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SOME FINITE DIMENSIONAL FILTERS DERIVED FROM THE STRUCTURE THEOREM FOR FIVE-DIMENSIONAL ESTIMATION ALGEBRAS*

WEN-LIN CHIOU†, SHAOPU LIN‡, AND STEPHEN S.-T. YAU§

Abstract. In this paper we apply the structure theorem for five-dimensional estimation algebras to construct a new class of five dimensional estimation algebras and hence a new class of finite dimensional filters.

Key words. Finite dimensional filters, Estimation algebras, Duncan-Mortensen-Zakai equation.

AMS subject classifications. 60G35, 93Ell, 17B30, 35J15, 58D25.

1. Introduction. In the late seventies, a basic approach to non-linear filtering theory was independently proposed by Mitter [Mi1]; Brockett [Br] and Brockett and Clark [Br-Cl]. They suggested that the construction of the filter should be divided into two parts: (i) a universal filter which is the evolution equation describing the unnormalized conditional density, the Duncan-Mortensen-Zakai equation and (ii) a state-output map, which depends on the statistics being computed, where the state of the filter is the unnormalized conditional density. The reason for focusing on the Duncan-Mortensen-Zakai equation is that it is a linear equation and is a much simpler object than the other non-linear conditional density equation and can be treated using geometric ideas. In 1983, Brockett formally proposed the problem of classifying all finite-dimensional estimation algebras in his lecture at the International Congress of Mathematicians. Recent works on estimation algebra have given us a deeper understanding of the Duncan-Mortensen-Zakai equation which was essential for progress in non-linear filtering as well as in stochastic control. Despite the usefulness of the Kalman-Bucy filter, however, it is not perfect. One of its weaknesses is that it is restricted to linear dynamical systems. Another weakness is that it needs a Gaussian assumption for the initial distribution. The advantage of the Brockett-Mitter approach of using the estimation algebra method to solve the Duncan-Mortensen-Zakai equation is the following. As long as the estimation algebra is finite dimensional, we will get a finite dimensional recursive filter and there is not a need to make any assumption on the initial distribution. Moreover, the approach applies well to non-linear dynamical systems. Wong ([Wo1], [Wo2]) introduced a fundamental notion of Wong matrix $\Omega = (\omega_{ij})$, an $n \times n$ skew-symmetric matrix with $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, where $f(x) = (f_1(x), \cdots, f_n(x))$ is the drift term, which plays an important role in the classification of finite-dimensional estimation algebras, and gave a structure theorem of estimation algebra in case $f(x)$ is real analytic and its first, second and third derivatives of $f(x)$ are bounded functions. Nevertheless, the structure and classification of finite-dimensional estimation algebras were studied in detail only in the early 1990s by Tam et al. [T-W-Y]; Chiou and Yau [Ch-Ya]; Yau [Ya1]; Chen and Yau ([Ch1-Ya],

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†Department of Mathematics, Fu-Jen University, Taipei, Taiwan (chiou@math.fju.edu.tw). Supported partially by Taiwan NSC grant.

‡Department of Mathematics, Fu-Jen University, Taipei, Taiwan.

§Corresponding author. IEEE Fellow. Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China (yau@uic.edu). Supported partially by Tsinghua University start up fund and NSFC grant (11471184).

[Ch2-Ya]); Chen et al. ([C-Y-L1], [C-Y-L2]); Wu et al. [W-Y-H], Yau [Ya2] and Yau and Hu [Ya-Hu2]. In particular, Yau [Ya2], and Yau and Hu [Ya-Hu2] finished the final step in classification of finite-dimensional estimation algebras with maximal rank. One of the consequences of the classification of finite dimensional estimation algebras with maximal rank is the following. In order for an estimation algebra with maximal rank to be finite dimensional, the dynamical system has to be quite special, i.e. the drift term f must be of the form $f(x) = (\ell_1, \ldots, \ell_n) + \nabla\phi$ where $\ell_1, \ldots, \ell_n$ are degree one polynomials in $x_1, \ldots, x_n$ and ϕ is a C^∞ function, or equivalently, Wong matrix $\Omega = (\omega_{ij})$ is a matrix with all entries being constants.

Although the classification of finite-dimensional estimation algebra of maximal rank was completed by Yau and his coworkers Chen, Chiou, Hu, Wong and Wu, the problem of classification of non-maximal rank finite-dimensional estimation algebra is still wide open except for the case of state space dimension 2. Due to the difficulty of the problem, Brockett suggested that one should understand the low dimensional estimation algebras first. Rasoulian and Yau [Ra-Ya] gave a general method to construct finite dimensional estimation algebras without maximal rank. But all their finite dimensional estimation algebras can be viewed as estimation algebras with maximal rank for certain filtering models. Wu and Yau [Wu-Ya] were able to classify all finite dimensional estimation algebras with state space dimension two. Their results are much deeper than the corresponding results of Chiou and Yau [Ch-Ya] in the maximal rank case. In [Ya-Ra], Yau and Rasoulian have classified estimation algebras of dimension at most four. In [C-C-Y1], Chiou et al. gave a structure theorem for estimation algebras of dimension five and a class of five-dimensional estimation algebras. Accordingly, in [C-C-Y2], they constructed a new class of finite dimensional nonlinear filters.

The purpose of this paper is to report the recent progress of classification of all estimation algebras of dimension at most 5. Using this structure theorem in [C-C-Y1], we have found other classes of finite dimensional estimation algebras. Accordingly, we construct a new class of finite dimensional nonlinear filters.

2. Basic concepts. The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t), & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t), & y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y and w are respectively $\mathbb{R}^n$, $\mathbb{R}^p$, $\mathbb{R}^m$ and $\mathbb{R}^m$ valued processes, and v and w have components that are independent, standard Brownian processes. We further assume that f, h are C^∞ smooth, and that g is an orthogonal matrix.

Let $\rho(t, x)$ denote the conditional density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known that $\rho(t, x)$ is given by normalizing a function $\sigma(t, x)$ which satisfies the Duncan-Mortensen-Zakai equation:

$$\begin{cases} d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t) \\ \sigma(0,x) = \sigma_0 \end{cases} \tag{2.2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and the zero degree differential operator of multiplication by h_i, L_i is defined by $L_i\phi = h_i\phi$, for any function ϕ. Here σ_0 is the probability density of the initial point x_0. Let

$$D_i = \frac{\partial}{\partial x_i} - f_i, \quad \eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2.$$

Then L_0 can be written as

$$L_0 = \frac{1}{2}\Big(\sum_{i=1}^{n} D_i^2 - \eta\Big).$$

Equation (2.2) is a stochastic partial differential equation. In real application, we are interested in constructing state estimators from observed sample paths with some property of robustness. M.H.A. Davis studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$u(t,x) = exp\Big(-\sum_{i=1}^{m} h_i(x)y_i(t)\Big)\sigma(t,x).$$

It is easy to show that $u(t,x)$ satisfies the following time-varying partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= L_0 u(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]u(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\ u(0,x) &= \sigma_0. \end{aligned} \tag{2.3}$$

We have used the following notation

Definition 2.1. If X and Y are differential operators, the Lie bracket of $X$and Y, $[X,Y]$ is defined by $[X,Y](\phi) = X(Y\phi) - Y(X\phi)$ for any C^∞ function ϕ.

Definition 2.2. The estimation algebra E of a filtering model (2.1) is defined to be the Lie algebra generated by $L_0, L_1, \ldots, L_m$ or $E = \langle L_0, L_1, \ldots, L_m\rangle_{\text{L.A.}}$. If $x_i \in E$ for every $1 \le i \le n$, then E is called an estimation algebra of maximal rank. If E as a vector space over R is a finite dimensional vector space, then E is called a finite dimensional estimation algebra.

Most of the known finite dimensional estimation algebras are of maximal rank. For example, if $h(x) = Cx + D$, where C is a $m \times n$ matrix with rank n, then the corresponding estimation algebra is of maximal rank.

In [Ya1], the following Proposition 2.1 is proven.

Proposition 2.1. *$\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions for all i and j if and only if $(f_1, \ldots, f_n) = (\ell_1, \ldots, \ell_n) + \big(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}\big)$, where $\ell_1, \ldots, \ell_n$ are polynomials of degree one and ϕ is a C^∞-function.*

The following theorem proved in [Ya1] plays a fundamental role in the classification of finite-dimensional estimation algebras.

THEOREM 2.1. *Let E be a finite-dimensional estimation algebra of* (2.1) *such that $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constant functions. If E is of maximal rank, then E is a real vector space of dimension $2n+2$ with a basis given by $1, x_1, \ldots, x_n, D_1, \ldots, D_n$ and L_0.*

Based on Theorem 2.1 and a series of papers by Chiou-Yau[Ch-Ya], Chen-Yau-Leung [C-Y-L1], Yau-Hu-Chiou[Y-H-C], Yau-Wu-Wong [Y-W-W], Yau-Hu [Ya-Hu2], Yau[Ya2] have proven the following theorem.

THEOREM 2.2. *If E is the finite dimensional estimation algebra of maximal rank associated to the filtering model* (2.1), *then the drift term f must be a linear vector field (i.e. each component is a polynomial of degree one) plus a gradient vector field. Furthermore E must be a real vector space of dimension $2n+2$ with a basis given by $1, x_1, \ldots, x_n, D_1, \ldots, D_n$ and L_0. Moreover, η is a polynomial of degree 2.*

The following theorem was proved by Yau-Hu [Ya-Hu1] which will be used in section 4.

THEOREM 2.3. *The general Kolmogorov equation*

$$\begin{cases} \frac{\partial u(t,x)}{\partial x} = L(x)u(t,x) \\ u(0,x) = \sigma_0 \end{cases} \tag{2.4}$$

where

$$L(x) = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} H_i(x)\frac{\partial}{\partial x_i} - P(x)$$

has a formal asympotic solution on $\mathbb{R}^n$

$$u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\sigma_0(\xi)}{(\sqrt{2\pi})^n} t^{n/2} \exp\left[-\frac{1}{2t}\sum_{i=1}^{n}(x_i-\xi_i)^2\right] b(t,x,\xi)d\xi_1, \cdots d\xi_n \tag{2.5}$$

where $b(t,x,\xi) = \sum_{k=0}^{\infty} a_k(x,\xi)t^k$. Here $a_k(x,\xi)$ are discribed explicitly as follows. Let

$$a(x,\xi) = \int_0^1 \sum_{i=1}^{n}(x_i-\xi_i)H_i[\xi+\tau(x-\xi)]d\tau. \tag{2.6}$$

Then

$$a_0(x,\xi) = e^{a(x,\xi)} \tag{2.7}$$

and for $k \geq 1$

$$a_k(x,\xi) = a_0(x,\xi)\int_0^1 \tau^{k-1}e^{-a(\xi+\tau(x-\xi),\xi)} \cdot g_k(\xi+\tau(x-\xi),\xi)d\tau \tag{2.8}$$

where $g_k = L(x)a_{k-1}(x,\xi)$. Notice that L_0 is a special $L(x)$, where $H_i(x) = f_i(x)$ and $P(x) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{m} h_i^2$.

3. Low dimensional estimation algebras. The initial approach of establishing Theorem 2.2 is to classify finite dimensional estimation algebras according to state space dimension. In the early nineties, Brockett communicated to the third author that it would be of interest to classify low dimensional estimation algebras. In this section, we list recent results of the classification for estimation algebras with dimension at most 5. The following Theorem 3.1 was proven in [Ch-Ya] and [T-W-Y]

THEOREM 3.1. *Suppose that the state space of the filtering model* (2.1) *is of dimension one. If the estimation algebra E is finite dimensional, then either (i) E is a real vector space of dimension 4 with a basis given by $1, x, D = \frac{\partial}{\partial x} - f(x)$, and $L_0 = \frac{1}{2}(D^2 - \eta)$, or (ii) E is a real vector space of dimension 2 with a basis given by 1, and L_0 or (iii) E is a real vector space of dimension 1 with a basis given by L_0. Here $\eta = f'(x) + f^2(x) + x^2 = \alpha x^2 + 2\beta x + \gamma$, where α, β, γ are constants, $\alpha - 1 \geq 0$ and $\sqrt{\alpha - 1} \geq \frac{\beta^2}{\alpha - 1} + \gamma$.*

The following Theorem 3.2, Theorem 3.3 were proven in [Ya-Ra].

THEOREM 3.2. *For any arbitrary state space dimension, there does not exist* 3*-dimensional estimation algebra.*

THEOREM 3.3. *Suppose that the state space of the filtering model* (2.1) *is of dimension greater than one. Then the* 4*-dimensional estimation algebra is isomorphic to a Lie algebra having the basis given by $1, x_1, D_1 = \frac{\partial}{\partial x_1} - f_1(x_1, \ldots, x_n)$ and $L_0 = \frac{1}{2}\big(\sum_{i=1}^{n} D_i^2 - \eta\big)$. Moreover $\omega_{12} = \omega_{13} = \cdots = \omega_{1n} = 0$, $[L_0, x_1] = D_1, [D_1, x_1] = 1$, $[L_0, D_1] = \frac{1}{2}\frac{\partial \eta}{\partial x_1} = \alpha x_1 + \beta$, where α, β are constants. Also, $\eta = \alpha x_1^2 + 2\beta x_1 + g(x_2, \ldots, x_n)$, where $g(x_2, \ldots, x_n)$ is in $C^\infty(\mathbb{R}^{n-1})$. In particular, $f_1, \ldots, f_n$ have to satisfy the equation*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = (\alpha - 1)x_1^2 + 2\beta x_1 + g(x_2, \ldots, x_n), \tag{3.1}$$

where $\alpha \geq 1$.

EXAMPLE 3.1. [Ya-Ra] If we take $f_1 = \sqrt{\alpha - 1}x_1 + \frac{\beta}{\sqrt{\alpha - 1}}$, $\alpha > 1, f_2 = f_2(x_2, \ldots, x_n), \cdots, f_n = f_n(x_2, \ldots, x_n), g(x_2, \ldots, x_n) = \sum_{i=2}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=2}^{n} f_i^2 + \frac{\beta^2}{\alpha - 1} + \sqrt{\alpha - 1}$, then $\omega_{12} = \omega_{13} = \cdots = \omega_{1n} = 0$ and (3.1) is satisfied.

REMARK. Notice that ω_{ij}'s are arbitrary for $i, j \geq 2$. This estimation algebra is not of maximal rank and does not belong to the class that is considered in Yau[Ya2]. However, it belongs to the class of nonmaximal rank finite dimensional estimation algebras constructed in Rasolulian and Yau[Ra-Ya].

The following structure theorem for five-dimensional estimation algebras was proven in Chiou et al.[C-C-Y1]

THEOREM 3.4. *Suppose that the state space of the filtering model* (2.1) *is of dimension at least two. Then the five-dimensional estimation algebra is isomorphic to a Lie algebra generated by L_0 and an observation function $h = x_1$ with a basis given by $1, x_1, D_1 = (\frac{\partial}{\partial x_1}) - f_1(x_1, ..., x_n), Y_1 = [L_0, D_1] = \sum_{i=1}^{n} \omega_{1i} D_i + \frac{1}{2}(\frac{\partial \eta}{\partial x_1}), L_0 =$*

$\frac{1}{2}(\sum\limits_{i=1}^{n} D_i^2 - \eta)$. *Moreover, the following holds:*
(1) $\omega_{1i} \neq 0$ *for some* $i = 2, ..., n$ *and each* ω_{1i} *is of the form*

$$\omega_{1i} = \sum_{k=2}^{n} e_{ik} x_k + e_i, 2 \leq i \leq n, \tag{3.2}$$

$$e_{ij} = -e_{ji}, 2 \leq i, j \leq n, \tag{3.3}$$

where e_{ij} *and* e_i *are constants.*
(2) η *is of the form*

$$\eta = \left(\sum_{j=2}^{n} \omega_{1j}^2 + C_3 \right) x_1^2 + \beta(x_2, ..., x_n) x_1 + \gamma(x_2, ..., x_n), \tag{3.4}$$

where $C_3 \geq 1$ *is a constant and* $\beta(x_2, ..., x_n)$ *and* $\gamma(x_2, ..., x_n)$ *are* C^∞ *functions.*
(3) *There exists a constant* C_1 *such that*

$$\sum_{j=1}^{n} \omega_{1j} \omega_{ji} + \frac{1}{2} \frac{\partial^2 \eta}{\partial x_i \partial x_1} = C_1 \omega_{1i}, 2 \leq i \leq n. \tag{3.5}$$

(4) *There exists constants* C_0 *and* C_2 *such that*

$$-\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial \omega_{1j}}{\partial x_i} \omega_{ji} + \frac{1}{2} \sum_{j=1}^{n} \omega_{1j} \frac{\partial \eta}{\partial x_j} = C_0 x_1 + \frac{C_1}{2} \frac{\partial \eta}{\partial x_1} + C_2. \tag{3.6}$$

In particular, $f_1, .., f_n$ *have to satisfy the following equation:*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \left(\sum_{j=2}^{n} \omega_{1j}^2 + C_3 - 1 \right) x_1^2 + \beta(x_2, ..., x_n) x_1 + \gamma(x_2, ..., x_n). \tag{3.7}$$

Moreover, this five-dimensional estimation algebra has the following multiplication table:

E	1	x_1	D_1	Y_1	L_0
1	0	0	0	0	0
x_1	0	0	-1	0	$-D_1$
D_1	0	1	0	C_3	$-Y_1$
Y_1	0	0	$-C_3$	0	$-C_0x_1 - C_1Y_1 - C_2 - C_3D_1$
L_0	0	D_1	Y_1	$C_0x_1 + C_1Y_1 + C_2 + C_3D_1$	0

Example 3.2. [C-C-Y1] If we take $f_1 = ax_1, f_2 = bx_1x_3, f_3 = -bx_1x_2, f_i = g_i(x_4, \ldots, x_n), 4 \leq i \leq n, h(x) = x_1$, where a, b are nonzero constants. Then $\omega_{12} = bx_3, \omega_{13} = -bx_2, \omega_{23} = -2bx_1, \omega_{1j} = 0, 4 \leq i \leq n$, and $\sum\limits_{i=1}^{n} f_i^2 + \sum\limits_{i=1}^{n} \frac{\partial f_i}{\partial x_i} = \left(\sum\limits_{i=1}^{n} \omega_{1i}^2 + a^2 \right) x_1^2 + \sum\limits_{i=4}^{n} g_i^2(x_4, \ldots, x_n) + a + \sum\limits_{i=4}^{n} \frac{\partial g_2}{\partial x_i}(x_4, \ldots, x_n)$. $\omega_{1i}, 1 \leq i \leq n$, satisfy (3.3), (3.4), η is of the form (3.5) satisfying (3.6) and (3.7) with $C_0 = 2b^2$, $C_1 = C_2 = 0$. The estimation algebra E is 5-dimensional with a basis $\{1, x_1, D_1, Y_1 = bx_3D_2 - bx_2D_3 + (a^2 + 1 + b^2x_2^2 + b^2x_3^2)x_1, L_0\}$.

EXAMPLE 3.3. [C-C-Y1] Consider the filtering model (2.1), where $f_1 = a_1x_1 + \frac{a_1^2}{a_2}x_2 + \sum_{i=1}^{n} a_ix_i + e, f_2 = a_2x_1 + a_1x_2 - \frac{a_1}{a_2}\sum_{i=3}^{n} a_ix_i + c, f_i = a_ix_1 + a_i\frac{a_1}{a_2}x_2 + g_i(x_3,\ldots,x_n), 3 \le i \le n, h(x) = x_1, a_1^2 \ne a_2^2, \sum_{i=1}^{n} a_i^2 > 0, \left(a_2 - \frac{a_1^2}{a_2}\right)^2 = a_1^2 + a_2^2 + \sum_{i=3}^{n} a_i^2, \sum_{i=3}^{n} a_ig_i(x_3,\ldots,x_n) = 0$. Then $\omega_{12} = a_2 - \frac{a_1^2}{a_2} \ne 0$, $\omega_{kj} = 0, k = 1,2;\ 3 \le j \le n$ and $\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} = \sum_{i=1}^{n} a_i^2x_1^2 + \beta(x_2)x_1 + \gamma(x_2,\ldots,x_n)$, where $\beta(x_2) = 2\left(\frac{a_1^3}{a_2} + a_2 + \frac{a_1}{a_2}\sum_{i=3}^{n} a_i^2\right)x_2 + 2(a_1e + a_2c)$ and $\gamma(x_2,\ldots,x_n) = \left[\left(\frac{a_1}{a_2}\right)^2 + a_1^2 + \left(\frac{a_1}{a_2}\right)^2\left(\sum_{i=3}^{n} a_i^2\right)\right]x_2^2 + \left[\left(1 + \frac{a_1}{a_2}\right)^2\right]\left(\sum_{i=3}^{n} a_ix_i\right)^2 + \sum_{i=3}^{n} g_i^2 + e^2 + c^2 + 2\left(\frac{a_1^2}{a_2}e + a_1c\right)x_2 + 2\left(e - c\frac{a_1}{a_2}\right)\sum_{i=3}^{n} a_ix_i + 2a_1 + \sum_{i=3}^{n}\frac{\partial g_i}{\partial x_i}$. $\omega_{1i}, 1 \le i \le n$, satisfy (3.3), (3.4), η is of the form (3.5) satisfying (3.6) and (3.7) for some constants C_0, C_1, C_2, C_3. The estimation algebra E is 5-dimensional with a basis $\{1, x_1, D_1, Y_1 = (\frac{a_1^2}{a_2} - a_2)D_2 + (\sum_{i=1}^{n} a_i^2 + 1)x_1 + (\frac{a_1^3}{a_2} + a_2a_1 + \frac{a_1}{a_2}\sum_{i=3}^{n} a_i^2)x_2 + (a_1e + a_2c), L_0\}$.

4. Finite dimensional filters. In this section we apply the structure theorem for five-dimensional estimation algebras to construct a new class of five dimensional estimation algebras and hence a new class of finite dimensional filters by Wei-Norman technique [We-No].

MAIN THEOREM. *Consider the filtering model (2.1) where*

$$\begin{aligned}
f_1 &= ax_1,\\
f_2 &= \sum_{i=3}^{m} a_{2i}x_1x_i,\\
&\vdots\\
f_k &= -\sum_{i=2}^{k-1} a_{ik}x_1x_i + \sum_{i=k+1}^{m} a_{ki}x_1x_i, 3 \le k \le m-1,\\
&\vdots\\
f_m &= -\sum_{i=2}^{m-1} a_{im}x_1x_i,\\
f_i &= g_i(x_{m+1},\cdots,x_n), m+1 \le i \le n,\\
h &= x_1,
\end{aligned}$$

where $m \ge 4$, and $a \ne 0$. For $m \ge j > i \ge 1$, let $a_{ji} = -a_{ij}$. Assume that for $3 \le k \le m-1, a_{ik} \ne 0$ for some $k \le i \le m$, and g_i are C^∞ functions in $x_{m+1},\cdots,x_n$ variables, for $m+1 \le i \le n$. Let $D_i = \frac{\partial}{\partial x_i} - f_i, \omega_{1i} = \frac{\partial f_i}{\partial x_1} - \frac{\partial f_1}{\partial x_i}, \eta = \sum_{i=1}^{n}(f_i^2 + \frac{\partial f_i}{\partial x_i}) + h^2, Y_1 = \sum_{i=1}^{n}\omega_{1i}D_i + \frac{1}{2}\frac{\partial \eta}{\partial x_1}$, and $L_0 = \frac{1}{2}(\sum_{i=1}^{n} D_i^2 - \eta)$, then the estimation algebra E for this filtering model is five-dimensional with a basis $\{1, x_1, D_1, Y_1, L_0\}$, and this following multiplication table is given here:

E	*1*	x_1	D_1	Y_1	L_0
1	*0*	*0*	*0*	*0*	*0*
x_1	*0*	*0*	-1	*0*	$-D_1$
D_1	*0*	*1*	*0*	C_3	$-Y_1$
Y_1	*0*	*0*	$-C_3$	*0*	$-C_0x_1 - C_3D_1$
L_0	*0*	D_1	Y_1	$C_0x_1 + C_3D_1$	*0*

where $C_0 = 2 \sum_{i,j=2,i<j}^{m} a_{ij}^2, C_3 = a^2 + 1$. *Moreover, let*

$$u(t,x) = e^{T(t)} e^{r(t)x_1} e^{s_2(t)D_1} e^{s_1(t)Y_1} e^{tL_0} \sigma_0 \tag{4.1}$$

be the solution of the robust Duncan-Mortensen-Zakai equation (2.3) *for all* $t \geq 0$. *Then* $r(t), s_1(t), s_2(t)$ *and* $T(t)$ *satisfy the following differential equations:*

$$\frac{ds_1}{dt}(t) = s_2(t), \tag{4.2}$$

$$\frac{ds_2}{dt}(t) = r(t) + C_3 s_1(t) + y(t), \tag{4.3}$$

$$\frac{dr}{dt}(t) = C_0 s_1(t), \tag{4.4}$$

$$\begin{aligned} \frac{dT}{dt}(t) = {} & \frac{C_3 s_2^2(t)}{2} + C_0 s_1(t) s_2(t) - \frac{C_3^2 s_1^2(t)}{2} \\ & - C_3 s_2(t) \frac{ds_1}{dt}(t) + r(t) \frac{ds_2}{dt}(t) - \frac{r^2(t)}{2} \\ & - C_3 r(t) s_1(t) + \frac{1}{2} y^2(t). \end{aligned} \tag{4.5}$$

Proof. Since $m \geq j \geq i \geq 1$, $a_{ji} = -a_{ij}$, we can rewrite f_k as

$$f_k = \left[\sum_{i=2}^{m} a_{ki} x_i \right] x_1, \quad \text{for } 2 \leq k \leq m,$$

We have

$$\begin{aligned} \omega_{1k} &= \sum_{\ell=2}^{m} a_{k\ell} x_\ell \Rightarrow f_k = \omega_{1k} x_1, \quad \text{for } 2 \leq k \leq m, \\ \omega_{1j} &= 0, \quad \text{for } m+1 \leq j \leq n, \\ \omega_{ij} &= -2a_{ij} x_1, \quad \text{for } 2 \leq i < j \leq m, \\ \omega_{ij} &= \omega_{ji} = 0, \quad \text{for } 2 \leq i \leq m, \quad j \geq m+1, \end{aligned}$$

and

$$\begin{aligned} \eta &= \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + h^2 \\ &= a^2 x_1 + \sum_{i=2}^{m} \omega_{1i}^2 x_1^2 + \sum_{i=m+1}^{n} g_i^2(x_{m+1}, \cdots, x_n) + a + \sum_{i=m+1}^{n} \frac{\partial g_i}{\partial x_i}(x_{m+1}, \cdots, x_n) + a x_1^2 \end{aligned}$$

$$= (\sum_{k=2}^{m} \omega_{1k}^2 + a^2 + 1)x_1^2 + \sum_{\ell=m+1}^{n} (g_\ell^2(x_{m+1}, \cdots, x_n) + \frac{\partial g_\ell}{\partial x_\ell}(x_{m+1}, \cdots, x_n)) + a,$$

it is easy to see that $\omega_{1i}, 1 \leq i \leq n$, satisfy (3.2), and η is of the form (3.4)(note: $\omega_{1j} = 0$, for $j \geq m+1$), where $C_3 = a^2 + 1 \geq 1$. Now observe

$$\frac{\partial \eta}{\partial x_i} = \frac{\partial}{\partial x_i}\Big\{\sum_{k=2}^{m}[\sum_{\ell=2}^{m} a_{k\ell}x_\ell]^2\Big\}x_1^2 + \frac{\partial}{\partial x_i}\sum_{\ell=m+1}^{n}[g_\ell^2(x_{m+1}, \cdots, x_n) + \frac{\partial g_\ell}{\partial x_\ell}(x_{m+1}, \cdots, x_n)]$$

$$= \begin{cases} 2\Big\{\sum\limits_{k=2}^{m}[\sum\limits_{\ell=2}^{m} a_{k\ell}x_\ell]a_{ki}\Big\}x_1^2, & \text{for } 2 \leq i \leq m \\ \frac{\partial}{\partial x_i}\sum\limits_{\ell=m+1}^{n}[g_\ell^2(x_{m+1}, \cdots, x_n) + \frac{\partial g_\ell}{\partial x_\ell}(x_{m+1}, \cdots, x_n)], & \text{for } i \geq m+1, \end{cases}$$

we have

$$\begin{aligned} \frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial \eta}{\partial x_j} &= \frac{1}{2}\sum_{j=2}^{m}\omega_{1j}\frac{\partial \eta}{\partial x_j} = \frac{1}{2}\sum_{j=2}^{m}[\sum_{s=2}^{m} a_{js}x_s]\cdot 2\Big\{\sum_{k=2}^{m}[\sum_{\ell=2}^{m} a_{k\ell}x_\ell]a_{kj}\Big\}x_1^2 \\ &= x_1^2 \sum_{j,s,k,\ell=2}^{m} a_{js}a_{k\ell}a_{kj}x_s x_\ell \\ &= x_1^2 \cdot \frac{1}{2}\Big[\sum_{s,\ell=2}^{m}\sum_{j,k=2}^{m} a_{js}a_{k\ell}a_{kj}x_s x_\ell + \sum_{\ell,s=2}^{m}\sum_{j,k=2}^{m} a_{j\ell}a_{ks}a_{kj}x_\ell x_s\Big] \\ &= x_1^2 \cdot \frac{1}{2}\Big[\sum_{s,\ell=2}^{m}\sum_{j,k=2}^{m} a_{js}a_{k\ell}a_{kj}x_s x_\ell + \sum_{\ell,s=2}^{m}\sum_{k,j=2}^{m} a_{k\ell}a_{js}a_{jk}x_\ell x_s\Big] \\ &= x_1^2 \cdot \frac{1}{2}\sum_{j,s,k,\ell=2}^{m} a_{js}a_{k\ell}[a_{kj} + a_{jk}]x_s x_\ell \qquad (\text{note: } a_{jk} = -a_{kj}) \\ &= 0, \end{aligned} \tag{4.6}$$

and

$$\text{for } 2 \leq i \leq m, \qquad \frac{1}{2}\frac{\partial^2 \eta}{\partial x_i \partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_1}(\frac{\partial \eta}{\partial x_i})$$

$$=\frac{1}{2}\frac{\partial}{\partial x_1}\Big\{\sum_{k=2}^{m}\frac{\partial}{\partial x_i}([\sum_{\ell=2}^{m} a_{\ell k}x_\ell]^2 x_1^2)\Big\} = \frac{1}{2}\frac{\partial}{\partial x_1}\Big\{\sum_{k=2}^{m} 2([\sum_{\ell=2}^{m} a_{\ell k}x_\ell]a_{ik}x_1^2$$

$$=2\sum_{k=2}^{m}[\sum_{\ell=2}^{m} a_{\ell k}x_\ell]a_{ik}x_1, \tag{4.7}$$

$$\text{for } i \geq m+1, \qquad \frac{1}{2}\frac{\partial^2 \eta}{\partial x_i \partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_1}(\frac{\partial \eta}{\partial x_i})$$

$$=\frac{1}{2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_i}\sum_{\ell=m+1}^{n}[g_\ell^2(x_{m+1}, \cdots, x_n) + \frac{\partial g_\ell}{\partial x_\ell}(x_{m+1}, \cdots, x_n)] = 0. \tag{4.8}$$

Next, observe

$$\sum_{j=1}^{n}\omega_{1j}\omega_{ji} = \sum_{k=2}^{m}\omega_{1k}\omega_{ki} = \begin{cases} \sum\limits_{k=2}^{m}\big([\sum\limits_{\ell=2}^{m} a_{k\ell}x_\ell](-2a_{ki}x_1)\big), & \text{for } 2 \leq i \leq m \\ \sum\limits_{k=2}^{m}[\sum\limits_{\ell=2}^{m} a_{k\ell}x_\ell]\cdot 0 = 0, & \text{for } i \geq m+1, \end{cases} \tag{4.9}$$

and

(4.10)

$$-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji}=-\frac{1}{2}\sum_{k=2}^{m}\frac{\partial\omega_{1k}}{\partial x_i}\omega_{ki}=\begin{cases}-\frac{1}{2}\sum\limits_{k=2}^{m}a_{ki}(-2a_{ki}x_1), & \text{for } 2\le i\le m\\ \sum\limits_{k=2}^{m}a_{ki}(-2a_{ki})\cdot 0=0, & \text{for } i\ge m+1.\end{cases}$$

Therefore by (4.7), (4.8) and (4.9) it is easy to get

$$\sum_{j=1}^{n}\omega_{1j}\omega_{ji}+\frac{1}{2}\frac{\partial^2\eta}{\partial x_i\partial x_1}=0=C_1\omega_{12}, \qquad \text{for } 2\le i\le n$$

which proves that (3.6) is satisfied, if we take $C_1=0$. And by (4.6) and (4.10)

$$\begin{aligned}&-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial\omega_{1j}}{\partial x_i}\omega_{ji}+\frac{1}{2}\sum_{j=1}^{n}\omega_{1j}\frac{\partial\eta}{\partial x_j}=-\frac{1}{2}\sum_{i=2}^{m}\sum_{k=2}^{m}a_{ki}(-2a_{ki}x_1)+0\\ =&\,2\Big(\sum_{2\le i<j\le m}a_{ij}^2\Big)x_1=C_0x_1+\frac{C_1}{2}\frac{\partial\eta}{\partial x_1}+C_2,\end{aligned}$$

which proves that (3.7) is satisfied, if we take $C_0=2\sum\limits_{2\le i<j\le m}a_{ij}^2$ and $C_2=0$. Hence the estimation algebra E for this filtering model is five-dimensional with basis $\{1,x_1,D_1,Y_1=\sum\limits_{i=1}^{n}\omega_{1i}D_i+\frac{1}{2}\frac{\partial\eta}{\partial x_1},L_0\}$, and we have this following multiplication table:

E	1	x_1	D_1	Y_1	L_0
1	0	0	0	0	0
x_1	0	0	-1	0	$-D_1$
D_1	0	1	0	C_3	$-Y_1$
Y_1	0	0	$-C_3$	0	$-C_0x_1-C_3D_1$
L_0	0	D_1	Y_1	$C_0x_1+C_3D_1$	0

Where $C_0=2\sum\limits_{2\le i<j\le m}a_{ij}^2, C_3=a^2+1$, and $\eta=\sum\limits_{i=1}^{n}(f_i^2+\frac{\partial f_i}{\partial x_i})+h^2$.

Let $u(t,x)=e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0$ be the solution of (2.3) for all $t\ge 0$, since L_0 is uniformly elliptic, for any $t>0, e^{tL_0}\sigma_0$ is C^∞, by differentiating $u(t,x)$, we have

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)\\ =&e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}L_0e^{tL_0}\sigma_0+\frac{ds_1}{dt}(t)e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}Y_1e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\ &+\frac{ds_2}{dt}(t)e^{T(t)}e^{r(t)x_1}D_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0+\frac{dr}{dt}(t)e^{T(t)}x_1e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\ &+\frac{dT}{dt}(t)e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\ =&e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}\Big(L_0+s_1(t)[Y_1,L_0]+\frac{s_1^2(t)}{2}[Y_1,[Y_1,L_0]]+\cdots\Big)e^{s_1(t)Y_1}e^{tL_0}\sigma_0\end{aligned}$$

$$
\begin{aligned}
&+\frac{ds_1}{dt}(t)e^{T(t)}e^{r(t)x_1}\Big(Y_1+s_2(t)[D_1,Y_1]+\frac{s_2^2(t)}{2}[D_1,[D_1,Y_1]]+\cdots\Big)\\
&\qquad\cdot e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_2}{dt}(t)e^{T(t)}\Big(D_1+r(t)[x_1,D_1]+\cdots\Big)e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{dr}{dt}(t)x_1u(t,x)+\frac{dT}{dt}(t)u(t,x)\\
=&e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}\left(L_0-s_1(t)\Big(C_0x_1+C_3D_1\Big)+\frac{C_3^2s_1^2(t)}{2}\right)e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_1}{dt}(t)e^{T(t)}e^{r(t)x_1}(Y_1+C_3s_2(t))e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_2}{dt}(t)e^{T(t)}(D_1-r(t))e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{dr}{dt}(t)x_1u(t,x)+\frac{dT}{dt}(t)u(t,x)\\
=&e^{T(t)}e^{r(t)x_1}\Big(L_0-s_2(t)Y_1-\frac{C_3s_2^2(t)}{2}-C_0s_1(t)x_1\\
&-C_0s_1(t)s_2(t)-C_3s_1(t)D_1+\frac{C_3^2s_1^2(t)}{2}\Big)e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_1}{dt}(t)e^{T(t)}\Big(Y_1+C_3s_2(t)\Big)e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_2}{dt}(t)(D_1-r(t))e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{dr}{dt}(t)x_1u(t,x)+\frac{dT}{dt}(t)u(t,x)\\
=&e^{T(t)}e^{r(t)x_1}L_0e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-s_2(t)e^{T(t)}e^{r(t)x_1}Y_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-C_0s_1(t)e^{T(t)}e^{r(t)x_1}x_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-C_3s_1(t)e^{T(t)}e^{r(t)x_1}D_1e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_1}{dt}(t)e^{T(t)}Y_1e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_2}{dt}D_1u(t,x)+\frac{dr}{dt}(t)x_1u(t,x)\\
&+\Big(\frac{dT}{dt}(t)-\frac{C_3s_2^2(t)}{2}-C_0s_1(t)s_2(t)+\frac{C_3^2s_1^2(t)}{2}\\
&+C_3s_2(t)\frac{ds_1}{dt}(t)-r(t)\frac{ds_2}{dt}(t)\Big)u(t,x)\\
=&e^{T(t)}\Big(L_0-r(t)D_1+\frac{r^2(t)}{2}\Big)e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-s_2(t)e^{T(t)}Y_1e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-C_3s_1(t)e^{T(t)}(D_1-r(t))e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\frac{ds_1}{dt}(t)Y_1u(t,x)+\frac{ds_2}{dt}(t)D_1u(t,x)\\
&+\Big(\frac{dr}{dt}(t)-C_0s_1(t)\Big)x_1u(t,x)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{dT}{dt}(t)-\frac{C_3 s_2^2(t)}{2}-C_0 s_1(t)s_2(t)+\frac{C_3^2 s_1^2(t)}{2}\\
&+C_3 s_2(t)\frac{ds_1}{dt}(t)-r(t)\frac{ds_2}{dt}(t)\Big)u(t,x)\\
=&e^{T(t)}L_0 e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-r(t)e^{T(t)}D_1 e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&-C_3 s_1(t)e^{T(t)}D_1 e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0\\
&+\Big(\frac{ds_1}{dt}(t)-s_2(t)\Big)Y_1 u(t,x)+\frac{ds_2}{dt}(t)D_1 u(t,x)\\
&+\Big(\frac{dr}{dt}(t)-C_0 s_1(t)\Big)x_1 u(t,x)\\
&+\Big(\frac{dT}{dt}(t)-\frac{C_3 s_2^2(t)}{2}-C_0 s_1(t)s_2(t)+\frac{C_3^2 s_1^2(t)}{2}\\
&+C_3 s_2(t)\frac{ds_1}{dt}(t)-r(t)\frac{ds_2}{dt}(t)+\frac{r^2(t)}{2}+C_3 r(t)s_1(t)\Big)u(t,x)\\
=&L_0 u(t,x)\\
&+\Big(\frac{ds_1}{dt}(t)-s_2(t)\Big)Y_1 u(t,x)\\
&+\Big(\frac{ds_2}{dt}(t)-r(t)-C_3 s_1(t)\Big)D_1 u(t,x)\\
&+\Big(\frac{dr}{dt}(t)-C_0 s_1(t)\Big)x_1 u(t,x)\\
&+\Big(\frac{dT}{dt}(t)-\frac{C_3 s_2^2(t)}{2}-C_0 s_1(t)s_2(t)+\frac{C_3^2 s_1^2(t)}{2}\\
&+C_3 s_2(t)\frac{ds_1}{dt}(t)-r(t)\frac{ds_2}{dt}(t)+\frac{r^2(t)}{2}+C_3 r(t)s_1(t)\Big)u(t,x).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial u}{\partial t}(u,x)&=L_0 u(t,x)+y(t)[L_0,x_1]u(t,x)\\
&\quad+\frac{1}{2}y^2(t)[[L_0,x_1],x_1]u(t,x)\\
&=L_0 u(t,x)+y(t)D_1 u(t,x)+\frac{1}{2}y^2(t)u(t,x).
\end{aligned}
$$

Since $u(t,x)$ is the solution of (2.3), these two equations must be equivalent, thus we complete the proof. ◻

Remark 4.1. The solution (4.1) is well-defined which is explained as follows.

Definition. Suppose X is a differential operator, ξ_0 is in the domain of X, r is a continuous function. We denote by $e^{\int_0^t r(s)dsX}\xi_0$ the solution at time t of the following equation:

$$
\frac{d\xi(t)}{dt}=r(t)X\xi(t),\qquad \xi(0)=\xi_0,
$$

if it is well-defined.

Now, by theorem 2.3 $e^{tL_0}\sigma_0$ is well-defined. Next, the following propositions 4.1, 4.2, and note 4.1 shows that $e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0$ is well-defined, and hence $u(t,x) = e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0$ is well-defined.

PROPOSITION 4.1. *The solution of the following differential equation*

$$\frac{d\xi(t,x)}{dt} = \Big[\sum_{j=1}^{n}\omega_{1j}(x)\frac{\partial}{\partial x_j} + u(x)\Big]\xi(t,x), \xi(0,x) = \xi_0 \tag{4.11}$$

is

$$e^{\sum\limits_{j=1}^{n}\omega_{1j}(x)\frac{\partial}{\partial x_j}+u(x)}\xi_0(x) = \exp\left(\int_0^t u(x+r\omega(x))dr\right)\xi_0(x+t\omega(x)),$$

where $\omega(x)$ denote the nth-dimensional vector-valued function whose jth component is $\omega_{1j}(x)$.

Proof. Since $\omega_{11}(x) = 0$ and for each $2 \le j \le n, \omega_{1j}(x) = \sum\limits_{k=2,k\ne j}^{n} e_{jk}x_j + e_j$ is a function of x indepedent of x_j, we have the following identity: For any smooth function $v(x)$

$$\frac{dv(x+r\omega(x))}{dr} = \sum_{j=1}^{n}\omega_{1j}(x)\frac{\partial v}{\partial x_j}(x_1+r\omega_{11}(x),\cdots,x_j+r\omega_{1j}(x),\cdots,x_n+r\omega_{1n}(x)) \tag{4.12}$$

Next observe that

$$\begin{aligned}
&\sum_{j=1}^{n}\omega_{1j}(x)\int_0^t\frac{\partial u}{\partial x_j}(x+r\omega(x))dr = \int_0^t\sum_{j=1}^{n}\omega_{1j}(x)\frac{\partial u}{\partial x_j}(x+r\omega(x))dr\\
&=\int_0^t\frac{du(x+r\omega(x))}{dr}dr \qquad \text{(by (4.12))}\\
&=\big[u(x+t\omega(x))-u(x)\big] \qquad \text{(Let } U(r) \text{ be an antiderivative of } u(x+r\omega(x)).)\\
&=\Big[\frac{d}{dt}(U(t)-U(0))\Big]-u(x)\\
&=\frac{d}{dt}\int_0^t u(x+r\omega(x))dr-u(x),
\end{aligned} \tag{4.13}$$

Let $\xi(t,x) = \exp\left(\int_0^t u(x+r\omega(x))dr\right)\xi_0(x+t\omega(x))$, then

$$\begin{aligned}
&\text{the R.H.S. of (4.11)}\\
&=\Big[\sum_{j=1}^{n}\omega_{1j}(x)\frac{\partial}{\partial x_j}+u(x)\Big]\exp\left(\int_0^t u(x+r\omega(x))dr\right)\xi_0(x+t\omega(x))\\
&=\sum_{j=1}^{n}\omega_{1j}(x)\Big\{\Big[\frac{\partial}{\partial x_j}\exp\left(\int_0^t u(x+r\omega(x))dr\right)\Big]\cdot\xi_0(x+t\omega(x))
\end{aligned}$$

$$
\begin{aligned}
&+\exp\left(\int_0^t u(x+r\omega(x))dr\right)\cdot\frac{\partial}{\partial x_j}\xi_0(x+t\omega(x))\Big\}+u(x)\xi(t,x)\\
=&\sum_{j=1}^n \omega_{1j}(x)\Big[\int_0^t \frac{\partial u}{\partial x_j}(x+r\omega(x))dr\Big]\cdot\exp\left(\int_0^t u(x+r\omega(x))dr\right)\cdot\xi_0(x+t\omega(x))\\
&+\exp\left(\int_0^t u(x+r\omega(x))dr\right)\cdot\sum_{j=1}^n\omega_{1j}(x)\frac{\partial}{\partial x_j}\xi_0(x+t\omega(x))+u(x)\xi(t,x)\\
=&\Big\{\Big[\frac{d}{dt}\int_0^t u(x+r\omega(x))dr-u(x)\Big]\cdot\exp\left(\int_0^t u(x+r\omega(x))dr\right)\Big\}\xi_0(x+t\omega(x)) \quad (\text{ by } (4.13)\ )\\
&+\exp\left(\int_0^t u(x+r\omega(x))dr\right)\cdot\frac{d\xi_0(x+t\omega(x))}{dt}+u(x)\xi(t,x) \qquad (\text{ by } (4.12)\ )\\
=&\frac{d}{dt}\exp\left(\int_0^t u(x+r\omega(x))dr\right)\xi_0(x+t\omega(x))-u(x)\xi(t,x)+u(x)\xi(t,x)\\
=&\text{the L.H.S. of } (4.11)
\end{aligned}
$$

The proof is complete. □

Similarly, we have

PROPOSITION 4.2. *The solution of the following differential equation*

$$\frac{d\xi(t,x)}{dt}=\big[\frac{\partial}{\partial x_1}-f_1\big]\xi(t,x), \qquad \xi(0,x)=\xi_0$$

is

$$
\begin{aligned}
e^{D_1}\xi_0(x)=&\exp\left(\int_0^t u(x_1+r,x_2,\cdots,x_n)dr\right)\\
&\cdot\xi_0(x_1+t,x_2,\cdots,x_n)
\end{aligned}
$$

where $D_1=\frac{\partial}{\partial x_1}-f_1$.

NOTE 4.1. Since $Y_1=\sum_{j=1}^n\omega_{1j}(x)\frac{\partial}{\partial x_j}+u(x)$, where $u(x)=-\sum_{j=1}^n\omega_{1j}(x)f_j(x)+\frac{1}{2}\frac{\partial\eta}{\partial x_1}$, by proposition 4.1 $e^{tY_1}\xi(x)$ is well-defined and hence can be expressed in the form $\int k(t,x,r)\xi(r)dr$, for some integrable kernel k. Therefore, we can extend the definition of $e^{s(t)Y_1}\xi(x)$ to $e^{s(t)Y_1}\xi(t,x)$ by defining

$$e^{s(t)Y_1}\xi(t,x)=\int k(s(t),x,r)\xi(t,r)dr,$$

where ξ is also a function of t. Similarly, we can define the expression $e^{r(t)D_1}\ell(t,x)$ for any C^∞ smooth function $\ell(t,x)$.

EXAMPLE 4.1. Consider the filtering model (2.1) where

$$
\begin{aligned}
f_1&=ax_1,\\
f_2&=bx_1x_3+cx_1x_4+dx_1x_5,\\
f_3&=-bx_1x_2+ex_1x_4+fx_1x_5,\\
f_4&=-cx_1x_2-ex_1x_3+gx_1x_5,\\
f_5&=-dx_1x_2-fx_1x_3-gx_1x_4,\\
f_i&=g_i(x_6,\dots,x_n), \quad \text{for } 6\le i\le n,
\end{aligned}
$$

$$h(x) = x_1;$$

$a \neq 0, c \neq 0, e \neq 0, g \neq 0$, and g_i are C^∞ functions in $x_6, \cdots, x_n$ variables, for $6 \leq i \leq n$. By Main Theorem the estimation algebra E is five-dimensional with a basis $\{1, x_1, D_1, Y_1 = (bx_3 + cx_4 + dx_5)D_2 + (-bx_2 + ex_4 + fx_5)D_3 + (-cx_2 - ex_3 + gx_5)D_4 + (-dx_2 - fx_3 - gx_4)D_5 + \Big(b^2(x_2^2 + x_3^2) + c^2(x_2^2 + x_4^2) + d^2(x_2^2 + x_5^2) + e^2(x_3^2 + x_4^2) + f^2(x_3^2 + x_5^2) + g^2(x_4^2 + x_5^2) + 2(ce + df)x_2x_3 + 2(dg - be)x_2x_4 - 2(cg + bf)x_2x_5 + 2(bc + fg)x_3x_4 + 2(bd + eg)x_3x_5 + 2(cd + ef)x_4x_5 + a^2 + 1\Big)x_1, L_0\}$,

and there is the following multiplication table:

E	1	x_1	D_1	Y_1	L_0
1	0	0	0	0	0
x_1	0	0	-1	0	$-D_1$
D_1	0	1	0	a^2+1	$-Y_1$
Y_1	0	0	$-a^2-1$	0	$-Cx_1 - (a^2+1)D_1$
L_0	0	D_1	Y_1	$Cx_1 + (a^2+1)D_1$	0

where $C = 2(b^2+c^2+d^2+e^2+f^2+g^2)$. Let $u(t,x) = e^{T(t)}e^{r(t)x_1}e^{s_2(t)D_1}e^{s_1(t)Y_1}e^{tL_0}\sigma_0$ be the solution of (2.3) for all $t \geq 0$, by Main Theorem, $r(t), s_1(t), s_2(t)$ and $T(t)$ satisfy the following differential equations:

$$\begin{aligned}
\frac{ds_1}{dt}(t) &= s_2(t),\\
\frac{ds_2}{dt}(t) &= r(t) + (a^2+1)s_1(t) + y(t),\\
\frac{dr}{dt}(t) &= 2(b^2+c^2+d^2+e^2+f^2+g^2)s_1(t),\\
\frac{dT}{dt}(t) &= \frac{(a^2+1)s_2^2(t)}{2} + 2(b^2+c^2+d^2+e^2+f^2+g^2)s_1(t)s_2(t) - \frac{(a^2+1)^2s_1^2(t)}{2}\\
&\quad -(a^2+1)s_2(t)\frac{ds_1}{dt}(t) + r(t)\frac{ds_2}{dt}(t) - \frac{r^2(t)}{2} - (a^2+1)r(t)s_1(t) + \frac{1}{2}y^2(t).
\end{aligned}$$

Remark 4.2. The above class of 5-dimensional estimation algebras in Main theorem includes example 3.2 as a special case. These estimation algebras are not of maximal rank and do not belong to the class that is considered in Yau [Ya2] and the class constructed in Rasolulian and Yau[Ra-Ya].

Remark 4.3. Our filters constructed in Main Theorem are of real applied significance for the following reasons:

(a) Observe that (4.2), (4.3), (4.4) and (4.5) are independent of the initial distribution of x_0. Therefore one can implement this filter in hardware for real application. These are the so called universal filters.

(b) It is interesting to observe that the state space dimension of our filters is arbitrarily large while the dimension of our filters is only four. The real time computation of (4.2), (4.3) and (4.4) is a trivial matter because these are linear equations. Once we find out what $r(t), s_1(t)$ and $s_2(t)$, we simply put them in (4.5). We only need to do simple integration to find out T. Therefore our filters are of real applied significance.

(c) Our filters are defined for all time t as one can see directly from (4.2), (4.3), (4.4) and (4.5).

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