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002

PROJECTIVE CONVERGENCE OF INHOMOGENEOUS 2×2 MATRIX PRODUCTS*

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Abstract. Each digit in a finite alphabet labels an element of a set $\mathcal{M}$ of 2×2 column-allowable matrices with nonnegative entries; the right inhomogeneous product of these matrices is made up to rank n, according to a given one-sided sequence of digits; then, the n-step matrix is multiplied by a fixed vector with positive entries. Our main result provides a characterization of those $\mathcal{M}$ for which the direction of the n-step vector is convergent toward a limit continuous w.r.t. to the digits sequence. The applications are concerned with Bernoulli convolutions and the Gibbs properties of linearly representable measures.

Key words. Inhomogeneous matrix product, joint spectral radius, Gibbs measure, weak Gibbs measure, sofic affine-invariant sets, measure with full dimension, an Erdős problem, Bernoulli convolutions.

AMS subject classifications. 34D08, 37-XX, 37C45, 37D35, 37F35, 37H15, 37L30, 28A80.

1. Introduction. The set $\mathcal{M} := \{M_0, \ldots, M_{\mathtt{s}-1}\}$ is made of 2×2 column-allowable matrices (i.e. with no null column) having nonnegative entries. We note $\mathcal{S}^{\mathbb{N}}$ the product space of the one-sided infinite sequences $\xi = \xi_0\xi_1 \cdots$ with digits in $\mathcal{S} = \{\mathtt{0}, \ldots, \mathtt{s}-1\}$ and consider the right inhomogeneous matrix product $M_n(\xi) := M_{\xi_0} \cdots M_{\xi_{n-1}}$. Given $0 \le \alpha \le 1$, we study the *limit direction map* $\mathtt{p}_\alpha : \mathcal{S}^{\mathbb{N}} \to [0\,;1]$, where (provided it exists) $\mathtt{p}_\alpha(\xi)$ is the limit of the first entry of the probability vector $M_n(\xi)U_\alpha/\|M_n(\xi)\|$: here and throughout, U_α is the probability vector whose first entry equals α and $\|\cdot\|$ stands for the matrix norm obtained by summing the modulus of the matrix entries. When the map $\xi \mapsto \mathtt{p}_\alpha(\xi)$ is well defined on the whole space $\mathcal{S}^{\mathbb{N}}$, we call $\mathcal{M} = \{M_0, \ldots, M_{\mathtt{s}-1}\}$ a *α-Right Projective Convergent Product (α-RPCP) set*; if in addition $\xi \mapsto \mathtt{p}_\alpha(\xi)$ is continuous on $\mathcal{S}^{\mathbb{N}}$ (endowed with the product topology), we call $\mathcal{M}$ a *continuous α-RPCP set*. This later notion is to be compared with the Right Convergent Product (RCP) sets of matrices introduced by Daubechies & Lagarias in [DL92, DL01]. Our main result in Theorem A provides a characterization of those $\mathcal{M}$ which are continuous RPCP sets (the case of the non-continuous RPCP set is developed in [OT13b, OT13c]). Theorem B shows how existence and continuity of the limit direction map $\xi \mapsto \mathtt{p}_\alpha(\xi)$ may be related to the Gibbs properties of linearly representable probability measures that we call $\mathcal{M}$-measures.

The motivation for studying this question originates in several works concerned with multifractal analysis [Oli99][FFW01][FL02][Tes06], the variational principle for Hausdorff dimension [McM84][Bed84][KP96b][KP96a][Yay09][Oli09][Oli10][Fen11] as well as Gibbs structures within different classes of Bernoulli convolutions [SV98][DST99][HL01][FO03][Fen05][OST05][Oli12].

The statements of Theorem A and Theorem B are given in § 1.1. In Section 2, the proof of Theorem B serves as an introductive illustration of the ideas developed throughout the paper, while Theorem A is completely established in Section 3. Special attention is given to applications of the continuous RPCP property. Section 4 shows

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how the multifractal analysis of the level sets for Birkhoff averages is related (in some cases) to the joint spectral radius of a finite set of matrices. Two other applications in Section 5 and Section 6 are concerned with Bernoulli convolutions whose characteristic scale are respectfully an integral basis and a quadratic Pisot Vijayaraghavan (PV) number.

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1.1. Statements of Theorem A and Theorem B. Recall that for any $0 \leq \alpha \leq 1$ we note

$$U_\alpha = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} \tag{1}$$

(so that $\{U_0, U_1\}$ is the canonical basis of the 2×1 vectors). Given V and W two linearly independent vectors $\mathcal{C}(V, W)$ stands for the cone of the linear combination $xV + yW$ where x, y are nonnegative real numbers such that $(x, y) \neq (0, 0)$. Hence, $\mathcal{C}(U_0, U_1)$ is the nonnegative orthant of the 2×1 vectors (with the origin removed): we shall denote by $\Pi : \mathcal{C}(U_0, U_1) \to [0\,; 1]$ the application such that $\Pi(V) = \alpha$, whenever V is proportional to U_α. Given $0 \leq \alpha \leq 1$, $\mathtt{p}_\alpha(n, \xi) := \Pi(M_n(\xi)U_\alpha)$ is called the n-step α-direction about ξ and (provided it makes sense)

$$\mathtt{p}_\alpha(\xi) := \lim_{n\to+\infty} \mathtt{p}_\alpha(n, \xi)$$

defines the limit α-direction about ξ. To clarify some aspects of Theorem A and Theorem B, we make two remarks. First, the transposed matrix $M^\star$ of a column-allowable matrix M is not necessarily column-allowable and thus, there are no simple relations between the RPCP property of $\mathcal{M} = \{M_{\mathtt{0}}, \ldots, M_{\mathtt{s}-1}\}$ and the one of $\mathcal{M}^\star = \{M_{\mathtt{0}}^\star, \ldots, M_{\mathtt{s}-1}^\star\}$. However,

$$\Pi\Big((\Delta M_{\xi_0}\Delta)\cdots(\Delta M_{\xi_{n-1}}\Delta)U_\alpha\Big) = 1 - \Pi\big(M_n(\xi)U_{1-\alpha}\big), \quad \text{where} \quad \Delta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Secondly, the existence of a *common invariant direction* for the matrices in $\mathcal{M}$ (i.e. a α for which $M_i U_\alpha$ is proportional to U_α, for any $i = \mathtt{0}, \ldots, \mathtt{s} - 1$) produces parasite situations where the n-step direction map $\mathtt{p}_\alpha(n, \cdot)$ is uniformly convergent independently of the configurations of the matrices in $\mathcal{M}$: a typical situation – with $\alpha = 1/2$ – arises when each matrix in $\mathcal{M}$ is stochastic, so that the limit direction map $\mathtt{p}_{1/2}(\cdot)$ is identically equal to $1/2$.

Throughout the paper we denote by $\mathbb{A}$ the set of 2×2 nonnegative matrices which are column-allowable, so that $M \in \mathbb{A}$ implies $M(\mathcal{C}(U_0, U_1)) \subset \mathcal{C}(U_0, U_1)$. We also consider the four applications $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} : \mathbb{A} \to [0\,; +\infty[$ such that

$$M = \begin{pmatrix} \mathrm{a}(M) & \mathrm{b}(M) \\ \mathrm{c}(M) & \mathrm{d}(M) \end{pmatrix}.$$

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The subsets of $\mathbb{A}$ specified by conditions on the entries are denoted in a special way: for instance, $\mathbb{A}\{\mathrm{b}=0\}$ stands for the set of those matrices $M \in \mathbb{A}$ for which $\mathrm{b}(M)=0$ (that is the lower triangular matrices in $\mathbb{A}$), while $\mathbb{A}\{\mathrm{b}=\mathrm{c}=0\}$ is the set of matrices which are diagonal with non zero diagonal entries, etc... Finally, we shall also write

$$\mathcal{M}_2 = \mathcal{M} \cup \Big\{ M_i M_j \ ; \ (M_i, M_j) \in \mathcal{M} \times \mathcal{M} \Big\}.$$

THEOREM A. *For $\mathcal{M} = \{M_0, \dots, M_{\mathtt{s}-1}\} \subset \mathbb{A}$ and $0 < \alpha < 1$, the following propositions hold:*

(i) : the maps $\mathtt{p}_\alpha(n,\cdot)$ converge uniformly on $\mathcal{S}^{\mathbb{N}}$ if one of the condition (1)-(4) below occurs:

$$\begin{aligned}
(1) \ &: & \mathcal{M}_2 &\subset \mathbb{A}\Big\{\mathrm{b}=0 \Rightarrow \mathrm{c}>0 \ \textit{and}\ \mathrm{a} \le \mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c}=0 \Rightarrow \mathrm{b}>0 \ \textit{and}\ \mathrm{a} \ge \mathrm{d}\Big\} \\
(2) \ &: & \mathcal{M}_2 &\subset \mathbb{A}\Big\{\mathrm{b}=0 \Rightarrow \mathrm{c}>0 \ \textit{and}\ \mathrm{a} > \mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c}=0 \Rightarrow \mathrm{b}>0 \ \textit{and}\ \mathrm{a} < \mathrm{d}\Big\} \\
(3) \ &: & \mathcal{M}_2 &\subset \mathbb{A}\Big\{\mathrm{a}>0\Big\} \cap \mathbb{A}\Big\{\mathrm{b}=0 \Rightarrow \mathrm{a}>\mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c}=0 \Rightarrow \mathrm{a} \ge \mathrm{d}\Big\} \\
(4) \ &: & \Delta\mathcal{M}_2\Delta &\subset \mathbb{A}\Big\{\mathrm{a}>0\Big\} \cap \mathbb{A}\Big\{\mathrm{b}=0 \Rightarrow \mathrm{a}>\mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c}=0 \Rightarrow \mathrm{a} \ge \mathrm{d}\Big\}
\end{aligned}$$

(ii) : conversely – provided that α is not a common invariant direction for $\mathcal{M}$ – the maps $\mathtt{p}_\alpha(n,\cdot)$ do not converge uniformly over $\mathcal{S}^{\mathbb{N}}$ if none of the above conditions (1)-(4) occurs.

As we shall see in the proof of part (ii) of Theorem A, the non uniform convergence of $\mathtt{p}_\alpha(n,\cdot)$ means either the non convergence of $\mathtt{p}_\alpha(n,\xi)$ for at least one $\xi \in \mathcal{S}^{\mathbb{N}}$ or the pointwise convergence of $\mathtt{p}_\alpha(n,\cdot)$ toward a non continuous limit function $\mathtt{p}_\alpha$. Hence, according to Theorem A, the continuous α-RPCP property of $\mathcal{M}$ means the uniform convergence of the n-step direction maps $\mathtt{p}_\alpha(n,\cdot)$, for each $0 < \alpha < 1$ (this does not hold for $d \times d$ matrices with $d \ge 3$). Furthermore, in the case of the uniform convergence of $\mathtt{p}_\alpha(n,\cdot)$, we stress that part (i) of Theorem A does not asserts equality of the limit direction maps $\mathtt{p}_\alpha$ and $\mathtt{p}_{\alpha'}$ when $0 < \alpha \neq \alpha' < 1$: this question is handled by Proposition 0 (for either pointwise and uniform convergence of the direction maps).

Our focus on RPCP properties roots in the multifractal analysis of the so-called $\mathcal{M}$-*linearly representable measures* (or $\mathcal{M}$-*measures* for short). A $\mathcal{M}$-measure is a Borel probability on $\mathcal{S}^{\mathbb{N}}$ defined by means of inhomogeneous products of matrices in $\mathcal{M}$: suppose for instance that $M_* := M_0 + \cdots + M_{\mathtt{s}-1}$ is irreducible and assume (without loss of generality) the spectral radius of M_* equal to 1: then, according to the Perron-Frobenius Theorem there exists $0 < \alpha_* < 1$ – the Perron-Frobenius direction of $\mathcal{M}$ – such that $M_* U_{\alpha_*} = U_{\alpha_*}$. For any $0 \le \beta \le 1$, the Kolmogorov Extension Theorem allows to define the $\mathcal{M}$-measure μ as the Borel probability such that for any $\xi \in \mathcal{S}^{\mathbb{N}}$ and any $n \ge 1$,

$$\mu[\xi_0 \cdots \xi_{n-1}] := U_\beta^\star M_n(\xi) U_{\alpha_*} / U_\beta^\star U_{\alpha_*}. \tag{2}$$

(here $[\xi_0 \cdots \xi_{n-1}]$ stands for the cylinder set of $\mathcal{S}^{\mathbb{N}}$ made of the ξ' s.t. $\xi'_0 \cdots \xi'_{n-1} = \xi_0 \cdots \xi_{n-1}$). If $0 < \beta < 1$, then we say that μ in (2) is a *positive* $\mathcal{M}$-*measure.* (For more details an references concerning the linearly representable measures we refer to [BP11].) Theorem B deals with the relationship between the RPCP property of $\mathcal{M}$ and the Gibbs properties of the positive $\mathcal{M}$-measures. Recall that a Borel probability

η over $\mathcal{S}^{\mathbb{N}}$ is a *weak Gibbs measure* (see [Yur98]) if there exists a continuous $\phi : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$ (usually called a potential) such that

$$\frac{1}{K_n} \leq \frac{\eta[\xi_0 \cdots \xi_{n-1}]}{e^{\sum_{k=0}^{n-1} \phi(\sigma^k \cdot \xi)}} \leq K_n, \tag{3}$$

where $n \mapsto K_n$ is a subexponential (i.e. $1/n \log K_n \to 0$ when $n \to +\infty$) and $\sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ is the one-sided shift map: if there exists a positive constant K such that $1/K \leq K_n \leq K$, for any $n \geq 1$, then η is called a *Gibbs measure* (in the sense of Bowen [Bow74]).

The existence of invariant directions for matrices in $\mathcal{M}$ (i.e. α for which there exists at least one i s.t. $M_i U_\alpha$ is proportional to U_α) produces special situations (for instance if each $M_i \in \mathcal{M}$ is stochastic then $\mathtt{p}_{1/2}(\xi) \equiv 1/2$). We call $0 \leq \alpha \leq 1$ a *regular direction* for $\mathcal{M}$ (or $\mathcal{M}$ a *α-regular set*) if $M_i U_\alpha$ is not proportional to U_α for each $i \in \mathcal{S}$.

Proposition 0. *Suppose $0 < \alpha_* < 1$ is a regular direction for $\mathcal{M}$ for which $\mathtt{p}_{\alpha_*}(n, \cdot) \to \mathtt{p}_{\alpha_*}$ pointwisely (resp. uniformly); then $\mathtt{p}_\alpha(n, \cdot) \to \mathtt{p}_{\alpha_*}$ pointwisely (resp. uniformly), for any $0 < \alpha < 1$; provided it exists, the function $\mathtt{p}_* := \mathtt{p}_{\alpha_*}$ is called the regular limit direction map of $\mathcal{M}$.*

Theorem B. *Let $\mathcal{M} = \{M_0, \ldots, M_{\mathtt{s}-1}\} \subset \mathbb{A}$ such that $M_* = \sum_k M_k$ is irreducible with Perron-Frobenius direction α_* supposed $\mathcal{M}$-regular; moreover, consider the propositions: (i) : $\mathcal{M}$ is continuous α_*-RPCP; (ii) : for any $0 < \alpha < 1$, the sequence $n \mapsto \mathtt{p}_\alpha(n, \cdot)$ is uniformly convergent over $\mathcal{S}^{\mathbb{N}}$ toward the same limit $\mathtt{p}_*$; (iii) : each positive $\mathcal{M}$-measure is weak-Gibbs; then*

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)}.$$

The equivalence (i) $\iff$ (ii) in Theorem B means that existence and continuity of $\mathtt{p}_* = \mathtt{p}_{\alpha_*}$ (when the Perron-Frobenius direction α_* is $\mathcal{M}$-regular) is equivalent to the uniform convergence $\mathtt{p}_\alpha(n, \cdot) \to \mathtt{p}_*$, for any $0 < \alpha < 1$. This is already implicitly in Theorem A (without assuming the $\mathcal{M}$-regular direction α_* is the Perron-Frobenius direction): however the proof given in § 2.3 of the special case in Theorem B is elementary (depending on Proposition 0 and a special property of the weak Gibbs measure in Lemma 2.1) while the corresponding argument, in the developed proof of Theorem A, does not clearly appear.

2. Proposition 0 and proof of Theorem B.

2.1. Notations. Given f and g two real-valued functions defined on a set X, we write (Xiangfan notations [Pey95]) either $f(x) \lhd g(x)$ or $g(x) \rhd f(x)$, when there exists $K > 0$ such that $f(x) \leq K\, g(x)$ for any $x \in X$ and $f(x) \bowtie g(x)$ means that both $f(x) \lhd g(x)$ and $f(x) \rhd g(x)$ hold.

Let X be a compact metric space and $\mathfrak{P}(X)$ be the weak-$*$ compact convex set of the probability measures defined on the borelian subsets of X. For $T : X \to X$ a continuous endomorphism, $\mathfrak{P}_T(X)$ stands for the set of $\mu \in \mathfrak{P}(X)$ which are T-invariant in the sense that $\mu \circ T^{-1} = \mu$; $\mathfrak{P}_T(X)$ is a non empty weak-$*$ compact Choquet simplex whose extremal points are the T-ergodic measures; moreover, if T is expansive, the Kolmogorov-Sinaï metric entropy map $\mu \mapsto \mathrm{h}_T(\mu)$ is affine and upper-semi continuous over $\mathfrak{P}_T(X)$ (see [DGS76] for a general introduction to ergodic theory and discrete dynamical systems).

For $\mathcal{S} = \{\mathtt{0}, \dots, \mathtt{s}-1\}$ (with $\mathtt{s} \geq 1$), we note $\mathcal{S}^0 = \{\not{o}\}$; an element in $\mathcal{S}^n$, written as a string of letters in $\mathcal{S}$, is called a word; the set of words $\mathcal{S}^* := \bigcup_{n=0}^{\infty} \mathcal{S}^n$ endowed with the concatenation, is a monoid whose unit element is the empty word $\not{o}$. Given $\mathcal{M} = \{M_{\mathtt{0}}, \dots, M_{\mathtt{s}-1}\}$ a set of $d \times d$ matrices, there exists a canonical morphism from $\mathcal{S}^*$ to the monoid generated by the finite matrix products of matrices in $\mathcal{M}$: any non empty word $i_1 \cdots i_n \in \mathcal{S}^*$ is sent to the product

$$M_{i_1 \cdots i_n} := M_{i_1} \cdots M_{i_n} \tag{4}$$

with the convention that $M_{\not{o}}$ is the $d \times d$ identity matrix.

The product space $\mathcal{S}^{\mathbb{N}}$ made of the one-sided infinite words of the form $\xi = \xi_0 \xi_1 \cdots$ (where each $\xi_i \in \mathcal{S}$) is endowed with the product topology: this makes the shift map $\sigma : \xi_0 \xi_1 \cdots \mapsto \xi_1 \xi_2 \cdots$ continuous over $\mathcal{S}^{\mathbb{N}}$. Let Σ be a compact subset of $\mathcal{S}^{\mathbb{N}}$ left invariant by the shift. A word w is said Σ-admissible when the cylinder set $\Sigma[w]$ of the $\xi \in \Sigma$ whose prefix is w is non empty (when Σ is the full shift $\mathcal{S}^{\mathbb{N}}$, we simply note $[w]$ instead of $\Sigma[w]$); moreover, $\Sigma^{(n)}$ stands for the collection of the Σ-admissible words of length n and $\Sigma^* := \bigcup_n \Sigma^{(n)}$ is the language of Σ.

2.2. Regular directions and Proposition 0. Because $\mathcal{M}$ has finite cardinality, *most of* $0 \leq \alpha \leq 1$ are regular directions for $\mathcal{M}$. Moreover, there is no loss of generality to consider the 1/2-regularity: indeed, the α-regularity of $\mathcal{M}$ for $0 < \alpha < 1$ is equivalent to the 1/2-regularity of $P_\alpha \mathcal{M} P_\alpha^{-1}$, where we have introduced

$$P_\alpha := \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}.$$

Proof of Proposition 0. Let A_k be a sequence of 2×2 matrices with non negative entries and V_0, V_1 be two non collinear vectors with nonnegative entries s.t. $A_k V_i / \|A_k V_i\|$ tends to the same probability vector U_β as $k \to +\infty$: we claim that, for any vector V with positive entries

$$\frac{A_k V}{\|A_k V\|} \to U_\beta \tag{5}$$

To see this, write $V = xV_0 + yV_1$ where $x, y \in \mathbb{R}$, so that, with $\theta_k := (x\|A_k V_0\| + y\|A_k V_1\|)/\|A_k V\|$ and any W,

$$\frac{A_k V}{\|A_k V\|} = \theta_k W + x\frac{\|A_k V_0\|}{\|A_k V\|}\left(\frac{A_k V_0}{\|A_k V_0\|} - W\right) + y\frac{\|A_k V_1\|}{\|A_k V\|}\left(\frac{A_k V_1}{\|A_k V_1\|} - W\right). \tag{6}$$

With $W = U_\beta$, the convergence in (5) comes, since (use the *quotient bound result* [Har02, (2.3)])

$$\min\left\{\frac{V_i(1)}{V(1)}, \frac{V_i(2)}{V(2)}\right\} \leq \frac{\|A_k V_i\|}{\|A_k V\|} \leq \max\left\{\frac{V_i(1)}{V(1)}, \frac{V_i(2)}{V(2)}\right\}.$$

Now, let $0 < \alpha_* < 1$ be a $\mathcal{M}$-regular direction s.t. $\mathtt{p}_{\alpha_*}(n, \xi) \to \mathtt{p}_{\alpha_*}(\xi) =: \mathtt{p}_*(\xi)$ pointwisely, as $n \to +\infty$: in other words, $M_n(\xi) U_{\alpha_*} / \|M_n(\xi) U_{\alpha_*}\| \to U_{\mathtt{p}_*(\xi)}$. Given any $0 < \alpha < 1$, the convergence $M_n(\xi) U_\alpha / \|M_n(\xi) U_\alpha\| \to U_{\mathtt{p}_*(\xi)}$ is equivalent to the convergence (as $k \to +\infty$)

$$\frac{M_{n_k}(\xi) U_\alpha}{\|M_{n_k}(\xi) U_\alpha\|} \to U_{\mathtt{p}_*(\xi)} \tag{7}$$

for any sequence $n_1 < n_2 < \cdots$ of ranks s.t. that $M_{\xi_{n_k}} = M_i$ (for a digit i satisfying $\xi_n = i$ for infinitely many n). With $n_1, n_2, \ldots$ and i fixed,

$$\frac{M_{n_k}(\xi)U_{\alpha_*}}{\|M_{n_k}(\xi)U_{\alpha_*}\|} \to U_{\mathtt{p}_*(\xi)} \quad \text{and} \quad \frac{M_{n_k}(\xi)M_iU_{\alpha_*}}{\|M_{n_k}(\xi)M_iU_{\alpha_*}\|} = \frac{M_{n_k+1}(\xi)U_{\alpha_*}}{\|M_{n_k+1}(\xi)U_{\alpha_*}\|} \to U_{\mathtt{p}_*(\xi)}.$$

Because α_* is a regular direction for $\mathcal{M}$, the two vectors $V_0 = U_{\alpha_*}$ and $V_1 = M_iU_{\alpha_*}$ are non collinear: hence, it is licit to apply the convergence result in (5) with $A_k = M_{n_k}(\xi)$, V_0, V_1, $U_\beta = U_{\mathtt{p}_*(\xi)}$ and to the vector $V = U_\alpha$, so that (7) holds: the pointwise convergence $M_n(\xi)U_\alpha/\|M_n(\xi)U_\alpha\| \to U_{\mathtt{p}_*(\xi)}$ is established. The case of uniform convergence is obtained with a closed similar argument using (6) to get uniform estimates. □

2.3. Proof of Theorem B. The *n-step potential* ($n \geq 1$) of a Borel probability μ fully supported by $\mathcal{S}^{\mathbb{N}}$ is $\phi_n : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$ such that ($\phi_1(\xi) = \log \mu[\xi_0]$ and for any $n \geq 2$):

$$\phi_n(\xi) = \log \frac{\mu[\xi_0 \cdots \xi_{n-1}]}{\mu[\xi_1 \cdots \xi_{n-1}]}.$$

Given any measurable $\phi : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$, it is always true that

$$\frac{1}{K_n} \leq \frac{\mu[\xi_0 \cdots \xi_{n-1}]}{e^{\sum_{k=0}^{n-1} \phi(\sigma^k \cdot \xi)}} \leq K_n, \quad \text{where} \quad K_n := e^{\sum_{k=1}^{n} \|\phi - \phi_k\|_\infty} \; ; \tag{8}$$

hence (Cesàro means lemma) if $\phi_k \to \phi$ uniformly on $\mathcal{S}^{\mathbb{N}}$, then $n \mapsto K_n$ is subexponential and (8) means that μ is a weak Gibbs measure of ϕ according to (3). The next lemma improves this remark. We note $\mathrm{Var}_n(f)$ the n-step variation of $f : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$, i.e. the supremum of $|f(\xi) - f(\xi')|$ for $\xi' \in [\xi_0 \cdots \xi_{n-1}]$, so that f is continuous if and only if $\mathrm{Var}_n(f) \to 0$ as $n \to +\infty$.

Lemma 2.1. *Let $\mu \in \mathfrak{P}(\mathcal{S}^{\mathbb{N}})$ be a fully supported measure whose n-step potential $\phi_n \to \phi$ pointwisely on $\mathcal{S}^{\mathbb{N}}$; then $\|\phi_n - \phi\|_\infty \leq \mathrm{Var}_n(\phi)$ and ϕ continuous implies μ is weak Gibbs.*

Proof. Fix a rank N and $\varepsilon > 0$. For any $\xi \in \mathcal{S}^{\mathbb{N}}$, the pointwise convergence of $\phi_n(\xi)$ toward $\phi(\xi)$ ensures the existence of a rank $N_{\xi,\varepsilon} \geq N$ for which $|\phi_{N_{\xi,\varepsilon}}(\xi) - \phi(\xi)| \leq \varepsilon$. Since $\mathcal{S}^{\mathbb{N}}$ is compact, there exists a covering of $\mathcal{S}^{\mathbb{N}}$ by cylinders of the form $[\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}]$, for ξ in a finite $X \subset \mathcal{S}^{\mathbb{N}}$. Moreover, X may be chosen so that to make this covering a partition, say: $\mathcal{S}^{\mathbb{N}} = \bigsqcup_{\xi \in X} [\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}]$ (the intersection of two cylinders is either empty or equal to one of them). For $\omega \in \mathcal{S}^{\mathbb{N}}$ let X_ω be the set of those $\xi \in X$ for which $[\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}] \cap [\omega_0 \cdots \omega_{N-1}] \neq \emptyset$. Because $N \leq N_{\xi,\varepsilon}$, the inclusion $[\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}] \subset [\omega_0 \cdots \omega_{N-1}]$ holds for any $\xi \in X_\omega$ and the partition in $\mathcal{S}^{\mathbb{N}} = \bigsqcup_{\xi \in X} [\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}]$ implies that $[\omega_0 \cdots \omega_{N-1}] = \bigsqcup_{\xi \in X_\omega} [\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}]$. Therefore,

$$e^{\phi_n(\omega)} = \frac{\mu[\omega_0 \ldots \omega_{N-1}]}{\mu[\omega_1 \ldots \omega_{N-1}]} = \frac{\sum_{\xi \in X_\omega} \mu[\xi_0 \ldots \xi_{N_{\xi,\varepsilon}-1}]}{\sum_{\xi \in X_\omega} \mu[\xi_1 \ldots \xi_{N_{\xi,\varepsilon}-1}]}$$

and since $\mu[\xi_0 \ldots \xi_{N_{\xi,\varepsilon}-1}]/\mu[\xi_1 \ldots \xi_{N_{\xi,\varepsilon}-1}] = e^{\phi_{N_{\xi,\varepsilon}}(\xi)}$, one obtains[2]

$$\min \left\{ \phi_{N_{\xi,\varepsilon}}(\xi) \; ; \; \xi \in X_\omega \right\} \leq \phi_N(\omega) \leq \max \left\{ \phi_{N_{\xi,\varepsilon}}(\xi) \; ; \; \xi \in X_\omega \right\} \; ;$$

[2]Here, we use the *quotient bound result*: if $a_i, b_i > 0$, then $\min_i\{a_i/b_i\} \leq (\sum_i a_i)/(\sum_i b_i) \leq \max_i\{a_i/b_i\}$.

let $\xi \in X_\omega$: because $|\phi_{N_{\xi,\varepsilon}}(\xi) - \phi(\xi)| \le \varepsilon$ one gets $|\phi_N(\omega) - \phi(\xi)| \le \varepsilon$, while $[\xi_0 \cdots \xi_{N_{\xi,\varepsilon}-1}] \subset [\omega_0 \cdots \omega_{N-1}]$ implies that $|\phi(\xi) - \phi(\omega)| \le \mathrm{Var}_N(\phi)$: by the triangular inequality, $|\phi_N(\omega) - \phi(\omega)| \le \mathrm{Var}_N(\phi) + \varepsilon$ and the lemma is proved because $\varepsilon > 0$ is arbitrary. □

Proof of Theorem B. Fix $\mathcal{M} = \{M_0, \dots, M_{\mathtt{s}-1}\} \subset \mathbb{A}$ for which $M_* = \sum_i M_i$ is irreducible and (without loss of generality) assume that its spectral radius is equal to 1; we note $0 < \alpha_* < 1$ the Perron-Frobenius direction meaning that $M_* U_{\alpha_*} = U_{\alpha_*}$. The argument depends on the equivalence in (11) below between the n-step direction maps $\mathtt{p}_{\alpha_*}(n,\cdot) = \Pi(M_n(\cdot)U_{\alpha_*})$ and the n-step potential of a positive $\mathcal{M}$-measure. More precisely, let $0 < \beta_0 < 1$ such that $U^\star_{\beta_0} M_i$ is not proportional to $U^\star_{\beta_0}$ (i.e. $\det(U^\star_{\beta_0} M_i, U^\star_{\beta_0}) \ne 0$), for any $i \in \mathcal{S}$ and let ϕ_n be the n-step potential associated with the positive $\mathcal{M}$-measure μ_0 associated with α_* and β_0 as in (2). For any $i = \mathtt{0}, \dots, \mathtt{s}-1$, the co-vector $U^\star_{\beta_0} M_i := \begin{pmatrix} a_i & b_i \end{pmatrix}$ is positive and for $\xi \in [i]$

(9)
$$e^{\phi_n(\xi)} = \frac{(U^\star_{\beta_0} M_i) M_{n-1}(\sigma\cdot\xi) U_{\alpha_*}}{U^\star_{\beta_0} M_{n-1}(\sigma\cdot\xi) U_{\alpha_*}} = \frac{\begin{pmatrix} a_i & b_i \end{pmatrix} U_{\mathtt{p}_{\alpha_*}(n-1,\sigma\cdot\xi)}}{\begin{pmatrix} \beta_0 & 1-\beta_0 \end{pmatrix} U_{\mathtt{p}_{\alpha_*}(n-1,\sigma\cdot\xi)}} = H_i\Big(\mathtt{p}_{\alpha_*}(n-1, \sigma\cdot\xi)\Big),$$

where the homography H_i is s.t.

$$H_i(x) = \frac{(a_i - b_i)x + b_i}{(2\beta_0 - 1)x + (1-\beta_0)}\ ; \tag{10}$$

H_i is finite on $[0\,;1]$ – because $0 < \beta_0 < 1$ – and non constant: indeed, the determinant of H_i is $\det(U^\star_{\beta_0} M_i, U^\star_{\beta_0})$ which is – by definition of β_0 – a nonzero quantity. Now, let $J \subset [0\,;1]$ be the closed convex hull of $\bigcup_{n=1}^\infty \mathtt{p}_{\alpha_*}(n, \mathcal{S}^{\mathbb{N}})$. Then, H_i forms a diffeomorphism from J onto $H_i(J)$; moreover, it is easy to check the existence of $0 < \varepsilon < 1/2$ (independant of i) such that $H_i(J) \subset [\varepsilon\,; 1-\varepsilon]$: applying the Mean Value Theorem, it follows from (9) that for any $p, q \ge 1$

$$\|\phi_p - \phi_q\|_\infty \bowtie \|\mathtt{p}_{\alpha_*}(p-1,\cdot) - \mathtt{p}_{\alpha_*}(q-1,\cdot)\|_\infty. \tag{11}$$

• Proof of $(i) \iff (ii)$: The implication (i) $\Longleftarrow$ (ii) is evident since $0 < \alpha_* < 1$ and each $\mathtt{p}_{\alpha_*}(n,\cdot)$ is continuous over $\mathcal{S}^{\mathbb{N}}$. To prove (i) $\Longrightarrow$ (ii), we assume the pointwise limit $\mathtt{p}_* = \mathtt{p}_{\alpha_*}$ (exists and) is continuous on $\mathcal{S}^{\mathbb{N}}$: from (9) the n-step potentials ϕ_n associated with the positive $\mathcal{M}$-measure μ_{β_0} converges (pointwisely) toward $\phi : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$ such that $\phi(\xi) = \log(H_{\xi_0} \circ \mathtt{p}_*(\sigma\cdot\xi))$. Because $\mathtt{p}_*$ is supposed continuous over $\mathcal{S}^{\mathbb{N}}$, the function ϕ is also continuous and Lemma 2.1 implies that $\phi_n(\xi) \to \phi(\xi)$ uniformly over $\mathcal{S}^{\mathbb{N}}$. Therefore, the inequalities in (11) ensure the convergence $\mathtt{p}_{\alpha_*}(n,\cdot) \to \mathtt{p}_*$ to be uniform: we use Proposition 0 (with uniform convergence) to conclude that, for any $0 < \alpha < 1$, the convergence $\mathtt{p}_\alpha(n,\cdot) \to \mathtt{p}_*$ is uniform as well.

• Proof of (ii) $\Longrightarrow$ (iii) : Because $0 < \alpha_* < 1$, condition (ii) ensures the uniform convergence of $n \mapsto \mathtt{p}_{\alpha_*}(n,\cdot)$ over $\mathcal{S}^{\mathbb{N}}$: by (11), the sequence $n \mapsto \phi_n$ of the n-step potentials of the positive $\mathcal{M}$-measure μ_{β_0} is also uniformly convergent over $\mathcal{S}^{\mathbb{N}}$: hence μ_{β_0} (and each positive $\mathcal{M}$-measure) is a weak Gibbs measure. □

3. Proof of Theorem A.

3.1. Projective metrics and contraction coefficient. The Hilbert projective distance $\delta_H(X,Y)$ $(= \delta_H(X^\star, Y^\star))$ of two positive vectors X and Y is defined by means of the *cross-ratio*, that is

$$\delta_H(X,Y) := \max_{i,j}\left\{\log\frac{X(i)Y(j)}{X(j)Y(i)}\right\} = \max_W\left\{\log\frac{X^\star W}{Y^\star W}\right\} + \max_W\left\{\log\frac{Y^\star W}{X^\star W}\right\},$$

where W runs over the set of the non zero nonnegative vectors. The Birkhoff's contraction coefficient of a nonnegative column-allowable $d \times d$ matrix A is by definition [Sen81, §-3.4] the supremum $\tau(A)$ of the $\delta_H(X^\star A, Y^\star A)/\delta_H(X^\star, Y^\star)$, where X and Y are non colinear positive vectors. We shall need a classical equivalent definition for $\tau(A)$, that is

$$\tau(A) = \frac{1-\sqrt{\Phi(A)}}{1+\sqrt{\Phi(A)}} \quad \text{where} \quad \Phi(A) = \min\left\{\frac{A(i,k)A(j,l)}{A(j,k)A(i,\ell)}\right\} \tag{12}$$

When A is a 2×2 matrix, we consider an other projective metric on $\mathcal{C}(U_0, U_1)$ i.e. $(X,Y) \mapsto |\Pi(X) - \Pi(Y)|$ and then define, for any $A \in \mathbb{A}$

$$\delta(A) := |\Pi(AU_0) - \Pi(AU_1)| = \frac{|\det(A)|}{(\mathrm{a}(A)+\mathrm{c}(A))(\mathrm{b}(A)+\mathrm{d}(A))}. \tag{13}$$

This leads to a second expression/upper-bound for $\tau(A)$ for $A \in \mathbb{A}$.

Proposition 3.1. *Given $A, B \in \mathbb{A}$, one has*

$$\tau(A) = \sup\left\{\frac{\delta(BA)}{\delta(B)}\ ;\ \delta(B) \neq 0\right\} = \frac{|\sqrt{\mathrm{a}(A)\mathrm{d}(A)} - \sqrt{\mathrm{b}(A)\mathrm{c}(A)}|}{\sqrt{\mathrm{a}(A)\mathrm{d}(A)} + \sqrt{\mathrm{b}(A)\mathrm{c}(A)}} \leq \frac{|\det(A)|}{\mathrm{a}(A)\mathrm{d}(A)}. \tag{14}$$

Proof. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ one writes

$$\begin{aligned}\delta(BA) &= \frac{|\det(AB)|}{\Big[(a'a+b'c)+(c'a+d'c)\Big]\Big[(a'b+b'd)+(c'b+d'd)\Big]}\\ &= \frac{|\det(A)\det(B)|}{\Big[a(a'+c')+c(b'+d')\Big]\Big[b(a'+c')+d(b'+d')\Big]} = \delta(B)\frac{|\det(A)|}{(a+c/x)(bx+d)},\end{aligned}$$

where $x = (a'+c')/(b'+d')$; then, provided that $\det(B) \neq 0$, one gets

$$\frac{\delta(BA)}{\delta(B)} = \frac{|\det(A)|}{(a+c/x)(bx+d)}\ ; \tag{15}$$

taking the maximum in (15) over x gives (with $x = \sqrt{cd}/\sqrt{ab}$)

$$\sup\left\{\frac{\delta(BA)}{\delta(B)}\ ;\ \delta(B) \neq 0\right\} = \frac{|\sqrt{\mathrm{a}(A)\mathrm{d}(A)} - \sqrt{\mathrm{b}(A)\mathrm{c}(A)}|}{\sqrt{\mathrm{a}(A)\mathrm{d}(A)} + \sqrt{\mathrm{b}(A)\mathrm{c}(A)}}.$$

In the last expression one recognizes the Birkhoff contraction coefficient $\tau(A)$ as given in (12) (for 2×2 matrices); the upper bound in (14) is obtained easily. □

Corollary 3.2. *For $A, B \in \mathbb{A}$, (i) : $\delta(AB) \leq \delta(A)\tau(B)$; (ii) : $0 \leq \tau(A) \leq 1$ and $\tau(A) < 1$ whenever A is positive; (iii) : $\tau(AB) \leq \tau(A)\tau(B)$.*

Proof. Part (i) and (ii) follows from (14); part (i) gives $\delta(CAB)/\delta(C) \leq \tau(A)\tau(B)$ for any C with $\det(C) \neq 0$: part (iii) is obtained by taking the supremum over C. □

3.2. Proof of Theorem A part (i). Fix $0 < \alpha < 1$: the main difficulty to prove Theorem A is to grab as many as possible configurations of $\mathcal{M}$ for which $n \mapsto \mathtt{p}_\alpha(n,\cdot) = \Pi(M_n(\cdot)U_\alpha)$ is uniformly convergent over $\mathcal{S}^{\mathbb{N}}$ (Theorem 3.3) and then to prove (see § 3.3) the non uniform convergence for each ones of the remaining possible configurations. In order to state the following theorem, recall that $\mathcal{M}_2 = \mathcal{M} \cup \{M_iM_j \ ; \ (M_i, M_j) \in \mathcal{M} \times \mathcal{M}\}$ and define $\mathfrak{A}_1 = \Delta\mathfrak{A}_1\Delta$, $\mathfrak{A}_2 = \Delta\mathfrak{A}_2\Delta$ and $\mathfrak{A}_3$ such that :

$$\mathcal{M} \in \mathfrak{A}_1 \iff \mathcal{M}_2 \subset \mathbb{A}\Big\{\mathrm{b} = 0 \Rightarrow \mathrm{c} > 0 \text{ and } \mathrm{a} \le \mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c} = 0 \Rightarrow \mathrm{b} > 0 \text{ and } \mathrm{a} \ge \mathrm{d}\Big\}$$

$$\mathcal{M} \in \mathfrak{A}_2 \iff \mathcal{M}_2 \subset \mathbb{A}\Big\{\mathrm{b} = 0 \Rightarrow \mathrm{c} > 0 \text{ and } \mathrm{a} > \mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c} = 0 \Rightarrow \mathrm{b} > 0 \text{ and } \mathrm{a} < \mathrm{d}\Big\}$$

$$\mathcal{M} \in \mathfrak{A}_3 \iff \mathcal{M}_2 \subset \mathbb{A}\Big\{\mathrm{a} > 0\Big\} \cap \mathbb{A}\Big\{\mathrm{b} = 0 \Rightarrow \mathrm{a} > \mathrm{d}\Big\} \cap \mathbb{A}\Big\{\mathrm{c} = 0 \Rightarrow \mathrm{a} \ge \mathrm{d}\Big\}$$

The key point in the following argument is to replace the uniform convergence of $n \mapsto \mathtt{p}_\alpha(n,\cdot)$ on $\mathcal{S}^{\mathbb{N}}$ by a pointwise convergence: to do this, we shall consider the smallest interval $[\underline{\alpha}\,;\overline{\alpha}]$ containing $\bigcup_{n=1}^{\infty} \mathtt{p}_\alpha(n, \mathcal{S}^{\mathbb{N}})$ and define the matrix (depending on α)

$$M_\diamond := \begin{pmatrix} \underline{\alpha} & \overline{\alpha} \\ 1-\underline{\alpha} & 1-\overline{\alpha} \end{pmatrix}. \tag{16}$$

We emphasize on the importance of the matrix $M_\diamond$ with the following remark: the inequality $\delta(M_n(\xi)M_\diamond) \le \delta(M_n(\xi))$ is always valid; however, it is possible that $\delta(M_n(\xi)M_\diamond) \to 0$, as $n \to +\infty$, while $\delta(M_n(\xi))$ tends to a positive limit: consider for instance $\alpha = 1/2$ and

$$\mathcal{M} = \left\{ M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix} \right\} \quad \text{so that} \quad M_\diamond = \begin{pmatrix} 1/3 & 2/5 \\ 2/3 & 3/5 \end{pmatrix}$$

(using (13), one checks that $\delta(M_0^n M_\diamond)$ converges to 0, while $\delta(M_0^n)$ has a positive limit).

Theorem 3.3. *Let* $0 < \alpha < 1$ *and consider (i)* : $\mathcal{M} \in \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup (\Delta\mathfrak{A}_3\Delta)$; *(ii)* : $\delta(M_n(\xi)M_\diamond) \to 0$, *as* $n \to +\infty$, *for any* $\xi \in \Sigma$; *(iii)* : $n \mapsto \mathtt{p}_\alpha(n,\cdot)$ *is uniformly convergent over* $\mathcal{S}^{\mathbb{N}}$; *then:*

$$(i) \implies (ii) \implies (iii)$$

Proof of (ii) $\implies$ *(iii) in Theorem 3.3.* Fix $0 < \alpha < 1$ and assume that $\delta(M_n(\xi)M_\diamond) \to 0$ as $n \to +\infty$, for any $\xi \in \mathcal{S}^{\mathbb{N}}$; for $\varepsilon > 0$, let $n_\varepsilon(\xi)$ be the (minimal) rank such that $\delta(M_n(\xi)M_\diamond) \le \varepsilon$, for any $n \ge n_\varepsilon(\xi)$. The cylinders $C_\varepsilon(\xi) := [\xi_0 \cdots \xi_{n_\varepsilon(\xi)-1}]$ form an open-covering of (the compact) $\mathcal{S}^{\mathbb{N}}$ and there exists a finite $X \subset \mathcal{S}^{\mathbb{N}}$ such that $\mathcal{S}^{\mathbb{N}} = \bigcup_{\xi \in X} C_\varepsilon(\xi)$. Let $\zeta \in C_\varepsilon(\xi)$ with $\xi \in X$; for $p, q \ge \max_{\xi \in X}\{n_\varepsilon(\xi)\}$ arbitrary given, the definition of $M_\diamond$ and of $\mathtt{p}_\alpha(\cdot,\cdot)$ ensures the existence of $0 \le a_p, a_q \le 1$ for which $\mathtt{p}_\alpha(p,\zeta) = \Pi(M_{n_\varepsilon(\xi)}(\xi)M_\diamond U_{a_p})$ and $\mathtt{p}_\alpha(q,\zeta) = \Pi(M_{n_\varepsilon(\xi)}(\xi)M_\diamond U_{a_q})$. It follows from the definition of $\delta(\cdot)$ in (13) that $|\mathtt{p}_\alpha(p,\zeta) - \mathtt{p}_\alpha(q,\zeta)| \le \delta(M_{n_\varepsilon(\xi)}(\xi)M_\diamond) \le \varepsilon$. □

Before we complete the proof of Theorem 3.3, we shall establish several intermediate lemmas.

LEMMA 3.4. *(i) : For any $k \geq 0$, let $A_k = \begin{pmatrix} a_k & b_k \\ ? & d_k \end{pmatrix}$ be a matrix with $a_k \geq d_k$: then,*

$$\sum\nolimits_k b_k/d_k = +\infty \Longrightarrow \lim_{n\to+\infty} \delta(A_0\cdots A_{n-1}) = 0\ ;$$

(ii) : for any $k \geq 0$, let $A_k = \begin{pmatrix} a_k & 0 \\ c_k & d_k \end{pmatrix}$ be a matrix with $a_k \leq d_k$: then,

$$\sum\nolimits_k c_k/a_k = +\infty \Longrightarrow \lim_{n\to+\infty} \delta(A_0\cdots A_{n-1}) = 0.$$

Proof. Using the definition of $\delta(\cdot)$ in (13), part (i) follows writing

$$\begin{aligned} A_0\cdots A_{n-1} &= d_0\cdots d_{n-1}\begin{pmatrix} \dfrac{a_0\cdots a_{n-1}}{d_0\cdots d_{n-1}} & \dfrac{\sum_{i=0}^{n-1} a_0\cdots a_{i-1}b_i d_{i+1}\cdots d_{n-1}}{d_0\cdots d_{n-1}} \\ 0 & 1 \end{pmatrix} \\ &\geq d_0\cdots d_{n-1}\begin{pmatrix} 1 & \sum_{i=0}^{n-1}\frac{b_i}{d_i} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

while part (ii) is deduced from (i) and the fact that $\delta(\Delta A\Delta) = \delta(A)$. □

LEMMA 3.5. *(i) : Given any matrix $A, B \in \mathbb{A}$, one has $\delta(AB) \leq \tau_1(A)\delta(B)$, where*

$$\tau_1(A) := \sup\left\{\frac{\delta(AB)}{\delta(B)}\ ;\ \delta(B) \neq 0\right\} = \frac{|\det(A)|}{\min\left\{\Big(\mathrm{a}(A)+\mathrm{c}(A)\Big)^2, \Big(\mathrm{b}(A)+\mathrm{d}(A)\Big)^2\right\}}\ ;$$

(ii) provided that $A \in \mathbb{A}\cap\mathbb{A}^\star$ and B is positive, one has $\delta(AB) \leq \delta(A^\star)\tau_2(B)$, where

$$\tau_2(B) := \frac{|\det(B)|}{\min\Big\{\mathrm{a}(A),\mathrm{c}(A)\Big\}\cdot\min\Big\{\mathrm{b}(A),\mathrm{d}(A)\Big\}} \geq \sup\left\{\frac{\delta(AB)}{\delta(A^\star)}\ ;\ \delta(A^\star) \neq 0\right\}.$$

Proof. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $\mathbb{A}$, parts (i) and (ii) follows from the identity:

$$\delta(AB) = \frac{|\det(A)\det(B)|}{\Big[(a+c)a' + (b+d)c'\Big]\Big[(a+c)b' + (b+d)d'\Big]}. \tag{17}$$

□

LEMMA 3.6. *Assume that the spectral radius of $A \in \mathbb{A}$ is an eigenvalue with eigenvector U_θ for $0 \leq \theta \leq 1$; if $\det(A) \geq 0$ then, $\theta \in [r\,;s] \subset [0\,;1]$ implies $A(\mathcal{C}(U_r,U_s)) \subset \mathcal{C}(U_r,U_s)$.*

Proof. For we assume the spectral radius ρ of A is an eigenvalue of A, it is necessary for the discriminant $(\mathrm{a}(A)-\mathrm{d}(A))^2 + 4\mathrm{b}(A)\mathrm{c}(A)$ of the characteristic polynomial of A to be non negative. If this discriminant equals 0, then it is necessary that $A \in \mathbb{A}\{\mathrm{bc} = 0\}$ and in that case, the assertion is obtained by direct computation.

Now, suppose the discriminant is positive; then, the second eigenvalue of A is necessarily a real number $\lambda \neq \rho$ such that $|\lambda| \leq \rho$ and the assumption that $\det(A) \geq 0$ implies $0 \leq \lambda < \rho$ (in particular the case $\lambda = -\rho$ is avoided). Given any vector W in the cone $\mathcal{C}(U_0, U_1)$, there exists a real number x and an eigenvector V of λ such that $W = V + xU_\theta$. Therefore, $1/\rho^n A^n W = (\lambda/\rho)^n V + xU_\theta$; since $1/\rho^n A^n W \in \mathcal{C}(U_0, U_1)$, the fact that $(\lambda/\rho)^n$ tends to 0 implies x to be non negative. Moreover

$$AW = \lambda V + \rho x U_\theta = \lambda(W - xU_\theta) + \rho x U_\theta = \lambda W + (\rho - \lambda) x U_\theta\ ; \tag{18}$$

notice that $AW \neq 0$ and λ together with $\rho - \lambda$ and x being nonnegative real numbers, it follows for (18) that AW is a non zero linear combination of W and U_θ with non negative coefficients; if W belongs to $\mathcal{C}(U_r, U_s)$, the condition that $\theta \in [r\,; s]$ implies that AW belongs to $\mathcal{C}(U_r, U_s)$. □

Proof of (i) $\implies$ (ii) in Theorem 3.3. Let $\mathcal{M} \in \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup (\Delta\mathfrak{A}_3\Delta)$ and $\xi \in \mathcal{S}^{\mathbb{N}}$; then, the convergence $\delta(M_n(\xi)M_\diamond) \to 0$ as $n \to +\infty$ is obvious when either (**C1**) : there exists k_0 such that $\det(M_{\xi_{k_0}}) = 0$, or (**C2**) : there exists infinitely many k for which M_{ξ_k} is a positive, or (**C3**) : there exists infinitely many k for which $M_{\xi_k} M_{\xi_{k+1}}$ is positive. From now on, and without loss of generalities, we shall assume that for any $\xi \in \mathcal{S}^{\mathbb{N}}$,

$$(\mathbf{C4})\ :\ \forall k \geq 0,\ M_{\xi_k} \in \mathbb{A}\{\det \neq 0\} \quad \text{and}$$

$$\exists N \geq 0,\ k \geq N \Rightarrow \begin{cases} M_{\xi_k} M_{\xi_{k+1}} \in \mathbb{A}\{\mathrm{abcd} = 0\}\ ; \\ M_{\xi_k} \in \mathbb{A}\{\mathrm{abcd} = 0\}. \end{cases}$$

• To begin with, consider that $\mathcal{M} \in \mathfrak{A}_3$ (and similarly for $\mathcal{M} \in \Delta\mathfrak{A}_3\Delta$). In order to apply Lemma 3.6, let $\mathcal{E}_+$ denote the set of the (normalized) Perron-eigenvectors of matrices in $\mathcal{M}\{\det > 0\}$. If $M \in \mathcal{M}\{\mathrm{abcd} > 0\}$ then (Perron-Frobenius Theorem) the unique Perron-eigenvector of M has positive entries and thus differs from $U_0 = (0\ \ 1)^\star$. Suppose that $M \in \mathcal{M}\{\mathrm{abcd} = 0\} \cap \{\det > 0\}$; because $\mathcal{M} \in \mathfrak{A}_3$, the condition $\det(M) > 0$ is equivalent to $\mathrm{d}(M) > 0$; hence, if $\mathrm{b}(M) > 0$ then U_0 is not an eigenvector of M; conversely, if $\mathrm{b}(M) = 0$ then $\mathrm{a}(M) > \mathrm{d}(M)$ and $\mathrm{a}(M)$ is the spectral radius of M: therefore U_0 is an eigenvector with eigenvalue $\mathrm{d}(M)$ and thus cannot be a Perron-eigenvector of M; we have proved that $U_0 \notin \mathcal{E}_+$. Now, consider $\mathcal{E}_-$ the set of the vectors of the form MU_i for $i = 0$ or 1 and $M \in \mathcal{M}\{\mathrm{abcd} = 0\} \cap \{\det \leq 0\}$; here, the fact that $\mathcal{M} \in \mathfrak{A}_3$ implies that both $\mathrm{a}(M) > 0$ and $\mathrm{d}(M) = 0$, so that $U_0 \notin \mathcal{E}_-$. One concludes that $\mathcal{E}_+ \cup \mathcal{E}_- \cup \{U_\alpha\} \subset \mathcal{C}(U_1, U_\gamma)$, for some $0 < \gamma < 1$: by construction, Lemma 3.6 implies $M_n(\xi)U_\alpha$ belong to $\mathcal{C}(U_1, U_\gamma)$. Therefore, the minimal cone $\mathcal{C}(U_{\underline{\alpha}}, U_{\overline{\alpha}})$ containing all the vectors $\Pi(M_n(\xi)U_\alpha)$ is itself contained in $\mathcal{C}(U_1, U_\gamma)$: hence $0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and in particular $\underline{\alpha}\overline{\alpha} > 0$.

In what follows we note $\mathbb{T} := \mathbb{A}\{\mathrm{b} = 0\}$ (resp. $\mathbb{D} = \mathbb{A}\{\mathrm{b} = 0 \text{ and } \mathrm{c} = 0\}$), that is the subset of $\mathbb{A}$ made of the lower triangular (resp. diagonal) matrices. Suppose first that $M_{\xi_k} \in \mathbb{T}$ for any $k \in \mathbb{N}$ (and similarly for $k \geq N$) and write

$$M_n(\xi)M_\diamond = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \underline{\alpha} \\ 1 - \overline{\alpha} & 1 - \underline{\alpha} \end{pmatrix} = \begin{pmatrix} a_n\overline{\alpha} & a_n\underline{\alpha} \\ c_n\overline{\alpha} + d_n(1 - \overline{\alpha}) & c_n\underline{\alpha} + d_n(1 - \underline{\alpha}) \end{pmatrix}\ ;$$

then, it follows from (13) and the identity $\det(M_n(\xi)M_\diamond) = a_n d_n(\overline{\alpha} - \underline{\alpha})$ that

$$\delta\left(M_n(\xi)M_\diamond\right) \leq \frac{|\det(M_n(\xi)M_\diamond)|}{a_n^2 \overline{\alpha}\underline{\alpha}} = \frac{d_n}{a_n} \cdot \left(\frac{\overline{\alpha} - \underline{\alpha}}{\overline{\alpha}\underline{\alpha}}\right) = \left(\prod_{k=0}^{n-1} \frac{\mathrm{d}(M_{\xi_k})}{\mathrm{a}(M_{\xi_k})}\right) \cdot \left(\frac{\overline{\alpha} - \underline{\alpha}}{\overline{\alpha}\underline{\alpha}}\right).$$

However $\mathcal{M} \in \mathfrak{A}_3$ implies $\mathrm{a}(M_{\xi_k}) > 0$ and for $M_{\xi_k} \in \mathbb{T}$ one gets $\mathrm{b}(M_{\xi_k}) = 0$ and $\mathrm{a}(M_{\xi_k}) > \mathrm{d}(M_{\xi_k})$: therefore, $\prod_{k=0}^{n-1} \mathrm{d}(M_{\xi_k})/\mathrm{a}(M_{\xi_k}) \to 0$ as $n \to +\infty$ and $\delta\,(M_n(\xi)M_\diamond) \to 0$ as well.

Assume now that $M_{\xi_k} \notin \mathbb{T}$ (i.e. $M_{\xi_k} \in \mathbb{A}\{\mathrm{ab} > 0\}$) for any $k \geq N$. We claim that $\mathrm{c}(M_{\xi_k}) = 0$, for any $k \geq N$. Indeed, $\mathcal{M} \in \mathfrak{A}_3$ and $\mathrm{c}(M_{\xi_k}) > 0$ implies that $M_{\xi_k} \in \mathbb{A}\{\mathrm{ac} > 0\}$; suppose for a contradiction, that $M_{\xi_k} \in \mathbb{A}\{\mathrm{ac} > 0\}$, for infinitely many $k \geq N$: for by assumption $M_{\xi_{k+1}} \in \mathbb{A}\{\mathrm{ab} > 0\}$, one would obtain $M_{\xi_k}M_{\xi_{k+1}} \in \mathbb{A}\{\mathrm{abcd} > 0\}$ for infinitely many $k \geq N$, in contradiction with (**C4**). Given $k \geq N$, the hypothesis that $M_{\xi_k} \in \mathcal{M} \in \mathfrak{A}_3$ together with $\mathrm{c}(M_{\xi_k}) = 0$ implies $M_{\xi_k} \in \mathbb{T}^\star\{\mathrm{ab} > 0\} \cap \{\mathrm{a} \geq \mathrm{d}\}$: it follows from part (i) of Lemma 3.4 that $\delta(M_n(\sigma^N \cdot \xi)) \to 0$ as $n \to +\infty$; since (use Proposition 3.1 and part (i) of Lemma 3.5) $\delta(M_{N+n}(\xi)M_\diamond) \leq \tau_1(M_N(\xi))\delta(M_n(\sigma^N \cdot \xi))\tau(M_\diamond)$, one concludes that $\delta(M_{N+n}(\xi)M_\diamond) \to 0$ as well.

• For the remaining cases (i.e. when $\mathcal{M} \in \mathfrak{A}_1 \cup \mathfrak{A}_2$), let $\gamma > 0$ and define

$$\Delta_\gamma := \begin{pmatrix} 0 & \gamma \\ 1 & 0 \end{pmatrix}$$

(we shall fix – throughout the argument – the suitable value of γ depending on either $\mathcal{M} \in \mathfrak{A}_1$ or $\mathcal{M} \in \mathfrak{A}_2$). For any $M \in \mathbb{A}\{\det \neq 0\}$ and $\varepsilon \in \{0, 1\}$, define $M^{(\varepsilon)}$ to be either $\Delta_\gamma^{-\varepsilon} M \Delta_\gamma^{1-\varepsilon}$ (i.e. $M^{(0)} = M\Delta_\gamma$ and $M^{(1)} = \Delta_\gamma^{-1}M$) if $\det(M) < 0$ or $\Delta_\gamma^{-\varepsilon} M \Delta_\gamma^{\varepsilon}$ (i.e. $M^{(0)} = M$ and $M^{(1)} = \Delta_\gamma^{-1} M \Delta_\gamma$) if $\det(M) > 0$ (in each case $\det(M^{(\varepsilon)}) > 0$). Now, define $A_k(\xi) = A_k := M_{\xi_k}^{(\varepsilon_k)}$, where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ is the sequence in $\{0, 1\}$ inductively defined by setting $\varepsilon_0 = 0$ and $\varepsilon_k = \varepsilon_{k-1}$ if $\det(M_{\xi_k} > 0)$ while $\varepsilon_k = 1 - \varepsilon_{k-1}$ otherwise. On the one hand, condition (**C4**) ensures that

$$k \geq N \Rightarrow A_k \in \mathbb{A}\{\mathrm{abcd} = 0\} \cap \{\det > 0\} = \mathbb{T}\{\mathrm{ad} > 0\} \cup \mathbb{T}^\star\{\mathrm{ad} > 0\}. \tag{19}$$

On the other hand, if $\det(M_{\xi_k}) > 0$ (resp. $\det(M_{\xi_k}) < 0$) then $\varepsilon_{k+1} = \varepsilon_k$ (resp. $\varepsilon_{k+1} = 1 - \varepsilon_k$), so that in any cases $\Delta_\gamma^{-\varepsilon_k} M_{\xi_k} \Delta_\gamma^{-\varepsilon_{k+1}} = A_k$ and thus,

$$M_{\xi_k} = \Delta_\gamma^{\varepsilon_k} (\Delta_\gamma^{-\varepsilon_k} M_{\xi_k} \Delta_\gamma^{-\varepsilon_{k+1}}) \Delta_\gamma^{-\varepsilon_{k+1}} = \Delta_\gamma^{\varepsilon_k} A_k \Delta_\gamma^{-\varepsilon_{k+1}}\ ; \tag{20}$$

a simple induction (using the fact that $\varepsilon_0 = 0$) gives:

$$M_k(\xi) = A_0 \cdots A_{k-1} \Delta_\gamma^{-\varepsilon_k}. \tag{21}$$

• Suppose that $\mathcal{M} \in \mathfrak{A}_1$; we shall first be interested in the matrices in $\mathcal{M}\{\det \leq 0\} = \mathcal{M}\{\mathrm{d} = 0\} \cup \mathcal{M}\{\mathrm{a} = 0\}$, so that one can fix the value of γ. If $\mathcal{M}\{\mathrm{d} = 0\} = \mathcal{M}\{\mathrm{a} = 0\} = \emptyset$ then we fix $\gamma = 1$. On the contrary, we fix γ to be either $\min\{\mathrm{b}(M_i)/\mathrm{c}(M_i)\ ;\ M_i \in \mathcal{M}\{\mathrm{d} = 0\}\}$, if $\mathcal{M}\{\mathrm{d} = 0\} \neq \emptyset$ and $\mathcal{M}\{\mathrm{a} = 0\} = \emptyset$, or $\max\{\mathrm{b}(M_j)/\mathrm{c}(M_j)\ ;\ M_j \in \mathcal{M}\{\mathrm{a} = 0\}\}$, if $\mathcal{M}\{\mathrm{a} = 0\} \neq \emptyset$ and $\mathcal{M}\{\mathrm{d} = 0\} = \emptyset$; finally, if both $\mathcal{M}\{\mathrm{d} = 0\}$ and $\mathcal{M}\{\mathrm{a} = 0\}$ are non empty and if $(M_i, M_j) \in \mathcal{M}\{\mathrm{d} = 0\} \times \mathcal{M}\{\mathrm{a} = 0\}$, then[3]

$$M_i M_j = \begin{pmatrix} \mathrm{b}(M_i)\mathrm{c}(M_j) & * \\ 0 & \mathrm{c}(M_i)\mathrm{b}(M_j) \end{pmatrix} \in \mathbb{A}\{\mathrm{c} = 0\}\ ;$$

[3]This is where we use the conditions on $\mathcal{M}_2$ (in the definitions of the sets $\mathfrak{A}_i$), rather than simply on $\mathcal{M}$.

by the condition that $\mathcal{M} \in \mathfrak{A}_1$, it is necessary that $\mathrm{b}(M_i)/\mathrm{c}(M_i) \geq \mathrm{b}(M_j)/\mathrm{c}(M_j)$; this last case proves that it is licit to assume the existence of $\gamma > 0$ such that

(22)
$$M_i \in \mathcal{M}\{\mathrm{d}=0\} \Rightarrow \gamma \leq \mathrm{b}(M_i)/\mathrm{c}(M_i) \quad \text{and} \quad M_j \in \mathcal{M}\{\mathrm{a}=0\} \Rightarrow \mathrm{b}(M_j)/\mathrm{c}(M_j) \geq \gamma.$$

From the definition of γ in (22) and the condition $\mathcal{M} \in \mathfrak{A}_1$, one deduces from (19) that

$$A_k \in \mathbb{T}\Big\{\mathrm{c} > 0 \text{ and } 0 < \mathrm{a} \leq \mathrm{d}\Big\} \cup \mathbb{T}^\star\Big\{\mathrm{b} > 0 \text{ and } \mathrm{a} \geq \mathrm{d} > 0\Big\}\ ;$$

actually the conditions in (**C4**) ensures the existence of N such that either $A_k \in \mathbb{T}\{\mathrm{c} > 0 \text{ and } 0 < \mathrm{a} \leq \mathrm{d}\}$, for any $k \geq N$ or $A_k \in \mathbb{T}^\star\{\mathrm{b} > 0 \text{ and } \mathrm{a} \geq \mathrm{d} > 0\}$, for any $k \geq N$ (otherwise $M_{\xi_k} M_{\xi_{k+1}} = \Delta_\gamma^{\varepsilon_k} A_k A_{k+1} \Delta_\gamma^{-\varepsilon_{k+2}}$ would be positive for infinitely many k, in contradiction with (**C4**)). Therefore, applying either part (i) or part (ii) of Lemma 3.4 gives in both cases that $\delta(A_N \ldots A_{n-1})$ tends to 0 when n goes to infinity; however, (use Proposition 3.1 and part (i) of Lemma 3.5) from the fact that

$$\delta(M_n(\xi)M_\diamond) = \delta(A_0 \cdots A_{n-1}\Delta_\gamma^{-\varepsilon_k} M_\diamond) \leq \tau_1(A_0 \cdots A_{N-1})\delta(A_N \cdots A_{n-1})\tau(\Delta_\gamma^{-\varepsilon_k} M_\diamond)),$$

one concludes $\delta(M_n(\xi)M_\diamond)$ tends to 0 as well.

• For $\mathcal{M} \in \mathfrak{A}_2$ we shall use part (ii) of Lemma 3.5 and we need prove that $M_\diamond$ is positive. To see this, we start from the matrix identities

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix} = \sqrt{bc}\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix},\ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} a-d \\ c \end{pmatrix} = a\begin{pmatrix} a-d \\ c \end{pmatrix},\ \begin{pmatrix} a & b \\ ? & d \end{pmatrix}\begin{pmatrix} b \\ d-a \end{pmatrix} = d\begin{pmatrix} b \\ d-a \end{pmatrix}$$

and (for an application of the Perron-Frobenius Theorem) the fact that for $acdb > 0$ the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad \begin{pmatrix} a & b \\ c & 0 \end{pmatrix},\quad \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$

are aperiodic; then, because $\mathcal{M} \in \mathfrak{A}_2$, each matrix $M \in \mathcal{M}$ has a Perron-eigenvector positive entries. Therefore, it is licit to consider $0 < x \leq y < 1$ such that $\mathcal{C}(U_x, U_y)$ is the (minimal) cone containing U_α and each ones of the positive Perron-eigenvectors of the matrices in $\mathcal{M}$. From (21) and Lemma 3.6, the vectors $M_n(\xi)U_\alpha$ belongs to $\mathcal{C}(U_x, U_y)$ for any $n \geq 0$, so that $\mathcal{C}(U_{\underline{\alpha}}, U_{\overline{\alpha}}) \subset \mathcal{C}(U_x, U_y)$: this implies in particular that $M_\diamond$ is a positive matrix.

By an analogous reasonning as the one leading to (22) we fix the value of $\gamma > 0$ (associated to the matrix Δ_γ) to be such that

(23)
$$M_i \in \mathcal{M}\{\mathrm{d}=0\} \Rightarrow \gamma \leq \mathrm{b}(M_i)/\mathrm{c}(M_i) \quad \text{and} \quad M_j \in \mathcal{M}\{\mathrm{a}=0\} \Rightarrow \mathrm{b}(M_j)/\mathrm{c}(M_j) \geq \gamma.$$

and consider again the matrices $A_0, A_1, \ldots$ satisfying (21). From the definition of γ in (23) together with (19), the condition $\mathcal{M} \in \mathfrak{A}_2$ implies that

$$A_k \in \mathbb{T}\Big\{\mathrm{c} > 0 \text{ and } \mathrm{a} > \mathrm{d} > 0\Big\} \cup \mathbb{T}^\star\Big\{\mathrm{b} > 0 \text{ and } 0 < \mathrm{a} < \mathrm{d}\Big\}\ ;$$

similarly to the case when $\mathcal{M} \in \mathfrak{A}_1$, the conditions (**C4**) imply the existence of N such that either $A_k \in \mathbb{T}\{\mathrm{c} > 0 \text{ and } \mathrm{a} > \mathrm{d} > 0\}$, for $k \geq N$ or $A_k \in \mathbb{T}^\star\{\mathrm{b} > 0 \text{ and } 0 < \mathrm{a} < \mathrm{d}\}$, for $k \geq N$. According to part (i) of Lemma 3.5 one has

$\delta\left(M_n(\xi)M_\diamond\right) \le \tau_1(A_0\cdots A_{N-1})\delta(A_N\cdots A_{n-1}\Delta_\gamma^{-\varepsilon_k}M_\diamond)$ and for $M_\diamond\Delta_\gamma^{-\varepsilon_k}$ being positive and $A_N\cdots A_{n-1}$ being in $\mathbb{A}\cap\mathbb{A}^\star$, it is licit to use part (ii) of Lemma 3.5, which gives

$$\delta\left(M_n(\xi)M_\diamond\right) \le \tau_1(A_0\cdots A_{N-1})\delta(A_{n-1}^\star\cdots A_N^\star)\tau_2(M_\diamond\Delta_\gamma^{-\varepsilon_k}). \tag{24}$$

Part (i) and (ii) of Lemma 3.4 ensures in each case that $\delta(A_n^\star\ldots A_{N+1}^\star)\to 0$ when $n\to+\infty$: one concludes with (24) that $\delta(M_n(\xi)M_\diamond)\to 0$ as well. □

3.3. Proof of Theorem A part (ii). Recall that $\mathbb{T} := \mathbb{A}\{\mathrm{b}=0\}$ (resp. $\mathbb{D} = \mathbb{A}\{\mathrm{b}=0 \text{ and } \mathrm{c}=0\}$); in what follows $\mathfrak{A}$ stands for set of finite subsets of $\mathbb{A}$ and we define:

$$\mathcal{M}\in\mathfrak{F}_1 \iff \begin{cases}\mathcal{M}_2\cap\mathbb{T}\{\mathrm{a}>\mathrm{d}\}\neq\emptyset\\ \mathcal{M}_2\cap\mathbb{T}\{\mathrm{a}\le\mathrm{d}\}\backslash\mathbb{D}\{\mathrm{a}=\mathrm{d}\}\neq\emptyset\end{cases}$$

$$\mathcal{M}\in\mathfrak{F}_2 \iff \begin{cases}\mathcal{M}_2\cap\mathbb{T}\{\mathrm{a}>\mathrm{d}\}\neq\emptyset\\ \mathcal{M}_2\cap\mathbb{T}^\star\{\mathrm{a}\le\mathrm{d}\}\backslash\mathbb{D}\{\mathrm{a}=\mathrm{d}\}\neq\emptyset\\ \mathcal{M}_2\cap\mathbb{T}\Delta\neq\emptyset\end{cases}$$

$$\mathcal{M}\in\mathfrak{F}_3 \iff \mathcal{M}_2\cap\mathbb{D}\{\mathrm{a}=\mathrm{d}\}\neq\emptyset.$$

Lemma 3.7. $\mathfrak{A}\backslash\Big(\mathfrak{A}_1\cup\mathfrak{A}_2\cup(\Delta\mathfrak{A}_3\Delta)\Big) = (\Delta\mathfrak{F}_1\Delta)\cup(\Delta\mathfrak{F}_2\Delta)\cup\mathfrak{F}_3$.

Proof. Let $\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}$ (resp. $\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}$) be the set of the $\mathcal{M}\in\mathfrak{A}$ such that $\mathcal{M}\cap\mathbb{T}\{\mathrm{a}>\mathrm{b}\}\neq\emptyset$ (resp. $\mathcal{M}\cap\mathbb{T}\{\mathrm{a}\le\mathrm{b}\}\neq\emptyset$) and $\mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\} := \Delta\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}\Delta$ (resp. $\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\} := \Delta\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}\Delta$); first, it is easily checked that

$$\begin{aligned}&\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}\cup\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}\cup\mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}\cup\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\}\cup\\ &\qquad\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}^c\cup\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}^c\cup\mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}^c\cup\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\}^c=\mathfrak{A}\ ;\end{aligned}$$

the second point holds with the following inclusions ($X\sqcup Y$ is the disjoint union of X and Y):

$$\begin{aligned}\mathfrak{F}_1\cup\mathfrak{F}_3 &\supset \mathfrak{T}\{\mathrm{a}>\mathrm{b}\}\cap\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}\\ \Delta\mathfrak{F}_1\Delta\cup\mathfrak{F}_3 &\supset \mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}\cap\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\}\\ \mathfrak{F}_3\sqcup\mathfrak{A}_1 &\supset \mathfrak{T}\{\mathrm{a}>\mathrm{b}\}^c\cap\mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}^c\\ \mathfrak{F}_3\sqcup\mathfrak{A}_2 &\supset \mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}^c\cap\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}^c\\ \mathfrak{F}_2\sqcup\mathfrak{A}_3 &\supset \mathfrak{T}\{\mathrm{a}>\mathrm{b}\}\cap\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}^c\cap\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\}\cap\mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}^c\\ \Delta\mathfrak{F}_2\Delta\sqcup\Delta\mathfrak{A}_3\Delta &\supset \mathfrak{T}^\star\{\mathrm{a}<\mathrm{b}\}^c\cap\mathfrak{T}^\star\{\mathrm{a}\ge\mathrm{b}\}^c\cap\mathfrak{T}\{\mathrm{a}\le\mathrm{b}\}^c\cap\mathfrak{T}\{\mathrm{a}>\mathrm{b}\}.\end{aligned}$$

Therefore,

$$\Big(\mathfrak{A}_1\cup\mathfrak{A}_2\cup(\Delta\mathfrak{A}_3\Delta)\Big)\cup\Big((\Delta\mathfrak{F}_1\Delta)\cup(\Delta\mathfrak{F}_2\Delta)\cup\mathfrak{F}_3\Big)=\mathfrak{A}$$

and one concludes, for $\mathfrak{A}_1\cup\mathfrak{A}_2\cup(\Delta\mathfrak{A}_3\Delta)$ and $(\Delta\mathfrak{F}_1\Delta)\cup(\Delta\mathfrak{F}_2\Delta)\cup\mathfrak{F}_3$ are disjoint. □

Proof of Theorem A part (ii). By direct computation, one gets for any $n\ge 1$,

$$\begin{pmatrix}a&0\\c&d\end{pmatrix}^n = \begin{pmatrix}a^n & 0\\ ca^{n-1}\Big(1+(d/a)+\cdots+(d/a)^{n-1}\Big) & d^n\end{pmatrix}. \tag{25}$$

(the column-allowable condition implies $d > 0$ and by convention $d/a = \infty$ if $a = 0$). To begin with, consider $A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\}$ (hence $\mathrm{a}(A) > \mathrm{d}(A) > 0$); we use (25) to write

$$\Pi(A^k U_{1/2}) = \frac{\mathrm{a}(A^k)}{\mathrm{a}(A^k) + \mathrm{c}(A^k) + \mathrm{d}(A^k)} = \left\{ 1 + \frac{\mathrm{c}(A)}{\mathrm{a}(A)} \left(\frac{1 - \left(\frac{\mathrm{d}(A)}{\mathrm{a}(A)}\right)^k}{1 - \frac{\mathrm{d}(A)}{\mathrm{a}(A)}} \right) + \left(\frac{\mathrm{d}(A)}{\mathrm{a}(A)}\right)^k \right\}^{-1},$$

and $\lim_k \Pi(A^k U_{1/2}) = [1 + \mathrm{c}(A)/(\mathrm{a}(A) - \mathrm{d}(A))]^{-1}$; in particular, the following implication holds:

$$A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\} \implies \lim_k \Pi(A^k U_{1/2}) > 0. \tag{26}$$

Given $B \in \mathbb{T}$, we use again (25) to get

$$\begin{aligned} \Pi(A^k B^n U_{1/2}) &= \frac{\mathrm{a}(A^k)\mathrm{a}(B^n)}{\mathrm{a}(A^k)\mathrm{a}(B^n) + \mathrm{c}(A^k)\mathrm{a}(B^n) + \mathrm{d}(A^k)\mathrm{c}(B^n) + \mathrm{d}(A^k)\mathrm{d}(B^n)} \\ &= \frac{\mathrm{a}(A^k)}{\mathrm{a}(A^k) + \mathrm{c}(A^k) + \mathrm{d}(A^k)\frac{\mathrm{c}(B^n)}{\mathrm{a}(B^n)} + \mathrm{d}(A^k)\frac{\mathrm{d}(B^n)}{\mathrm{a}(B^n)}} \le \frac{\frac{\mathrm{a}(A^k)}{\mathrm{d}(A^k)}}{\frac{\mathrm{c}(B^n)}{\mathrm{a}(B^n)} + \frac{\mathrm{a}(B^n)}{\mathrm{d}(B^n)}} \end{aligned}$$

and with the convention that $1/0 = \infty$ when $\mathrm{a}(B) = 0$,

$$\Pi(A^k B^n U_{1/2}) \le \frac{\mathrm{a}(A^k)}{\mathrm{d}(A^k)} \cdot \min \left\{ \left(\frac{\mathrm{d}(B)}{\mathrm{a}(B)}\right)^n, \frac{\mathrm{a}(B)}{\mathrm{c}(B)\left(1 + \mathrm{d}(B)/\mathrm{a}(B) + \cdots + (\mathrm{d}(B)/\mathrm{a}(B))^{n-1}\right)} \right\};$$

therefore one can write the implication valid for any $k \ge 1$

$$\left.\begin{aligned} A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\} \\ B \in \mathbb{T}\{\mathrm{a} \le \mathrm{d}\} \backslash \mathbb{D}\{\mathrm{a} = \mathrm{d}\} \end{aligned}\right\} \Longrightarrow \lim_n \Pi(A^k B^n U_{1/2}) = 0. \tag{27}$$

By a similar application of (25) one gets, for any $k \ge 1$

$$\left.\begin{aligned} A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\} \\ B \in \mathbb{T}^\star\{\mathrm{a} \le \mathrm{d}\} \backslash \mathbb{D}\{\mathrm{a} = \mathrm{d}\} \\ C \in \mathbb{T}\Delta \end{aligned}\right\} \Longrightarrow \lim_n \Pi(A^k C B^n U_{1/2}) = \lim_n \Pi(A^k (C\Delta)(\Delta B \Delta)^n U_{1/2}) = 0. \tag{28}$$

Now, let $\mathcal{M} \subset \mathbb{A}$ and consider the special case of $\alpha = 1/2$ (which does not produce any loss of generality): according to part (i) of Theorem A and Lemma 3.7, we must show that $\mathcal{M}$ is not continuous RPCP whenever $\mathcal{M} \in (\Delta\mathfrak{F}_1\Delta) \cup (\Delta\mathfrak{F}_2\Delta) \cup \mathfrak{F}_3$. To see this, let $A, B, C \in \mathcal{M}_2$. On the one hand, if $A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\}$ and $B \in \mathbb{T}\{\mathrm{a} \le \mathrm{d}\} \backslash \mathbb{D}\{\mathrm{a} = \mathrm{d}\}$, then (26) and (27) imply that that $\mathcal{M}$ is not continuous RPCP if $\mathcal{M} \in \mathfrak{F}_1$ (and similarly when $\mathcal{M} \in \Delta\mathfrak{F}_1\Delta$). On the other hand, if $A \in \mathbb{T}\{\mathrm{a} > \mathrm{d}\}$, $B \in \mathbb{T}^\star\{\mathrm{a} \le \mathrm{d}\} \backslash \mathbb{D}\{\mathrm{a} = \mathrm{d}\}$ and $C \in \mathbb{T}\Delta$, then (26) and (28) imply that $\mathcal{M}$ is not continuous RPCP if $\mathcal{M} \in \mathfrak{F}_2$ (and similarly if $\mathcal{M} \in \Delta\mathfrak{F}_2\Delta$). Finally, suppose that $A \in \mathbb{D}\{\mathrm{a} = \mathrm{d}\}$; since (hypothesis) $1/2$ is not a common invariant direction for $\mathcal{M}$, there exists at least one $B \in \mathcal{M}$ s.t. $BU_{1/2}$ is proportional to U_γ for $0 \le \gamma \ne 1/2 \le 1$: then, for any $k \ge 1$:

$$\lim_{n \to +\infty} \Pi(A^n U_{1/2}) = \Pi(U_{1/2}) = 1/2 \ne \gamma = \Pi(BU_{1/2}) = \lim_{n \to +\infty} \Pi(A^k B A^n U_{1/2});$$

this proves that $\mathcal{M}$ is not continuous RPCP when $\mathcal{M} \in \mathfrak{F}_3$. □

4. Multifractal analysis and estimation of the joint spectral radius.

4.1. Generalities. The multifractal analysis of Lyapunov exponents for inhomogeneous matrix products is studied in [Fen03][Fen04][Fen09][FH10] (see also [BPS97] for an analogous analysis w.r.t. local entropy). In this paragraph we show (with an example) how the Gibbs properties of a suitable $\mathcal{M}$-measure (as in (32) below) may be used to estimate the joint spectral radius $\rho(\mathcal{M})$ of $\mathcal{M} = \{M_0, \ldots, M_{\mathtt{s}-1}\}$. We start with Rota & Strang definition in [RS60] which gives

$$\rho(\mathcal{M}) = e^{\Lambda(\mathcal{M})} \quad \text{where} \quad \Lambda(\mathcal{M}) = \lim_{n\to+\infty} \frac{1}{n} \log \max \left\{ \|M_w\| \; ; \; w \in \mathcal{S}^n \right\} \tag{29}$$

(where M_w is as in (4)). A way to estimate $\Lambda(\mathcal{M})$ roots in the seminal work by Furstenberg & Kesten [FK60]. Given any $\xi \in \mathcal{S}^{\mathbb{N}}$, the *upper Lyapunov exponent* $\lambda(\xi)$ of $\mathcal{M}$ at ξ satisfies

$$\lambda(\xi) := \limsup_{n\to+\infty} \frac{1}{n} \log \|M_n(\xi)\| \leq \Lambda(\mathcal{M}). \tag{30}$$

Recall that $\sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ is the shift map and notice that $C : (\xi, n) \mapsto \log \|M_n(\xi)\|$, from $\mathcal{S}^{\mathbb{N}} \times \mathbb{N}$ to $\mathbb{R}$, is a *subadditive process* in the sense that $C(\xi, n+m) \leq C(\xi, n) + C(\sigma^n \cdot \xi, m)$. If μ is a σ-ergodic probability measure on $\mathcal{S}^{\mathbb{N}}$, it follows from Kingman's Subadditive Ergodic Theorem that $\lambda(\xi) = \int \lambda(\omega)\mu(d\omega)$, for μ-a.e. $\xi \in \mathcal{S}^{\mathbb{N}}$; hence, (30) implies $\Lambda(\mathcal{M})$ is bounded from below by the supremum of the $\int \lambda(\xi)\mu(d\xi)$, for $\mu \in \mathfrak{P}_\sigma(\mathcal{S}^{\mathbb{N}})$; actually Fend & Huang [FH10, Lemma A3] proved (see also [DHX11]):

$$\Lambda(\mathcal{M}) = \sup \left\{ \int \lambda(\xi)\mu(d\xi) \; ; \; \mu \in \mathfrak{P}_\sigma(\mathcal{S}^{\mathbb{N}}) \right\}, \tag{31}$$

the supremum in (31) being attained for at least one ergodic measure. By Theorem B it is reasonable to consider the existence of a positive $\mathcal{M}$-measure μ satisfying the Gibbs estimates

$$\|M_n(\xi)\| \bowtie \mu[\xi_0 \cdots \xi_{n-1}] \approx \exp(S_n\phi(\xi)), \tag{32}$$

where $\phi : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$ is a continuous function and

$$S_n\phi(\xi) = \sum\nolimits_{k=0}^{\infty} \phi(\sigma^k \cdot \xi).$$

In particular, for any ergodic $\mu \in \mathfrak{P}_\sigma(\mathcal{S}^{\mathbb{N}})$, it follows from (30) (and Birkhoff Individual Ergodic Theorem) that $\lambda(\xi) = \mu(\phi)$, for μ-a.e. $\xi \in \mathcal{S}^{\mathbb{N}}$: in view of (31), this relates the estimation of $\Lambda(\mathcal{M})$ with the multifractal analysis of the level sets for the Birkhoff averages of ϕ, that is the set $E_\phi(\alpha)$ of the ξ s.t. $S_n\phi(\xi)/n \to \alpha$, as $n \to +\infty$. If ν is the Parry measure on $\mathcal{S}^{\mathbb{N}}$ (i.e. the Bernoulli measure with parameter $(1/\mathtt{s}, \ldots, 1/\mathtt{s})$) and if $M \mapsto \dim_\nu(M)$ is the Billingsley dimension w.r.t. ν (see [Bil65]), then the multifractal spectrum (for Birkhoff averages of ϕ) is the map $\alpha \mapsto \dim_\nu E_\phi(\alpha)$. Let $\mathbf{P}(f)$ be the pressure of a continuous $f : \mathcal{S}^{\mathbb{N}} \to \mathbb{R}$, that is

$$\mathbf{P}(f) := \lim_{n\to+\infty} \frac{1}{n} \log \sum\nolimits_{w\in\mathcal{S}^n} \exp(S_n f[w]), \tag{33}$$

where $S_n f[w]$ is the maximum of $S_n f(\xi)$ for ξ taken in the cylinder set $[w]$. The map $f \mapsto \mathbf{P}(f)$ is a convex function on the space of the continuous real-valued functions defined on $\mathcal{S}^{\mathbb{N}}$ and lipschitzian w.r.t. the norm of the uniform convergence (see

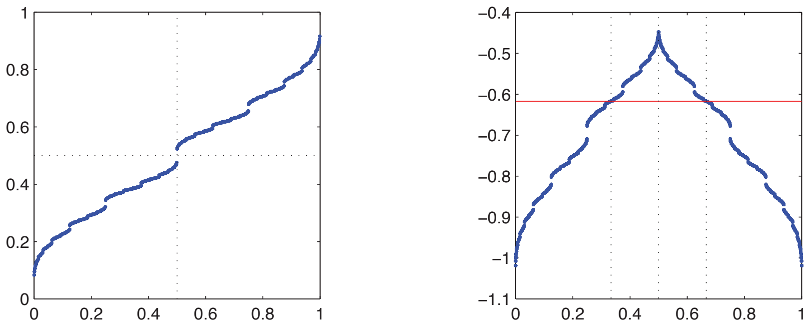

FIG. 1. *(left) : The function* $x \mapsto \mathtt{p}_*(x_0x_1\cdots)$ *(where* $x = \sum_{k=0}^{\infty} x_k/2^{k+1}$*) associated with* $\mathcal{M} = \{M_0, M_1\}$ *in (36) ; (right) : the potential* $x \mapsto \phi(x_0x_1\cdots)$ *in (41) and associated with* μ *defined in (40); the constant function (horizontal line) represents the value* $\sup_\sigma(\phi) = \log(\beta/3)$ *where* $\beta = (1+\sqrt{5})/2$.

[Wal82]). It is known [Oli99][FFW01], that the multifractal domain of ϕ, i.e. set of the $\alpha \in \mathbb{R}$ for which $E_\phi(\alpha) \neq \emptyset$, is the compact interval $[\inf_\sigma(\phi)\,;\sup_\sigma(\phi)]$, where [Oli99, Lemma 3.2]

$$\begin{cases} \inf_\sigma(\phi) := \inf\{\mu(\phi)\ ;\ \mu \in \mathfrak{P}_\sigma(\mathcal{S}^{\mathbb{N}})\} = \lim\limits_{q\to-\infty} \mathbf{P}(q\phi)/q \\ \sup_\sigma(\phi) := \sup\{\mu(\phi)\ ;\ \mu \in \mathfrak{P}_\sigma(\mathcal{S}^{\mathbb{N}})\} = \lim\limits_{q\to+\infty} \mathbf{P}(q\phi)/q. \end{cases} \tag{34}$$

Actually, for any $\inf_\sigma(\phi) \leq \alpha \leq \sup_\sigma(\phi)$, the value of $\dim_\nu E_\phi(\alpha)$ is related to the pressure function $q \mapsto \mathbf{P}(q\phi)$ by a Legendre transform formula (see [Oli99, § 4]). Comparing (34) with (31) in [DHX11] makes the link between $\Lambda(\mathcal{M})$ and the pressure map $q \mapsto \mathbf{P}(q\phi)$, so that

$$\Lambda(\mathcal{M}) = \sup_\sigma(\phi) \quad \text{and} \quad \rho(\mathcal{M}) = \exp(\sup_\sigma(\phi)). \tag{35}$$

4.2. Illustration. As an illustration consider

$$\mathcal{M} := \left\{ M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}. \tag{36}$$

Theorem A ensures $\mathcal{M}$ to be continuous RPCP; however, it is informative to give an explicit computation of the regular limit direction map $\xi \mapsto \mathtt{p}_*(\xi) = \mathtt{p}_{1/2}(\xi)$ about $\xi = \mathtt{0}^{a_1}\mathtt{1}^{a_2}\mathtt{0}^{a_3}\cdots$, where $a_1, a_2, \ldots$ is an infinite sequence of integers (with $a_1 \geq 0$ and $a_i > 0$, for $i \geq 2$). Indeed, for any $n \geq 1$ and $\varepsilon \in \{\mathtt{0}, \mathtt{1}\}$ (depending on the parity of n), a classical induction gives

$$M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\varepsilon = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\varepsilon ; \tag{37}$$

by convention $(q_0, p_0) = (1, 0)$, while for $n \geq 0$ the two integers p_n and q_n are such that

$$\frac{p_n}{q_n} = \cfrac{1}{a_0 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} =: [\![a_1, \ldots, a_n]\!]. \tag{38}$$

Moreover, the ratio p_n/q_n converges toward an irrational real number $x = [\![a_1, a_2, \dots]\!] \in [0\,;1]$. We use (37) and put $\theta_n(x) := q_n/q_{n-1}$, so that by the approximation $p_n \approx xq_n$, one gets

$$\frac{M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n} U}{\|M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n}\|} \approx \frac{1}{(1+x)(1+\theta_n(x))} \begin{pmatrix} 1 & \theta_n(x) \\ x & x\theta_n(x) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = U_{H(x)}$$

where $H(x) = 1/(1+x)$ (and recall that $U = 2U_{1/2}$); then, it is simple to deduce that

$$\mathtt{p}_{1/2}(\xi) = \mathtt{p}_*(\xi) = [\![1 + a_1, a_2, \dots]\!]. \tag{39}$$

Proposition 4.1 ([FO03]). *The (shift-ergodic) positive $\mathcal{M}$-measure μ defined on $\{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}$ and s.t.*

$$\mu[\xi_0 \cdots \xi_{n-1}] = \frac{\|M_n(\xi)\|}{2 \cdot 3^n} \tag{40}$$

is a weak Gibbs measure of $\phi : \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}} \to \mathbb{R}$ s.t. for any $\xi = \varepsilon^{a_1}(\mathtt{1} - \varepsilon)^{a_2} \varepsilon^{a_3} \cdots$ (with $a_i > 0$),

$$\phi(\xi) = \log\left(\frac{U^\star M_{\xi_0} U_{\mathtt{p}_*(\sigma\cdot\xi)}}{3}\right) = \log\left(\frac{(2-\xi_0)\mathtt{p}_*(\sigma\cdot\xi) + 1 + \xi_0}{3}\right) = \log\left(\frac{1 + [\![a_1, a_2, \cdots]\!]}{3}\right). \tag{41}$$

We now use Proposition 4.1 to get the joint spectral radius of $\mathcal{M}$ in (36).

Proposition 4.2 (Folklore). *The joint spectral radius of $\mathcal{M} = \{M_\mathtt{0}, M_\mathtt{1}\}$ is $\beta = (1 + \sqrt{5})/2$.*

Sketched proof. According to Proposition 4.1, we know that μ is a weak Gibbs measure satisfying the Gibbs estimates

$$\mu[\xi_0 \cdots \xi_{n-1}] = \frac{\|M_n(\xi)\|}{2 \cdot 3^n} \approx \exp(S_n\phi(\xi)), \tag{42}$$

where $\phi : \{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}} \to \mathbb{R}$ has a simple expression described in (41). Analogously to (35), it follows from (42) that $\rho(\mathcal{M}) = 3\exp(\sup_\sigma(\phi))$. In order to estimate $\sup_\sigma(\phi)$, we use the thermodynamic formalism: since $\|M_n(\xi)/3^n\| \approx \exp(S_n\phi(\xi))$, an application of Walters Variational Principle for the pressure (see [Wal82]) gives, for any $q \in \mathbb{R}$:

$$\mathbf{P}(q\phi) = \lim_{n\to+\infty} \frac{1}{n} \log\left(\sum_{w\in\{\mathtt{0},\mathtt{1}\}^n} \left(\frac{\|M_w\|}{3^n}\right)^q\right) = \sup\left\{\mathrm{h}_\sigma(\mu) + q\mu(\phi)\ ;\ \mu \in \mathfrak{P}_\sigma\left(\{\mathtt{0}, \mathtt{1}\}^{\mathbb{N}}\right)\right\}. \tag{43}$$

With the ergodic measure $\mu_0 = (\delta_{\overline{\mathtt{01}}} + \delta_{\overline{\mathtt{10}}})/2$, one deduces from the expression of ϕ in (41) together with (43) that $\mathrm{h}_\sigma(\mu_0) + q\mu_0(\phi) = q\log(\beta/3) \le \mathbf{P}(q\phi)$, where $\beta = (1+\sqrt{5})/2 = 1 + [\![1, 1, \cdots]\!]$ is the Golden Number. Hence, by (34) one concludes that $\sup_\sigma(\phi) \ge \log(\beta/3)$. The converse inequality depends on an upper bound $\|M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n}\| \le K\beta^{a_1+\cdots+a_n}$ ($\varepsilon = \mathtt{0}$ or $\mathtt{1}$ depending on the parity of n) for a finite K. We use the fact that for integral $x, y \ge 1$

$$(x\beta^y + 1) \le \beta^{x+y} \tag{44}$$

According to (37) the maximal entry of $\|M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n}\|$ is q_n. The sequence $q_0, q_1, \ldots$ is defined by the initial condition $(q_0, q_1) = (1, a_1)$ and the induction $q_{n+2} = a_{n+2} q_{n+1} + q_n$. Notice that $q_1 = a_1 \leq \beta^{a_1}$ while with (44) $q_2 = a_2 a_1 + 1 \leq a_2 \beta^{a_1} + 1 \leq \beta^{a_1 + a_2}$. For a $n \geq 1$ and $1 \leq k \leq n+1$ such that $q_k \leq \beta^{a_1 + \cdots + a_k}$, the inequality (44) proves the induction, since

$$q_{n+2} \leq \left(a_{n+2} \beta^{a_{n+1}} + 1\right) \beta^{a_1 + \cdots + a_n} \leq \beta^{a_{n+2} + a_{n+1}} \beta^{a_1 + \cdots + a_n}.$$

Finally, because p_{n-1}, p_n and q_{n-1} in (37) are bounded by q_n,

$$\|M_0^{a_1} M_1^{a_2} \cdots M_\varepsilon^{a_n}\| \leq 4 \beta^{a_1 + \cdots + a_n},$$

where $\varepsilon \in \{0, 1\}$, $a_1 \geq 0$ and $a_2 \cdots a_n > 0$. With (43), one gets for any $n \geq 1$ and any $q > 0$:

$$\frac{\mathbf{P}(q\phi)}{q} \leq \lim_{n \to +\infty} \frac{1}{nq} \log \left(2^n \cdot \frac{4^q \cdot \beta^{nq}}{3^{nq}} \right) = \frac{\log 2}{q} + \log \left(\frac{\beta}{3} \right).$$

Using (34) once more gives $\sup_\sigma(\phi) \leq \log(\beta/3)$: hence $\sup_\sigma(\phi) = \log(\beta/3)$ and $\rho(\mathcal{M}) = \beta$. □

Remark 4.3. *Notice that the maximum of ϕ over $\{0, 1\}^{\mathbb{N}}$ is $\phi(0\bar{1}) = \phi(1\bar{0}) = \log(2/3)$ and*

$$\log(\beta/3) = \sup_\sigma(\phi) < \sup_\xi\{\phi(\xi)\} = \log(2/3).$$

5. Bernoulli convolutions in integral basis. In this paragraph $\Sigma_{\mathtt{c}} := \{0, \ldots, \mathtt{c}\}^{\mathbb{N}}$ and we consider that $\Sigma_{\mathtt{c}} \subset \Sigma_{\mathtt{c}'}$ as soon as $1 \leq \mathtt{c} \leq \mathtt{c}'$. Fix $\beta \geq 2$ an integer and note $\mathtt{b} := \beta - 1$. Given $\mathtt{n} \geq 1$ an integer, let $X : \Sigma_{\mathtt{nb}} \to \mathbb{R}$ and $Y : \Sigma_{\mathtt{nb}} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the maps such that $X(\xi) = \sum_{k=0}^{\infty} \xi_k / \mathtt{b}^{k+1}$ and $Y(\xi) = \mathrm{fr}\{X(\xi)\}$ (here $\mathrm{fr}\{\cdot\} : \mathbb{R} \to [0\,;1[$ stands for the fractional part and $\mathbb{T} \equiv [0\,;1[$). The β-numeration is related to the multiplication by β modolo 1, that is $T : \mathbb{T} \to \mathbb{T}$ s.t. $T(x) = \mathrm{fr}\{\beta x\}$; because

$$Y(\sigma \cdot \xi) = \mathrm{fr}\{\beta X(\xi)\} = \mathrm{fr}\{\beta \lfloor X(\xi) \rfloor + \beta Y(\xi)\} = \mathrm{fr}\{\beta Y(\xi)\} = T(Y(\xi)),$$

one deduces that $Y : \Sigma_{\mathtt{nb}} \to \mathbb{T}$ makes $T : \mathbb{T} \to \mathbb{T}$ a factor of $\sigma : \Sigma_{\mathtt{nb}} \to \Sigma_{\mathtt{nb}}$. Let $\mathbb{P}$ be a probability on $\Sigma_{\mathtt{nb}}$ (to be specified) and let μ and $\mu_\circ$ be the distribution of $\xi \mapsto X(\xi)$ and $\xi \mapsto Y(\xi) = \mathrm{fr}\{X(\xi)\}$, for $\Sigma_{\mathtt{nb}}$ weighted by $\mathbb{P}$. Notice that $\mu_\circ(B) = \mu(B) + \mu(B+1) + \mu(B+2) + \cdots$, for any borelian $B \subset \mathbb{T}$. One advantage of $\mu_\circ$ is given by the following proposition.

Proposition 5.1. *$\mu_\circ = \mathbb{P} \circ Y^{-1}$ is T-invariant as soon as $\mathbb{P}$ is σ-invariant.*

Let $(i, j) \mapsto P(i, j)$ be a $\mathtt{n} \times (\mathtt{b}+1)$ stochastic matrix (i.e. $P(i, 0) + \cdots + P(i, \mathtt{b}) = 1$, for each $i = 1, \ldots \mathtt{n}$) and let $\mathbb{P} \in \mathfrak{P}(\Sigma_{\mathtt{nb}})$ be the Bernoulli measure of parameter $p = (p_0, \ldots, p_{\mathtt{nb}})$, where

$$p_k := \sum_{i_1 + \cdots + i_{\mathtt{n}} = k} P(1, i_1) \cdots P(\mathtt{n}, i_{\mathtt{n}}).$$

For instance, if $\beta = 3$ (i.e. $\mathtt{b} = 2$) and $\mathtt{n} = 3$ (i.e. $\mathtt{nb} = 6$), then

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix} \implies p = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 \end{pmatrix},$$

so that μ is the 3-fold convolution of the Cantor measure studied by Hu & Lau in [HL01], while $\mu_\circ$ is the 3-fold convolution of the Cantor measure (mod 1) considered in [FHJ11] (see § 5.2).

5.1. Matrix decomposition. For any i, define $R_r : \mathbb{R} \to \mathbb{R}$ the affine contraction such that $R_r(x) = x/\beta + r/\beta$. Given $\mathbb{P}$ the Bernoulli measure over $\{\mathtt{0}, \dots, \mathtt{nb}\}^{\mathbb{N}} = \Sigma_{\mathtt{nb}}$ with parameter $p = (p_\mathtt{0}, \dots, p_{\mathtt{nb}})$, the probability μ has support in a minimal intervall $[0\,;\gamma]$ for $\gamma \leq \mathtt{nb}/(\beta-1) = \mathtt{n}$ and is characterized by the self-similarity equation $\mu = \sum_{r=0}^{\mathtt{nb}} p_r \cdot \mu \circ R_r^{-1}$. We shall always assume $0 \leq p_r < 1$ for any $i = \mathtt{0}, \dots, \mathtt{nb}$ so that the *law of pure type* holds: μ is non atomic and either purely singular or absolutely continuous. When the IFS $\{R_\mathtt{0}, \dots, R_{\mathtt{nb}}\}$ displays overlaps, the self-similar measure μ belongs to the class of *Bernoulli convolutions* as studied for instance in [HL01][FO03][FLW05][Sch05][OST05][OT10][FHJ11]. We shall focus our attention on the measure $\mu_\circ$ for two reasons: firstly (Proposition 5.1) $\mu_\circ$ is invariant w.r.t. multiplication by β (mod 1) and secondly $\mu_\circ$ gives nice examples of $\mathcal{M}$-measures: the exact relationship between μ and $\mu_\circ$ is of much interests and is studied in § 5.2 below in the case of the 3-fold convolution of the Cantor measure (see [OT13a] for developments).

To decompose $\mu_\circ$ as a $\mathcal{M}$-measure, fix $k \in \{\mathtt{0}, \dots, \mathtt{b}\}$; given $\xi \in \Sigma_{\mathtt{nb}}$, any integer i and B a Borel set of the real line, $X(\xi) \in B + i$ if and only if $X(\sigma \cdot \xi) \in R_k^{-1}(B) + k + \beta i - \xi_0$: for $\mathbb{P}$ being Bernoulli with parameter $p = (p_\mathtt{0}, \dots, p_{\mathtt{nb}})$, one gets

$$\mu(B+i) = \sum_{r=0}^{\mathtt{nb}} p_r \cdot \mu\Big(R_k^{-1}(B) + k + \beta i - r\Big).$$

Recall that μ is non atomic and supported by a subset of $[0\,;\gamma]$, where $\gamma \leq \mathtt{n}$; when $B \subset [0\,;1[$, one has $\mu(B+i) = 0$ whenever $i \notin \{0, \dots, \mathtt{c}\}$ where $\mathtt{c}$ is the integer such that $\mathtt{c} < \gamma \leq \mathtt{c}+1$ ($\mathtt{c}$ coincides with the integral part of γ when γ is not an integer). Accordingly,

$$\begin{aligned}
&\forall k \in \{\mathtt{0}, \dots, \mathtt{b}\},\ \forall i \in \{\mathtt{0}, \dots, \mathtt{c}\},\\
&B \cup R_k^{-1}(B) \subset [0\,;1[\implies \mu(B+i) = \sum_{j=0}^{\mathtt{c}} M_k(i,j)\mu\Big(R_k^{-1}(B) + j\Big),
\end{aligned}$$

where M_k is the $(\mathtt{c}+1) \times (\mathtt{c}+1)$ matrix whose coefficient of row index i and column index j is $M_k(i,j) = p_r$ if $r = k + \beta i - j \in \{\mathtt{0}, \dots, \mathtt{nb}\}$ and $M_k(i,j) = 0$ otherwise: in other words,

$$\forall k \in \{\mathtt{0}, \dots, \mathtt{b}\}, \quad M_k = \begin{pmatrix} p_k & p_{k-1} & \cdots & p_{k-\mathtt{c}} \\ p_{k+\beta} & p_{k+\beta-1} & \cdots & p_{k+\beta-\mathtt{c}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k+\beta\mathtt{c}} & p_{k+\beta\mathtt{c}-1} & \cdots & p_{k+\beta\mathtt{c}-\mathtt{c}} \end{pmatrix}, \tag{45}$$

where by convention we assume that $p_r = 0$ if $r \notin \{\mathtt{0}, \dots, \mathtt{nb}\}$. Given any $i \in \{\mathtt{0}, \cdots, \mathtt{c}\}$, we note μ_i the Borel probability measure on $\mathbb{R}$, such that

$$\mu_i(B) = \mu(B \cap \mathbb{T} + i). \tag{46}$$

By definition, the β-addic interval (of $\mathbb{T} \equiv [0,1[$) coded by $w \in \{\mathtt{0}, \dots, \mathtt{b}\}^*$ is $I_w := R_w([0\,;1[)$. Given any $x \in \mathbb{T}$, there exists $\xi \in \Sigma_\mathtt{b}$ such that $x \in I_{\xi_0 \cdots \xi_{n-1}} =: I_n(x)$, for any $n \geq 0$ (by convention $I_\phi = \mathbb{T}$): $I_n(x)$ is called the n-step β-addic interval about x.

THEOREM 5.2. *For $\xi_0 \cdots \xi_{n-1} \in \{0, \dots, b\}^*$, and M_{ξ_i} as defined in (45),*

$$\begin{pmatrix} \mu_0(I_{\xi_0\cdots\xi_{n-1}}) \\ \vdots \\ \mu_c(I_{\xi_0\cdots\xi_{n-1}}) \end{pmatrix} = M_{\xi_0\cdots\xi_{n-1}} R \quad \textit{for} \quad R = \begin{pmatrix} \mu_0(\mathbb{T}) \\ \vdots \\ \mu_c(\mathbb{T}) \end{pmatrix}$$

which allows to define $\mu_\circ$ as a linearly representable measure that is, with $U^\star = (1 \cdots 1)$,

$$\mu_\circ(I_{\xi_0\cdots\xi_{n-1}}) = U^\star M_{\xi_0\cdots\xi_{n-1}} R. \tag{47}$$

In the uniform case, i.e. with $p_0 = \cdots = p_{nb} = 1/(nb + 1)$, one recovers (up to slight changes) the set of matrices A_k in [Pro00, eq. (33)] found (independently) by Protasov.

5.2. The 3-fold convolution of the Cantor measure. We now consider the case when $\beta = 3$ (i.e. $b = 2$) and $n = 3$. Assuming the product space Σ_6 endowed with the Bernoulli measure $\mathbb{P}$ of parameter $p = \begin{pmatrix}1/8 & 3/8 & 3/8 & 1/8 & 0 & 0 & 0\end{pmatrix}$, we know that $\mu = \mathbb{P} \circ X^{-1}$ (resp. $\mu_\circ = \mathbb{P} \circ Y^{-1}$) coincides with the 3-fold convolution of the Cantor measure (resp. the 3-fold convolution of the Cantor measure modulo 1). The support of μ is fully supported by the interval $[0\,;3/2]$: according to Theorem 5.2, $\mu_\circ = \mu_0 + \mu_1$ and for any $w \in \{0, 1, 2\}^*$:

$$\mu_\circ(I_w) = U^\star M_w U_{4/5} \quad \text{where}$$

$$M_0 = \begin{pmatrix} 1/8 & 0 \\ 1/8 & 3/8 \end{pmatrix}, \ M_1 = \begin{pmatrix} 3/8 & 1/8 \\ 0 & 1/8 \end{pmatrix}, \ M_2 = \begin{pmatrix} 3/8 & 3/8 \\ 0 & 0 \end{pmatrix}.$$

We point out that neither μ_0 nor μ_1 are weak Gibbs w.r.t. the triadic intervals on $\mathbb{T}$. This is easily checked for instance with μ_0 (and similarly for μ_1). Indeed, one can check that

$$\frac{1}{8^n} \cdot \begin{pmatrix} 0 \\ 3^n \end{pmatrix} \lhd \frac{1}{8^n} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} U_{4/5} \leq M_0^n U_{4/5} \leq \frac{1}{8^n} \cdot \begin{pmatrix} 1 & 0 \\ 3^n & 3^n \end{pmatrix} U_{4/5} \leq \frac{1}{8^n} \cdot \begin{pmatrix} 1 \\ 3^n \end{pmatrix}$$

and thus μ_0 cannot be weak Gibbs, since

$$\frac{\mu_0(I_{20^n})}{\mu_0(I_2)\mu_0(I_{0^n})} = \frac{\begin{pmatrix}1 & 1\end{pmatrix} M_0^n U_{4/5}}{\begin{pmatrix}1 & 0\end{pmatrix} M_0^n U_{4/5}} \rhd 3^n \implies \left(\frac{\mu_0(I_{20^n})}{\mu_0(I_2)\mu_0(I_{0^n})}\right)^{1/n} \not\to 1.$$

PROPOSITION 5.3. *Let $p_{1/2} = p_* : \Sigma_2 \to [0\,;1]$ be the regular limit direction map for $\mathcal{M} = \{M_0, M_1, M_2\}$ and $F : [0\,;1] \to \mathbb{R}$ the function s.t. $F(x) = p_*(\xi_0\xi_2\cdots)$ as soon as $x \in I_{\xi_0\cdots\xi_{n-1}}$, for any n: then, (i) : $\mu_0(B) = \mu(B)$, for any $B \subset [0\,;1[$ and μ_0 is equivalent to $\mu_\circ$ with*

$$x \in [0\,;1[\implies \lim_{n\to+\infty} \frac{\mu_0\big(I_n(x)\big)}{\mu_\circ\big(I_n(x)\big)} = F(x) = \frac{\mu_0(dx)}{\mu_\circ(dx)} \quad \textit{for } \mu_\circ\textit{-a.e. } x\,; \tag{48}$$

(ii) : $\mu_1(B) = \mu(B + 1)$, for any $B \subset [0\,;1[$ and μ_1 is absolutely continuous w.r.t $\mu_\circ$ with

$$x \in [0\,;1[\implies \lim_{n\to+\infty} \frac{\mu_1\big(I_n(x)\big)}{\mu_\circ\big(I_n(x)\big)} = 1 - F(x) = \frac{\mu_1(dx)}{\mu_\circ(dx)} \quad \textit{for } \mu_\circ\textit{-a.e. } x. \tag{49}$$

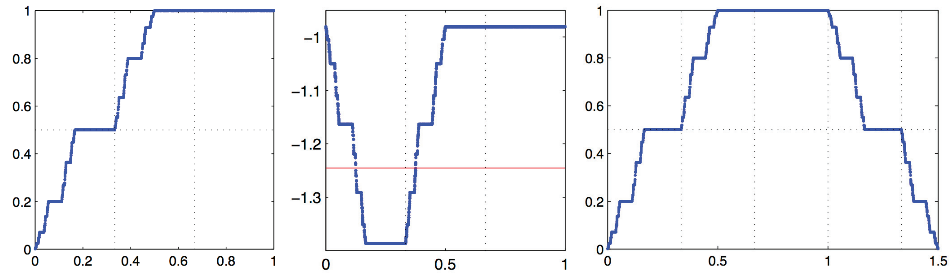

FIG. 2. *(left) : The function* $x \mapsto \mathtt{p}_*(x_0x_1\cdots)$ *(where* $x = \sum_{k=0}^{\infty} x_k/3^{k+1}$*) for the matrices* M_0, M_1, M_2 *associated with* $\mu_\circ$ *; (middle) : the potential* $x \mapsto \phi(x_0x_1\cdots)$ *for* ϕ *as in (51) – the horizontal line represent* $\inf_\sigma(\phi) = \log(\theta)/3$*; (right) : the Radon-Nikodym derivative* $\mu(dx)/\tilde{\mu}_\circ(dx)$ *for* $\tilde{\mu}_\circ = \sum_{k\in\mathbb{Z}} \mu_\circ(\cdot + k)$.

Sketched proof. (i) : The identity $\mu_0(B) = \mu(B)$, for any $B \subset [0\,;1[$ is just the definition of μ_0. Suppose that $x \in I_{\xi_0\cdots\xi_{n-1}}$, for any n; then

$$\frac{\mu_0\big(I_n(x)\big)}{\mu_\circ\big(I_n(x)\big)} = \frac{\mu_0\big(I_{\xi_0\cdots\xi_{n-1}}\big)}{\mu_\circ\big(I_{\xi_0\cdots\xi_{n-1}}\big)} = \frac{U_0^\star M_n(\xi)U_{4/5}}{U^\star M_n(\xi)U_{4/5}} = \frac{U_0^\star M_n(\xi)U_{4/5}}{\|M_n(\xi)U_{4/5}\|}$$

The convergence $\mu_0\big(I_n(x)\big)/\mu_\circ\big(I_n(x)\big) \to \mathtt{p}_{1/2}(\xi_0\xi_2\cdots)$ in part (i) comes with Proposition 0, since $1/2$ is regular for $\mathcal{M}$ and $0 < 4/5 < 1$. Part (ii) concerning μ_1 is similar since

$$\frac{\mu_1\big(I_n(x)\big)}{\mu_\circ\big(I_n(x)\big)} = \frac{U_1^\star M_n(\xi)U_{4/5}}{\|M_n(\xi)U_{4/5}\|}.$$

(The coincidence of the limits in (48) and (49) with the Radon-Nikodym derivatives is simply justified by the fact that $x \mapsto F(x)$ is $\mu_\circ$-a.e. locally constant.) □

REMARK 5.4. *Consider the* $\mathbb{Z}$*-periodic distribution* $\tilde{\mu}_\circ := \sum_{k\in\mathbb{Z}} \mu_\circ(\cdot + k)$*; then the 3-fold convolution of the Cantor measure* μ *is absolutely continuous w.r.t.* $\tilde{\mu}_\circ$*, with (see Figure 2-(left))*

$$x \in [0\,;3/2] \implies \frac{\mu(dx)}{\tilde{\mu}_\circ(dx)} = \mathbb{I}_{[0\,;1]}(x)F(x) + \mathbb{I}_{[1\,;3/2]}(x)F(-2x+3)$$

We know that $\mu_\circ$ is invariant w.r.t. $T : \mathbb{T} \to \mathbb{T}$ such that $x \mapsto \mathrm{fr}\{3x\}$; furthermore, by a straightforward application of Theorem B, one verifies that $\mu_\circ$ is gibbsian w.r.t. the net $\{I_w\,;\ w \in \{0,1,2\}^*\}$ of the triadic intervals. The Gibbs properties are known to ensure the multifractal formalism for the local dimension to hold. To be more precise, let η be a Borel probability measure on the real line which is supported by the unit interval $[0\,;1]$; for simplicity we assume $\eta\{0\} = \eta\{1\}$ (so that η identifies with a measure on $\mathbb{T}$). By definition

$$F_\eta(\alpha) := \Big\{x\,;\ D_\eta(x) = \alpha\Big\} \tag{50}$$

is the level set of η w.r.t. the triadic local dimension $x \mapsto D_\eta(x) := \liminf_n \log\eta(I_n(x))/\log 1/3^n$; then, the corresponding multifractal domain is $\mathrm{Dom}(\eta) := \{\alpha \in \mathbb{R}\,;\ F_\eta(\alpha) \neq \emptyset\}$.

THEOREM 5.5. *(i) : $\mu_\circ$ is a T-ergodic (with $T(x) = \mathrm{fr}\{3x\}$) Gibbs measure in the sense that $\mu_\circ(I_n(\xi)) \bowtie \exp(\sum_{k=0}^{n-1} \phi(\sigma^k \cdot \xi))$, where the Hőlder continuous potential $\phi : \Sigma_2 \to \mathbb{R}$ is function of the regular limit direction map $\mathtt{p}_* : \Sigma_2 \to [0\,;1]$ of $\mathcal{M} = \{M_0, M_1, M_2\}$, that is for any $\xi \in \Sigma_2$,*

$$\phi(\xi) = \log\left(\frac{3 - \mathtt{p}_*(\xi)}{8}\right)\mathbb{I}_{[0]}(\xi) + \log\left(\frac{2 + \mathtt{p}_*(\xi)}{8}\right)\mathbb{I}_{[1]}(\xi) + \log\left(\frac{3}{8}\right)\mathbb{I}_{[2]}(\xi)\ ; \tag{51}$$

(ii) : the scale spectrum of $\mu_\circ$ coincides with pressure in the sense that, for any $q \in \mathbf{R}$,

$$\tau(q) := \lim_n -\frac{1}{n\log 3}\log\left(\sum\nolimits_{w\in\{\mathtt{0},\mathtt{1},\mathtt{2}\}^n} \mu_\circ[w]^q\right) = -\mathbf{P}(q\phi)/\log 3\ ; \tag{52}$$

(iii) : τ is concave, real analytic on $\mathbb{R}$; (iv) : the multifractal domain $\mathrm{DOM}(\mu_\circ)$ is a compact interval $[\underline{a}\,;\overline{a}]$ and the multifractal formalism is fully satisfied, since for any $\underline{a} \le \alpha \le \overline{a}$

$$\dim_H F_{\mu_\circ}(\alpha) = \inf\Big\{q\alpha - \tau(q)\ ;\ q \in \mathbb{R}\Big\}.$$

It was initially proved by Hu & Lau [HL01, Theorem 1.2] that $\mathrm{DOM}(\mu) = [\underline{a};\overline{a}] \cup \{D_\mu(0)\}$, were $\overline{a} < D_\mu(0)$: actually the authors give the explicit values, say $\underline{a} = \log(8/3)/\log 3 \approx 0,89278$, $\overline{a} = \log(8/\theta)/\log 3 \approx 1,1335$, with $\theta = (1+\sqrt{13})/2$ and $D_\mu(0) = \log 8/\log 3 \approx 1,89278$. The theoretical importance of this result is to show that multifractal domains of self-similar measures need not reduce to intervals (see also [OST05][Sch05][FLW05][FHJ11]). This question is also related to the multifractal analysis developed in [FL02][Fen03][Fen04][Tes06][Fen09]. The Gibbs estimates $\mu_\circ(I_n(\xi)) \bowtie \exp(\sum_{k=0}^{n-1}(\phi(\sigma^k \cdot \xi)))$ in Theorem 5.5 connects the levels sets $F_{\mu_\circ}(\alpha)$ as defined in (50) to the level sets for the Birkhoff averages of ϕ in (51) – see Figure 2 (middle) – since $\xi \in F_{\mu_\circ}(\alpha)$ if and only if $1/S_n\phi(\xi) \to -\alpha/\log 3$ as $n \to +\infty$; hence (use (34) in § 4.1) the multifractal domain $\mathrm{Dom}(\mu_\circ)$ is the closed interval $[\underline{a}\,;\overline{a}]$, where $\underline{a} = -\sup_\sigma(\phi)/\log 3$ while $\overline{a} = -\inf_\sigma(\phi)/\log 3$. One obtains Hu & Lau's value of $\underline{a}$ with the following proposition.

PROPOSITION 5.6. *Let $\phi : \Sigma_2 \to \mathbb{R}$ be the potential associated with $\mu_\circ$ in (51); then*

$$\log 1/4 = \inf_{\xi\in\Sigma_2}\{\phi(\xi)\} < \inf_\sigma(\phi) = \log\theta/8 \quad \textit{and} \quad \sup_\sigma(\phi) = \sup_{\xi\in\Sigma_2}\{\phi(\xi)\} = \log 3/8.$$

Proof. On the one hand it is easy checked that $\sup_\sigma(\phi) = \phi(\overline{2})$: indeed the Dirac mass $\delta_{\overline{2}}$ is σ-invariant and $\sup\{\phi(\xi)\ ;\ \xi \in \Sigma_2\} = \phi(\overline{2}) \le \sup_\sigma(\phi)$. The difficult part is to compute explicitly $\inf_\sigma(\phi)$. By analogy with the computation of the joint spectral radius of the two matrices in (36), one is lead to prove that $1/2(\delta_{\overline{01}}+\delta_{\overline{10}})(\phi) = \phi(\overline{01}) = \phi(\overline{10}) = \inf_\sigma(\phi)$. To begin with it is rather simple to compute $\phi(\overline{01})$; by a classical computation (Binet formula for linear recurrence)

$$(M_1M_0)^nU_1 = \begin{pmatrix}1 & 3\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}1\\ ?\end{pmatrix} = \gamma\begin{pmatrix}\theta^{n+1}\\ \theta^n\end{pmatrix} - \gamma\begin{pmatrix}(-3/\theta)^{n+1}\\ (-3/\theta)^n\end{pmatrix} = \gamma\theta^n\begin{pmatrix}\theta(1-(-3/\theta^2)^{n+1})\\ 1-(-3/\theta^2)^n\end{pmatrix}$$

where $\theta = (1+\sqrt{13})/2$ and $-3/\theta = (1-\sqrt{13})/2$ are the two solutions of $x^2 = x+3$ and γ is a suitable constant. Then, by definition of ϕ and because $0 \le 3/\theta^2 < 1$,

$$\exp(\phi(\overline{01})) = \lim_{n\to+\infty}\frac{1}{8}\cdot\frac{\begin{pmatrix}2 & 3\end{pmatrix}(M_1M_0)^nU_1}{\begin{pmatrix}1 & 1\end{pmatrix}(M_1M_0)^nU_1} = \frac{1}{8}\cdot\frac{2\theta+3}{\theta+1} = \frac{1}{8}\cdot\frac{8+2\sqrt{13}}{3+\sqrt{13}} = \frac{\theta}{8}$$

so that $\inf_\sigma(\phi) \le \log\theta/8$. To obtain the converse inequality we use the Gibbs estimates $\exp(\sum_{k=0}^{n-1}\phi(\sigma^k\cdot\xi)) \bowtie \|M_{\xi_0}\cdots M_{\xi_{n-1}}\|$, together with a minoration of $\|M_{\xi_0}\cdots M_{\xi_{n-1}}\|$ w.r.t. the decomposition $\xi_0\cdots\xi_{n-1} = \mathtt{0}^{a_0}\mathtt{1}^{a_1}\cdots\mathtt{i}^{a_k}$ with $a_0 \ge 0$, $a_1,\dots,a_k > 0$ (if $k \ge 0$) and $\mathtt{i} = \mathtt{0}$ or $\mathtt{1}$ depending on the parity of k. Because $\theta := (1+\sqrt{13})/2$ is the spectral radius of $M_\mathtt{0}$ and $M_\mathtt{1}$ (the corresponding eigenvectors having positive entries), there exists $\eta > 0$ s.t. $\|M_\mathtt{0}^{a_0}M_\mathtt{1}^{a_1}\cdots M_\mathtt{i}^{a_k}\| \le \eta^k\cdot(\theta/8)^{a_0+\cdots+a_k}$. Therefore, $\mathbf{P}(q\phi) = \lim_n 1/n\log Z_n(q)$ where

$$Z_n(q) = \sum_{k=0}^{n}\sum_{a_0+\cdots+a_k=n}\|M_\mathtt{0}^{a_0}M_\mathtt{1}^{a_1}\cdots M_\mathtt{i}^{a_k}\|^q \le \left(\frac{\theta}{8}\right)^{nq}\sum_{k=0}^{n}\frac{n!\cdot\eta^{kq}}{(n-k)!k!} = \left(\frac{\theta}{8}\right)^{nq}(1+\eta^q)^n$$

and $1/n\log Z_n(q) \le q\log(\theta/8)+\log(1+\eta^q)$ so that $\mathbf{P}(q\phi)/q \ge \log(\theta/8)+1/q\log(1+\eta^q)$, for any $q<0$. Taking the limit as $q\to-\infty$,

$$\inf_\sigma(\phi) = \lim_{q\to-\infty}\frac{\mathbf{P}(q\phi)}{q} \ge \log\left(\frac{\theta}{8}\right) + \lim_{q\to-\infty}\frac{\log(1+\eta q)}{q} = \log\left(\frac{\theta}{8}\right).$$

□

6. Bernoulli convolutions in a quadratic PV-basis.

6.1. Generalities. Let $\beta > 1$ be a non integral real number, $\mathtt{b}$ its integral part and $T_\beta : \mathbb{T}\to\mathbb{T}$ such that $T_\beta(x) = \mathrm{fr}\{\beta x\}$ the β-transformation; the β-shift $\sigma : \Omega_\beta\to\Omega_\beta$ is the symbolic dynamic of T_β w.r.t. the partition $\mathbb{T} = \bigsqcup_{i=0}^{\mathtt{b}} I_i$, where $I_i = [i/\beta\,;(i+1)/\beta[$ for $0\le i<\mathtt{b}$ and $I_\mathtt{b} = [\mathtt{b}/\beta\,;1[$. Let $x\mapsto R_i(x) = x/\beta+i/\beta$ ($i = 0,\dots,\mathtt{b}$) be the inverse branches of T_β; for any word w such that $wi\in\Omega_\beta^*$ we define $I_{wi} := R_w(I_i)$ in such a way that for any $n\ge 0$, one has the partition $\mathbb{T} = \bigsqcup_w I_w$, where w runs over the β-admissible words of length n. Given $p = (p_\mathtt{0},\dots,p_\mathtt{b})$ a positive probability vector, the p-distributed $(\beta,\mathtt{b})$-Bernoulli convolution μ is the distribution of the random variable $\{\mathtt{0},\dots,\mathtt{b}\}^\mathbb{N}\ni\xi\mapsto X(\xi) = \sum_{n=0}^\infty \xi_n/\beta^{n+1}$ when $\{\mathtt{0},\dots,\mathtt{b}\}^\mathbb{N}$ is weighted with the Bernoulli measure of parameter p. Because each $p_i > 0$, the measure μ is continuous, fully supported by the interval $[0\,;\alpha_*]$, where $\alpha_* = \mathtt{b}/(\beta-1)$ and satisfies the law of pure type (it is either purely singular or absolutely continuous). The case when $1<\beta<2$ and $p = (1/2,1/2)$ has attracted much attention and is now known as *an Erdős problem* (see the seminal works [JW35][Erd39, Erd40] and [PSS00] for historical notes and references). We begin with $\beta>1$ being an algebraic integer whose minimal polynomial is $P(Z) = Z^{d+1}-(a_dZ^d+\cdots+a_1Z+a_0)$ (i.e. each $a_i\in\mathbb{Z}$), in such a way that

$$\beta\left(x_0\beta^0+\cdots+x_d\beta^d\right) = \begin{pmatrix}\beta^d\\ \vdots\\ \beta^1\\ \beta^0\end{pmatrix}^\star \begin{pmatrix}a_d & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_1 & 0 & \cdots & 0 & 1\\ a_0 & 0 & \cdots & 0 & 0\end{pmatrix}\begin{pmatrix}x_d\\ \vdots\\ x_1\\ x_0\end{pmatrix}. \tag{53}$$

Now, fix $i\in\{\mathtt{0},\dots,\mathtt{b}\}$: then, for $B\subset[0\,;1]$, $x\in\mathbb{R}$ and $\xi = \xi_0\xi_1\cdots\in\Omega_\beta$,

$$X(\xi)\in B+x \iff X(\sigma\cdot\xi)\in R_i^{-1}(B)+(i+\beta x-\xi_0)\ ; \tag{54}$$

for $x = \sum_{k=0}^{d} x_k \beta^k \in \mathbb{Z}[\beta]$ we use the *companion* equation in (53) to get $i + \beta x - \xi_0 = (i, \xi_0) * x$, where for any $(i,j) \in \{\mathtt{0}, \dots, \mathtt{b}\} \times \{\mathtt{0}, \dots, \mathtt{b}\}$,

$$(i,j) * x = \begin{pmatrix} \beta^d \\ \vdots \\ \beta^1 \\ \beta^0 \end{pmatrix}^{\star} \left(\begin{pmatrix} a_d & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & 0 & \cdots & 0 & 1 \\ a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_d \\ \vdots \\ x_1 \\ x_0 \end{pmatrix} + (i-j) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right).$$

Let $i \in \{\mathtt{0}, \dots, \mathtt{b}\}$ and suppose that iw is a β-admissible word; if $B = I_{iw}$, then both B and $R_i^{-1}(B) = I_w$ are subsets of $[0\,;1]$. Given any $x \in \mathbb{R}$, we identify $\Omega_\beta^* \ni w \mapsto \mu_x(w) := \mu(I_w + x)$, with a positive measure on Ω_β (i.e. $\mu_x(w)$ is an abusive notation for $\mu_x(\Omega_\beta[w])$); the Bernoulli measure of parameter $p = (p_\mathtt{0}, \dots, p_\mathtt{b})$ being a product measure, the equivalence in (54) gives

$$\mu_x(iw) = \sum_{j=\mathtt{0}}^{\mathtt{b}} p_j \cdot \mu_{(i,j)*x}(w). \tag{55}$$

In order to remove from the sum in (55) the vanishing terms, we use the fact that μ is fully supported by $[0\,;\alpha_*]$; let $B \subset [0\,;1]$: because μ is non atomic, the condition $x \notin]-1\,;\alpha_*[$ implies $\mu(B+x) = 0$. Therefore, there exists a finite or countable (minimal) set $\mathcal{V}$ such that

$$0 \in \mathcal{V} = \{\mathtt{x}_1, \mathtt{x}_2, \dots\} \subset]-1\,;\alpha_*[\quad \text{and} \quad (i,j) \in \{\mathtt{0}, \dots, \mathtt{b}\}^2 \Rightarrow (i,j) * \mathcal{V} \subset \mathcal{V}. \tag{56}$$

The following proposition is a corollary of *Garsia separation Lemma [Gar63].*

PROPOSITION 6.1 (See [OST05]). *$\mathcal{V}$ is finite whenever β is a PV-number.*

Now, suppose β is a PV-number with $\mathcal{V} = \{0 = \mathtt{x}_1, \dots, \mathtt{x}_s\}$ and let $\Lambda_{(i,j)} : \mathcal{V} \times \mathcal{V} \to \{0,1\}$ such that $\Lambda_{(i,j)}(\mathtt{x}_u, \mathtt{x}_v) = 1$ if and only if $\mathtt{x}_u = (i,j) * \mathtt{x}_v$. With the $s \times s$ matrix M_i defined by

$$M_i(u,v) := \sum_{j=\mathtt{0}}^{\mathtt{b}} p_j \Lambda_{(i,j)}(\mathtt{x}_u, \mathtt{x}_v), \tag{57}$$

it follows from (55) that, for any $\xi \in \Omega_\beta$

$$\begin{pmatrix} \mu_{\mathtt{x}_1}(\xi_0 \cdots \xi_{n-1}) \\ \vdots \\ \mu_{\mathtt{x}_s}(\xi_0 \cdots \xi_{n-1}) \end{pmatrix} = M_{\xi_0} \begin{pmatrix} \mu_{\mathtt{x}_1}(\xi_1 \cdots \xi_{n-1}) \\ \vdots \\ \mu_{\mathtt{x}_s}(\xi_1 \cdots \xi_{n-1}) \end{pmatrix} = M_{\xi_0 \cdots \xi_{n-1}} \begin{pmatrix} \mathrm{r}_1 \\ \vdots \\ \mathrm{r}_s \end{pmatrix}. \tag{58}$$

where $\mathrm{r}_i := \mu([0\,;1] + \mathtt{x}_i)$ is always positive. The Bernoulli convolution μ is known to be self-similar (we shall not need this property) and it is reasonable to expect that each $\mu_{\mathtt{x}_i}$ *contains* the complete *fractal structure* of μ. This however is not clear, because the system of affine contractions of the self similar equation satisfied by μ displays overlaps. To enlighten this question, we shall consider the Gibbs properties of an intermediate probability.

DEFINITION 6.2. $\nu = 1/\mathrm{r}_*(\mu_{\mathtt{x}_1} + \cdots + \mu_{\mathtt{x}_s})$, *where* $\mathrm{r}_* = \mathrm{r}_1 + \cdots + \mathrm{r}_s$.

REMARK 6.3. *The main point is given by a heuristic argument suggesting that one may reasonably expect each $\mu_{\mathtt{x}_i}$ $(i = 1, \dots, s)$ is absolutely continuous w.r.t. an ergodic measure $\nu' \in \mathfrak{P}_\sigma(\Omega_\beta)$ (we write $\mu_{\mathtt{x}_i} \ll \nu'$), with ν' being itself equivalent to ν*

(we write $\nu' \sim \nu$). To see this, suppose for instance that ν is a Gibbs measure of a potential $\phi : \Omega_\beta \to \mathbb{R}$ in the sense that

$$\nu(\Omega_\beta[\xi_0 \cdots \xi_{n-1}]) \bowtie \exp\Big(\textstyle\sum_{k=0}^{n-1}\phi(\sigma^k \cdot \xi)\Big). \tag{59}$$

If – in addition – ϕ is assumed to have summable variations, then (RPF-Theorem) the unique equilibrium state, say ν' of ϕ is the unique ergodic measure satisfying the Gibbs property

$$\nu'(\Omega_\beta[\xi_0 \cdots \xi_{n-1}]) \bowtie \exp\Big(\textstyle\sum_{k=0}^{n-1}\phi(\sigma^k \cdot \xi)\Big)\ ; \tag{60}$$

(59) and (60) ensures the equivalence $\nu \sim \nu'$ and one concludes with the fact that $\mu_{\mathtt{x}_i} \ll \nu$ for any $i = 1, \ldots, s$. The actual situation is not exactly to above one: in Theorem 6.5 we prove that for quadratic PV numbers β (and a range of probability vectors p) ν is a weak Gibbs measure.

6.2. Quadratic PV numbers. Given $1 \le \mathtt{c} \le \mathtt{b}$ two integers, the dominant solution β of the equation $x^2 = \mathtt{b}x + \mathtt{c}$ satisfies $\mathtt{b} < \beta < \mathtt{b}+1$ (i.e. $\mathtt{b}$ is the integral part of β), while its conjugate – that is $-\mathtt{c}/\beta$ – belongs to $]-1\,;0[$: in other words β is a quadratic PV-number. According to the previous subsection the computation of $\mathcal{V}$ gives

$$\mathcal{V} = \{0, 1, \beta - \mathtt{b} = \mathtt{c}/\beta\}. \tag{61}$$

By Definition 6.2 of ν and of the matrices M_i in (57) it follows from (58) that for any $w \in \Omega_\beta^*$,

$$\nu(w) = \frac{1}{\mathrm{r}_*}\Big(\mu_0(w) + \mu_1(w) + \mu_{\mathtt{c}/\beta}(w)\Big) = U^\star M_w R, \tag{62}$$

where, for any $i = \mathtt{0}, \ldots, \mathtt{b}$,

$$M_i = \begin{pmatrix} p_i & p_{i-1} & 0 \\ 0 & 0 & p_{\mathtt{b}+i} \\ p_{\mathtt{c}+i} & p_{\mathtt{c}+i-1} & 0 \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad R := \begin{pmatrix} \mathrm{r}_1/\mathrm{r}_* \\ \mathrm{r}_2/\mathrm{r}_* \\ \mathrm{r}_3/\mathrm{r}_* \end{pmatrix} \tag{63}$$

($\mathrm{r}_* = \mathrm{r}_1 + \mathrm{r}_2 + \mathrm{r}_3$ and we use the convention that $p_i = 0$ if $i \notin \{\mathtt{0}, \ldots, \mathtt{b}\}$).

Remark 6.4. *Notice that R in (63) is well defined and has positive entries. To compute R we may use the fact that Ω_β is a subshift of finite type. More precisely, let $\mathcal{W}$ be the set of the words*

$$\mathrm{w}_i = \begin{cases} i & \text{if } i \in \{\mathtt{0}, \ldots, \mathtt{b}-1\} \\ \mathtt{b}(i-\mathtt{b}) & \text{if } i \in \{\mathtt{b}, \ldots, \mathtt{b}+\mathtt{c}-1\} \end{cases} \tag{64}$$

then, there exists a one to one onto map $\chi : \Omega_\beta \to \mathcal{W}^{\mathbb{N}}$ for which $\chi(\xi) = \mathrm{w}_{i_0}\mathrm{w}_{i_1}\cdots$ means the identity $\xi = \mathrm{w}_{i_0}\mathrm{w}_{i_1}\cdots$ to be valid in $\{\mathtt{0}, \ldots, \mathtt{b}\}^{\mathbb{N}}$. The matrix $\sum_{i=0}^{\mathtt{b}+\mathtt{c}-1} M_{\mathrm{w}_i}$, is primitive with a spectral radius necessarily equal to 1 and R is obtained as the unique normalized Perron vector.

The main result of the present section is the following theorem.

Theorem 6.5. *(i) : ν is weak Gibbs whenever $p_\mathtt{0}^2 \ge p_\mathtt{b} p_{\mathtt{c}-1}$ and $p_\mathtt{0} p_{\mathtt{b}-\mathtt{c}+1} \ge p_\mathtt{b}^2$; (ii) : ν is not Gibbs (and thus not weak Gibbs) whenever $p_\mathtt{0} p_{\mathtt{b}-\mathtt{c}+1} < p_\mathtt{b}^2$.*

6.3. Proof of Theorem 6.5. We shall need an asymptotic estimate of the power M_0^n when n goes to infinity; $(X-p_0)(X^2-p_b p_{b-c+1})$ is the characteristic polynomial of M_0 and we write

$$M_0 = \left(\begin{array}{c|cc} p_0 & 0 & 0 \\ \hline \begin{array}{c} 0 \\ p_c \end{array} & \multicolumn{2}{c}{\sqrt{p_b p_{c-1}}\begin{pmatrix} 0 & \sqrt{p_b/p_{c-1}} \\ \sqrt{p_{c-1}/p_b} & 0 \end{pmatrix}} \end{array}\right).$$

Lemma 6.6. *(i) : If $p_0^2 > p_b p_{c-1}$, then the spectral radius of M_0 is $\rho = p_0$ and*

$$M_0^n \approx \rho^n \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline \begin{array}{c} p_b p_c/(p_0^2 - p_b p_{c-1}) \\ p_0 p_c/(p_0^2 - p_b p_{c-1}) \end{array} & \multicolumn{2}{c}{\left(\frac{p_b p_{c-1}}{p_0^2}\right)^{n/2}\begin{pmatrix} 0 & \sqrt{p_b/p_{c-1}} \\ \sqrt{p_{c-1}/p_b} & 0 \end{pmatrix}^n} \end{array}\right);$$

(ii) : if $p_0^2 = p_b p_{c-1}$, then the spectral radius of M_0 is $\rho = p_0 = \sqrt{p_b p_{c-1}}$ and

$$M_0^n \approx \frac{n\rho^n}{2} \left(\begin{array}{c|cc} 2/n & 0 & 0 \\ \hline \begin{array}{c} p_b p_c \\ p_0 p_c \end{array} & \multicolumn{2}{c}{\frac{2}{n}\begin{pmatrix} 0 & \sqrt{p_b/p_{c-1}} \\ \sqrt{p_{c-1}/p_b} & 0 \end{pmatrix}^n} \end{array}\right);$$

(iii) : if $p_0^2 < p_b p_{c-1}$, then the spectral radius of M_0 is $\rho = \sqrt{p_b p_{c-1}}$ and

$$M_0^n \approx \rho^n \left(\begin{array}{c|cc} (p_0^2/p_b p_{c-1})^{n/2} & 0 & 0 \\ \hline \begin{array}{c} p_c/(p_b p_{c-1} - p_0^2) \\ p_c/(p_b p_{c-1} - p_0^2) \end{array} & \multicolumn{2}{c}{\begin{pmatrix} 0 & \sqrt{p_b/p_{c-1}} \\ \sqrt{p_{c-1}/p_b} & 0 \end{pmatrix}^n} \end{array}\right).$$

The measure ν being not shift-invariant, the main argument leading to Theorem 6.5 is based on Lemma 2.1 and the fact that the weak Gibbs property follows from the uniform convergence on Ω_β of n-step potentials

$$(65) \qquad \Omega_\beta \ni \xi \mapsto \phi_n(\xi) = \log \frac{\nu(\xi_0 \cdots \xi_{n-1})}{\nu(\xi_1 \cdots \xi_{n-1})} = \log \frac{U^\star M_{\xi_0} \cdots M_{\xi_{n-1}} R}{U^\star M_{\xi_1} \cdots M_{\xi_{n-1}} R}$$

Proof of Theorem 6.5. According to Lemma 2.1, we shall prove the pointwise convergence of the n-step potential ϕ_n toward ϕ, where the map $\xi \mapsto \phi(\xi)$ is defined and continuous on Ω_β. In order to apply Theorem A we consider the 2×3 matrix Y and for each $i \in \{0, \dots, 2b\}$ the words $m_i \in \{0, \dots, b\}^*$ respectively defined by setting:

$$(66) \qquad Y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad m_i = \begin{cases} 0i & \text{if } i \in \{0, \dots, b\} \text{ or} \\ i - b & \text{if } i \in \{b+1, \dots, 2b\} \end{cases}$$

and by direct computation, one verifies the following commutation relations:
(67)

$$YM_{\mathrm{m}_i} = P_iY \quad\text{where}\quad P_i = \begin{cases} \begin{pmatrix} p_{\mathtt{0}}^2 & 0 \\ p_{\mathtt{b}}p_{\mathtt{c}} & p_{\mathtt{b}}p_{\mathtt{c}-1}\end{pmatrix} \in \mathbb{A}\{\mathrm{acd}>0\} & \text{if } i=\mathtt{0} \\ \begin{pmatrix} p_{\mathtt{0}}p_i & p_{\mathtt{0}}p_{i-1} \\ p_{\mathtt{b}}p_{\mathtt{c}+i} & p_{\mathtt{b}}p_{\mathtt{c}+i-1}\end{pmatrix} \in \mathbb{A}\{\mathrm{abcd}>0\} & \text{if } \mathtt{1}\le i\le \mathtt{b}-\mathtt{c} \\ \begin{pmatrix} p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1} & p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}} \\ 0 & p_{\mathtt{b}}^2\end{pmatrix} \in \mathbb{A}\{\mathrm{abd}>0\} & \text{if } i=\mathtt{b}-\mathtt{c}+1 \\ \begin{pmatrix} p_{\mathtt{0}}p_i & p_{\mathtt{0}}p_{i-1} \\ 0 & 0\end{pmatrix} \in \mathbb{A}\{\mathrm{ab}>0\} & \text{if } \mathtt{b}-\mathtt{c}+1< i\le \mathtt{b} \\ \begin{pmatrix} p_{i-\mathtt{b}} & p_{i-\mathtt{b}-1} \\ 0 & 0\end{pmatrix} \in \mathbb{A}\{\mathrm{ab}>0\} & \text{if } \mathtt{b}< i\le \mathtt{2b} \end{cases}$$

($p_j = 0$ whenever $j \notin \{\mathtt{0},\dots,\mathtt{b}\}$). Hence, given $\xi \in \Omega_\beta$, we use (67) to replace (partially) the product $M_{\xi_0}\cdots M_{\xi_{n-1}}$ by a product of 2×2 matrices. For any $\xi \in \Omega_\beta\backslash\{\overline{\mathtt{0}}, \mathtt{1}\overline{\mathtt{0}},\dots,\mathtt{b}\overline{\mathtt{0}}\}$ there exists $k\ge 0$ and $\varepsilon\in\{\mathtt{1},\dots,\mathtt{b}\}$ such that $\mathtt{0}^k\varepsilon$ is a prefix of $\sigma\cdot\xi$; then, for such a ξ, we write
(68)

$$\nu(\Omega_\beta[\xi_0\cdots\xi_{n-1}]) = U^\star M_{\xi_0}M_0^kQ_\varepsilon(YM_{\xi_{k+2}}\cdots M_{\xi_{n-1}}R) \quad\text{with}\quad Q_\varepsilon := \begin{pmatrix} p_\varepsilon & p_{\varepsilon-1} \\ 0 & 0 \\ p_{\mathtt{c}+\varepsilon} & p_{\mathtt{c}+\varepsilon-1}\end{pmatrix},$$

where we use the fact that $M_\varepsilon = Q_\varepsilon Y$. Given $\xi\in\Omega_\beta$, the definition of the words m_i ensures the existence of a unique sequence $i_0i_1\cdots\in\{\mathtt{0},\dots,\mathtt{2b}\}^{\mathbb{N}}$ such that $\xi = \mathrm{m}_{i_0}\mathrm{m}_{i_1}\cdots$. In what follows we note $\psi:\Omega_\beta\to\{\mathtt{0},\dots,\mathtt{2b}\}^{\mathbb{N}}$ the coding map s.t.

$$\psi(\xi) = i_0i_1\cdots \tag{69}$$

Proposition 6.7. *For $0<\alpha := \mathrm{r}_1/(\mathrm{r}_1+\mathrm{r}_2) < 1$ – with $\mathrm{r}_1,\mathrm{r}_2$ as in (63) – the limit direction map $\mathtt{p}_\alpha : \{\mathtt{0},\dots,\mathtt{2b}\}^{\mathbb{N}} \to [0\,;1]$ of $\{P_{\mathtt{0}},\dots,P_{\mathtt{2b}}\}$ is well defined and continuous; moreover,*

$$p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1}\ge p_{\mathtt{b}}^2 \implies \lim_{n\to+\infty}\Pi(YM_{\xi_0}\cdots M_{\xi_{n-1}}R) = \mathtt{p}_\alpha\circ\psi(\xi) =: \theta(\xi) \tag{70}$$

the convergence being uniform over Ω_β.

Proof. According to (67) we know that for $\mathtt{1}\le i\le \mathtt{b}-\mathtt{c}$ (resp. $\mathtt{b}-\mathtt{c}+1 < i\le \mathtt{2b}$) $P_i\in\mathbb{A}\{\mathrm{abcd}>0\}$ (resp. $P_i\in\mathbb{A}\{\mathrm{c}=0\}\cap\{\mathrm{a}>\mathrm{d}=0\}$) while

$$P_{\mathtt{0}} = \begin{pmatrix} p_{\mathtt{0}}^2 & 0 \\ p_{\mathtt{b}}p_{\mathtt{c}} & p_{\mathtt{b}}p_{\mathtt{c}-1}\end{pmatrix} \quad\text{and}\quad P_{\mathtt{b}-\mathtt{c}+1} = \begin{pmatrix} p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1} & p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}} \\ 0 & p_{\mathtt{b}}^2\end{pmatrix}. \tag{71}$$

The condition $p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1}\ge p_{\mathtt{b}}^2$ ensures

$$\{P_{\mathtt{0}},\dots,P_{\mathtt{2b}}\}\subset\begin{cases} \mathbb{A}\Big\{\mathrm{b}=0\Rightarrow\mathrm{c}>0 \text{ and } \mathrm{a}\le\mathrm{d}\Big\}\cap\mathbb{A}\Big\{\mathrm{c}=0\Rightarrow\mathrm{b}>0\text{ and }\mathrm{a}\ge\mathrm{d}\Big\} & \text{if } p_{\mathtt{0}}^2\le p_{\mathtt{b}}p_{\mathtt{c}-1} \\ \mathbb{A}\Big\{\mathrm{a}>0\Big\}\cap\mathbb{A}\Big\{\mathrm{b}=0\Rightarrow\mathrm{a}>\mathrm{d}\Big\}\cap\mathbb{A}\Big\{\mathrm{c}=0\Rightarrow\mathrm{a}\ge\mathrm{d}\Big\} & \text{if } p_{\mathtt{0}}^2> p_{\mathtt{b}}p_{\mathtt{c}-1}\end{cases}$$

Let's denote $P_n(i_0i_1\cdots) := P_{i_0}\cdots P_{i_{n-1}}$; then, part (i) of Theorem A ensures that $n\mapsto \mathtt{p}_\alpha(n,\cdot) := \Pi(P_n(\cdot)U_\alpha)$ is a sequence of continuous maps uniformly convergent over $\{\mathtt{0},\dots,\mathtt{2b}\}^{\mathbb{N}}$, it limit, that is the limit direction map $\mathtt{p}_\alpha:\{\mathtt{0},\dots,\mathtt{2b}\}^{\mathbb{N}}\to[0\,;1]$

associated with $\{P_{\mathtt{0}}, \ldots, P_{2\mathtt{b}}\}$, being well defined and continuous. The convergence in (70) is then a consequence of the definition of the function ψ in (69). □

• **Part (i).** – Suppose that $p_{\mathtt{0}}^2 \geq p_{\mathtt{b}}p_{\mathtt{c}-1}$ and $p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1} \geq p_{\mathtt{b}}^2$. Let K_n be either equal to $p_{\mathtt{0}}^n/(p_{\mathtt{0}}^2 - p_{\mathtt{b}}p_{\mathtt{c}-1})$ or $np_{\mathtt{0}}^n/2$ when either $p_{\mathtt{0}} > p_{\mathtt{b}}p_{\mathtt{c}-1}$ or $p_{\mathtt{0}} = p_{\mathtt{b}}p_{\mathtt{c}-1}$, respectfully; then, it follows from part (i) and (ii) of Lemma 6.6 that for any $0 \leq \gamma \leq 1$

$$M_{\mathtt{0}}^n(Q_\varepsilon U_\gamma) \approx K_n R' \begin{pmatrix}1 & 0 & 0\end{pmatrix} (Q_\varepsilon U_\gamma)$$

where

$$R' = \begin{pmatrix} p_{\mathtt{0}}^2 - p_{\mathtt{b}}p_{\mathtt{c}-1} \\ p_{\mathtt{b}}p_{\mathtt{c}} \\ p_{\mathtt{0}}p_{\mathtt{c}} \end{pmatrix} \quad \text{and} \quad Q_\varepsilon U_\gamma = \begin{pmatrix} \gamma p_\varepsilon + (1-\gamma)p_{\varepsilon-1} \\ ? \\ \gamma p_{\mathtt{c}+\varepsilon} + (1-\gamma)p_{\mathtt{c}+\varepsilon-1} \end{pmatrix}.$$

Because $\gamma p_\varepsilon + (1-\gamma)p_{\varepsilon-1} > 0$ (recall that $\varepsilon \in \{\mathtt{1}, \ldots, \mathtt{b}\}$ so that both p_ε and $p_{\varepsilon-1}$ are positive), one deduces the following lemma:

LEMMA 6.8. *Suppose that $p_{\mathtt{0}}^2 \geq p_{\mathtt{b}}p_{\mathtt{c}-1}$; then, for any $\varepsilon \in \{\mathtt{1}, \ldots, \mathtt{b}\}$ and any $0 \leq \gamma \leq 1$*

$$\lim_{n\to+\infty} \frac{M_{\mathtt{0}}^n(Q_\varepsilon U_\gamma)}{\|M_{\mathtt{0}}^n(Q_\varepsilon U_\gamma)\|_1} = \frac{R'}{\|R'\|_1}.$$

the convergence being uniform for $\gamma \in [0\,;1]$.

From now on, consider that $\xi \in \Omega_\beta \backslash \{\overline{\mathtt{0}}, \mathtt{1}\overline{\mathtt{0}}, \ldots, \mathtt{b}\overline{\mathtt{0}}\}$ and let $k \geq 0$ and $\varepsilon \in \{\mathtt{1}, \ldots, \mathtt{b}\}$ for which $\mathtt{0}^k\varepsilon$ is a prefix of $\sigma \cdot \xi$ (i.e. $\xi_1 \cdots \xi_{k+1} = \mathtt{0}^k\varepsilon$); then, according to (68) one has

$$\exp(\phi_n(\xi)) = \frac{U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon (Y M_{\xi_{k+2}} \cdots M_{\xi_{n-1}} R)}{U^\star M_{\mathtt{0}}^k Q_\varepsilon (Y M_{\xi_{k+2}} \cdots M_{\xi_{n-1}} R)}. \tag{72}$$

The inequality $p_{\mathtt{0}}p_{\mathtt{b}-\mathtt{c}+1} \geq p_{\mathtt{b}}^2$ being satisfied, by Proposition 6.7, the numerator and the denominator in (72) converge respectively toward $U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}$ and $U^\star M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}$ and the convergence is uniform on each cylinder of the form $\Omega_\beta[\mathtt{a}\mathtt{0}^k\varepsilon]$, with $\mathtt{a} \in \{\mathtt{0}, \ldots, \mathtt{b}\}$ and $\varepsilon \neq \mathtt{0}$. The first entry of $Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}$ is bounded from below by $\min\{p_\varepsilon, p_{\varepsilon-1}\} > 0$, so that both $U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}$ and $U^\star M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}$ are positive and the map

$$\xi \mapsto \lim_{n\to+\infty} \phi_n(\xi) = \log \frac{U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}}{U^\star M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}}$$

is continuous on $\Omega_\beta[\mathtt{a}\mathtt{0}^k\varepsilon]$; moreover, with the inequality $p_{\mathtt{0}}^2 \geq p_{\mathtt{b}}p_{\mathtt{c}-1}$, we use Lemma 6.8, to get

$$\lim_{n\to+\infty} \frac{U^\star M_{\xi_0} M_{\mathtt{0}}^n Q_\varepsilon U_\gamma}{U^\star M_{\mathtt{0}}^n Q_\varepsilon U_\gamma} = \frac{U^\star M_{\xi_0} R'}{U^\star R'},$$

with a uniform convergence over $\gamma \in [0\,;1]$; therefore, the function $\phi : \Omega_\beta \to \mathbb{R}$ such that

$$\phi(\xi) = \lim_{n\to+\infty} \phi_n(\xi) = \begin{cases} \log\left(\dfrac{U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}}{U^\star M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2}\cdot\xi)}}\right) & \text{if } \xi_0 \cdots \xi_{k+1} = \xi_0 \mathtt{0}^k \varepsilon \\ \\ \log\left(\dfrac{U^\star M_{\xi_0} R'}{U^\star R'}\right) & \text{if } \xi = \xi_0 \overline{\mathtt{0}} \end{cases} \tag{73}$$

is well defined and continuous on Ω_β. According to Lemma 2.1, one concludes that the n-step potential ϕ_n of ν converges toward the potential ϕ defined in (73), uniformly on Ω_β, whenever $p_0^2 \geq p_b p_{c-1}$ and $p_0 p_{b-c+1} \geq p_b^2$, ensuring that ν is weak gibbs in this case.

• **Part (ii).** – Suppose that $p_0 p_{b-c+1} < {p_b}^2$. To prove that ν is not Gibbs, we write

$$\frac{\nu\Big(1(0(b-c+1))^n b\Big)}{\nu\Big(1(0(b-c+1))^n\Big)\nu(b)} = \frac{U^\star \begin{pmatrix} p_1 & p_0 \end{pmatrix} \begin{pmatrix} p_0 p_{b-c+1} & p_0 p_c \\ ? & p_b^2 \end{pmatrix}^n \begin{pmatrix} p_b & p_{b-1} \\ ? & 0 \end{pmatrix} YR}{U^\star \begin{pmatrix} p_1 & p_0 \end{pmatrix} \begin{pmatrix} p_0 p_{b-c+1} & p_0 p_c \\ ? & p_b^2 \end{pmatrix}^n YR} \cdot \frac{1}{\nu(b)}$$
$$\bowtie \left(\frac{p_0 p_{b-c+1}}{{p_b}^2}\right)^n ;$$

if ν is gibbsian, then it is necessary that $(p_0 p_{b-c+1}/{p_b}^2)^n \bowtie 1$, a contradiction. □

6.4. Some remarks about Theorem 6.5. Let β be the *golden number* (i.e. $\beta^2 = \beta + 1$), so that the β-shift is the subshift $\Omega_\beta \subset \{0,1\}^{\mathbb{N}}$ whose elements are the $\xi = \xi_0 \xi_1, \cdots$ not factorized by 11. If $p_0 = p_1 = 1/2$, our analysis gives for any β-admissible word w

$$\begin{pmatrix} \mu_0(w) \\ \mu_1(w) \\ \mu_{1/\beta}(w) \end{pmatrix} = M_w R \quad \text{where} \quad M_0 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix},$$

and R (see Remark 6.4) is proportional to the Perron vector of

$$M_* := M_0 + M_{10} = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{pmatrix} ;$$

one recovers that M_* has a spectral radius equal to 1 and thus R is proportional to

$$R' = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

A way to determine an ergodic measure equivalent to ν is to look at $\nu' \in \mathfrak{P}(\Omega_\beta)$ such that

$$\nu'(\Omega_\beta[\xi_0 \cdots \xi_{n-1}]) = L^\star_{\xi_0} M_n(\xi) R, \tag{74}$$

where the two vectors L_0 and L_1 satisfy (if possible) the three following conditions:

$$(i) \ : \ L_0^\star M_0 + L_1^\star M_1 = L_0^\star, \quad (ii) \ : \ L_0^\star M_0 = L_1^\star \quad \text{and} \quad (iii) \ : \ L_0^\star R = 1.$$

Such a measure does exists, since by a direct computation one finds

$$L_0 = \begin{pmatrix} 3/10 \\ 2/10 \\ 1/10 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 2/10 \\ 1/20 \\ 1/10 \end{pmatrix}.$$

By a similar proof as for ν, the sequence of the n-step potential of ν' converges uniformly toward a (non Hölder) potential ψ (see [Oli12]). Using Berbee criterium (see

[Ber87]), one proves that ψ has a unique equilibrium state which coincides necessarily with ν' (in particular, ν' is ergodic): this is equivalently to say that ν' is the unique ergodic weak Gibbs measure of ψ (c.f. Sidorov & Vershik questions about the invariant Erdős measure in [SV98] and [Oli12]). Finally, ν and ν' being clearly equivalent, one recovers here that μ_0, μ_1 and $\mu_{1/\beta}$ are absolutely continuous w.r.t the ergodic (weak Gibbs measure) ν'.

Theorem 6.5 does not consider the case when $p_{\mathtt{0}}^2 < p_{\mathtt{b}} p_{\mathtt{c}-1}$ and $p_{\mathtt{0}} p_{\mathtt{b}-\mathtt{c}+1} \geq p_{\mathtt{b}}^2$. This situation does not hold when $\mathtt{c} = 1$, because $p_{\mathtt{0}} p_{\mathtt{b}-\mathtt{c}+1} \geq p_{\mathtt{b}}^2$ implies $p_{\mathtt{0}} p_{\mathtt{b}} \geq p_{\mathtt{b}}^2$ and $p_{\mathtt{0}}^2 \geq p_{\mathtt{b}} p_{\mathtt{0}} = p_{\mathtt{b}} p_{\mathtt{c}-1}$. However, if $\mathtt{c} \in \{2, \dots, \mathtt{b}\}$ then, the two problematic inequalities may hold simultaneously. In particular, since $p_{\mathtt{0}} p_{\mathtt{b}-\mathtt{c}+1} \geq p_{\mathtt{b}}^2$, we know (Proposition 6.7) that

$$\xi \mapsto \lim_{n \to +\infty} \phi_n(\xi) = \log \frac{U^\star M_{\xi_0} M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2} \cdot \xi)}}{U^\star M_{\mathtt{0}}^k Q_\varepsilon U_{\theta(\sigma^{k+2} \cdot \xi)}}$$

is a well defined map, continuous on $\Omega_\beta[\mathtt{a}\mathtt{0}^k \varepsilon]$ for any $\mathtt{a} \in \{\mathtt{0}, \dots, \mathtt{b}\}$, $k \geq 0$ and $\varepsilon \neq \mathtt{0}$. But the convergence of $\phi_n(\xi)$ may fails for $\xi \in \{\overline{\mathtt{0}}, \mathtt{1}\overline{\mathtt{0}}, \dots, \mathtt{b}\overline{\mathtt{0}}\}$: this should deserve a special analysis.

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