three dimensional complex projective space with homogeneous coordinates $[x_1 : x_2 : x_3 : x_4]$.

EXAMPLE 6.1. On $\mathbb{P}^3$ we consider the vector field which is defined in the affine chart $x_4 \neq 0$ with coordinates $x = x_1/x_4, y = x_2/x_4, z = x_3/x_4$ by

$$X(x, y, z) := x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

The singularities are the line L given by $x_1 = x_2 = 0$ and the point at infinity given by $Q := [1 : 1 : 1 : 0]$ (see the next expression (6.2)).

The vector field X generates a one-dimensional foliation $\mathcal{F}$ given by $X : \mathbb{P}^3 \times \mathbb{C} \to T\mathbb{P}^3$ on $\mathbb{P}^3$. By the Baum-Bott theorem, we can localize $\varphi(T\mathbb{P}^3/\mathcal{F})$ for homogeneous symmetric polynomials φ of degree 3. Such polynomials are essentially given by c_1^3, c_1c_2 and c_3. Moreover, since $\mathcal{F}$ is trivial, we see that $\varphi(T\mathbb{P}^3/\mathcal{F}) = \varphi(T\mathbb{P}^3)$. Let $\mathcal{O}(1)$ be the hyperplane bundle on $\mathbb{P}^3$ and let $\xi := c_1(\mathcal{O}(1)) \in H^2_{\mathrm{dR}}(\mathbb{P}^3)$. From the Euler exact sequence, it follows that $c(T\mathbb{P}^3) = (1 + \xi)^4$, from which

$$\int_{\mathbb{P}^3} c_1^3(T\mathbb{P}^3) = 64, \quad \int_{\mathbb{P}^3} c_1c_2(T\mathbb{P}^3) = 24, \quad \int_{\mathbb{P}^3} c_3(T\mathbb{P}^3) = 4. \tag{6.1}$$

Changing coordinates, in the affine chart $x_3 \neq 0$ with coordinates $\tilde{x} = x_1/x_3, \tilde{y} = x_2/x_3, \tilde{z} = x_4/x_3$ the vector field X has the expression:

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y})\frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2)\frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z}\frac{\partial}{\partial \tilde{z}}. \tag{6.2}$$

From this it follows that the first jet of X at Q is given by the non-degenerate matrix

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence since Q is a non-degenerate isolated singularity for X it follows (see, *e.g.* [4, (0.7)] or [11])

$$\mathrm{BB}_\varphi(X; Q) = \frac{\varphi(A)}{\det A}, \tag{6.3}$$

that is

$$\mathrm{BB}_{c_1^3}(X; Q) = 27 \quad \mathrm{BB}_{c_1c_2}(X; Q) = 9 \quad \mathrm{BB}_{c_3}(X; Q) = 1. \tag{6.4}$$

By the Baum-Bott theorem,

$$\int_{\mathbb{P}^3} \varphi(T\mathbb{P}^3) = \mathrm{BB}_\varphi(X; Q) + \mathrm{BB}_\varphi(X; L).$$

From this and by (6.1) and (6.4) we obtain

$$\mathrm{BB}_{c_1^3}(X; L) = 37 \quad \mathrm{BB}_{c_1c_2}(X; L) = 15 \quad \mathrm{BB}_{c_3}(X; L) = 3. \tag{6.5}$$

However, it sometimes happens that we need to compute such residues only from the local data near the singularity, without using the Baum-Bott theorem, and it is usually very complicated to do so particularly if the singular set is non-isolated.

We present now a deformation procedure which allows to compute the previous residues and explain in practice how our Theorem 1.1 works.

Let $\widetilde{M} := \mathbb{P}^3 \times (-1,1)$ and let $\widetilde{\mathcal{F}}$ be the deformation of foliations defined by the vector fields X_t, $t \in (-1,1)$, which on the chart $x_4 \neq 0$ are defined as

$$X_t(x,y,z) = (x+tz)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

On the chart $x_3 \neq 0$ the vector field X_t is given by

$$X_t(\tilde{x},\tilde{y},\tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y} + t)\frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2)\frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z}\frac{\partial}{\partial \tilde{z}}.$$

The singularities of X_t for $t \neq 0$ are given by $O := [0:0:0:1]$ and $P_i(t) := [u_{t,i}^2 : u_{t,i} : 1 : 0]$ for $i = 1,2,3$, where the $u_{t,i}$'s are the three roots of the equation $\lambda^3 - \lambda^2 - t = 0$.

At the point O the first jet of X_t, $t \neq 0$, is non-degenerate and it is given by the matrix

$$\begin{pmatrix} 1 & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

From this and (6.3),

$$\mathrm{BB}_{c_1^3}(X_t; O) = \frac{1}{t} \quad \mathrm{BB}_{c_1c_2}(X_t; O) = 0 \quad \mathrm{BB}_{c_3}(X_t; O) = 1. \tag{6.6}$$

Remark 6.2. It is interesting to note that $\lim_{t\to 0} \mathrm{BB}_{c_1^3}(X_t; O) = \infty$, namely the residue by itself may not be continuous. Only the sum of the residues for all the singularities belonging to one connected component in the ambient space $\widetilde{M}$ is guaranteed to be continuous.

At the point $P_i(t)$ the vector field X_t has first jet given by the matrix

$$B(t,i) := \begin{pmatrix} 1 - u_{t,i} & -u_{t,i}^2 & 0 \\ 1 & -2u_{t,i} & 0 \\ 0 & 0 & -u_{t,i} \end{pmatrix},$$

with determinant $\det B(t,i) = u_{t,i}^2(2 - 3u_{t,i})$. Thus, for $t \to 0$, $t \neq 0$ the points $P_i(t)$ are isolated non-degenerate singularities for X_t and one can use (6.3) to compute the residues:

$$\mathrm{BB}_{c_1^3}(X_t; P_i(t)) = \frac{(4u_{t,i}-1)^3}{u_{t,i}^2(3u_{t,i}-2)} \quad \mathrm{BB}_{c_1c_2}(X_t; P_i(t)) = \frac{3(2u_{t,i}-1)(4u_{t,i}-1)}{u_{t,i}(3u_{t,i}-2)}$$
$$\mathrm{BB}_{c_3}(X_t; P_i(t)) = 1.$$

Now, as $t \to 0$, two of the roots of of the equation $\lambda^3 - \lambda^2 - t = 0$ tend to 0 and one tends to 1. We assume that $u_{t,1}, u_{t,2} \to 0$ and $u_{t,3} \to 1$. Hence, if $S'(\widetilde{\mathcal{F}})$ is the connected component which contains the line L in the manifold deformation $M \times (-1,1)$, the intersection of $S'(\widetilde{\mathcal{F}})$ with $M \times \{t\}$ is given by the points $O, P_1(t), P_2(t)$. While, the connected component in $M \times (-1,1)$ which contains Q contains all the points $P_3(t)$.

A direct computation, taking into account that $u_{t,1}+u_{t,2}+u_{t,3}=1$, $u_{t,1}u_{t,2}+u_{t,1}u_{t,3}+u_{t,2}u_{t,3}=0$ and $u_{t,1}u_{t,2}u_{t,3}=t$, shows that

$$\mathrm{BB}_\varphi(X_t;P_1(t))+\mathrm{BB}_\varphi(X_t;P_2(t))=\begin{cases}37-\frac{1}{t}, & \varphi=c_1^3\\ 15, & \varphi=c_1c_2\\ 2, & \varphi=c_3.\end{cases} \tag{6.7}$$

By Theorem 1.1, we have

$$\mathrm{BB}_\varphi(X;L)=\lim_{t\to 0}[\mathrm{BB}_\varphi(X_t;O)+\mathrm{BB}_\varphi(X_t;P_1(t))+\mathrm{BB}_\varphi(X_t;P_2(t))]$$

and we recover (6.5) from (6.6) and (6.7).

We note that the residues at $P_3(t)$ remain the same for $\varphi=c_1^3,c_1c_2,c_3$:

$$\mathrm{BB}_\varphi(X_t;P_3(t))=\mathrm{BB}_\varphi(X;Q).$$

We may also apply our method to the following example in [5], where the residues are computed by a rather involved way. We thank D. Lehmann for drawing our attention to this.

Example 6.3. Again on $\mathbb{P}^3$ we consider the vector field

$$X(x,y,z):=z\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

The singularities are the line L given by $x_2=x_3=0$ and the point $Q:=[0:1:0:0]$. The residues at Q are the same as (6.4). To compute the residues at L, we consider the deformation

$$X_t(x,y,z)=z\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+tx\frac{\partial}{\partial z}.$$

On the chart $x_1\neq 0$ with coordinates $x'=x_2/x_1, y'=x_3/x_1, z'=x_4/x_1$ the vector field X_t is given by

$$X_t(x',y',z')=x'(1-y')\frac{\partial}{\partial x'}+(t-y'^2)\frac{\partial}{\partial y'}-y'z'\frac{\partial}{\partial z'}.$$

Also on the chart $x_2\neq 0$ with coordinates $x''=x_1/x_2, y''=x_3/x_2, z''=x_4/x_2$ it is given by

$$X_t(x'',y'',z'')=(y''-x'')\frac{\partial}{\partial x''}+(tx''-y'')\frac{\partial}{\partial y''}-z''\frac{\partial}{\partial z''}.$$

The singularities of X_t for $t\neq 0$ are the four points given by $O:=[0:0:0:1]$, Q and $P_i(t):=[1:0:u_{t,i}:0]$ for $i=1,2$, where the $u_{t,i}$'s are the roots of the equation $\lambda^2-t=0$.

At the point O the first jet of X_t, $t\neq 0$, is non-degenerate and it is given by the matrix

$$\begin{pmatrix}0&0&1\\0&1&0\\t&0&0\end{pmatrix}.$$

From this and (6.3),

$$\mathrm{BB}_{c_1^3}(X_t;O) = -\frac{1}{t} \quad \mathrm{BB}_{c_1c_2}(X_t;O) = 1 \quad \mathrm{BB}_{c_3}(X_t;O) = 1. \tag{6.8}$$

At the point $P_i(t)$ the vector field X_t has first jet given by the matrix

$$B(t,i) := \begin{pmatrix} 1-u_{t,i} & 0 & 0 \\ 0 & -2u_{t,i} & 0 \\ 0 & 0 & -u_{t,i} \end{pmatrix}.$$

Thus, for $t \to 0$, $t \neq 0$ one can use (6.3) to compute the residues:

$$\mathrm{BB}_{c_1^3}(X_t;P_i(t)) = \frac{(1-4u_{t,i})^3}{2t(1-u_{t,i})} \quad \mathrm{BB}_{c_1c_2}(X_t;P_i(t)) = \frac{(1-4u_{t,i})(5t-3u_{t,i})}{2t(1-u_{t,i})}$$
$$\mathrm{BB}_{c_3}(X_t;P_i(t)) = 1.$$

If $S'(\widetilde{\mathcal{F}})$ is the connected component which contains the line L in the manifold deformation $M \times (-1,1)$, the intersection of $S'(\widetilde{\mathcal{F}})$ with $M \times \{t\}$ is given by the points $O, P_1(t), P_2(t)$. While, the connected component in $M \times (-1,1)$ which contains Q equals $Q \times (-1,1)$.

A direct computation, taking into account that $u_{t,1} + u_{t,2} = 0$ and $u_{t,1}u_{t,2} = -t$, shows that

$$\mathrm{BB}_\varphi(X_t;P_1(t)) + \mathrm{BB}_\varphi(X_t;P_2(t)) = \begin{cases} \frac{-64t^2+36t+1}{t(1-t)}, & \varphi = c_1^3 \\ \frac{2(7-10t)}{1-t}, & \varphi = c_1c_2 \\ 2, & \varphi = c_3. \end{cases} \tag{6.9}$$

By Theorem 1.1, we have

$$\mathrm{BB}_\varphi(X;L) = \lim_{t\to 0}[\mathrm{BB}_\varphi(X_t;O) + \mathrm{BB}_\varphi(X_t;P_1(t)) + \mathrm{BB}_\varphi(X_t;P_2(t))]$$

and using (6.8) and (6.9) we see that we have the same values as (6.5) for the residues at L.

The residues at Q are given

$$\mathrm{BB}_{c_1^3}(X_t;Q) = \frac{27}{1-t} \quad \mathrm{BB}_{c_1c_2}(X_t;Q) = \frac{3(3-t)}{1-t} \quad \mathrm{BB}_{c_3}(X_t;Q) = 1.$$

Note that they depend on t and as the limits as $t \to 0$, we have the same values as (6.4).

REFERENCES

[1] M. Abate, F. Bracci, T. Suwa, and F. Tovena, *Localization of Atiyah classes*, Revista Matematica Iberoamericana, 29:2 (2013), pp. 547–578.

[2] M. Abate, F. Bracci, and F. Tovena, *Index theorems for holomorphic maps and foliations*, Indiana Univ. Math. J., 57 (2008), pp. 2999–3048.

[3] M. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology, 1 (1961), pp. 25–45.

[4] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geom., 7 (1972), pp. 279–342.

[5] El H. M. Dia, *Sur les résidus de Baum-Bott*, Ann. Fac. Sci. Toulouse (6), 19:2 (2010), pp. 363–403.

[6] X. Gómez-Mont, *The transverse dynamics of a holomorphic flow*, Ann. of Math. (2), 127:1 (1988), pp. 49–92.

[7] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures I, II*, Ann. of Math. (2), 67 (1958), pp. 328–466.

[8] D. Lehmann, *Variétés stratifiées C^∞: intégration de Čech-de Rham, et théorie de Chern-Weil*, Geometry and topology of submanifolds, II (Avignon, 1988), pp. 205–248, World Sci. Publ., Teaneck, NJ, 1990.

[9] G. Pourcin, *Deformations of singular holomorphic foliations on reduced compact $\mathbb{C}$-analytic spaces*, Holomorphic dynamics (Mexico, 1986), pp. 246–255, Lecture Notes in Math., 1345, Springer, Berlin, 1988.

[10] H.-J. Reiffen, *The variety of a moduli of foliations on a complex space*, Enseign. Math. (2), 33 (1987), no. 3-4, pp. 191–197.

[11] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann, Paris, 1998.