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TOPOLOGICAL CLASSIFICATION OF SIMPLEST GORENSTEIN NON-COMPLETE INTERSECTION SINGULARITIES OF DIMENSION 2*

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Dedicated to Henry Laufer at the occasion of his 70th birthday

Abstract. Let p be normal singularity of the 2-dimensional Stein space V. Let $\pi\colon M \to V$ be a minimal good resolution of V, such that the irreducible components A_i of $A = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to A is weighted dual graph Γ which, along with the genera of the A_i, fully describes the topology and differentiable structure of A and the topological and differentiable nature of the embedding of A in M. It is well known that the simplest Gorenstein non-complete intersection singularities of dimension two are exactly those minimal elliptic singularities with fundamental cycle self intersection number -5. In this paper we classify all weighted dual graphs of these singularities. In particular, we prove that there is no integral homology link structure in the class of simplest Gorenstein non-complete intersection singularities of dimension two.

Key words. Normal singularities, topological classification, weighted dual graph.

AMS subject classifications. Primary 32S25; Secondary 58K65, 14B05.

1. Introduction. Let p be a normal singularity of the 2-dimensional Stein space V. Let $\pi\colon M \to V$ be a resolution of V such that the irreducible components A_i, $1 \le i \le n$, of $A = \pi^{-1}(p)$ are nonsingular and have only normal crossings. Associated to A is a weighted dual graph Γ (e.g., see [HNK] or [La1]) which, along with the genera of the A_i, fully describes the topology and differentiable structure of A and the topological and differentiable nature of the embedding of A in M. One of the famous important questions in normal two dimensional singularities asks: What conditions are imposed on the abstract topology of (V, p) by the *complete intersection* hypothesis? Recall a theorem of Milnor [Mi, Theorem 2, p. 18] essentially says that any isolated singularity is a cone over its link L which is the intersection of V with a small sphere centered at p. L is a compact real 3-manifold whose oriented homeomorphism type determines and is determined by the weighted dual graph Γ of a canonically determined resolution (cf. [Ne]). So, we may equivalently ask: What conditions will the existence of a complete intersection representative (V, p) put on a weighted dual graph Γ. A complete intersection singularity (V, p) is Gorenstein [Ba], [Gr-Ri]. So there exists an integral cycle K on Γ which satisfies the adjunction formula [Se].

M. Artin has studied the rational singularities (those for which $R^1\pi_*(\mathcal{O}) = 0$). It is well known that rational complete intersection singularities are hypersurfaces (cf. Theorem 4.3 below). Artin has shown that all hypersurface rational singularities have multiplicities two and the graphs associated with those singularities are one of the graphs A_k, $k \ge 1$; D_k, $k \ge 4$; E_6, E_7 and E_8 which arise in the classification of simple Lie groups. In [La4], Laufer examines a class of elliptic singularities which satisfy a minimality condition. These minimally elliptic

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singularities have a theory much like the theory for rational singularities. Laufer [La4] proved that p is minimally elliptic if and only if $H^1(M,\mathcal{O}) = \mathbb{C}$ and $\mathcal{O}_{V,p}$ is a Gorenstein ring. Let Z be the fundamental cycle [Ar, p. 132] of the minimal resolution of a minimally elliptic singularity. If $Z^2 = -1$ or -2, then p is a double point [La4]. Laufer [La4] proved that if $Z^2 = -3$, then p is a hypersurface singularity with multiplicity 3. In fact he shows that for a minimally elliptic singularity $Z^2 \geq -4$ if and only if p is a complete intersection singularity.

Now let p be an arbitrary singularity in the Stein normal 2-dimensional space V having p as its only singularity. Let Γ denote the weighted dual graph of the exceptional set of the minimal good resolution $\pi\colon M \to V$. In [La3], Laufer developed a deformation theory preserving Γ. This theory allows him to introduce the notion of a property of the associated singularity holding generically for Γ. Now suppose that Γ is a weighted dual graph which does not correspond to a rational double point or to a minimally elliptic singularity. Then a deep theorem of Laufer [La4] asserts that the corresponding singularity is generically non-Gorenstein. In particular, it is generically not a complete intersection. As a consequence we can characterize those weighted dual graphs which have only complete intersection singularities associated to them. These are precisely rational double point graphs and minimally elliptic graphs with $Z^2 = -1$, -2, -3 or -4. Notice that rational double point graphs and minimally elliptic graphs with $Z^2 = -1, -2$ or -3 are precisely those graphs which have only hypersurface singularities associated with them. Laufer [La4] has completely classified minimally elliptic graphs with $Z^2 = -1, -2$, or -3. In [C-X-Y] the authors classified minimally elliptic graphs with $Z^2 = -4$. These are minimally elliptic complete intersection singularities which are not hypersurface singularities. Therefore weighted dual graphs which have only complete intersection singularities associated with them have been figured out. In this paper, what we do is the classification of minimally elliptic graphs with $Z^2 = -5$. This gives a topological classification of the simplest Gorenstein singularities that are not complete intersection singularities. Recall that the link L of a normal singularity is called a rational homology sphere (RHS) if $H_1(L,\mathbb{Q}) = 0$. L is called an integral homology sphere (IHS) if $H_1(L,\mathbb{Z}) = 0$. It is well known that L is an RHS if and only if the weighted dual graph Γ is a tree and the genus of each vertex equals to zero. L is IHS if additionally the determinant of the intersection matrix $(A_i \cdot A_j)$ is $\pm$ 1. In [Ne-Wa], Neumann and Wahl made the Splice Type Conjecture that any Gorenstein surface singularity with integral homology sphere link is a complete intersection of splice type. While this conjecture is false as shown by Luengo, Melle-Hernandez and Nemethi [L-M-N] (see some discussions in [Ne-Wa1]). It is a natural question to ask whether there is a integral homology sphere link Gorenstein but not complete intersection singularities. Our result below suggests that this is unlikely to happen. Consequently, our main theorem is interesting, not only in its own right, but also because it recovers Okuma's result [Ok], in which Okuma proved that if the link of a minimally elliptic singularity is an integral homology sphere, then that singularity is a complete intersection.

Main Theorem. *There are exactly 222 weighted dual graphs of minimally elliptic singularities with $Z^2 = -5$ where Z is the fundamental cycle. Furthermore there is no integral homology sphere link for this class of simplest Gorenstein singularities which are not complete intersection.*

The proof of the main theorem is in section 8. The crucial part is the classification of all minimally elliptic singularity graphs with $Z^2 = -5$. Our strategy of the classification is quite tricky. We first introduce the concept of effective component which is a irreducible component A_* of the exceptional set such that $A_* \cdot Z < 0$. It turns out that there are at most 5 effective components and their coefficients in the fundamental cycle are determined (Proposition 6.2). Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Suppose A_* is an effective component of Γ. Let Γ_1 be any connected component of Γ' which intersect with A_*. Then Γ_1 is necessarily one of the rational double point graphs appearing in Theorem 4.2. Let Z_1 be the fundamental cycle of Γ_1. Then $A_* \cdot Z_1 \leq 2$. If $A_* \cdot Z_1 = 2$, then $\Gamma = A_* \cup \Gamma_1$ and $Z = A_* + Z_1$; moreover for any $A_j \in \Gamma_1$, $A_j \cdot A_k > 0$ if and only if $A_j \cdot Z_1 < 0$ (Proposition 6.3). In order to find out how one can add A_* to the rational double point graphs, we use Theorem 3.5 and adjunction formula (2.3).

2. Preliminaries. Let $\pi\colon M \to V$ be a resolution of the normal two-dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = A = \cup A_i$, $1 \leq i \leq n$, be the decomposition of the exceptional set A into irreducible components.

A cycle $D = \Sigma d_i A_i$, $1 \leq i \leq n$ is an integral combination of the A_i, with d_i an integer. There is a natural partial ordering denoted by $<$, between cycles defined by comparing the coefficients. We let supp $D = \cup A_i$, $d_i \neq 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on A_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. Define

$$\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{2.1}$$

The Riemann-Roch theorem [Se, Proposition IV.4, p. 75] says

$$\chi(D) = -\frac{1}{2}(D^2 + D \cdot K), \tag{2.2}$$

where K is the canonical divisor on M. $D \cdot K$ may be defined as follows. Let ω be a meromorphic 2-form on M. Let (ω) be the divisor of ω. Then $D \cdot K = D \cdot (\omega)$ and this number is independent of the choice of ω. In fact, let g_i be the geometric genus of A_i, i.e., the genus of the desingularization of A_i. Then the adjunction formula [Se, Proposition IV, 5, p. 75] says

$$A_i \cdot K = -A_i^2 + 2g_i - 2 + 2\delta_i \tag{2.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. It follows immediately from (2.2) that if B and C are cycles, then

$$\chi(B + C) = \chi(B) + \chi(C) - B \cdot C. \tag{2.4}$$

Definition 2.1. Associated to π is a unique fundamental cycle Z [Ar, pp. 131-132] such that $Z > 0$, $A_i \cdot Z \leq 0$ all A_i and such that Z is minimal with respect to those two properties. Z may be computed from the intersection as follows via a computation sequence for Z in the sense of Laufer [La2, Proposition 4.1, p. 607].

$$Z_0 = 0, Z_1 = A_{i_1}, Z_2 = Z_1 + A_{i_2}, \ldots, Z_j = Z_{j-1} + A_{i_j}, \ldots,$$
$$Z_\ell = Z_{\ell-1} + A_{i_\ell} = Z$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leq \ell$.

$\mathcal{O}(-Z_{j-1})/(\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j} \cdot Z_{j-1}$. So

$$H^0(M, \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)) = 0$$

for $j > 1$.

$$0 \to \mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j) \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0 \tag{2.5}$$

is an exact sheaf sequence. From the long exact cohomology sequence for (2.5), it follows by induction that

$$H^0(M, \mathcal{O}_{Z_k}) = \mathbb{C}, \quad 1 \leq k \leq \ell \tag{2.6}$$

$$\dim H^1(M, \mathcal{O}_{Z_k}) = \sum \dim H^1\big(M, \mathcal{O}(-Z_{j-1}))/\mathcal{O}(-Z_j)\big), \quad 1 \leq j \leq k. \tag{2.7}$$

Lemma 2.2 ([La4]). *Let Z_k be part of a computation sequence for Z and such that $\chi(Z_k) = 0$. Then* $\dim H^1(M, \mathcal{O}_D) \leq 1$ *for all cycles D such that $0 \leq D \leq Z_k$. Also $\chi(D) \geq 0$.*

3. Minimally elliptic singularities. In this section we shall recall some of the properties of minimally elliptic singularities which we need for our classification problem.

Definition 3.1. A cycle $E > 0$ is *minimally elliptic* if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles D such that $0 < D < E$.

Wagreich [Wa] defined the singularity p to be elliptic if $\chi(D) \geq 0$ for all cycles $D \geq 0$ and $\chi(F) = 0$ for some cycles $F > 0$. He proved that this definition is independent of the resolution. It is easy to see that under this hypothesis, $\chi(Z) = 0$. The converse is also true [La4]. Henceforth, we shall adopt the following definition:

Definition 3.2. p is said to be weakly elliptic if $\chi(Z) = 0$.

The following Proposition and Lemma holds for weakly elliptic singularity.

Proposition 3.3 ([La4]). *Suppose that $\chi(D) \geq 0$ for all cycles $D > 0$. Let $B = \Sigma b_i A_i$ and $C = \Sigma c_i A_i$, $1 \leq i \leq n$, be any cycles such that $0 < B, C$ and $\chi(B) = \chi(C) = 0$. Let $F = \Sigma \min(b_i, c_i) A_i$, $1 \leq i \leq n$. Then $F > 0$ and $\chi(F) = 0$. In particular, there exists a unique minimally elliptic cycle E.*

Lemma 3.4 ([La4]). *Let E be a minimally elliptic cycle. Then for $A_i \subset \operatorname{supp} E$, $A_i \cdot E = -A_i \cdot K$. Suppose additionally that π is the minimal resolution. Then E is the fundamental cycle for the singularity having* supp E *as its exceptional set. Also, if E_k is part of a computation sequence for E as a fundamental cycle and $A_j \subset \operatorname{supp}\,(E - E_k)$, then the computation sequence may be continued past E_k so as to terminate at $E = E_\ell$ with $A_{i_\ell} = A_j$.*

Theorem 3.5 ([La4]). *Let $\pi\colon M \to V$ be the minimal resolution of the normal two dimensional variety V with one singular point p. Let Z be the fundamental cycle on the exceptional set $A = \pi^{-1}(p)$. Then the following are equivalent:*

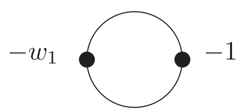
(1) *Z is a minimally elliptic cycle,*
(2) *$A_i \cdot Z = -A_i \cdot K$ for all irreducible components A_i in A,*
(3) *$\chi(Z) = 0$ and any connected proper subvariety of A is the exceptional set for a rational singularity.*

In [La4], Laufer introduced the notion of minimally elliptic singularity.

Definition 3.6. Let p be a normal two-dimensional singularity. p is minimally elliptic if the minimal resolution $\pi\colon M \to V$ of a neighborhood of p satisfies one of the conditions of Theorem 3.5.

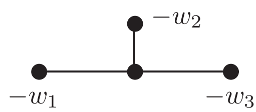
Proposition 3.7 ([La4]). *Let $\pi\colon M \to V$ and $\pi'\colon M' \to V$ be the minimal resolution and minimal good resolution respectively for a minimally elliptic singularity p. Then $\pi = \pi'$ and all the A_i are rational curves except for the following cases:*
(1) *A is an elliptic curve. π is a minimal good resolution.*
(2) *A is a rational curve with a node singularity.*
(3) *A is a rational curve with a cusp singularity.*
(4) *A is two non-singular rational curves which have first order tangential contact at one point.*
(5) *A is three non-singular ratonal curves all meeting transversely at the same point.*

In case (2), the weighted dual graph of the minimal good resolution is



with $w_1 \geq 5$

In cases (3)–(5), π' has the following weighted dual graph



with $w_i \geq 2$, $1 \leq i \leq 3$

Minimally elliptic singularities can be characterized without explicit use of the resolution as follows because $H^1(M, \mathcal{O})$ can be described in terms of V [La2, Theorem 3.4, p. 604].

Theorem 3.8 ([La4]). *Let V be a Stein normal two-dimensional space with p as its only singularity. Let $\pi\colon M \to V$ be a resolution of V. Then p is minimally elliptic singularity if and only if $H^1(M, \mathcal{O}) = \mathbb{C}$ and $\mathcal{O}_{V,p}$ is a Gorenstein ring.*

4. Weighted dual graphs admitting no complete intersection singularities structures. In this section, we shall show that a large class of weighted dual graphs do not admit any complete intersection singularity structure. Let (V, p) be a normal 2-dimensional singularity. Let $\pi\colon M \to V$ be the minimal resolution. Let Z be the fundamental cycle.

Definition 4.1. p is a rational singularity if $\chi(Z) = 1$.

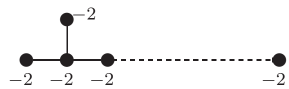
If p is a rational singularity, then π is also a minimal good resolution, i.e., exceptional set with nonsingular A_i and normal crossings. Moreover each A_i is a rational curve [Ar].

Theorem 4.2 ([Ar]). *If p is a hypersurface rational singularity, then p is a rational double point. Moreover the set of weighted dual graphs of hypersurface rational singularities consists of the following graphs*

(1) $A_n, n \geq 1$

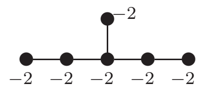
$Z = 1\ 1 \ldots 1$

(2) $D_n, n \geq 4$



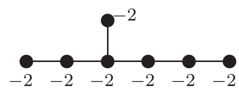
$$Z = \begin{matrix} & 1 & & & \\ 1 & 2 & 2 \ldots & 2 & 1 \end{matrix}$$

(3) E_6



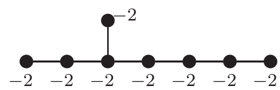
$$Z = \begin{matrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{matrix}$$

(4) E_7



$$Z = \begin{matrix} & & 2 & & & \\ 2 & 3 & 4 & 3 & 2 & 1 \end{matrix}$$

(5) E_8



$$Z = \begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{matrix}$$

Theorem 4.3 ([C-X-Y]). *Let Γ be a weighted dual graph of a rational singularity. If Γ is not one of five types in Theorem 4.2, then Γ does not admit any Gorenstein singularity structure, in particular Γ does not admit any complete intersection singularity structure.*

Proof. Since in the definition of rational singularity, $\chi(Z)$ can be computed from the weighted dual graph, any singularity associated to Γ is a rational singularity. To prove the theorem, we only need to prove that if p is a Gorenstein rational singularity, then its graph is one of the five types in Theorem 4.2. Suppose (V, p) is a Gorenstein rational singularity. Then $\dim H^1(M, \mathcal{O}) = 0$ [Ar]. By a result of Laufer [La2], $\dim H^1(M, \mathcal{O}) = \dim H^0(M - A, \Omega^2) \big/ H^0(M, \Omega^2)$ where Ω^2 is the sheaf of germs of holomorphic 2-forms on M. Therefore there exists an effective canonical divisor $K = \Sigma k_i A_i$, k_i nonnegative integer, on M. Since M is a minimal resolution, by adjunction formula, we have

$$A_i \cdot K \geq 0 \text{ for all } A_i \subseteq A. \tag{4.1}$$

It follows that

$$K^2 = \Sigma k_i (A_i \cdot K) \geq 0. \tag{4.2}$$

On the other hand, the intersection matrix is a negative definition [Gr]. Therefore $K^2 \leq 0$. This together with (4.2) implies $K^2 = 0$. The negative definiteness of the intersection matrix implies $K = 0$. The adjunction formula tells us that $A_i^2 = -2$ for all A_i. Then as an easy exercise, one can show that the weighted dual graph of the exceptional set is one of the five types listed in Theorem 4.2. □

5. Characterization of weighted dual graphs admitting only complete intersection singularities structures. In this section we shall give characterization of weighted dual graphs admitting only hypersurface singularities structures.

THEOREM 5.1 ([La4]). *Let p be a minimally elliptic singularity. Let $\pi\colon M \to V$ be a resolution of a Stein neighborhood V of p with p as its only singular point. Let m be the maximal ideal in $\mathcal{O}_{V,p}$. Let Z be the fundamental cycle on $A = \pi^{-1}(p)$.*

(1) If $Z^2 \leq -2$, then $\mathcal{O}(-Z) = m\mathcal{O}$ on A.
(2) If $Z^2 = -1$, and π is the minimal resolution or the minimal resolution with non-singular A_i and normal crossings, $\mathcal{O}(-Z)/m\mathcal{O}$ is the structure sheaf for an embedded point.
(3) If $Z^2 = -1$ or -2, then p is a double point.
(4) If $Z^2 = -3$, then for all integers $n \geq 1$, $m^n \approx H^0\big(A, \mathcal{O}(-nZ)\big)$ and $\dim m^n/m^{n+1} = -nZ^2$.
(5) If $-3 \leq Z^2 \leq -1$, then p is a hypersurface singularity.
(6) If $Z^2 = -4$, then p is a complete intersection and in fact a tangential complete intersection.
(7) If $Z^2 \leq -5$, then p is not a complete intersection.

Let p be a normal two-dimensional singularity. Choose the minimal resolution of p having non-singular A_i and normal crossings. Let Γ denote the weighted dual graph along with the genera. See [HNK] or [La1] for a more detailed description of Γ. Γ may be described abstractly. Given Γ, we say that p is a singularity associated to Γ. As in [La1, Theorem 6.20, p. 132] we may choose a suitably large infinitesimal neighborhood B of the exceptional set such that B depends only on Γ and determines p. We can deform B in such a way that Γ is preserved. See [La3] for the general theory in this situation.

DEFINITION 5.2. Let Γ be a weighted dual graph, including genera for the vertices. A property is *generically* true for an associated singularity of Γ if given any normal two-dimensional singularity p having Γ as the weighted dual graph of its minimal resolution with non-singular A_i and normal crossings, then the poperty is true for all singularities near the p and off a proper subvariety of the parameter space of a complete deformation of a suitable large infinitesimal neighborhood B of the exceptional set for P.

The following deep theorem is due to Laufer.

THEOREM 5.3 ([La4]). *All rational double points and all minimally elliptic singularities are Gorenstein. Let Γ be a weighted dual graph, including genera for the vertices, associated to a minimal resolution with non-singular A_i and normal crossings of a singularity p. Suppose that p is not a rational double point or minimally elliptic. Then an associated singularity of Γ is generically non-Gorenstein.*

Now we are ready to give a characterization of weighted dual graphs admitting only complete intersection singularities structures (respectively hypersurface singularities structures). Recall that rational and minimally elliptic singularities have topological definitions, i.e., they can be defined in terms of their weighted dual graphs.

THEOREM 5.4 ([C-X-Y]).

(1) The weighted dual graphs which have only hypersurface singularities associated to them are precisely those graphs coming from rational double points, minimally elliptic double points ($Z^2 = -1$, or -2), minimally elliptic triple points ($Z^2 = -3$).
(2) The weighted dual graphs which have only complete intersection singularities associated to them are precisely those graphs coming from rational double

points, minimally elliptic double points ($Z^2 = -1$, *or* -2), *minimally elliptic triple points* ($Z^2 = -3$), *minimally elliptic quadruple points* ($Z^2 = -4$).

(3) The weighted dual graphs which have only complete intersection but not hypersurface singularities associated to them are precisely those graphs coming from minimally elliptic quadruple points ($Z^2 = -4$).

Proof. We only need to observe that hypersurface or complete intersection singularities are Gorenstein. Theorem 5.4 follows directly from Theorem 5.1 and Theorem 5.3. □

6. Classification of weighted dual graphs without complete intersection singularities. In view of Theorem 3.8, Theorem 4.3 and Theorem 5.1, the simplest class of Gorenstein non-complete intersection singularities of dimension two is precisely non-complete intersection singularities with $Z^2 = -5$. In this section, we shall give a complete classification of the weighted dual graphs of these singularities.

Definition 6.1. Let (V, p) be a germ of weakly elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution with $\pi^{-1}(p) = A = \cup A_i$, $1 \leq i \leq n$ the irreducible decomposition of the exceptional set. Let Z be the fundamental cycle. The set of effective components $\{A_{*1}, \dots, A_{*m}\}$ is the set $\{A_i\colon A_i \cdot Z < 0\}$.

Proposition 6.2. *Let (V,p) be a germ of minimally elliptic singularity. Let* $\pi : M \to V$ *be the minimal good resolution of* p*. If* π *is also a minimal good resolution and* $Z^2 = -5$*, then the set of effective components* $\{A_{*1}, \dots, A_{*m}\}$ *must be one of the following:*

(1) $\{A_{*1}\}$, $A_{*1}^2 = -3$, $z_{*1} = 5$
(2) $\{A_{*1}\}$, $A_{*1}^2 = -7$, $z_{*1} = 1$
(3) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = A_{*2}^2 = -3$, $z_{*1} = 3, z_{*2} = 2$
(4) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = A_{*2}^2 = -3$, $z_{*1} = 4, z_{*2} = 1$
(5) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = -3, A_{*2}^2 = -4$, $z_{*1} = 3, z_{*2} = 1$
(6) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = -3$, $A_{*2}^2 = -5$, $z_{*1} = 2, z_{*2} = 1$
(7) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = -4$, $A_{*2}^2 = -3$, $z_{*1} = 2, z_{*2} = 1$
(8) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = -3$, $A_{*2}^2 = -6$, $z_{*1} = z_{*2} = 1$
(9) $\{A_{*1}, A_{*2}\}$, $A_{*1}^2 = -4$, $A_{*2}^2 = -5$, $z_{*1} = z_{*2} = 1$
(10) $\{A_{*1}, A_{*2}, A_{*3}\}$, $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = -3$, $z_{*1} = z_{*2} = 2, z_{*3} = 1$
(11) $\{A_{*1}, A_{*2}, A_{*3}\}$, $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = -3$, $z_{*1} = z_{*2} = 1, z_{*3} = 3$
(12) $\{A_{*1}, A_{*2}, A_{*3}\}$, $A_{*1}^2 = -4$, $A_{*2}^2 = -3 = A_{*3}^2$, $z_{*1} = z_{*2} = 1, z_{*3} = 2$
(13) $\{A_{*1}, A_{*2}, A_{*3}\}$, $A_{*1}^2 = -3$, $A_{*2}^2 = A_{*3}^2 = -4$, $z_{*1} = z_{*2} = z_{*3} = 1$
(14) $\{A_{*1}, A_{*2}, A_{*3}\}$, $A_{*1}^2 = A_{*2}^2 = -3$, $A_{*3}^2 = -5$, $z_{*1} = z_{*2} = z_{*3} = 1$
(15) $\{A_{*1}, A_{*2}, A_{*3}, A_{*4}\}$, $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = A_{*4}^2 = -3$, $z_{*1} = z_{*2} = z_{*3} = 1, z_{*4} = 2$
(16) $\{A_{*1}, A_{*2}, A_{*3}, A_{*4}\}$, $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = -3$, $A_{*4}^2 = -4$, $z_{*1} = z_{*2} = z_{*3} = z_{*4} = 1$
(17) $\{A_{*1}, A_{*2}, A_{*3}, A_{*4}, A_{*5}\}$, $A_{*i}^2 = -3$, $z_{*i} = 1, i = 1, 2, 3, 4, 5$

where $A_{*i} \neq A_{*j}$ *if* $i \neq j$ *and* z_i *is the coefficient of* A_{*i} *in* Z.

Proof. Let $\{A_{*1}, \dots, A_{*m}\}$ be the set of effective components. Then, by Theorem 3.5, we have

$$-Z^2 = -\sum_{i=1}^{n} z_i(A_i \cdot Z) = -\sum_{i=1}^{m} z_i(A_{*i} \cdot Z)$$

$$= \sum_{i=1}^{m} z_i(A_{*i} \cdot K).$$

This implies $5 = \sum_{i=1}^{m} z_i(-A_{*i}^2 - 2)$. By definition of the effective component, we have $-A_{*i}^2 - 2 = A_{*i} \cdot K = -A_{*i} \cdot Z > 0$. Hence we have $1 \leq m \leq 5$. If $m = 1$, then $-5 = z_1(A_{*1}^2 + 2)$ and we are in case (1) or case (2) . If $m = 2$, then $-5 = z_1(A_{*1}^2 + 2) + z_2(A_{*2}^2 + 2)$. It follows easily that we are in case (3), case (4), case (5), case (6), case (7), case (8) or case (9). If $m = 3$, then $-5 = z_1(A_{*1}^2 + 2) + z_2(A_{*2}^2 + 2) + z_3(A_{*3}^2 + 2)$. It is easy to see that we are in case (10), case (11), case (12), case (13) or case (14). If $m = 4$, then $-5 = z_1(A_{*1}^2 + 2) + z_2(A_{*2}^2 + 2) + z_3(A_{*3}^2 + 2) + z_4(A_{*4}^2 + 2)$. So we are in case (15) or case (16). If $m = 5$, then $-5 = (A_{*1}^2 + 2) + (A_{*2}^2 + 2) + (A_{*3}^2 + 2) + (A_{*4}^2 + 2) + (A_{*5}^2 + 2)$. Then we are in case (17). □

PROPOSITION 6.3. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Suppose that A_* is an effective component of Γ, and let $\{\Gamma_1, \ldots, \Gamma_n\}$ be the set of connected components of Γ' which intersect with A_*. Then $\Gamma_1, \ldots, \Gamma_n$ are necessarily one of the rational double point graphs appearing in Theorem 4.2. Let $Z_1, \ldots, Z_n$ be the fundamental cycles of $\Gamma_1, \ldots, \Gamma_n$ respectively. Then $A_* \cdot Z_1 \leq 2$. If $A_* \cdot Z_1 = 2$, then $\Gamma = A_* \cup \Gamma_1$ and $Z = A_* + Z_1$; moreover for any $A_1 \in \Gamma_1$, $A_1 \cdot A_* > 0$ if and only if $A_1 \cdot Z_1 < 0$.*

Proof. For any $A_j \in \Gamma_i$, $0 = A_j \cdot Z = A_j \cdot (-K) = A_j^2 + 2$. Hence $A_j^2 = -2$. It follows that Γ_i are rational double point graphs.

Since Γ is the graph of a minimally elliptic singularity, we have

$$\begin{aligned} 0 &\leq \chi(A_* + Z_1) \\ &= \chi(A_*) + \chi(Z_1) - A_* \cdot Z_1 \end{aligned} \tag{6.1}$$

which implies

$$A_* \cdot Z_1 \leq \chi(A_*) + \chi(Z_1) = 2. \tag{6.2}$$

Observe that if $\Gamma \neq A_* \cup \Gamma_1$ or $Z > A_* + Z_1$, then the inequalities in (6.1) and (6.2) are strict inequalities. Hence $A_* \cdot Z_1 = 1$. We have proved that if $A_* \cdot Z_1 = 2$, then $\Gamma = A_* \cup \Gamma_1$ and $Z = A_* + Z_1$.

We shall assume from now on that $A_* \cdot Z_1 = 2$. Let $A_1 \in \Gamma_1$ such that $A_1 \cdot A_* > 0$. $A_1 \cdot Z_1 = 0$ would imply $A_1 \cdot (Z_1 + A_*) > 0$ and hence $A_1 \cdot Z > 0$, which is absurd. It follows that $A_1 \cdot Z_1 < 0$.

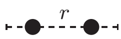
Conversely, if $A_1 \in \Gamma_1$ and $A_1 \cdot Z_1 < 0$, but $A_* \cdot A_1 = 0$, then there is a $A_2 \in \Gamma_1$ such that $A_2 \cdot A_* > 0$ and $A_2 \cdot Z_1 < 0$. Since $Z_1^2 = -2$, we have $A_2 \cdot Z_1 = A_1 \cdot Z_1 = -1$ and the coefficient of A_2 in Z_1 is one. It follows that A_2 is the only component in Γ_1 which intersects with A_* and $A_2 \cdot A_* = 2$. Observe that $\chi(A_* + A_2) = 0$ and $A_* + A_2 < Z$. This contradicts the fact that Z is the minimally elliptic cycle. So we have shown that $A_* \cdot A_1 > 0$ if and only if $A_1 \cdot Z_1 < 0$. □

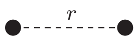
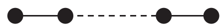
NOTATION. From now on, we shall denote ● a nonsingular rational curve with -2 weight.

COROLLARY 6.4. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ_1 be a rational double point subgraph of Γ with fundamental cycle Z_1 in Proposition 6.3. Let A_* be an effecitve component attaching on Γ_1. Suppose that $A_* \cdot Z_1 = 2$. Then one of the following cases holds*

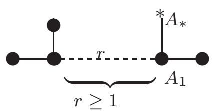
(1) Γ is of the following form:

A_*  $r \geq 1$ $\quad Z = A_* + A_1 + \cdots + A_r$

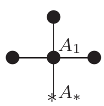
where  *denote*  *with r vertices and $r+1$ edges.* ● *is a nonsingular rational curve with weight -2.*

 *denote*  *with $r+1$ vertices and r edges.* ● *is a nonsingular rational curve with weight -2. (r can be zero when it denotes only one vertice.)*

(2) Γ_1 is either D_n, E_6, E_7 or E_8. There exists an unique A_1 in Γ_1 such that $A_1 \cdot A_* = 1$ and $A_1 \cdot Z_1 < 0$. The coefficient of A_1 in Z_1 is 2. $\Gamma = A_* \cup \Gamma_1$ and $Z = A_* + Z_1$. Γ is one of the following forms.

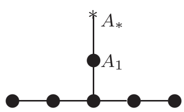
(a) (i) 

$$Z = \begin{array}{ccccc} & 1 & & 1 & \\ 1 & 2 \dots & & 2 & 1 \end{array}$$

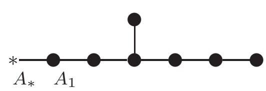
(ii) 

$$Z = \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array}$$

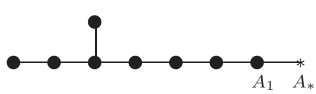
It is also the special case for r=0 in (i)

(b) 

$$Z = \begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

(c) 

$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$$

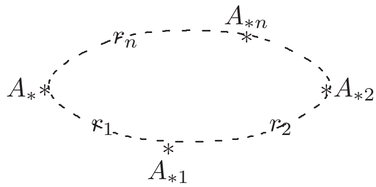
(d) 

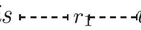
$$Z = \begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

Proof. This follows from Proposition 6.3 and Theorem 4.2. □

Definition 6.5. Let A_1 be an irreducible component in a weighted dual graph Γ. Degree of A_1 is defined to be the number of distinct irreducible components in Γ intersecting with A_1 positively.

Lemma 6.6. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ_1 be a subgraph of Γ in Proposition 6.3 with fundamental cycle Z_1. Let A_* be an effective component attaching on Γ_1. Suppose* *that the coefficient* z_* *of* A_* *in* Z *is one. Then either* A_* *has degree one or* Γ *is of the following form*



where $n \geq 1$ *and* Γ_1 *is* 

which denotes  *with* r_1 *vertices and* $r_1 + 1$ *edges.*

Proof. By Proposition 6.3, $A_* \cdot Z_1$ equals to either 1 or 2. If $A_* \cdot Z_1 = 2$, then the lemma follows from Corollary 6.4.

From now on, we shall assume that $A_* \cdot Z_1 = 1$. To prove the lemma, we only need to prove that if $\deg A_* > 1$, then Γ must be of circular form shown as above. If $\deg A_* > 1$, then there exists A_2 not in Γ_1 such that $A_2 \cdot A_* > 0$. Clearly $A_2 \cdot A_* = 1$ by minimally ellipticity of Γ. We claim that A_2 is connected to Γ_1 via a path in Γ which is disjoint from A_*.

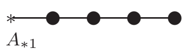
By Theorem 3.4, we can choose a computation sequence of the fundamental cycle Z starting from A_* continuing to Γ_1 and ending at A_2. Now $z_* = 1$, $A_*^2 + 2 = A_* \cdot Z$ and $deg\, A_* > 1$ implies that the computation sequence contains A_* only once and the coefficient of A_2 in Z must also be one. Hence the computation sequence must contain A_2 only once. Moreover $A_2^2 + 2 = A_2 \cdot Z$ implies that $deg\, A_2 = 2$. Repeating the same argument, we see that for every component in that computation sequence its coefficient in Z is one, its degree is 2 and the computation sequence passes it only once. Therefore Γ must be the form shown in the lemma. ∎

Remark 6.7. With the same assumption and notations in Lemma 6.6, so long as the interseciton matrix remains negative definite, A_*^2 can be given any value at most -2 and Z remains unchanged and Γ still corresponds to a minimally elliptic singularity.

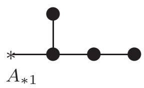
Proposition 6.8. *Let* Γ *be the minimal resolution graph of minimally elliptic singularity with fundamental cycle* Z*. Suppose that there is no effective component with coefficient in* Z *strictly bigger than* 1*. Set all* A_*^2 *of effective components of* Γ *but one to* -2 *and the remaining weight to* -3*. Then the new weighted dual graph* $\widetilde{\Gamma}$*, which coincide with* Γ *except the weights, is obtained from a rational double point weighted dual graph by the addition of one additional vertex* A_**. In fact* $\widetilde{\Gamma}$ *corresponds to a minimally elliptic double point with* $Z^2 = -1$*.*

Proof. Since $A_* \cdot Z = -A_* \cdot K = A_*^2 + 2$, after setting all A_*^2 of effective components of Γ but one to -2 and the remaining weight to -3, it is still true that $A_i \cdot Z \leq 0$ for all i and that $A_* Z < 0$ for one A_*. Therefore Z is also the fundamental cycle for $\widetilde{\Gamma}$ and the intersection matrix of $\widetilde{\Gamma}$ is still negative definite [Ar, Proposition 2, pp. 130–131]. By Lemma 6.6, Γ is obtained from a rational double point weighted dual graph by the addition of one additional vertex A_*. Clearly $Z_{\widetilde{\Gamma}}^2 = -1$. ∎

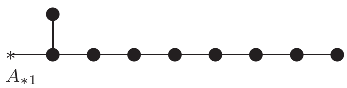
Proposition 6.9. *Let* Γ *be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle* Z*. Let* Γ' *be the subgraph of* Γ *by removing all the* *effective components of* Γ*. Let* A_* *be an effective component of* Γ*. Suppose that* Γ_1 *is a connected component of* Γ' *which corresponds to* A_n *graph in case* (1) *of Theorem 4.2. Suppose also that* Γ_1 *intersects with* A_* *but disjoint from other effective component. Let* Z_1 *be the fundamental cycle on* Γ_1*. Suppose* $A_{*1} \cdot Z_1 = 1$*. If the coefficient* z_{*1} *of* A_{*1} *in* Z *is* 5 *and* $A_{*1}^2 = -3$*, then* $A_{*1} \cup \Gamma_1$ *and restriction of* Z *on* $A_{*1} \cup \Gamma_1$ *must be one of the following form.*

(1) 

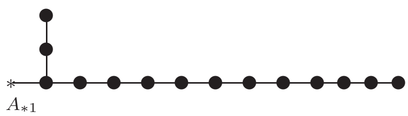
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{5} \quad 4 \quad 3 \quad 2 \quad 1$$

(2) 

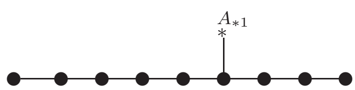
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{5} \quad \begin{array}{c} 3 \\ 6 \end{array} \; 4 \; 2$$

(3) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{5} \quad \begin{array}{c} 4 \\ 8 \end{array} \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1$$

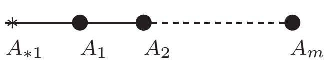
(4) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{5} \quad \begin{array}{c} 4 \\ 8 \\ 12 \end{array} \quad 11 \; 10 \; 9 \; 8 \; 7 \; 6 \; 5 \; 4 \; 3 \; 2 \; 1$$

(5) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad \begin{array}{c} \underline{5} \\ 12 \end{array} \quad 9 \quad 6 \quad 3$$

Proof. Consider A_{*1} attaching on Γ_1 in the following form



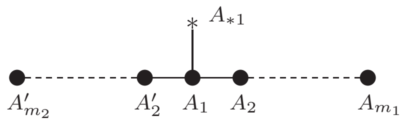
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 5 \quad n_1 \quad n_2 \ldots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 5 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{5}{m+1} \end{cases}.$$

Therefore $m = 4$ and we are in case (1).

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = \overset{}{n'_{m_2}} \dots n'_2 \overset{5}{n_1}\, n_2 \dots n_{m_1}.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \le j \le m_2$, we have the following system of equations

$$(6.3) \qquad \begin{cases} -2n_{m_1} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \quad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \qquad \begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \quad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases}$$

$$(6.4) \qquad 5 - 2n_1 + n_2 + n'_2 = 0$$

(6.3) implies

$$(6.5) \qquad n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \le i \le m_1$$

$$(6.6) \qquad n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \le j \le m_2$$

$$(6.7) \qquad m_1 n_{m_1} = m_2 n'_{m_2}.$$

Putting (6.5) and (6.6) into (6.4), we get

$$\begin{aligned} 0 &= 5 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2} \\ &= 5 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} \end{aligned} \qquad (6.8)$$

(6.7) and (6.8) imply

$$(6.9) \qquad n_{m_1} + n'_{m_2} = 5$$

(6.9) implies that either $n_{m_1} = 3$, $n'_{m_2} = 2$ or $n_{m_1} = 4, n'_{m_2} = 1$.

Case I. $n_{m_1} = 3$ and $n'_{m_2} = 2$. By (6.7), we have $3m_1 = 2m_2$. Observe that

$$\begin{aligned} &-1 = A_*^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \ge 5(-3) + n_1 = -15 + 3m_1 \\ &\Rightarrow 3m_1 \le 14 \\ &\Rightarrow m_1 \le 4. \end{aligned}$$

If $m_1 = 2$, or 4, then we are in case (2) or case (5) respectively in the statement of the proposition.

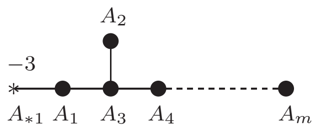
Case II. $n_{m_1} = 4, n'_{m_2} = 1$. By (6.7), we have $4m_1 = m_2$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \ge 5(-3) + n_1 = -15 + 4m_1 \\ &\Rightarrow 4m_1 \le 14 \\ &\Rightarrow m_1 \le 3. \end{aligned}$$

If $m_1 = 2$, or 3, then we are in case (3) or case (4) respectively in the statement of the proposition. $\square$

Proposition 6.10. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to D_m graph in case (2) of Theorem 4.2. Suppose also that Γ_1 intersects with A_* but disjoint with other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 5 and $A_{*1}^2 = -3$, then such a graph does not exist.*

Proof. Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = 5\ n_1\ \begin{matrix} n_2 \\ n_3 \end{matrix}\ n_4\ \dots\ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations

$$-2n_1 + 5 + n_3 = 0 \tag{6.10}$$

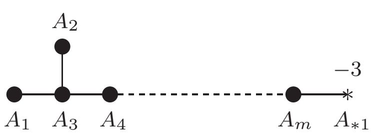
$$\begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.11}$$

(6.11) implies

$$n_1 = \frac{m}{2} n_m,\ n_2 = \frac{m-2}{2} n_m,\ n_j = (m-j+1)n_m,\ 3 \le j \le m \tag{6.12}$$

(6.10) and (6.12) imply $n_m = \frac{5}{2}$. This contradicts the fact that n_m is an integer.

We next consider A_{*1} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ \begin{matrix} n_2 \\ n_3 \end{matrix}\ n_4\ \dots\ n_m\ 5.$$

Since $A_i \cdot Z = 0$, $1 \le i \le m$, we have the following system of equations

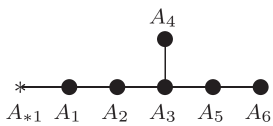
$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} + 5 = 0 \end{cases} \tag{6.13}$$

$$-2n_m + n_{m-1} + 5 = 0 \tag{6.14}$$

(6.13) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.14), we know that $n_1 = \frac{5}{2}$. This contradicts the fact that n_1 is an integer. □

Proposition 6.11. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to either E_6, E_7 or E_8 graph in case* (3)*–case* (5) *of Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} but disjoint with other effective component. Let Z_1 be the fundamental cycle of Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 5 and $A_{*1}^2 \leq -3$, then such a graph does not exist.*

Proof. By Theorem 4.2, A_{*1} attaching on E_6 must be of the following form.



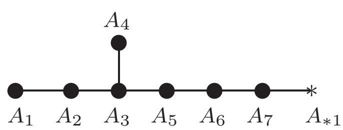
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 5\ n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6.$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \leq i \leq 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 5 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = \frac{10}{3}$. This contradicts to the fact that n_6 is an integer.

By Theorem 4.2, A_{*1} attaching on E_7 must be of the following form.



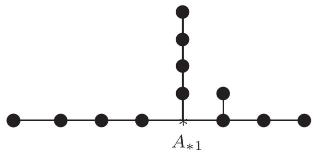
$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6\ n_7\ 5.$$

Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \leq i \leq 7$, we have the following system of equations.

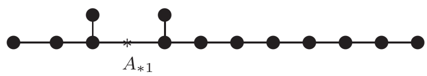
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 5 = 0. \end{cases}$$

We get $n_1 = 5$, and $n_4 = \frac{15}{2}$, which contradicts the fact that n_4 is an integer. By Theorem 4.2, A_{*1} cannot attach on E_8 because $A_{*1} \cdot Z_1 \geq 2$. □

Theorem 6.12. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi \colon M \longrightarrow V$ be the minimal resolution of p. If case* (1) *of Proposition 6.2 holds, i.e., there exists only one effective component A_{*1}, and $A_{*1}^2 = -3$, $z_{*1} = 5$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

$$\begin{array}{cccccccccc} & & & & & 1 & & & \\ & & & & & 2 & & & \\ & & & & & 3 & & & \\ & & & & & 4 & 3 & & \\ Z= & 1 & 2 & 3 & 4 & \underline{5} & 6 & 4 & 2 \end{array}$$

(2) 

$$\begin{array}{ccccccccccccc} & & & 3 & & 4 & & & & & & & \\ Z= & 2 & 4 & 6 & \underline{5} & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

Proof. Let Γ' be the graph obtained by deleting A_{*1} from Γ. Let $\Gamma_1, \dots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \dots, Z_m$ respectively. Since $z_{*1} = 5$, in view of Proposition 6.3 and Corollary 6.4, we have $A_* \cdot Z_i = 1$ for $1 \le i \le m$. By Proposition 6.9, Proposition 6.10 and Proposition 6.11, we have

$$\{A_{*1} \cdot Z\Big|_{\Gamma_1}, \dots, A_{*1} \cdot Z\Big|_{\Gamma_m}\} \subseteq \{4, 6, 8, 12\}.$$

Since the singularity is minimally elliptic, we have

$$A_* \cdot (Z - 5A_{*1}) = -A_{*1} \cdot (K + 5A_{*1}) = A_{*1}^2 + 2 - 5A_{*1}^2 = 14.$$

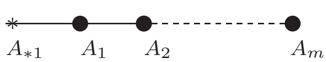
Observe that we can write

$$\begin{aligned} 14 &= 4 + 4 + 6 \\ &= 8 + 6. \end{aligned}$$

By Propositions 6.9, 6.10 and 6.11 together with Definition 2.1, in case of $14 = 4 + 4 + 6$, we have case (1). In case of $14 = 8 + 6$, we only have case (2). $\square$

Proposition 6.13. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 1 and $A_{*1}^2 \le -3$, then such a graph does not exist.*

Proof. The proof is similar to those of Propositions 6.9, 6.10 and 6.11. Consider A_{*1} attaching on Γ_1 in the following form



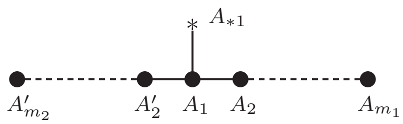
$$Z\Big|_{A_{*1} \cup \Gamma_1} = 1 \quad n_1 \quad n_2 \dots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m - i + 1)n_m \quad 1 \le i \le m \\ n_m = \frac{1}{m+1} \end{cases}.$$

We get $n_m = \frac{1}{m+1}$, which contradicts with the fact that n_m is an integer.

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n'_{m_2} \dots n'_2 \overset{1}{n_1} n_2 \dots n_{m_1}.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \le j \le m_2$, we have the following system of equations

$$\begin{cases} -2n_{m_1}+n_{m_1-1}=0 \\ -2n_{m_1-1}+n_{m_1-2}+n_{m_1}=0 \\ \quad\vdots \\ -2n_3+n_2+n_4=0 \\ -2n_2+n_1+n_3=0 \end{cases} \qquad \begin{cases} -2n'_{m_2}+n'_{m_2-1}=0 \\ -2n'_{m_2-1}+n'_{m_2-2}+n'_{m_2}=0 \\ \quad\vdots \\ -2n'_3+n'_2+n'_4=0 \\ -2n'_2+n_1+n'_3=0 \end{cases} \tag{6.15}$$

$$1 - 2n_1 + n_2 + n'_2 = 0 \tag{6.16}$$

(6.15) implies

$$n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \le i \le m_1 \tag{6.17}$$

$$n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \le j \le m_2 \tag{6.18}$$

$$m_1 n_{m_1} = m_2 n'_{m_2}. \tag{6.19}$$

Putting (6.17) and (6.18) into (6.16), we get

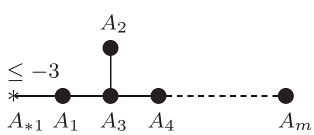
$$\begin{aligned} 0 &= 1 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2} \\ &= 1 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} = 0 \end{aligned} \tag{6.20}$$

(6.19) and (6.20) imply

$$n_{m_1} + n'_{m_2} = 1. \tag{6.21}$$

But (6.21) contradicts with the fact that $n_{m_1} \ge 1$, $n'_{m_2} \ge 1$.

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = 1\ n_1\ \overset{n_2}{n_3}\ n_4\ \dots\ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations

$$-2n_1 + 1 + n_3 = 0 \tag{6.22}$$

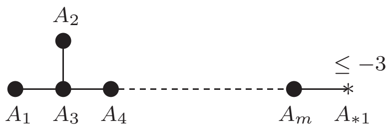
$$\begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.23}$$

(6.23) implies

$$n_1 = \frac{m}{2}n_m,\ n_2 = \frac{m-2}{2}n_m,\ n_j = (m-j+1)n_m,\ 3 \le j \le m \tag{6.24}$$

(6.22) and (6.24) imply $n_m = \frac{1}{2}$. This contradicts the fact that n_m is an integer.

We next consider A_{*1} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ \overset{n_2}{n_3}\ n_4\ \dots\ n_m\ 1.$$

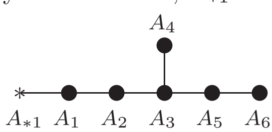
Since $A_i \cdot Z = 0$, $1 \le i \le m$, we have the following system of equations

$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.25}$$

$$-2n_m + n_{m-1} + 1 = 0 \tag{6.26}$$

(6.25) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.26), we know that $n_1 = \frac{1}{2}$. This contradicts the fact that n_1 is an integer.

By Theorem 4.2, A_{*1} attaching on E_6 must be of the following form.



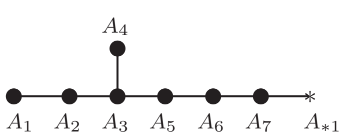
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 1\ n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6.$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \le i \le 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = \frac{2}{3}$. This contradicts to the fact that n_6 is an integer.

By Theorem 4.2, A_{*1} attaching on E_7 must be of the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6\ n_7\ 1.$$

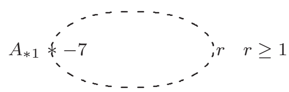
Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \leq i \leq 7$, we have the following system of equations.

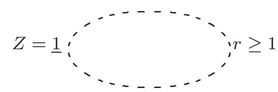
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 1 = 0. \end{cases}$$

We get $n_1 = 1$, and $n_4 = \frac{3}{2}$, which contradicts the fact that n_4 is an integer. By Theorem 4.2, A_{*1} cannot attach on E_8 because $A_{*1} \cdot Z_1 \geq 2$. ☐

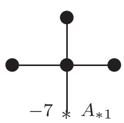
THEOREM 6.14. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi \colon M \to V$ be the minimal resolution of p. If case (2) of Proposition 6.2 holds, i.e., there exists one effective component A_{*1}, and $A_{*1}^2 = -7$, $z_{*1} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1)



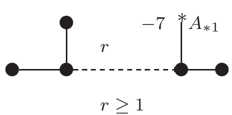


(2) (i)



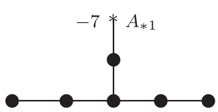
$$Z = \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & \underline{1} & \end{array}$$

(ii)



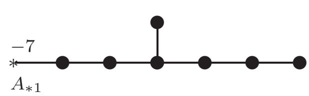
$$Z = \begin{array}{cccccc} & 1 & & \underline{1} & \\ 1 & 2 & \ldots & 2 & 1 \end{array}$$

(3)



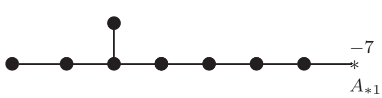
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

(4)



$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$$

(5)

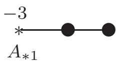


$$Z = \begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

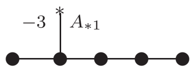
Proof. This follows easily from Proposition 6.3, Corollary 6.4 and Proposition 6.13. ☐

PROPOSITION 6.15. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that*

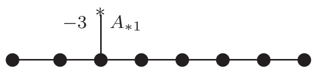
Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 3 and $A_{*1}^2 = -3$, then $A_{*1} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup \Gamma_1$ must be one of the following forms.

(1) 

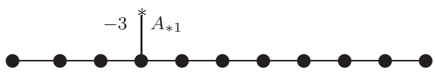
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{3}\ 2\ 1$$

(2) 

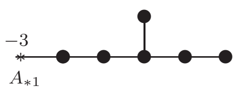
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{3} & & & \\ 2 & 4 & 3 & 2 & 1 \end{array}$$

(3) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccccc} & & \underline{3} & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

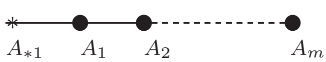
(4) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccccccccccc} & & & \underline{3} & & & & & & & \\ 2 & 4 & 6 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

(5) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccc} & & & 3 & & \\ \underline{3} & 4 & 5 & 6 & 4 & 2 \end{array}$$

Proof. The proof is similar to those of Proposition 6.9, 6.10 and 6.11. Consider A_{*1} attaching on Γ_1 in the following form



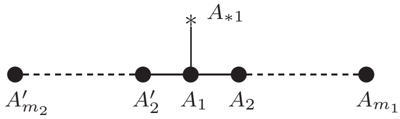
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 3 \quad n_1 \quad n_2 \ldots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 3 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{3}{m+1} \end{cases}.$$

Therefore $m = 2$ and we are in case (1).

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{array}{c} \phantom{n'_{m_2}\ldots n'_2\ } 3 \phantom{\ n_2\ldots n_{m_1}} \\ n'_{m_2}\ldots n'_2\ n_1\ n_2 \ldots n_{m_1}. \end{array}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \le j \le m_2$, we have the following system of equations

$$(6.27) \qquad \begin{cases} -2n_{m_1}+n_{m_1-1}=0 \\ -2n_{m_1-1}+n_{m_1-2}+n_{m_1}=0 \\ \quad\vdots \\ -2n_3+n_2+n_4=0 \\ -2n_2+n_1+n_3=0 \end{cases} \qquad \begin{cases} -2n'_{m_2}+n'_{m_2-1}=0 \\ -2n'_{m_2-1}+n'_{m_2-2}+n'_{m_2}=0 \\ \quad\vdots \\ -2n'_3+n'_2+n'_4=0 \\ -2n'_2+n_1+n'_3=0 \end{cases}$$

$$(6.28) \qquad 3 - 2n_1 + n_2 + n'_2 = 0$$

(6.27) implies

$$(6.29) \qquad n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \le i \le m_1$$

$$(6.30) \qquad n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \le j \le m_2$$

$$(6.31) \qquad m_1 n_{m_1} = m_2 n'_{m_2}.$$

Putting (6.29) and (6.30) into (6.28), we get

$$\begin{aligned} 0 &= 3 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2} \\ (6.32) \qquad &= 3 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} = 0 \end{aligned}$$

(6.31) and (6.32) imply

$$(6.33) \qquad n_{m_1} + n'_{m_2} = 3$$

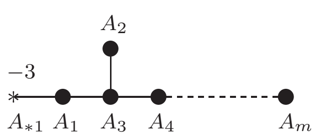
(6.33) implies that $n_{m_1} = 2$, $n'_{m_2} = 1$.

Therefore $n_{m_1} = 2$ and $n'_{m_2} = 1$. By (6.31), we have $2m_1 = m_2$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \ge 3(-3) + n_1 = -9 + 2m_1 \\ &\Rightarrow 2m_1 \le 8 \\ &\Rightarrow m_1 \le 4. \end{aligned}$$

If $m_1 = 2$, 3, or 4, then we are in case (2) case (3) or case (4) respectively in the statement of the proposition.

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = 3 \ n_1 \ \begin{matrix} n_2 \\ n_3 \end{matrix} \ n_4 \ \ldots \ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations

$$(6.34) \qquad -2n_1 + 3 + n_3 = 0$$

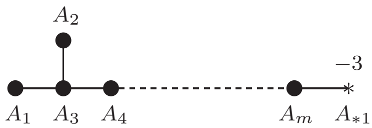
$$(6.35) \qquad \begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases}$$

(6.35) implies

$$n_1 = \frac{m}{2}n_m,\ n_2 = \frac{m-2}{2}n_m,\ n_j = (m-j+1)n_m,\ 3 \le j \le m \tag{6.36}$$

(6.34) and (6.36) imply $n_m = \frac{3}{2}$. This contradicts the fact that n_m is an integer.

We next consider A_{*1} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ \overset{n_2}{n_3}\ n_4\ \ldots\ n_m\ 3.$$

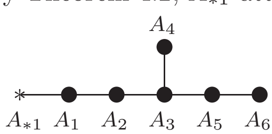
Since $A_i \cdot Z = 0$, $1 \le i \le m$, we have the following system of equations

$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.37}$$

$$-2n_m + n_{m-1} + 3 = 0 \tag{6.38}$$

(6.37) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.38), we know that $n_1 = \frac{3}{2}$. This contradicts the fact that n_1 is an integer.

By Theorem 4.2, A_{*1} attaching on E_6 must be of the following form.



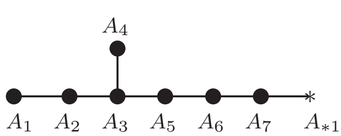
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 3\ n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6.$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \le i \le 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 3 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = 2$. Then we are in case (5)

By Theorem 4.2, A_{*1} attaching on E_7 must be of the following form.



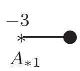
$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6\ n_7\ 3.$$

Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \leq i \leq 7$, we have the following system of equations.

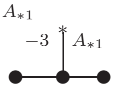
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 3 = 0. \end{cases}$$

We get $n_1 = 3$, and $n_4 = \frac{9}{2}$, which contradicts the fact that n_4 is an integer. By Theorem 4.2, A_{*1} cannot attach on E_8 because $A_{*1} \cdot Z_1 \geq 2$. ☐

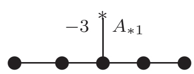
PROPOSITION 6.16. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 2 and $A_{*1}^2 = -3$, then $A_{*1} \cup \Gamma_1$ and restriction of Z on $A_* \cup \Gamma_1$ must be one of the following forms.*

(1) 

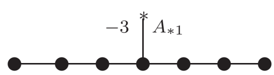
$Z\big|_{A_{*1}\cup\Gamma_1} = \underline{2}\ 1$

(2) 

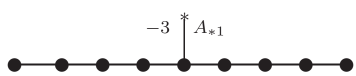
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccc} & \underline{2} & \\ 1 & 2 & 1 \end{array}$

(3) 

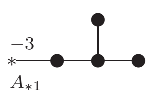
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccccc} & & \underline{2} & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$

(4) 

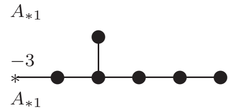
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccccccc} & & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$

(5) 

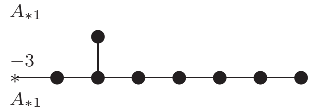
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{ccccccccc} & & & & \underline{2} & & & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{array}$

(6) 

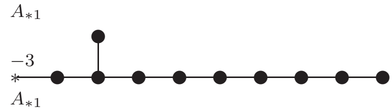
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccc} & & 1 & \\ \underline{2} & 2 & 2 & 1 \end{array}$

(7) 

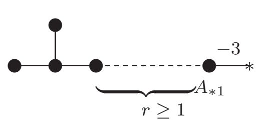
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccc} & & 2 & & & \\ \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$

(8) 

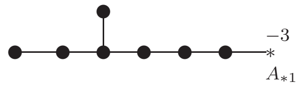
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccccc} & & 3 & & & & & \\ \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$

(9) 

$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccccccc} & & 4 & & & & & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$

(10) 

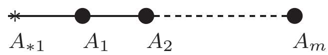
$Z\big|_{A_{*1}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & & & \\ 1 & 2 & 2 & \ldots & 2 & \underline{2} \end{array}$

(11) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} \end{matrix}$$

Proof. The proof is similar to those of Proposition 6.9, 6.10 and 6.11.

Consider A_{*1} attaching on Γ_1 in the following form



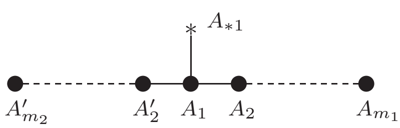
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2 \quad n_1 \quad n_2 \dots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{2}{m+1} \end{cases}.$$

Therefore $m = 1$ and we are in case (1).

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n'_{m_2} \dots n'_2 \overset{2}{n_1}\; n_2 \dots n_{m_1}.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \le j \le m_2$, we have the following system of equations

$$\begin{cases} -2n_{m_1} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \quad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \qquad \begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \quad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.39}$$

$$2 - 2n_1 + n_2 + n'_2 = 0 \tag{6.40}$$

(6.39) implies

$$n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \le i \le m_1 \tag{6.41}$$

$$n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \le j \le m_2 \tag{6.42}$$

$$m_1 n_{m_1} = m_2 n'_{m_2}. \tag{6.43}$$

Putting (6.41) and (6.42) into (6.40), we get

$$0 = 2 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2}$$

$$= 2 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} = 0 \tag{6.44}$$

(6.43) and (6.44) imply

$$n_{m_1} + n'_{m_2} = 2 \tag{6.45}$$

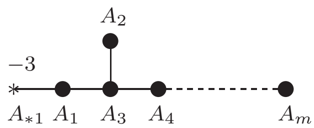
(6.45) implies that $n_{m_1} = 1$, $n'_{m_2} = 1$.

Therefore $n_{m_1} = 1$ and $n'_{m_2} = 1$. By (6.43), we have $m_1 = m_2$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 2(-3) + n_1 = -6 + m_1 \\ &\Rightarrow m_1 \leq 5. \end{aligned}$$

If $m_1 = 2$, 3, 4, or 5, then we are in case (2) case (3) case (4) case (5) respectively in the statement of the proposition.

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\ n_1\ \overset{n_2}{n_3}\ n_4\ \ldots\ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have the following system of equations

$$-2n_1 + 2 + n_3 = 0 \tag{6.46}$$

$$\begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.47}$$

(6.47) implies

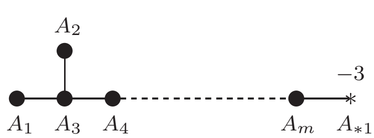
$$n_1 = \frac{m}{2}n_m,\ n_2 = \frac{m-2}{2}n_m,\ n_j = (m - j + 1)n_m,\ 3 \leq j \leq m \tag{6.48}$$

(6.46) and (6.48) imply $n_m = 1$.

Therefore $n_1 = \frac{m}{2}$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 2(-3) + n_1 = -6 + \frac{m}{2} \\ &\Rightarrow m \leq 10. \end{aligned}$$

Then m=4, 6, 8, 10, we are in case (6), case (7), case (8), case (9). We next consider A_{*1} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ \overset{n_2}{n_3}\ n_4\ \ldots\ n_m\ 2.$$

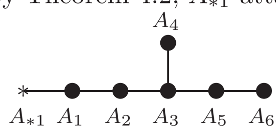
Since $A_i \cdot Z = 0$, $1 \leq i \leq m$, we have the following system of equations

$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.49}$$

$$-2n_m + n_{m-1} + 2 = 0 \tag{6.50}$$

(6.49) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.50), we know that $n_1 = 1$. Then we are in case (10).

By Theorem 4.2, A_{*1} attaching on E_6 must be of the following form.



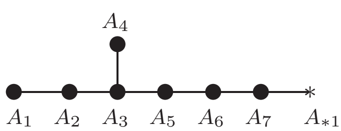
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\ n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \leq i \leq 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = \frac{4}{3}$, which contradicts the fact that n_6 is an integer.

By Theorem 4.2, A_{*1} attaching on E_7 must be of the following form.



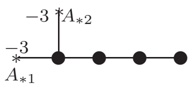
$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6\ n_7\ 2.$$

Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \leq i \leq 7$, we have the following system of equations.

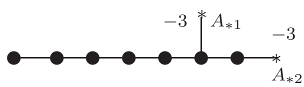
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 2 = 0. \end{cases}$$

We get $n_1 = 2$, then we are in case (11). By Theorem 4.2, A_{*1} cannot attach on E_8 because $A_{*1} \cdot Z_1 \geq 2$. $\square$

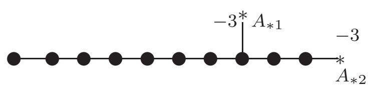
PROPOSITION 6.17. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all* *the effective components of* Γ. *Let* A_{*1} *and* A_{*2} *be two effective components of* Γ. *Suppose that* Γ_1 *is a connected component of* Γ' *which corresponds to a rational double point graph in Theorem 4.2. Suppose also that* Γ_1 *intersects with both* A_{*1} *and* A_{*2}, *but no other effective component. Let* Z_1 *be the fundamental cycle on* Γ_1. *Suppose* $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. *If* $A_{*1} \cdot A_{*2} = 0$ *and the coefficients* $z_{*1} = 3$, $z_{*2} = 2$ *and* $A_{*1}^2 = A_{*2}^2 = -3$, *then* $A_{*1} \cup A_{*2} \cup \Gamma_1$ *and restriction of* Z *on* $A_{*1} \cup A_{*2} \cup \Gamma_1$ *must be one of the following forms.*

(1) 

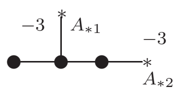
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{2} & & & \\ \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

(2) 

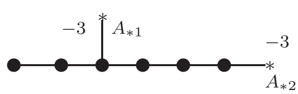
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccc} & & & & & \underline{3} & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & \underline{2} \end{array}$$

(3) 

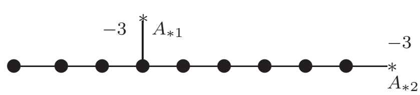
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccccccc} & & & & & & & \underline{3} & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 6 & 4 & \underline{2} \end{array}$$

(4) 

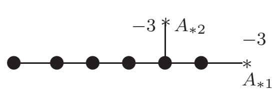
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & \underline{3} & & \\ 2 & 4 & 3 & \underline{2} \end{array}$$

(5) 

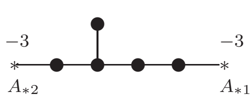
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & \underline{3} & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} \end{array}$$

(6) 

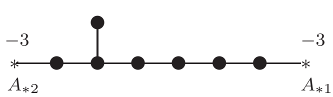
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccccc} & & & \underline{3} & & & & & & \\ 2 & 4 & 6 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2} \end{array}$$

(7) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & & \underline{2} & & \\ 1 & 2 & 3 & 4 & 5 & 4 & \underline{3} \end{array}$$

(8) 

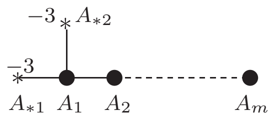
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & 3 & & & \\ \underline{2} & 4 & 6 & 5 & 4 & \underline{3} \end{array}$$

(9) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccc} & & 4 & & & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} \end{array}$$

Proof. (I) Assume that Γ_1 is of the form of case (1) in Theorem 4.2.

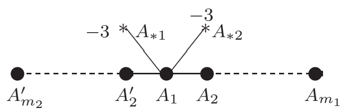
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{matrix} & 2 & & & \\ 3 & n_1 & n_2 & \ldots\ldots & n_m. \end{matrix}$$

As in the proof of Proposition 6.9, we have $m = 4$. If $m = 4$, then we are in case (1).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



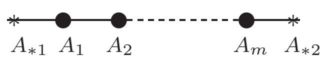
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{matrix} & 3 & 2 & \\ n'_{m_2}\ldots n'_2 & n_1 & & n_2\ldots n_{m_1}. \end{matrix}$$

As in the proof of Proposition 6.9, we have either $n_{m_1} = 3$, $n'_{m_2} = 2$ or $n_{m_1} = 4, n'_{m_2} = 1$.

If $n_{m_1} = 3$, $n'_{m_2} = 2$, then $3m_1 = 2m_2$ and $n_1 = 3m_1$. Since $-1 = A_{*2}^2 + 2 = A_{*2}\cdot(-K) = A_{*2}\cdot Z \geq 2(-3) + n_1 \Rightarrow n_1 = 3m_1 \leq 5$. Therefore $m_1 = 1$, $m_2 = \frac{3}{2}$, which contradicts that m_2 is an integer.

If $n_{m_1} = 4, n'_{m_2} = 1$, then $4m_1 = m_2$ and $n_1 = 4m_1$. The same argument as above shows that $4m_1 \leq 5$ i.e., $m_1 \leq \frac{5}{4}$. So $m_1 = 1$, then we are in case (1).

Consider A_{*1} and A_{*2} attaching an Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 3\ n_1\ n_2\ldots n_m\ 2.$$

Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have

$$\begin{cases} -2n_1 + 3 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.51}$$

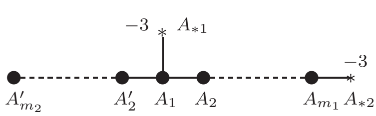
$$-2n_m + n_{m-1} + 2 = 0 \tag{6.52}$$

(6.51) implies

$$n_j = jn_1 - 3(j-1) \qquad 2 \leq j \leq m \tag{6.53}$$

(6.52) and (6.53) imply $n_1 = 3 - \frac{1}{m+1}$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{matrix} & 3 & & \\ n'_{m_2}\ldots n'_2 & n_1 & n_2\ldots n_{m_1} & 2. \end{matrix}$$

Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m_1$ and $A'_j\cdot Z = A'_j\cdot(-K) =$

$A_j'^2 + 2 = 0$, $2 \leq j \leq m_2$, set $n_1' = n_1$ and we have

$$\begin{cases} -2n_{m_1} + n_{m_1-1} + 2 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad \vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \tag{6.54}$$

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.55}$$

$$3 - 2n_1 + n_2 + n'_2 = 0 \tag{6.56}$$

(6.54) implies

$$n_j = (m_1 - j + 1)n_{m_1} - 2(m_1 - j), \qquad 1 \leq j \leq m_1 - 1 \tag{6.57}$$

(6.55) implies

$$n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \leq j \leq m_2 - 1 \tag{6.58}$$

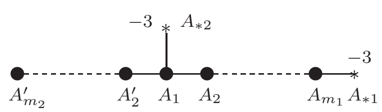
(6.57), (6.58) and $n'_1 = n_1$ imply

$$m_1 n_{m_1} - 2(m_1 - 1) = m_2 n'_{m_2} = n_1 \tag{6.59}$$

(6.56), (6.57) and (6.59) imply $n_{m_1} + n'_{m_2} = 5$. We have either $n_{m_1} = 4$, $n'_{m_2} = 1$ or $n_{m_1} = 3, n'_{m_2} = 2$.

If $n_{m_1} = 4$ and $n'_{m_2} = 1$, then (6.59) implies $m_2 = 2m_1 + 2 = n_1$. $-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 3(-3) + n_1$ implies $2m_1 + 2 = n_1 \leq 8$ i.e., $m_1 \leq 3$. If $m_1 = 1$, then $m_2 = 4$ and we are in case (1). If $m_1 = 2$, then $m_2 = 6$ and we are in case (2). If $m_1 = 3$, then $m_2 = 8$ and we are in case (3).

If $n_{m_1} = 3, n'_{m_2} = 2$, then (6.59) implies $m_1 + 2 = 2m_2 = n_1$. $-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 3(-3) + n_1$ implies $m_1 + 2 = n_1 \leq 8$ i.e., $m_1 = 2, 4$ or 6. If $m_1 = 2$, then $m_2 = 2$ and we are in case (4). If $m_1 = 4$, then $m_2 = 3$ and we are in case (5). If $m_1 = 6$, then $m_2 = 4$ and we are in case (6). Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup \Gamma_1} = n'_{m_2} \ldots n'_2 \ \overset{2}{n_1} \ n_2 \ldots n_{m_1} \ 3.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$ and $A'_j \cdot Z = A'_j \cdot (-K) =$

$A_j'^2 + 2 = 0$, $2 \leq j \leq m_2$, set $n_1' = n_1$ and we have

$$\begin{cases} -2n_{m_1} + n_{m_1-1} + 3 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \tag{6.60}$$

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.61}$$

$$2 - 2n_1 + n_2 + n'_2 = 0 \tag{6.62}$$

(6.60) implies

$$n_j = (m_1 - j + 1)n_{m_1} - 3(m_1 - j), \qquad 1 \leq j \leq m_1 - 1 \tag{6.63}$$

(6.61) implies

$$n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \leq j \leq m_2 - 1 \tag{6.64}$$

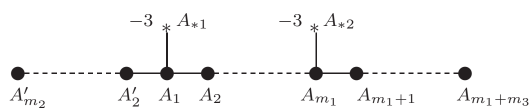
(6.63), (6.64) and $n'_1 = n_1$ imply

$$m_1 n_{m_1} - 3(m_1 - 1) = m_2 n'_{m_2} = n_1 \tag{6.65}$$

(6.62), (6.63) and (6.64) imply $n_{m_1} + n'_{m_2} = 5$. We have either $n_{m_1} = 4$, $n'_{m_2} = 1$ or $n_{m_1} = 3, n'_{m_2} = 2$.

If $n_{m_1} = 4$ and $n'_{m_2} = 1$, then (6.65) implies $m_2 = m_1 + 3 = n_1$. $-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 2(-3) + n_1$ implies $m_1 + 3 = n_1 \leq 5$ i.e., $m_1 \leq 2$. If $m_1 = 1$, then $m_2 = 4$ and we are in case (1). If $m_1 = 2$, then $m_2 = 5$ and we are in case (7). If $n_{m_1} = 3, n'_{m_2} = 2$, then (6.65) implies $3 = 2m_2 = n_1$ .Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2} \ldots n'_2 \ \overset{3}{n_1} \ \ n_2 \cdots \overset{2}{n_{m_1}} \ \ n_{m_1+1} \ldots n_{m_1+m_3}.$$

And $m_1 \geq 2$, $m_3 \geq 1$, $m_2 \geq 2$. By the same argument as before, we have the following equations

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.66}$$

$$-2n_1 + n_2' + n_2 + 3 = 0 \tag{6.67}$$

$$\begin{cases} -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 = 0 \\ \qquad \vdots \\ -2n_{m_1-2} + n_{m_1-3} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \end{cases} \tag{6.68}$$

$$-2n_{m_1} + n_{m_1-1} + n_{m_1+1} + 2 = 0 \tag{6.69}$$

$$\begin{cases} -2n_{m_1+1} + n_{m_1} + n_{m_1+2} = 0 \\ -2n_{m_1+2} + n_{m_1+1} + n_{m_1+3} = 0 \\ \qquad \vdots \\ -2n_{m_1+m_3-1} + n_{m_1+m_3-2} + n_{m_1+m_3} = 0 \\ -2n_{m_1+m_3} + n_{m_1+m_3-1} = 0 \end{cases} \tag{6.70}$$

(6.66) implies

$$n_j' = (m_2 - j + 1)n_{m_2}' \qquad 1 \leq j \leq m_2 \tag{6.71}$$

(6.67) and (6.71) imply

$$n_2 = (m_2 + 1)n_{m_2}' - 3 \tag{6.72}$$

(6.72) and (6.68) imply

$$n_j = (m_2 + j - 1)n_{m_2}' - 3(j-1), \qquad 2 \leq j \leq m_1 \tag{6.73}$$

(6.70) implies

$$n_{m_1+j} = (m_3 - j + 1)n_{m_1+m_3}, \qquad 0 \leq j \leq m_3 \tag{6.74}$$

(6.73) and (6.74) imply

$$n_{m_1} = (m_2 + m_1 - 1)n_{m_2}' - 3(m_1 - 1) = (m_3 + 1)n_{m_1+m_3} \tag{6.75}$$

(6.73), (6.74) and (6.69) imply

$$(m_2 + m_1)n_{m_2}' - 3m_1 = m_3 n_{m_1+m_3} + 2 \tag{6.76}$$

(6.75) and (6.76) imply $n_{m_2}' + n_{m_1+m_3} = 5$. Therefore we have four possible cases: (1)$n_{m_2}' = 2$, $n_{m_1+m_3} = 3$; (2) $n_{m_2}' = 3, n_{m_1+m_3} = 2$; (3) $n_{m_2}' = 4$, $n_{m_1+m_3} = 1$; (4) $n_{m_2}' = 1, n_{m_1+m_3} = 4$.

If $n_{m_2}' = 2$ and $n_{m_1+m_3} = 3$, then $2m_2 = m_1 + 3m_3 + 2$ by (6.76) and $n_1 = 2m_2$ by (6.71). Since $-1 = A_{*1}^2 + 2 = A_{*1}(-K) = A_{*1} \cdot Z \geq 3(-3) + n_1$, we have $m_2 \leq 4$. Hence $m_1 + 3m_3 \leq 6$. Since $m_1 \geq 2$, we have $m_3 = 1$, and $m_1 = 3$, $m_2 = 4$. And $-1 = A_{*2}^2 + 2 = A_{*2}(-K) = A_{*2} \cdot Z \geq 2(-3) + n_{m_1}$. If $m_3 = 1$, $m_1 = 3$, $m_2 = 4$, then by (6.72) $n_{m_1} = 6$ and we have $-1 \geq -6 + 6 = 0$. Contradiction!

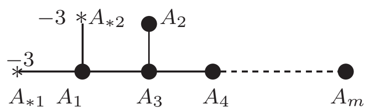
If $n'_{m_2} = 3, n_{m_1+m_3} = 2$, then $n_1 = 3m_2$ and $3m_2 = 2(m_3 + 1)$ by (6.76). Since $-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 3(-3) + n_1$, we have $3m_2 \leq 8$ which implies $m_2 = 2$ and $m_3 = 2$. And $-1 = A_{*2}^2 + 2 = A_{*2}(-K) = A_{*2} \cdot Z \geq 2(-3) + n_{m_1}$. If $m_3 = 2$ and $m_2 = 2$, then by (6.72) $n_{m_1} = 6$ and we we have $-1 \geq -6 + 6 = 0$. Contradiction!

If $n'_{m_2} = 4$, $n_{m_1+m_3} = 1$, then $n_1 = 4m_2$ and $4m_2 = m_3 + 2 - m_1$ by (6.76). Since $-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 3(-3) + n_1$, we have $4m_2 \leq 8$ which implies $m_2 = 2$ and $n_{m_1} = m_1 + 7$ by (6.75). And $-1 = A_{*2}^2 + 2 = A_{*2}(-K) = A_{*2} \cdot Z \geq 2(-3) + n_{m_1}$. So $m_1 \leq -2$. This case cannot occur.

If $n'_{m_2} = 1, n_{m_1+m_3} = 4$, then $n_{m_1} = 4(m_3 + 1)$ by (6.75). Since $-1 = A_{*2}^2 + 2 = A_{*2} \cdot (-K) = A_{*2} \cdot Z \geq 2(-3) + n_{m_1}$, we have $4m_3 \leq 1$. This contradicts to the condition $m_3 \geq 1$.

(II) Assume that Γ_1 is of the form $D_m (m \geq 4)$ of case (2) in Theorem 4.2.

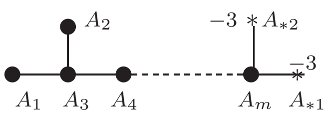
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & 2 & n_2 & & \\ 3 & n_1 & n_3 & n_4 \ldots n_m. \end{array}$$

As in the proof of Proposition 6.10, we have $n_m = \frac{5}{2}$, which contradicts to the fact that n_m is an integer.

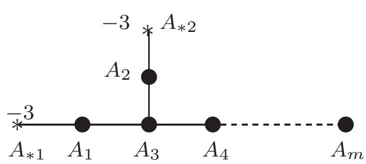
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & n_2 & & & 2 & \\ n_1 & n_3 & n_4 \ldots & & n_m & 3. \end{array}$$

As in the proof of Proposition 6.10, we have $n_1 = n_2 = \frac{5}{2}$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & & 2 & \\ & & n_2 & \\ 3 & n_1 & n_3 & n_4 \ldots n_m. \end{array}$$

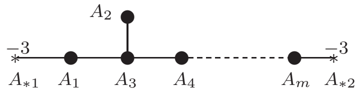
By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 3 + n_3 = 0 \\ -2n_2 + 2 + n_3 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.77}$$

$$-2n_3 + n_1 + n_2 + n_4 = 0. \tag{6.78}$$

From (6.77) we get $n_1 - n_2 = \frac{1}{2}$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{array}{cccccc} & & n_2 & & & \\ 3 & n_1 & n_3 & n_4 \ldots n_m & 2. \end{array}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 3 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.79}$$

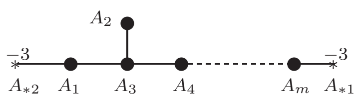
$$-2n_m + n_{m-1} + 2 = 0 \tag{6.80}$$

(6.79) implies

$$n_1 = \frac{3}{2} + n_2, \tag{6.81}$$

Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{array}{cccccc} & & n_2 & & & \\ 2 & n_1 & n_3 & n_4 \ldots n_m & 3. \end{array}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 2 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.82}$$

$$-2n_m + n_{m-1} + 3 = 0 \tag{6.83}$$

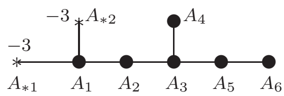
(6.82) implies

$$n_1 = 1 + n_2, n_j = 2n_2 - (j-3), 3 \leq j \leq m \tag{6.84}$$

(6.83) and (6,84) imply $n_2 = \frac{m+1}{2}$. In particular m is odd. Since$-1 = A_{*2}^2 + 2 = A_{*2} \cdot (-K) = A_{*2} \cdot Z \geq 2(-3) + n_1$, we have $4 \leq m \leq 7$. If $m = 5, 7$, then we are in case (8) and case (9) respectively.

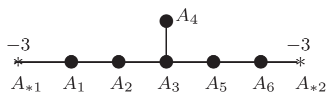
(III) Assume that Γ_1 is of the form E_6 of case (3) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 2 & & n_4 & & \\ 3 & n_1 & n_2 & n_3 & n_5 & n_6. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible. Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A*1\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & n_4 & & & \\ 3 & n_1 & n_2 & n_3 & n_5 & n_6 & 2. \end{array}$$

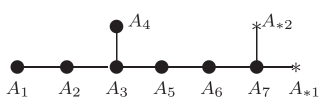
By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 3 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + 2 = 0 \end{cases}$$

We have $n_6 = \frac{14}{3}$. Contradiction!

(IV) Assume that Γ_1 is of the form E_7 of case (4) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_7 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & n_4 & & & 2 & \\ n_1 & n_2 & n_3 & n_5 & n_6 & n_7 & 3. \end{array}$$

By the same argument as before, we have the following equations

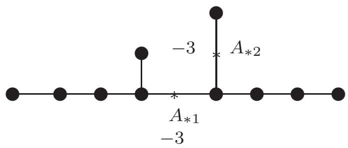
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 5 = 0 \end{cases} \tag{6.85}$$

We get $n_7 = \frac{15}{2}$. Contradiction!

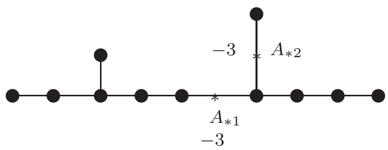
(V) Assume that Γ_1 is of the form E_8 of case (5) in Theorem 4.2. This case cannot happen because $A_{*1} \cdot Z_1 \geq 2$. □

Theorem 6.18. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (3) of Proposition 6.2 holds,*

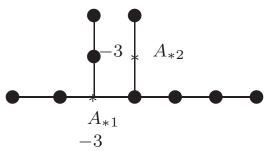
i.e., there exists two effective components A_{*1} *and* A_{*2} *with* $A_{*1}^2 = -3 = A_{*2}^2$ *and* $z_{*1} = 3, z_{*2} = 2$*, then the weighted dual graph* Γ *of the exceptional set is one of the following forms.*

$(1a_1)$ 

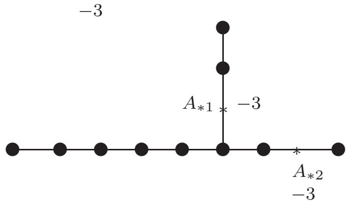
$$Z = \begin{array}{ccccccccc} &&&&&1&&& \\ &&&2&&\underline{2}&&& \\ 1&2&3&4&\underline{3}&4&3&2&1 \end{array}$$

$(1a_2)$ 

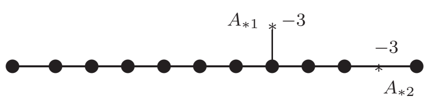
$$Z = \begin{array}{cccccccccc} &&&&&&1&&& \\ &&3&&&&\underline{2}&&& \\ 2&4&6&5&4&\underline{3}&4&3&2&1 \end{array}$$

$(1b)$ 

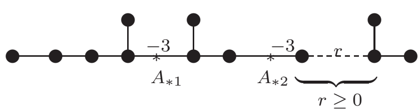
$$Z = \begin{array}{ccccccc} &&1&1&&& \\ &&2&\underline{2}&&& \\ 1&2&\underline{3}&4&3&2&1 \end{array}$$

(2) 

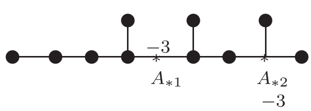
$$Z = \begin{array}{ccccccccc} &&&&&1&&& \\ &&&&&2&&& \\ &&&&&\underline{3}&&& \\ 1&2&3&4&5&6&4&\underline{2}&1 \end{array}$$

(3) 

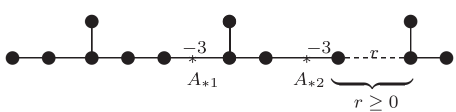
$$Z = \begin{array}{cccccccccccc} &&&&&&&\underline{3}&&&& \\ 1&2&3&4&5&6&7&8&6&4&\underline{2}&1 \end{array}$$

$(4a_1)$ 

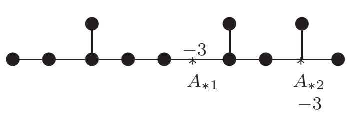
$$Z = \begin{array}{cccccccccccc} &&&2&&2&&&&&1& \\ 1&2&3&4&\underline{3}&4&3&\underline{2}&2&\dots&2&1 \end{array}$$

$(4a_2)$ 

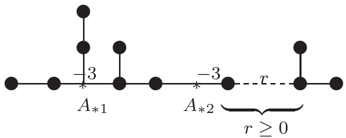
$$Z = \begin{array}{ccccccccc} &&&2&&2&&1& \\ 1&2&3&4&\underline{3}&4&3&\underline{2}&1 \end{array}$$

$(4a_3)$ 

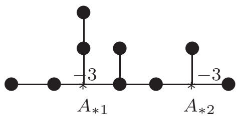
$$Z = \begin{array}{cccccccccccccc} &&3&&&&2&&&&&1& \\ 2&4&6&5&4&\underline{3}&4&3&\underline{2}&2&\dots&2&1 \end{array}$$

$(4a_4)$ 

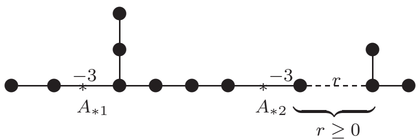
$$Z = \begin{array}{cccccccccc} &&3&&&&2&&1& \\ 2&4&6&5&4&\underline{3}&4&3&\underline{2}&1 \end{array}$$

$(4b_1)$ 

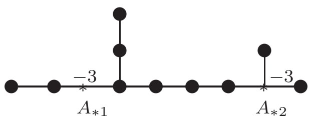
$$\begin{array}{rcccccccccc} & & & 1 & & & & & & & \\ & & & 2 & 2 & & & & & 1 & \\ Z = & 1 & 2 & \underline{3} & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1 \end{array}$$

$(4b_2)$ 

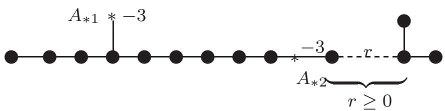
$$\begin{array}{rccccccc} & & & 1 & & & & \\ & & & 2 & 2 & & 1 & \\ Z = & 1 & 2 & \underline{3} & 4 & 3 & \underline{2} & 1 \end{array}$$

$(5a_1)$ 

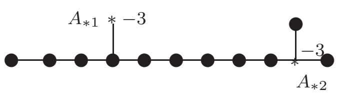
$$\begin{array}{rcccccccccccc} & & & & 2 & & & & & & & & \\ & & & & 4 & & & & & & & 1 & \\ Z = & 1 & 2 & \underline{3} & 6 & 5 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1 \end{array}$$

$(5a_2)$ 

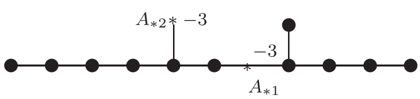
$$\begin{array}{rccccccccc} & & & & 2 & & & & & \\ & & & & 4 & & & & 1 & \\ Z = & 1 & 2 & \underline{3} & 6 & 5 & 4 & 3 & \underline{2} & 1 \end{array}$$

$(6a_1)$ 

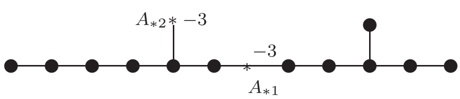
$$\begin{array}{rcccccccccccccc} & & & & \underline{3} & & & & & & & & & 1 & \\ Z = & 2 & 4 & 6 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1 \end{array}$$

$(6a_2)$ 

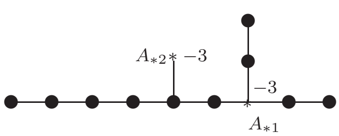
$$\begin{array}{rccccccccccc} & & & & \underline{3} & & & & & & 1 & \\ Z = & 2 & 4 & 6 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2} & 1 \end{array}$$

$(7a_1)$ 

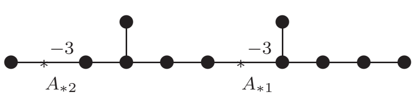
$$\begin{array}{rccccccccccc} & & & & & \underline{2} & & & 2 & & & \\ Z = & 1 & 2 & 3 & 4 & 5 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

$(7a_2)$ 

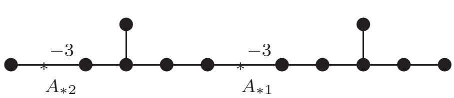
$$\begin{array}{rcccccccccccc} & & & & & \underline{2} & & & & & 3 & & \\ Z = & 1 & 2 & 3 & 4 & 5 & 4 & \underline{3} & 4 & 5 & 6 & 4 & 2 \end{array}$$

$(7b)$ 

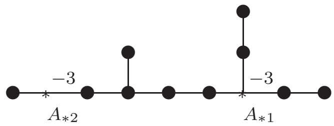
$$\begin{array}{rccccccccc} & & & & & & & 1 & & \\ & & & & & \underline{2} & & 2 & & \\ Z = & 1 & 2 & 3 & 4 & 5 & 4 & \underline{3} & 2 & 1 \end{array}$$

$(8a_1)$ 

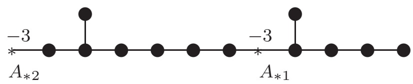
$$\begin{array}{rccccccccccc} & & & & 3 & & & & 2 & & & \\ Z = & 1 & \underline{2} & 4 & 6 & 5 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

$(8a_2)$ 

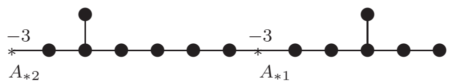
$$\begin{array}{rcccccccccccc} & & & & 3 & & & & & & 3 & & \\ Z = & 1 & \underline{2} & 4 & 6 & 5 & 4 & \underline{3} & 4 & 5 & 6 & 4 & 2 \end{array}$$

(8b) 

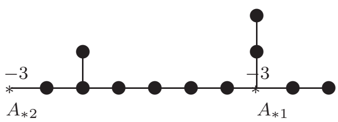
$$\begin{array}{lccccccccc} & & & & & & & 1 & & \\ & & & & 3 & & & 2 & & \\ Z = & 1 & \underline{2} & 4 & 6 & 5 & 4 & \underline{3} & 2 & 1 \end{array}$$

($9a_1$) 

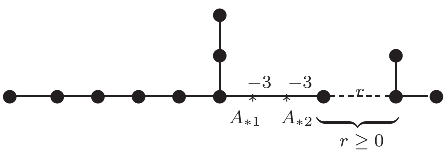
$$\begin{array}{lcccccccccccc} & & & 4 & & & & & & 2 & & & \\ Z = & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

($9a_2$) 

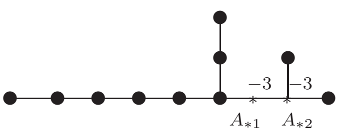
$$\begin{array}{lccccccccccccc} & & & 4 & & & & & & & & 3 & & \\ Z = & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 4 & 5 & 6 & 4 & 2 \end{array}$$

(9b) 

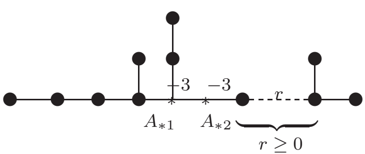
$$\begin{array}{lcccccccccc} & & & & & & & & 1 & & \\ & & & 4 & & & & & 2 & & \\ Z = & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 2 & 1 \end{array}$$

($10a_1$) 

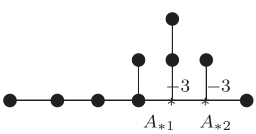
$$\begin{array}{lcccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & 4 & & & & & 1 & \\ Z = & 1 & 2 & 3 & 4 & 5 & 6 & \underline{3} & \underline{2} & 2 & \dots & 2 & 1 \end{array}$$

($10a_2$) 

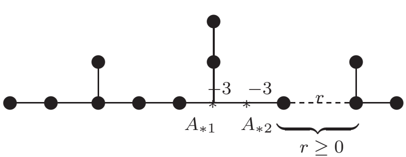
$$\begin{array}{lccccccccc} & & & & & & 2 & & & \\ & & & & & & 4 & & 1 & \\ Z = & 1 & 2 & 3 & 4 & 5 & 6 & \underline{3} & \underline{2} & 1 \end{array}$$

($10b_1$) 

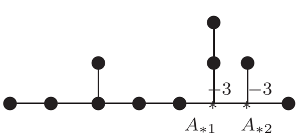
$$\begin{array}{lcccccccccc} & & & & & 1 & & & & & \\ & & & & 2 & 2 & & & & 1 & \\ Z = & 1 & 2 & 3 & 4 & \underline{3} & \underline{2} & 2 & \dots & 2 & 1 \end{array}$$

($10b_2$) 

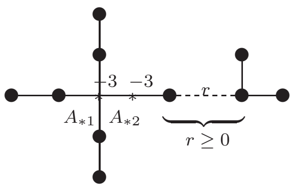
$$\begin{array}{lccccccc} & & & & & 1 & & \\ & & & & 2 & 2 & 1 & \\ Z = & 1 & 2 & 3 & 4 & \underline{3} & \underline{2} & 1 \end{array}$$

($10b_3$) 

$$\begin{array}{lccccccccccc} & & & & & & 1 & & & & & \\ & & & 3 & & & 2 & & & & 1 & \\ Z = & 2 & 4 & 6 & 5 & 4 & \underline{3} & \underline{2} & 2 & \dots & 2 & 1 \end{array}$$

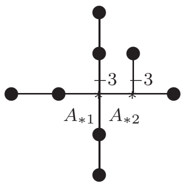
($10b_4$) 

$$\begin{array}{lcccccccc} & & & & & & 1 & & \\ & & & 3 & & & 2 & 1 & \\ Z = & 2 & 4 & 6 & 5 & 4 & \underline{3} & \underline{2} & 1 \end{array}$$



$(10c_1)$

$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & 1 & \\ 1 & 2 & \underline{3} & \underline{2} & 2 & \ldots & 2 & 1 \\ & & 2 & & & & & \\ & & 1 & & & & & \end{array}$$



$(10c_2)$

$$Z = \begin{array}{ccccc} & & 1 & & \\ & & 2 & 1 & \\ 1 & 2 & \underline{3} & \underline{2} & 1 \\ & & 2 & & \\ & & 1 & & \end{array}$$

Proof. Since the singularity is minimally elliptic, $A_{*i}^2 = -3$, $z_{*1} = 3, z_{*2} = 2$, we have

$$A_{*1} \cdot (Z - 3A_{*1}) = -A_{*1} \cdot (K + 3A_{*1}) = A_{*1}^2 + 2 - 3A_{*1}^2 = 8. \tag{6.86}$$

$$A_{*2} \cdot (Z - 2A_{*2}) = -A_{*2} \cdot (K + 2A_{*2}) = A_{*2}^2 + 2 - 2A_{*2}^2 = 5. \tag{6.87}$$

Let Γ' be the graph obtained by deleting A_{*1} and A_{*2} from Γ. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \ldots, Z_m$ respectively. (6.86) and (6.87) imply that

$$\sum_{j=1}^{m} A_{*1} \cdot Z \Big|_{\Gamma_j} = 8, \tag{6.88}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z \Big|_{\Gamma_j} = 5. \tag{6.89}$$

Since we have two effective component, by Corollary 6.4 we have

$$A_{*i} \cdot Z_j = 1 \qquad \text{for } i = 1, 2 \text{ and } 1 \le j \le m. \tag{6.90}$$

Consider first that A_{*1} and A_{*2} do not meet. Then Proposition 6.17 applies. In case (1) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 4 + 4$ and the decomposition (6.89) of A_{*2} is $5 = 4 + 1$, then we are in case $(1a_1)$ and case $(1a_2)$. If the decomposition (6.88) at A_{*1} is $8 = 4 + 2 + 2$ and the decomposition of (6.89) at A_{*2} is $5 = 4 + 1$, then we are in case $1(b)$.

In case (2) of Proposition 6.17, the decomposition (6.88) at A_{*1} must be $8 = 6 + 2$ and the decomposition (6.89) of A_{*2} must be $5 = 4 + 1$, then we are in case (2).

In case (3) of Proposition 6.17, the decomposition (6.88) at A_{*1} must be $8 = 8 + 0$ and the decomposition (6.89) of A_{*2} must be $5 = 4 + 1$, then we are in case (3).

Here I will use one graph to cover three graphs: (2), (6) and (10) in Proposition 6.16, in which $r = 0, 1$ or $r \ge 2$ respectively.

In case (4) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 4+4$ and the decomposition (6.89) of A_{*2} is $5 = 3+2$, then we are in case $(4a_1)$ and case $(4a_3)$. If the decomposition (6.88) at A_{*1} is $8 = 4+4$ and the decomposition of (6.89) at A_{*2} is $5 = 3+1+1$, then we are in case $4(a_2)$ and case $(4a_4)$. If the decomposition (6.88) at A_{*1} is $8 = 4+2+2$ and the decomposition of (6.89) at A_{*2} is $5 = 3+2$, then we are in case $4(b_1)$. If the decomposition (6.88) at A_{*1} is $8 = 4+2+2$ and the decomposition of (6.89) at A_{*2} is $5 = 3+1+1$, then we are in case $4(b_2)$.

In case (5) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 6+2$ and the decomposition (6.89) of A_{*2} is $5 = 3+2$, then we are in case $(5a_1)$. If the decomposition (6.88) at A_{*1} is $8 = 6+2$ and the decomposition of (6.89) at A_{*2} is $5 = 3+1+1$, then we are in case $5(a_2)$.

In case (6) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 8+0$ and the decomposition (6.89) of A_{*2} is $5 = 3+2$, then we are in case $(6a_1)$. If the decomposition (6.88) at A_{*1} is $8 = 8+0$ and the decomposition of (6.89) at A_{*2} is $5 = 3+1+1$, then we are in case $6(a_2)$.

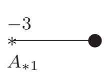
In case (7) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 4+4$ and the decomposition (6.89) of A_{*2} is $5 = 5+0$, then we are in case $(7a_1)$ and case $(7a_2)$. If the decomposition (6.88) at A_{*1} is $8 = 4+2+2$ and the decomposition of (6.89) at A_{*2} is $5 = 5+0$, then we are in case $7(b)$.

In case (8) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 4+4$ and the decomposition (6.89) of A_{*2} is $5 = 4+1$, then we are in case $(8a_1)$ and case $(8a_2)$. If the decomposition (6.88) at A_{*1} is $8 = 4+2+2$ and the decomposition of (6.89) at A_{*2} is $5 = 4+1$, then we are in case $8(b)$.

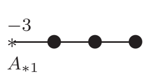
In case (9) of Proposition 6.17, if the decomposition (6.88) at A_{*1} is $8 = 4+4$ and the decomposition (6.89) of A_{*2} is $5 = 5+0$, then we are in case $(9a_1)$ and $(9a_2)$. If the decomposition (6.88) at A_{*1} is $8 = 4+2+2$ and the decomposition of (6.89) at A_{*2} is $5 = 5+0$, then we are in case $9(b)$.

We next consider the case $A_{*1} \cdot A_{*2} > 0$. Since the singularity is minimally elliptic and $z_{*1} = 3, z_{*2} = 2$, it follows that $A_{*1} \cdot A_{*2} = 1$ from (6.87). Then we are in case $10(a_1)$ - $10(c_2)$. ☐

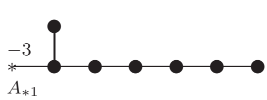
PROPOSITION 6.19. *Let Γ be the minimal resolution graph of a minimallyl elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = 1$. If the coefficient z_{*1} of A_{*1} in Z is 4 and $A_{*1}^2 = -3$, then $A_{*1} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup \Gamma_1$ must be one of the following form.*

(1) 

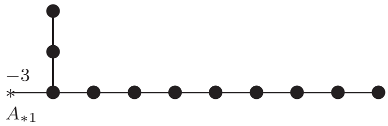
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\quad 2$$

(2) 

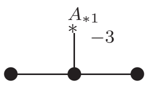
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\quad 3\quad 2\quad 1$$

(3) 

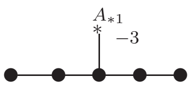
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\quad \begin{array}{l} 3 \\ 6\ 5\ 4\ 3\ 2\ 1 \end{array}$$

(4) 

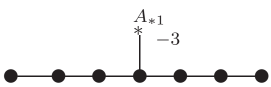
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\quad \begin{array}{c}3\\6\\9\end{array}\quad 8\quad 7\quad 6\quad 5\quad 4\quad 3\quad 2\quad 1$$

(5) 

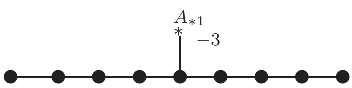
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\quad \begin{array}{c}\underline{4}\\4\end{array}\quad 2$$

(6) 

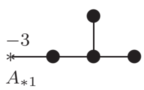
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\quad 4\quad \begin{array}{c}\underline{4}\\6\end{array}\quad 4\quad 2$$

(7) 

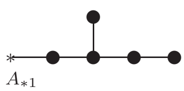
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\quad 4\quad 6\quad \begin{array}{c}\underline{4}\\8\end{array}\quad 6\quad 4\quad 2$$

(8) 

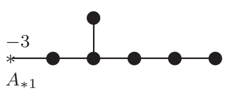
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\quad 4\quad 6\quad 8\quad \begin{array}{c}\underline{4}\\10\end{array}\quad 8\quad 6\quad 4\quad 2$$

(9) 

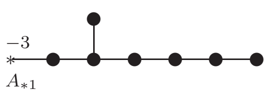
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 4\ \begin{array}{c}2\\4\end{array}\ 2$$

(10) 

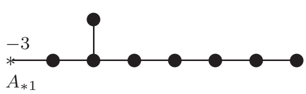
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 5\ \begin{array}{c}3\\6\end{array}\ 4\ 2$$

(11) 

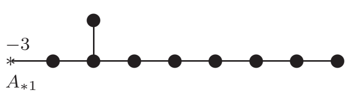
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 6\ \begin{array}{c}4\\8\end{array}\ 6\ 4\ 2$$

(12) 

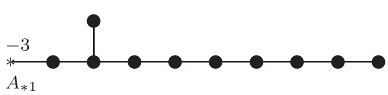
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 7\ \begin{array}{c}5\\10\end{array}\ 8\ 6\ 4\ 2$$

(13) 

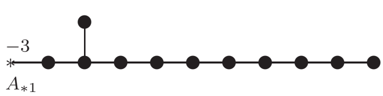
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 8\ \begin{array}{c}6\\12\end{array}\ 10\ 8\ 6\ 4\ 2$$

(14) 

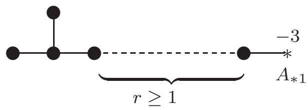
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 9\ \begin{array}{c}7\\14\end{array}\ 12\ 10\ 8\ 6\ 4\ 2$$

(15) 

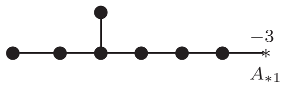
$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 10\ \begin{array}{c}8\\16\end{array}\ 14\ 12\ 10\ 8\ 6\ 4\ 2$$

(16) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = \underline{4}\ 11\ \begin{array}{c}9\\18\end{array}\ 16\ 14\ 12\ 10\ 8\ 6\ 4\ 2$$

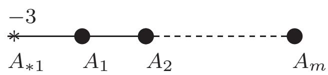
(17) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = 2\ \overset{2}{4}\ 4\ \ldots\ 4\ \underline{4}$$

(18) 

$$Z\Big|_{A_{*1}\cup\Gamma_1} = 4\ 8\ \overset{6}{12}\ \ 10\ 8\ 6\ \underline{4}$$

Proof. The proof is similar to those of Proposition 6.9, 6.10 and 6.11. Consider A_{*1} attaching on Γ_1 in the following form



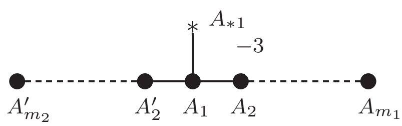
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 4 \quad n_1 \quad n_2 \ldots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 4 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \leq i \leq m \\ n_m = \frac{4}{m+1} \end{cases}.$$

Therefore $m = 1$ or $m = 3$ and we are in case (1) and case (2).

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n'_{m_2} \ldots n'_2\ \overset{4}{n_1}\ n_2 \ldots n_{m_1}.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \leq j \leq m_2$, we have the following system of equations

$$\begin{cases} -2n_{m_1} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \quad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \qquad \begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \quad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.91}$$

$$4 - 2n_1 + n_2 + n'_2 = 0 \tag{6.92}$$

(6.91) implies

$$n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \leq i \leq m_1 \tag{6.93}$$

$$n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \leq j \leq m_2 \tag{6.94}$$

$$m_1 n_{m_1} = m_2 n'_{m_2} \tag{6.95}$$

Putting (6.93) and (6.94) into (6.92), we get

$$\begin{aligned} 0 &= 4 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2} \\ &= 4 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} = 0 \end{aligned} \tag{6.96}$$

(6.95) and (6.96) imply

$$n_{m_1} + n'_{m_2} = 4 \tag{6.97}$$

(6.97) implies that $n_{m_1} = 1$, $n'_{m_2} = 3$ or $n_{m_1} = 2$, $n_{m_2} = 2$

Case 1. $n_{m_1} = 1$ and $n'_{m_2} = 3$. By (6.95), we have $m_1 = 3m_2$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 4(-3) + n_1 = -12 + m_1 \\ &\Rightarrow m_1 \leq 11. \end{aligned}$$

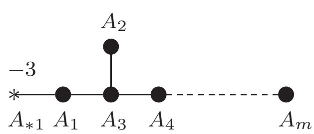
If $m_1 = 6$, or 9, then we are in case (3) case (4). respectively in the statement of the proposition.

Case 2. $n_{m_1} = 2$ and $n'_{m_2} = 2$. By (6.95), we have $m_1 = m_2$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 4(-3) + n_1 = -12 + 2m_1 \\ &\Rightarrow m_1 \leq \frac{11}{2}. \end{aligned}$$

If $m_1 = 2$, 3, 4, 5, then we are in case (5) case (6) case (7) case (8) respectively in the statement of the proposition.

Consider A_{*1} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup\Gamma_1} = 4\ n_1\ \begin{matrix} n_2 \\ n_3 \end{matrix}\ n_4\ \ldots\ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have the following system of equations

$$-2n_1 + 4 + n_3 = 0 \tag{6.98}$$

$$\begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.99}$$

(6.99) implies

$$n_1 = \frac{m}{2} n_m,\ n_2 = \frac{m-2}{2} n_m,\ n_j = (m - j + 1)n_m,\ 3 \leq j \leq m \tag{6.100}$$

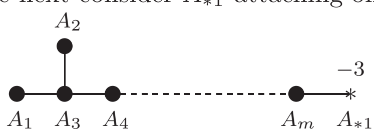
(6.98) and (6.100) imply $n_m = 2$.

Therefore. $n_1 = m$. Observe that

$$\begin{aligned} &-1 = A_{*1}^2 + 2 = A_{*1} \cdot (-K) = A_{*1} \cdot Z \geq 4(-3) + n_1 = -12 + m \\ &\Rightarrow m \leq 11. \end{aligned}$$

If m=4, 5, 6, 7, 8, 9, 10, 11, then we are in case (9), case (10), case (11), case (12), case (13), case (14), case (15), case (16).

We next consider A_{*1} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1 \ \begin{matrix} n_2 \\ n_3 \end{matrix} \ n_4 \ \ldots \ n_m \ 4.$$

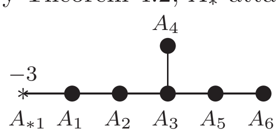
Since $A_i \cdot Z = 0$, $1 \leq i \leq m$, we have the following system of equations

$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.101}$$

$$-2n_m + n_{m-1} + 4 = 0 \tag{6.102}$$

(6.101) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.102), we know that $n_1 = 2$. Then we are in case (17).

By Theorem 4.2, A_* attaching on E_6 must be of the following form.



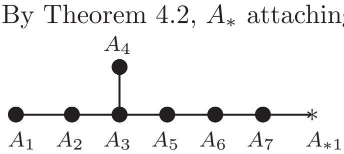
$$Z\Big|_{A_{*1}\cup\Gamma_1} = 4 \ n_1 \ n_2 \ \begin{matrix} n_4 \\ n_3 \end{matrix} \ n_5 \ n_6$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \leq i \leq 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 4 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = \frac{8}{3}$, which contradicts the fact that n_6 is an integer.

By Theorem 4.2, A_* attaching on E_7 must be of the following form.



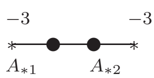
$$Z\Big|_{A_{*1}\cup\Gamma_1} = n_1 \ n_2 \ \begin{matrix} n_4 \\ n_3 \end{matrix} \ n_5 \ n_6 \ n_7 \ 4$$

Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \leq i \leq 7$, we have the following system of equations.

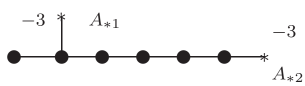
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 4 = 0 \end{cases}$$

We get $n_1 = 4$, then we are in case (18). By Theorem 4.2, A_{*1} cannot attach on E_8 because $A_{*1} \cdot Z_1 \geq 2$. $\square$

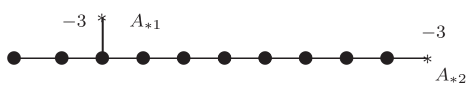
Proposition 6.20. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients $z_{*1} = 4$, $z_{*2} = 1$ and $A_{*1}^2 = A_{*2}^2 = -3$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following forms.*

(1) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{4}\ 3\ 2\ \underline{1}$$

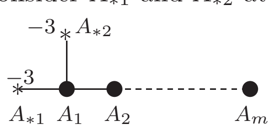
(2) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & \underline{4} & & & & & \\ 3 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(3) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccccccc} & & \underline{4} & & & & & & & & \\ 3 & 6 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. (I) Assume that Γ_1 is of the form of case (1) in Theorem 4.2. Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



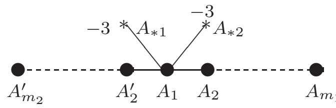
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & & & \\ 4 & n_1 & n_2 & \ldots\ldots & n_m. \end{array}$$

Observe that

$$\begin{aligned} & -1 = A_{*2}^2 + 2 = A_{*2} \cdot (-K) = A_{*2} \cdot Z \geq (-3) + n_1 = -3 + m \\ & \Rightarrow m \leq 2. \end{aligned}$$

As in the proof of Proposition 6.9, we have $m = 4$, which contradicts with the fact that $m \le 2$.

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.

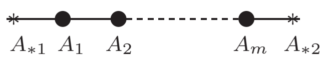


$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2 \overset{4}{\quad} n_1 \overset{1}{\quad} n_2\dots n_{m_1}.$$

As in the proof of Proposition 6.9, we have either $n_{m_1} = 3$, $n'_{m_2} = 2$ or $n_{m_1} = 4, n'_{m_2} = 1$.

If $n_{m_1} = 3$, $n'_{m_2} = 2$, then $3m_1 = 2m_2$ and $n_1 = 3m_1$. Since $-1 = A_{*2}^2 + 2 = A_{*2}\cdot(-K) = A_{*2}\cdot Z \ge (-3) + n_1 \Rightarrow n_1 = 3m_1 \le 2$. Therefore $m_1 \le \frac{3}{2}$, which contradicts with the fact that $m \ge 1$. If $n_{m_1} = 4, n'_{m_2} = 1$, then $4m_1 = m_2$ and $n_1 = 4m_1$. The same argument as above shows that$4m_1 \le 2$ i.e., $m_1 \le \frac{1}{2}$. Contradiction!

Consider A_{*1} and A_{*2} attaching an Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 4\ n_1\ n_2\dots n_m\ 1.$$

Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have

$$\begin{cases} -2n_1 + 4 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.103}$$

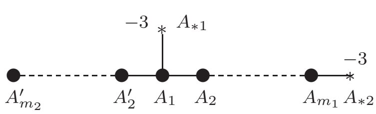
$$-2n_m + n_{m-1} + 1 = 0 \tag{6.104}$$

(6.103) implies

$$n_j = jn_1 - 4(j-1) \qquad 2 \le j \le m \tag{6.105}$$

(6.104) and (6.105) imply $n_1 = 3$. We are in case (1).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2\ \overset{4}{n_1}\ n_2\dots n_{m_1}\ 1.$$

Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and $A'_j\cdot Z = A'_j\cdot(-K) = A_j'^2 + 2 = 0$, $2 \le j \le m_2$, we have

$$\begin{cases} -2n_{m_1} + n_{m_1-1} + 1 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \tag{6.106}$$

$$\begin{cases} -2n'_{m_2}+n'_{m_2-1}=0 \\ -2n'_{m_2-1}+n'_{m_2-2}+n'_{m_2}=0 \\ \qquad\vdots \\ -2n'_3+n'_2+n'_4=0 \\ -2n'_2+n_1+n'_3=0 \end{cases} \tag{6.107}$$

$$4-2n_1+n_2+n'_2=0 \tag{6.108}$$

(6.106) implies

$$n_j=(m_1-j+1)n_{m_1}-(m_1-j), \qquad 1\le j\le m_1-1 \tag{6.109}$$

(6.107) implies

$$n'_j=(m_2-j+1)n'_{m_2}, \qquad 1\le j\le m_2-1 \tag{6.110}$$

(6.109) and (6.110) imply

$$m_1n_{m_1}-(m_1-1)=m_2n'_{m_2}=n_1 \tag{6.111}$$

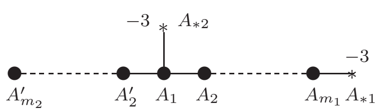
(6.108), (6.109) and (6.110) imply $n_{m_1}+n'_{m_2}=5$.

Observe that $-1=A_{*2}^2+2=A_{*2}\cdot(-K)=A_{*2}\cdot Z\ge(-3)+n_{m_1}$ implies $n_{m_1}\le 2$. So We have two cases:(1) $n_{m_1}=2$, $n'_{m_2}=3$; (2) $n_{m_1}=1, n'_{m_2}=4$

If $n_{m_1}=1$ and $n'_{m_2}=4$. Then $n_1=1=4m_2$, which contradicts with the fact that $m_2\ge 2$.

If $n_{m_1}=2, n'_{m_2}=3$, then (6.111) implies $m_1+1=3m_2=n_1$. $-1=A_{*1}^2+2=A_{*1}\cdot(-K)=A_{*1}\cdot Z\ge 4(-3)+n_1$implies $3m_2=n_1\le 11$ i.e., $m_2=2,3$. If $m_2=2$, then $m_1=5$ and we are in case (2). If $m_2=3$, then $m_1=8$ and we are in case (3).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1}=n'_{m_2}\dots n'_2\ \overset{1}{n_1}\ \ n_2\dots n_{m_1}\ 4.$$

Since $A_i\cdot Z=A_i\cdot(-K)=A_i^2+2=0$, $1\le i\le m$ and $A'_j\cdot Z=A'_j\cdot(-K)=A_j^{'2}+2=0$, $2\le j\le m_2$, we have

$$\begin{cases} -2n_{m_1}+n_{m_1-1}+4=0 \\ -2n_{m_1-1}+n_{m_1-2}+n_{m_1}=0 \\ \qquad\vdots \\ -2n_3+n_2+n_4=0 \\ -2n_2+n_1+n_3=0 \end{cases} \tag{6.112}$$

$$\begin{cases} -2n'_{m_2}+n'_{m_2-1}=0 \\ -2n'_{m_2-1}+n'_{m_2-2}+n'_{m_2}=0 \\ \qquad\vdots \\ -2n'_3+n'_2+n'_4=0 \\ -2n'_2+n_1+n'_3=0 \end{cases} \tag{6.113}$$

(6.114) $$1 - 2n_1 + n_2 + n_2' = 0$$

(6.112) implies

(6.115) $$n_j = (m_1 - j + 1)n_{m_1} - 4(m_1 - j), \qquad 1 \le j \le m_1 - 1$$

(6.113) implies

(6.116) $$n_j' = (m_2 - j + 1)n_{m_2}', \qquad 1 \le j \le m_2 - 1$$

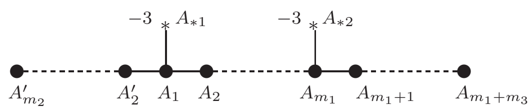
(6.115) and (6.116) imply

(6.117) $$m_1 n_{m_1} - 4(m_1 - 1) = m_2 n_{m_2}' = n_1$$

(6.114), (6.115) and (6.116) imply $n_{m_1} + n_{m_2}' = 5$. Observe that $-1 = A_{*2}^2 + 2 = A_{*2} \cdot (-K) = A_{*2} \cdot Z \ge (-3) + n_1$, we get $n_1 \le 2$. We have either $n_{m_1} = 4$, $n_{m_2}' = 1$ or $n_{m_1} = 3, n_{m_2}' = 2$.

If $n_{m_1} = 4$ and $n_{m_2}' = 1$, then $n_1 = 4 > 2$. Contradiction! If $n_{m_1} = 3, n_{m_2}' = 2$, then $4 - m_1 = 2m_2 = n_1 \le 2$. But $m_2 \ge 2$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2} \ldots n'_2 \;\overset{4}{n_1}\; n_2 \cdots \overset{1}{n_{m_1}} \; n_{m_1+1} \ldots n_{m_1+m_3}.$$

And $m_1 \ge 2$, $m_3 \ge 1$, $m_2 \ge 2$. By the same argument as before, we have the following equations

(6.118) $$\begin{cases} -2n_{m_2}' + n_{m_2-1}' = 0 \\ -2n_{m_2-1}' + n_{m_2-2}' + n_{m_2}' = 0 \\ \qquad\vdots \\ -2n_3' + n_2' + n_4' = 0 \\ -2n_2' + n_1 + n_3' = 0 \end{cases}$$

(6.119) $$-2n_1 + n_2' + n_2 + 4 = 0$$

(6.120) $$\begin{cases} -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 = 0 \\ \qquad\vdots \\ -2n_{m_1-2} + n_{m_1-3} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \end{cases}$$

(6.121) $$-2n_{m_1} + n_{m_1-1} + n_{m_1+1} + 1 = 0$$

$$\begin{cases} -2n_{m_1+1}+n_{m_1}+n_{m_1+2}=0 \\ -2n_{m_1+2}+n_{m_1+1}+n_{m_1+3}=0 \\ \qquad\qquad \vdots \\ -2n_{m_1+m_3-1}+n_{m_1+m_3-2}+n_{m_1+m_3}=0 \\ -2n_{m_1+m_3}+n_{m_1+m_3-1}=0 \end{cases} \tag{6.122}$$

(6.118) implies

$$n'_j=(m_2-j+1)n'_{m_2} \qquad 1\le j\le m_2 \tag{6.123}$$

(6.119) and (6.123) imply

$$n_2=(m_2+1)n'_{m_2}-4 \tag{6.124}$$

(6.124) and (6.120) imply

$$n_j=(m_2+j-1)n'_{m_2}-4(j-1), \qquad 2\le j\le m_1 \tag{6.125}$$

(6.122) implies

$$n_{m_1+j}=(m_3-j+1)n_{m_1+m_3}, \qquad 0\le j\le m_3 \tag{6.126}$$

(6.125) and (6.126) imply

$$n_{m_1}=(m_2+m_1-1)n'_{m_2}-4(m_1-1)=(m_3+1)n_{m_1+m_3} \tag{6.127}$$

(6.125), (6.126) and (6.121) imply

$$(m_2+m_1)n'_{m_2}-4m_1=m_3n_{m_1+m_3}+1 \tag{6.128}$$

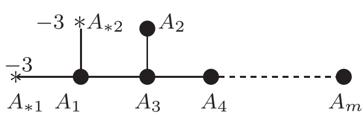
(6.127) and (6.128) imply $n'_{m_2}+n_{m_1+m_3}=5$.

$n_{m_1}=(m_3+1)n_{m_1+m_3}$. Observe that $-1=A_{*2}^2+2\ge -3+n_{m_1}$, so $(m_3+1)n_{m_1+m_3}=n_{m_1}\le 2$. Because $m_3\ge 1$, we have $m_3=1$ and $n_{m_1+m_3}=1$. Therefore we have : $n'_{m_2}=4$, $n_{m_1+m_3}=1$.

If $n'_{m_2}=4$, $n_{m_1+m_3}=1$, then $n_{m_1}=4m_2=2$. This case cannot occur.

(II) Assume that Γ_1 is of the form $D_m(m\ge 4)$ of case (2) in Theorem 4.2.

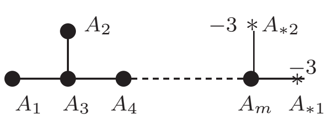
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & 1 & n_2 & \\ 4 & n_1 & n_3 & n_4\dots n_m \end{array}.$$

As in the proof of Proposition 6.10, we have $n_m=\frac{5}{2}$, which contradicts to the fact that n_m is an integer.

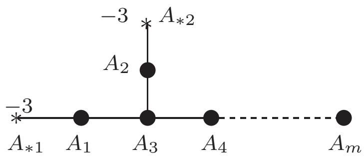
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & n_2 & & & 1 & \\ n_1 & n_3 & n_4\dots & & n_m & 4 \end{array}.$$

As in the proof of Proposition 6.10, we have $n_1=n_2=\frac{5}{2}$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & & 1 & & \\ & & n_2 & & \\ 4 & n_1 & n_3 & n_4\ldots n_m. \end{array}$$

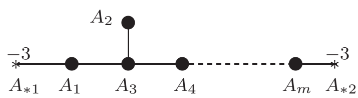
By the same argument as before, we have the following equations.

(6.129)
$$\begin{cases} -2n_1+4+n_3=0 \\ -2n_2+1+n_3=0 \\ -2n_4+n_3+n_5=0 \\ \qquad\vdots \\ -2n_{m-1}+n_{m-2}+n_m=0 \\ -2n_m+n_{m-1}=0 \end{cases}$$

(6.130)
$$-2n_3+n_1+n_2+n_4=0$$

From (6.129) we get $n_1-n_2=\frac{3}{2}$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{array}{ccccc} & & n_2 & & \\ 4 & n_1 & n_3 & n_4\ldots n_m & 1. \end{array}$$

By the same argument as before, we have the following equations.

(6.131)
$$\begin{cases} -2n_1+4+n_3=0 \\ -2n_2+n_3=0 \\ -2n_3+n_1+n_2+n_4=0 \\ -2n_4+n_3+n_5=0 \\ \qquad\vdots \\ -2n_{m-1}+n_{m-2}+n_m=0 \end{cases}$$

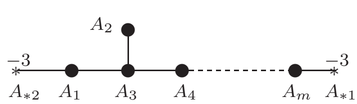
(6.132)
$$-2n_m+n_{m-1}+1=0$$

(6.131) implies

(6.133)
$$n_m=2n_1-2(m-1), n_{m-1}=2n_1-2(m-2).$$

(6.133)and (6.132) get that $n_1=m+\frac{1}{2}$, which contradicts with the fact that n_1 is an integer.

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{array}{ccccc} & & n_2 & & \\ 1 & n_1 & n_3 & n_4\ldots n_m & 4. \end{array}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1+1+n_3=0 \\ -2n_2+n_3=0 \\ -2n_3+n_1+n_2+n_4=0 \\ -2n_4+n_3+n_5=0 \\ \qquad\vdots \\ -2n_{m-1}+n_{m-2}+n_m=0 \end{cases} \tag{6.134}$$

$$-2n_m+n_{m-1}+4=0 \tag{6.135}$$

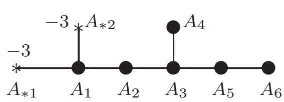
(6.134) implies

$$n_1=\frac{1}{2}+n_2. \tag{6.136}$$

Contradiction!

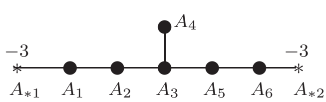
(III) Assume that Γ_1 is of the form E_6 of case (3) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & n_4 & & \\ 4 & n_1 & n_2 & n_3 & n_5 & n_6. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible. Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A*1\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & n_4 & & & \\ 4 & n_1 & n_2 & n_3 & n_5 & n_6 & 1. \end{array}$$

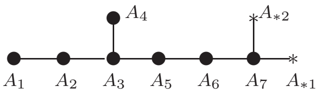
By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1+4+n_2=0 \\ -2n_2+n_1+n_3=0 \\ -2n_3+n_2+n_4+n_5=0 \\ -2n_4+n_3=0 \\ -2n_5+n_3+n_6=0 \\ -2n_6+n_5+1=0 \end{cases}$$

We have $n_6=4$. But $-1=A_{*2}^2+2=A_{*2}\cdot(-K)=A_{*2}\cdot Z\geq -3+n_6=1$. This is not possible.

(IV) Assume that Γ_1 is of the form E_7 of case (4) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_7 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & n_4 & & & 1 & \\ n_1 & n_2 & n_3 & n_5 & n_6 & n_7 & 4 \end{array}.$$

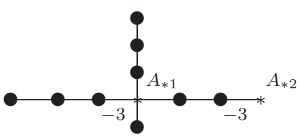
By the same argument as before, we have the following equations

$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 5 = 0 \end{cases} \tag{6.137}$$

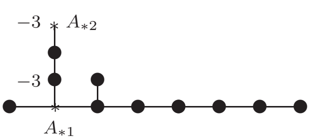
We get $n_7 = \frac{15}{2}$.Contradiction!

(V) Assume that Γ_1 is of the form E_8 of case (5) in Theorem 4.2. This case cannot happen because $A_{*1} \cdot Z_1 \geq 2$. ▯

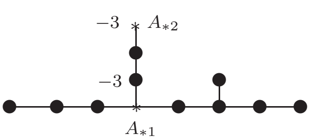
THEOREM 6.21. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (4) of Proposition 6.2 holds, i.e., there exists two effective components A_{*1} and A_{*2} with $A_{*1}^2 = -3 = A_{*2}^2$ and $z_{*1} = 4, z_{*2} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1a) 

$$Z = \begin{array}{ccccccc} & & & 1 & & & \\ & & & 2 & & & \\ & & & 3 & & & \\ 1 & 2 & 3 & \underline{4} & 3 & 2 & \underline{1} \\ & & & 2 & & & \end{array}$$

(1b) 

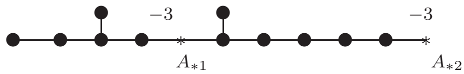
$$Z = \begin{array}{cccccccc} & \underline{1} & & & & & & \\ & 2 & & & & & & \\ & 3 & 3 & & & & & \\ 2 & \underline{4} & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

(1c) 

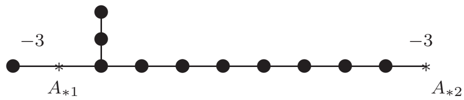
$$Z = \begin{array}{cccccccc} & & & \underline{1} & & & & \\ & & & 2 & & & & \\ & & & 3 & & 3 & & \\ 1 & 2 & 3 & \underline{4} & 5 & 6 & 4 & 2 \end{array}$$

(2a) 

$$Z = \begin{array}{cccccccc} & 1 & & & & & & \\ & 2 & & & & & & \\ & 3 & 3 & & & & & \\ 2 & \underline{4} & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(2*b*) 

$$\begin{array}{l} \qquad\quad\ \ 3 \quad\ \ \ 3 \\ Z = 2\ 4\ \ 6\ \ 5\ \underline{4}\ \ 6\ \ 5\ 4\ 3\ 2\ \underline{1} \end{array}$$

(3) 

$$\begin{array}{l} \qquad\quad\ \ 3 \\ \qquad\quad\ \ 6 \\ Z = 2\ \underline{4}\ \ 9\ \ 8\ 7\ 6\ 5\ 4\ 3\ 2\ \underline{1} \end{array}$$

Proof. Since $z_{*1} = 4, z_{*2} = 1, A_{*1}^2 = A_{*2}^2 = -3$, and $A_{*2} \cdot Z = A_{*2}^2 + 2 = -1$, hence $A_{*1} \cdot A_{*2} = 0$.

Since the singularity is minimally elliptic, $A_{*i}^2 = -3$, $z_{*1} = 4, z_{*1} = 2$, we have

$$A_{*1} \cdot (Z - 4A_{*1}) = -A_{*1} \cdot (K + 4A_{*1}) = A_{*1}^2 + 2 - 4A_{*1}^2 = 11. \tag{6.138}$$

$$A_{*2} \cdot (Z - A_{*2}) = -A_{*2} \cdot (K + A_{*2}) = A_{*2}^2 + 2 - A_{*2}^2 = 2. \tag{6.139}$$

Let Γ' be the graph obtained by deleting A_{*1} and A_{*2} from Γ. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \ldots, Z_m$ respectively. (6.138) and (6.139) imply that

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 11, \tag{6.140}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z\Big|_{\Gamma_j} = 2. \tag{6.141}$$

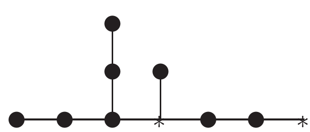
Since we have two effective component, by Corrollay 6.4 we have

$$A_{*i} \cdot Z_j = 1 \qquad \text{for } i = 1, 2 \text{ and } 1 \leq j \leq m. \tag{6.142}$$

Since A_{*1} and A_{*2} do not meet by (6.139). Then Proposition 6.20 applies.

In case (1) of Proposition 6.20, if the decomposition (6.140) at A_{*1} is $11 = 3 + 3 + 3 + 2$, then we are in case (1*a*).

If the decomposition (6.140) at A_{*1} is $11 = 3 + 6 + 2$, then we are in case 1(*b*) and other three graphs :



$$\begin{array}{l} \qquad\quad\ \ 2 \\ \qquad\quad\ \ 4\ 2 \\ Z = 2\ 4\ \ 6\ \underline{4}\ \ 3\ 2\ \underline{1} \end{array}$$

But it is not a fundamental cycle because the fundamental cycle is as follow:

1

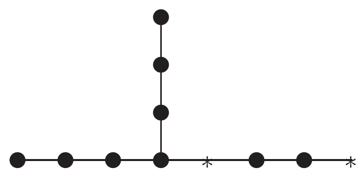
$2\ 1$

$Z = 1\ 2\ \ 3\ \underline{2}\ \ 2\ 2\ \underline{1}$

So are the other two graphs. They are all not fundamental cycles.

If the decomposition (6.140) at A_{*1} is $11 = 3 + 3 + 5$, then we are in case $1(c)$.

If the decomposition (6.140) at A_{*1} is $11 = 3 + 8$, then we have two graphs :



2

4

6

$Z = 2\ 4\ 6\ \ 8\ \ \underline{4}\ 3\ 2\ \underline{1}$

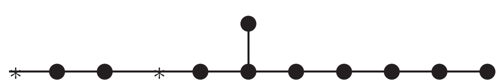
But it is not a fundamental cycle because there is a smaller cycle:

1

2

3

$Z = 1\ 2\ 3\ \ 4\ \ \underline{2}\ 2\ 2\ \underline{1}$



6

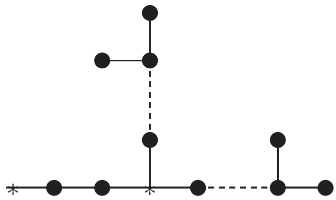
$Z = \ \underline{1}\ 2\ 3\ \underline{4}\ 8\ \ 12\ \ 10\ 8\ 6\ 4\ 2$

But it is not a fundamental cycle because there is a smaller cycle:

3

$Z = \ \underline{1}\ 2\ 2\ \underline{2}\ 4\ \ 6\ \ 5\ 4\ 3\ 2\ 1$

If the decomposition (6.140) at A_{*1} is $11 = 3 + 4 + 4$, then we have one graphs (Here we use one graph to cover three different graphs in 6.19, which are case (5), case (9) and case (17)):

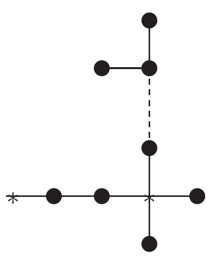


$$
\begin{array}{cccccccc}
 & & & & 2\ \ 2 & & & \\
 & & & & 4 & & & \\
 & & & & \vdots & & & \\
 & & & & 4 & & 2 & \\
Z= & \underline{1} & 2 & 3 & \underline{4} & 4\cdots & 4 & 2
\end{array}
$$

But it is not a fundamental cycle because there is a smaller cycle:

$$
\begin{array}{cccccccc}
 & & & & 1\ \ 1 & & & \\
 & & & & 2 & & & \\
 & & & & \vdots & & & \\
 & & & & 2 & & 1 & \\
Z= & \underline{1} & 2 & 2 & \underline{2} & 2\cdots & 2 & 1
\end{array}
$$

If the decomposition (6.140) at A_{*1} is $11 = 3+4+2+2$, then we have three graphs (which are cover by one graph :



$$
\begin{array}{cccccc}
 & & & & 2\ \ 2 & \\
 & & & & 4 & \\
 & & & & \vdots & \\
 & & & & 4 & \\
Z= & \underline{1} & 2 & 3 & \underline{4} & 2 \\
 & & & & 2 &
\end{array}
$$

But it is not a fundamental cycle because there is a smaller cycle:

$$\begin{array}{cccccc} & & & 1 \quad 1 & & \\ & & & 2 & & \\ & & & \vdots & & \\ & & & 2 & & \\ Z= & \underline{1} & 2\ 2 & \underline{2} & & 1 \\ & & & 1 & & \end{array}$$

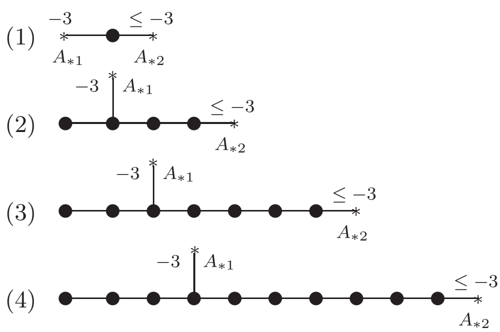
If the decomposition (6.140) at A_{*1} is $11 = 3+2+2+2+2$, then we have one graph and it is not a fundmental cycle.

In case (2) of Proposition 6.20, if the decomposition (6.140) at A_{*1} is $11 = 6+3+2$, then we are in case (2a).

If the decomposition (6.140) at A_{*1} is $11 = 6+5$, then we are in case (2b).

In case (3) of Proposition 6.17, the decomposition (6.140) at A_{*1} must be $11 = 9+2$, then we are in case (3). ∎

Proposition 6.22. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients $z_{*1} = 3$, $z_{*2} = 1$ and $A_{*1}^2 = -3, A_{*2}^2 \leq -3$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following forms.*



$Z\Big|_{A_*\cup\Gamma_1} = \underline{3}\ 2\ \underline{1}$

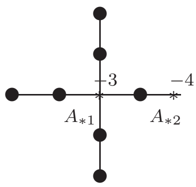
$Z\Big|_{A_*\cup\Gamma_1} = 2\ \overset{\underline{3}}{4}\ 3\ 2\ \underline{1}$

$Z\Big|_{A_*\cup\Gamma_1} = 2\ 4\ \overset{\underline{3}}{6}\ 5\ 4\ 3\ 2\ \underline{1}$

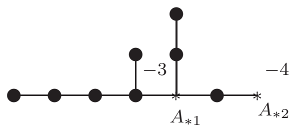
$Z\Big|_{A_*\cup\Gamma_1} = 2\ 4\ 6\ \overset{\underline{3}}{8}\ 7\ 6\ 5\ 4\ 3\ 2\ \underline{1}$

Proof. The proof is similar to those of Propostion 6.20. And the result can also be found in [C-X-Y], Proposition 6.22. ∎

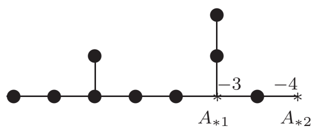
Theorem 6.23. *Let (V,p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (5) of Proposition 6.2 holds, i.e., there exists two effective components A_{*1} and A_{*2} with $A_{*1}^2 = -3, A_{*2}^2 = -4$ and $z_{*1} = 3, z_{*2} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1a) 

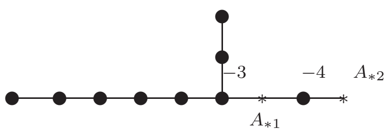
$$Z = \begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ 1 & 2 & \underline{3} & 2 & \underline{1} \\ & & 2 & & \\ & & 1 & & \end{array}$$

$(1b_1)$ 

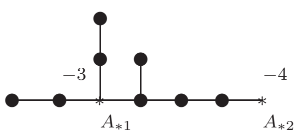
$$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & & & 2 & 2 & & \\ 1 & 2 & 3 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

$(1b_2)$ 

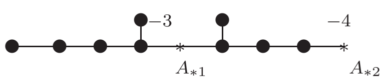
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & & 3 & & & 2 & & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

(1c) 

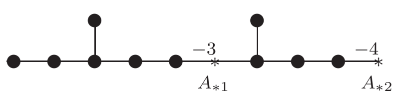
$$Z = \begin{array}{ccccccccc} & & & & & 2 & & & \\ & & & & & 4 & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & \underline{3} & 2 & \underline{1} \end{array}$$

(2a) 

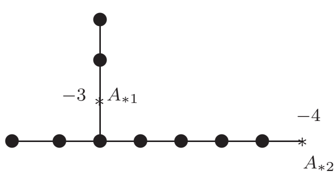
$$Z = \begin{array}{ccccccc} & & 1 & & & & \\ & & 2 & 2 & & & \\ 1 & 2 & \underline{3} & 4 & 3 & 2 & \underline{1} \end{array}$$

$(2b_1)$ 

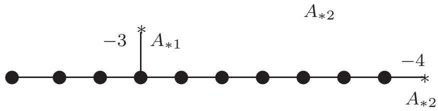
$$Z = \begin{array}{ccccccccc} & & & 2 & & 2 & & & \\ 1 & 2 & 3 & 4 & \underline{3} & 4 & 3 & 2 & \underline{1} \end{array}$$

$(2b_2)$ 

$$Z = \begin{array}{cccccccccc} & & 3 & & & & 2 & & & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & 4 & 3 & 2 & \underline{1} \end{array}$$

(3) 

$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & & \\ & & \underline{3} & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(4) 

$$Z = \begin{array}{ccccccccccc} & & & \underline{3} & & & & & & & \\ 2 & 4 & 6 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. Since $z_{*1} = 3$, $z_{*2} = 1$, $A_{*1}^2 = -3, A_{*2}^2 = -4$, and $A_{*1} \cdot Z = -1$, $A_{*2} \cdot Z = -2$, hence $A_{*1} \cdot A_{*2} = 0$.

Since the singularity is minimally elliptic, $A_{*1}^2 = -3, A_{*2}^2 = -4$, $z_{*1} = 3, z_{*2} = 1$, we have

$$A_{*1} \cdot (Z - 3A_{*1}) = -A_{*1} \cdot (K + 3A_{*1}) = A_{*1}^2 + 2 - 3A_{*1}^2 = 8. \tag{6.143}$$

$$A_{*2} \cdot (Z - A_{*2}) = -A_{*2} \cdot (K + A_{*2}) = A_{*2}^2 + 2 - A_{*2}^2 = 2. \tag{6.144}$$

Let Γ' be the graph obtained by deleting A_{*1} and A_{*2} from Γ. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \ldots, Z_m$ respectively. (6.143) and (6.144) imply that

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 8, \tag{6.145}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z\Big|_{\Gamma_j} = 2. \tag{6.146}$$

Since we have two effective component, by Corrollay 6.4 we have

$$A_{*i} \cdot Z_j = 1 \qquad \text{for } i = 1, 2 \text{ and } 1 \le j \le m. \tag{6.147}$$

Since A_{*1} and A_{*2} do not meet, then Proposition 6.22 applies.

For case (1) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 2 + 2 + 2 + 2$, according to Proposition 6.15, then we are in case $(1a)$.

For case (1) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 4+2+2$, according to Proposition 6.15, then we are in case $(1b_1)$, case$(1b_2)$.

For case (1) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 6 + 2$, according to Proposition 6.15, then we are in case $(1c)$.

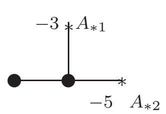
For case (2) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 4+2+2$, according to Proposition 6.15, then we are in case $(2a)$.

For case (2) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 4 + 4$, according to Proposition 6.15, then we are in case $(2b_1)$, case$(2b_2)$.

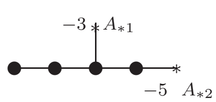
For case (3) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 6 + 2$, according to Proposition 6.15, then we are in case (3).

For case (4) of Proposition 6.22 if the decomposition (6.145) at A_{*1} is $8 = 8 + 0$, according to Proposition 6.15, then we are in case (4). □

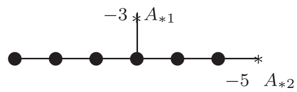
PROPOSITION 6.24. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients $z_{*1} = 2$, $z_{*2} = 1$ and $A_{*1}^2 = -3, A_{*2}^2 = -5$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following forms.*

(1) 

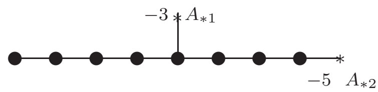
$$Z\Big|_{A_{*1} \cup A_{*2} \cup \Gamma_1} = \begin{array}{ccc} & \underline{2} & \\ 1 & 2 & \underline{1} \end{array}$$

(2) 

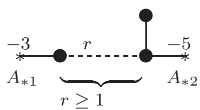
$$Z\Big|_{A_{*1} \cup A_{*2} \cup \Gamma_1} = \begin{array}{ccccc} & & \underline{2} & & \\ 1 & 2 & 3 & 2 & \underline{1} \end{array}$$

(3) 

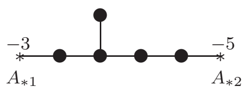
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & \underline{2} & & & \\ 1&2&3&4&3&2&\underline{1}\end{array}$$

(4) 

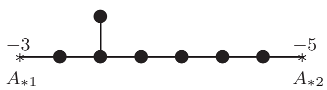
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccccc} & & & & \underline{2} & & & & \\ 1&2&3&4&5&4&3&2&\underline{1}\end{array}$$

(5) 

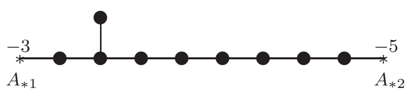
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & & & 1 & \\ \underline{2}&2&\ldots&2&\underline{1}\end{array}$$

(6) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & 2 & & & \\ \underline{2}&3&4&3&2&\underline{1}\end{array}$$

(7) 

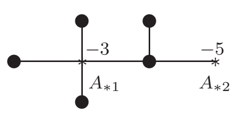
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccc} & & 3 & & & & & \\ \underline{2}&4&6&5&4&3&2&\underline{1}\end{array}$$

(8) 

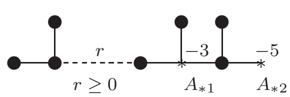
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccccc} & & 4 & & & & & & & \\ \underline{2}&5&8&7&6&5&4&3&2&\underline{1}\end{array}$$

Proof. The proof is similar to those of Propostion 6.20. And the result can also be found in [C-X-Y], Proposition 6.24. ∎

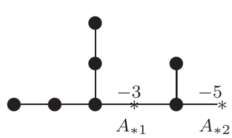
THEOREM 6.25. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (6) of Proposition 6.2 holds, i.e., there exists two effective components A_{*1} and A_{*2} with $A_{*1}^2 = -3, A_{*2}^2 = -5$ and $z_{*1} = 2, z_{*2} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1a) 

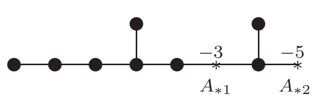
$$Z = \begin{array}{cccc} & 1 & 1 & \\ 1 & \underline{2} & 2 & \underline{1} \\ & 1 & & \end{array}$$

(1b) 

$$Z = \begin{array}{ccccccc} & 1 & & & 1 & 1 & \\ 1 & 2 & \ldots & 2 & \underline{2} & 2 & \underline{1}\end{array}$$

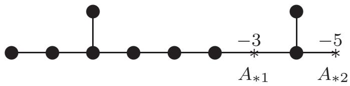
(1c₁) 

$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \underline{1}\end{array}$$

(1c₂) 

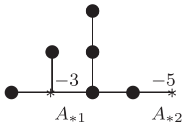
$$Z = \begin{array}{cccccccc} & & & 2 & & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \underline{1}\end{array}$$

(1c₃)



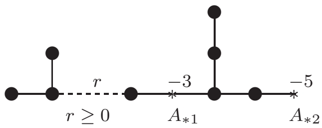
$$Z = \begin{array}{ccccccccc} & & 3 & & & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \underline{1} \end{array}$$

(2a)



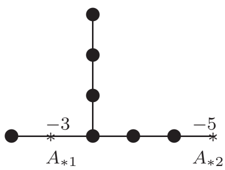
$$Z = \begin{array}{ccccc} & & 1 & & \\ & 1 & 2 & & \\ 1 & \underline{2} & 3 & 2 & \underline{1} \end{array}$$

(2b)



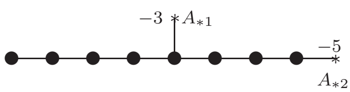
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & & 2 & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3 & 2 & \underline{1} \end{array}$$

(3)



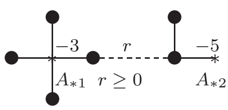
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & & \\ & & 3 & & & \\ 1 & \underline{2} & 4 & 3 & 2 & \underline{1} \end{array}$$

(4)



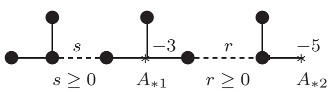
$$Z = \begin{array}{ccccccccc} & & & & \underline{2} & & & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(5a)



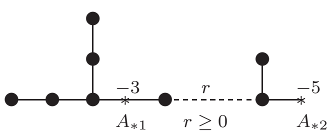
$$Z = \begin{array}{cccccc} & 1 & & & 1 & \\ 1 & \underline{2} & 2 & \dots & 2 & \underline{1} \\ & 1 & & & & \end{array}$$

(5b)



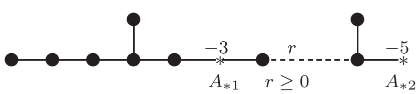
$$Z = \begin{array}{ccccccccc} & 1 & & & 1 & & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

(5c₁)



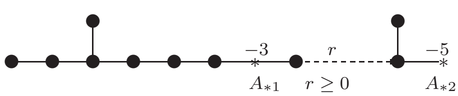
$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

(5c₂)



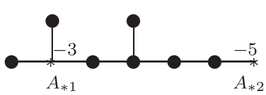
$$Z = \begin{array}{cccccccccc} & & & 2 & & & & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

(5c₃)



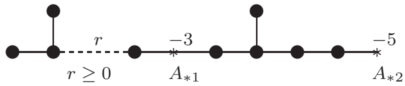
$$Z = \begin{array}{ccccccccccc} & & 3 & & & & & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

(6a)



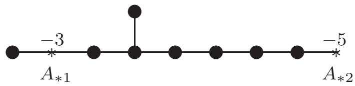
$$Z = \begin{array}{ccccccc} & 1 & & 2 & & & \\ 1 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(6b)



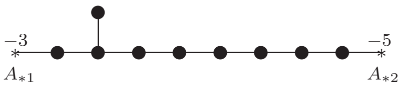
$$\begin{array}{ccccccccccc} & & 1 & & & & & 2 & & & \\ Z = & 1 & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(7)



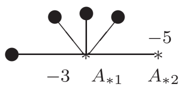
$$\begin{array}{cccccccccc} & & & & 3 & & & & & \\ Z = & 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(8)



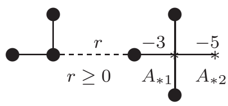
$$\begin{array}{ccccccccccc} & & & 4 & & & & & & & \\ Z = & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(9)



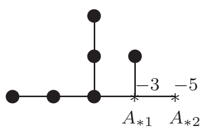
$$\begin{array}{cccccc} & & 1 & 1 & 1 & \\ Z = & 1 & & \underline{2} & & \underline{1} \end{array}$$

(10)



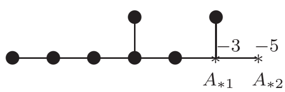
$$\begin{array}{ccccccc} & & 1 & & & 1 & \\ Z = & 1 & 2 & \dots & 2 & \underline{2} & \underline{1} \\ & & & & & 1 & \end{array}$$

(11)



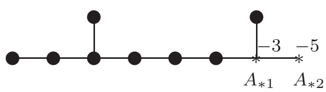
$$\begin{array}{cccccc} & & & 1 & & \\ & & & 2 & 1 & \\ Z = & 1 & 2 & 3 & \underline{2} & \underline{1} \end{array}$$

(12)



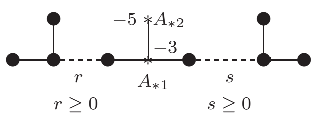
$$\begin{array}{cccccccc} & & & & 2 & & 1 & \\ Z = & 1 & 2 & 3 & 4 & 3 & \underline{2} & \underline{1} \end{array}$$

(13)



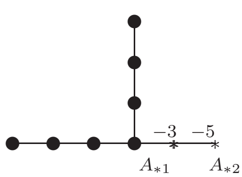
$$\begin{array}{ccccccccc} & & & 3 & & & & 1 & \\ Z = & 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & \underline{1} \end{array}$$

(14)



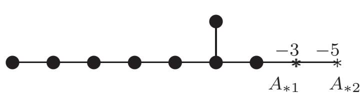
$$\begin{array}{ccccccccccc} & & 1 & & & \underline{1} & & & 1 & \\ Z = & 2 & 2 & \dots & 2 & \underline{2} & 2 & \dots & 2 & 1 \end{array}$$

(15)



$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 2 & & \\ & & & & 3 & & \\ Z = & 1 & 2 & 3 & 4 & \underline{2} & \underline{1} \end{array}$$

(16)



$$\begin{array}{cccccccccc} & & & & & & 3 & & & \\ Z = & 1 & 2 & 3 & 4 & 5 & 6 & 4 & \underline{2} & \underline{1} \end{array}$$

Proof. In the proof, I will use one graph to cover two graphs: (1), (5) in Proposition 6.24, in which $r = 0$ or $r \geq 1$ respectively.

Since the singularity is minimally elliptic, $A_{*1}^2 = -3, A_{*2}^2 = -5$, $z_{*1} = 2, z_{*2} = 1$, we have

$$A_{*1} \cdot (Z - 2A_{*1}) = -A_{*1} \cdot (K + 2A_{*1}) = A_{*1}^2 + 2 - 2A_{*1}^2 = 5. \tag{6.148}$$

$$A_{*2} \cdot (Z - A_{*2}) = -A_{*2} \cdot (K + A_{*2}) = A_{*2}^2 + 2 - A_{*2}^2 = 2. \tag{6.149}$$

Let Γ' be the graph obtained by deleting A_{*1} and A_{*2} from Γ. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \ldots, Z_m$ respectively. (6.148) implies that

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 5, \tag{6.150}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z\Big|_{\Gamma_j} = 2. \tag{6.151}$$

Since we have two effective component, by Corrollay 6.4 we have

$$A_{*i} \cdot Z_j = 1 \qquad \text{for } i = 1, 2 \text{ and } 1 \leq j \leq m. \tag{6.152}$$

Consider first that A_{*1} and A_{*2} do not meet. Then Proposition 6.24 applies.

In case (1) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+1+1+1$, according to Proposition 6.16, then we are in case $(1a)$.

In case (1) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+2+1$, according to Proposition 6.16, then we are in case $(1b)$.

In case (1) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+3$, according to Proposition 6.16, then we are in case $(1c_1)$, case$(1c_2)$, case$(1c_3)$.

In case (2) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 3+1+1$, according to Proposition 6.16, then we are in case $(2a)$.

In case (2) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 3+2$, according to Proposition 6.16, then we are in case $(2b)$.

In case (3) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 4+1$, according to Proposition 6.16, then we are in case (3).

In case (4) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 5+0$, according to Proposition 6.16, then we are in case (4).

In case (5) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+1+1+1$, according to Proposition 6.16, then we are in case $(5a)$.

In case (5) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+2+1$, according to Proposition 6.16, then we are in case $(5b)$.

In case (5) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 2+3$, according to Proposition 6.16, then we are in case $(5c_1)$, case$(5c_2)$, case$(5c_3)$.

In case (6) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 3+1+1$, according to Proposition 6.16, then we are in case $(6a)$.

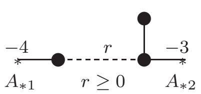
In case (6) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 3+2$, according to Proposition 6.16, then we are in case $(6b)$.

In case (7a) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 4 + 1$, according to Proposition 6.16, then we are in case (7).

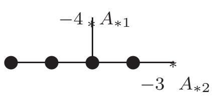
In case (8) of Proposition 6.24, if the decomposition (6.150) at A_{*1} is $5 = 5 + 0$, according to Proposition 6.16, then we are in case (1*b*).

If $A_{*1} \cdot A_{*2} \neq 0$, then $A_{*1} \cdot A_{*2} = 1$. It follows that $A_{*1} \cdot (Z - 2A_{*1} - A_{*2}) = -A_{*1} \cdot (K + 2A_{*1} + A_{*2}) = -A_{*1}^2 + 1 = 4$. For $4 = 1 + 1 + 1 + 1$, we are in case (9). For $4 = 1 + 1 + 2$, we are in case (10). For $4 = 1 + 3$, we are in case (11), case (12) and case (13). For $4 = 2 + 2$, we are in case (14). For $4 = 4$, we are in case (15) and case (16). ▯

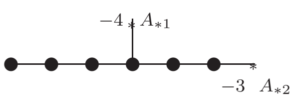
Proposition 6.26. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients $z_{*1} = 2$, $z_{*2} = 1$ and $A_{*1}^2 = -4, A_{*2}^2 = -3$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following forms.*

(1) 

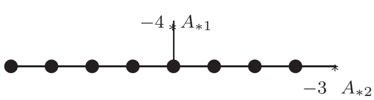
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 2\ \dots\ \overset{1}{2}\ \underline{1}$$

(2) 

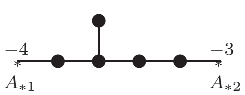
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ 2\ \overset{\underline{2}}{3}\ 2\ \underline{1}$$

(3) 

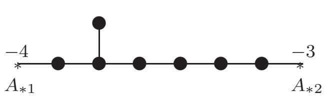
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ 2\ 3\ \overset{\underline{2}}{4}\ 3\ 2\ \underline{1}$$

(4) 

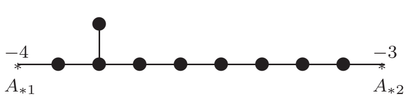
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ 2\ 3\ 4\ \overset{\underline{2}}{5}\ 4\ 3\ 2\ \underline{1}$$

(5) 

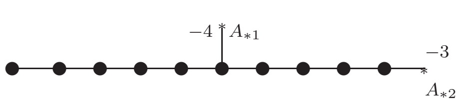
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 3\ \overset{2}{4}\ 3\ 2\ \underline{1}$$

(6) 

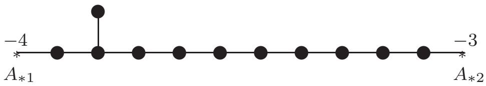
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ \underline{1}$$

(7) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 5\ \overset{4}{8}\ 7\ 6\ 5\ 4\ 3\ 2\ \underline{1}$$

(8) 

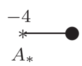
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ 2\ 3\ 4\ 5\ \overset{\underline{2}}{6}\ 5\ 4\ 3\ 2\ \underline{1}$$

(9) 

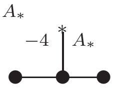
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 6\ \overset{5}{10}\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ \underline{1}$$

Proof. The proof is similar to those of Propostion 6.24. ▯

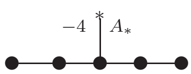
PROPOSITION 6.27. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_* be an effective component of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_* but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_* \cdot Z_1 = 1$. If the coefficient z_* of A_* in Z is 2 and $A_*^2 = -4$, then $A_* \cup \Gamma_1$ and restriction of Z on $A_* \cup \Gamma_1$ must be one of the following forms.*

(1) 

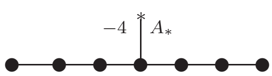
$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 1$$

(2) 

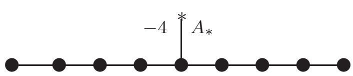
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ \overset{\underline{2}}{2}\ 1$$

(3) 

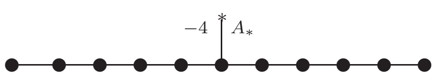
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ 2\ \overset{\underline{2}}{3}\ 2\ 1$$

(4) 

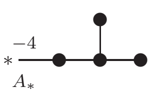
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ 2\ 3\ \overset{\underline{2}}{4}\ 3\ 2\ 1$$

(5) 

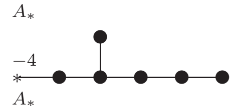
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ 2\ 3\ 4\ \overset{\underline{2}}{5}\ 4\ 3\ 2\ 1$$

(6) 

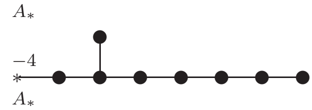
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ 2\ 3\ 4\ 5\ \overset{\underline{2}}{6}\ 5\ 4\ 3\ 2\ 1$$

(7) 

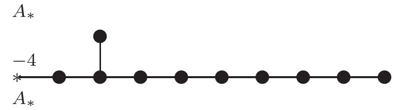
$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 2\ \overset{1}{2}\ 1$$

(8) 

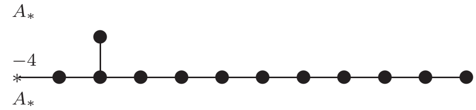
$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 3\ \overset{2}{4}\ 3\ 2\ 1$$

(9) 

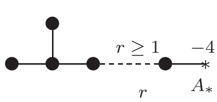
$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2\ 1$$

(10) 

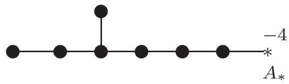
$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 5\ \overset{4}{8}\ 7\ 6\ 5\ 4\ 3\ 2\ 1$$

(11) 

$$Z\Big|_{A_*\cup\Gamma_1} = \underline{2}\ 6\ \overset{5}{10}\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1$$

(12) 

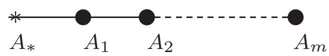
$$Z\Big|_{A_*\cup\Gamma_1} = 1\ \overset{1}{2}\ 2\ \ldots\ldots\ \underline{2}$$

(13) 

$$Z\Big|_{A_*\cup\Gamma_1} = \begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} \end{matrix}$$

Proof. The proof is similar to those of Proposition 6.9, 6.10 and 6.11.

Consider A_* attaching on Γ_1 in the following form



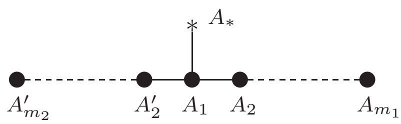
$$Z\Big|_{A_*\cup\Gamma_1} = 2 \quad n_1 \quad n_2 \ldots n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{2}{m+1} \end{cases}.$$

Therefore $m = 1$ and we are in case (1).

Consider A_* attaching on Γ_1 in the following form



$$Z\Big|_{A_*\cup\Gamma_1} = \begin{matrix} & & & 2 & & \\ n'_{m_2} \ldots & n'_2 & & n_1 & n_2 \ldots n_{m_1}. \end{matrix}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m_1$ and similarly $A'_j \cdot Z = 0$ for $2 \le j \le m_2$, we have the following system of equations

$$\begin{cases} -2n_{m_1} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \quad \vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \qquad \begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \quad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.153}$$

$$2 - 2n_1 + n_2 + n'_2 = 0 \tag{6.154}$$

(6.153) implies

$$n_i = (m_1 - i + 1)n_{m_1} \qquad 1 \le i \le m_1 \tag{6.155}$$

$$n'_j = (m_2 - j + 1)n'_{m_2} \qquad 2 \le j \le m_2 \tag{6.156}$$

$$m_1 n_{m_1} = m_2 n'_{m_2} \tag{6.157}$$

Putting (6.155) and (6.156) into (6.154), we get

$$0 = 2 - 2m_1 n_{m_1} + (m_1 - 1)n_{m_1} + (m_2 - 1)n'_{m_2}$$

$$= 2 - (m_1 + 1)n_{m_1} + (m_2 - 1)n'_{m_2} = 0 \tag{6.158}$$

(6.157) and (6.158) imply

$$n_{m_1} + n'_{m_2} = 2 \tag{6.159}$$

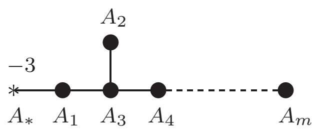
(6.159) implies that $n_{m_1} = 1$, $n'_{m_2} = 1$.

Therefore $n_{m_1} = 1$ and $n'_{m_2} = 1$. By (6.157), we have $m_1 = m_2$. Observe that

$$\begin{aligned} &-2 = A_*^2 + 2 = A_* \cdot (-K) = A_* \cdot Z \geq 2(-4) + n_1 = -8 + m_1 \\ &\Rightarrow m_1 \leq 6. \end{aligned}$$

If $m_1 = 2$, 3, 4, 5,or 6 , then we are in case (2), case (3), case (4), case (5), case (6) respectively in the statement of the proposition.

Consider A_* attaching on Γ_1 in the following form



$$Z\Big|_{A_*\cup\Gamma_1} = 2\ n_1\ \begin{matrix} n_2 \\ n_3 \end{matrix}\ n_4\ \ldots\ n_m.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have the following system of equations

$$-2n_1 + 2 + n_3 = 0 \tag{6.160}$$

$$\begin{cases} -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.161}$$

(6.161) implies

$$n_1 = \frac{m}{2} n_m,\ n_2 = \frac{m-2}{2} n_m,\ n_j = (m - j + 1)n_m,\ 3 \leq j \leq m \tag{6.162}$$

(6.160) and (6.162) imply $n_m = 1$. Therefore $n_1 = \frac{m}{2}$.

Observe that

$$\begin{aligned} &-2 = A_*^2 + 2 = A_* \cdot (-K) = A_* \cdot Z \geq 2(-4) + n_1 = -8 + \frac{m}{2} \\ &\Rightarrow m \leq 12. \end{aligned}$$

Then m=4, 6, 8, 10, 12, we are in case (7), case (8), case (9), case (10) and case (11). We next consider A_* attaching on Γ_1 in the following form.

$$Z\Big|_{A_*\cup\Gamma_1} = n_1\ \begin{matrix} n_2 \\ n_3 \end{matrix}\ n_4\ \ldots\ n_m\ 2.$$

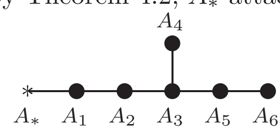
Since $A_i \cdot Z = 0$, $1 \le i \le m$, we have the following system of equations

$$\begin{cases} -2n_1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.163}$$

$$-2n_m + n_{m-1} + 2 = 0 \tag{6.164}$$

(6.163) implies $n_3 = n_4 = \cdots = n_m = 2n_1 = 2n_2$. By (6.164), we know that $n_1 = 1$. Then we are in case (12).

By Theorem 4.2, A_* attaching on E_6 must be of the following form.



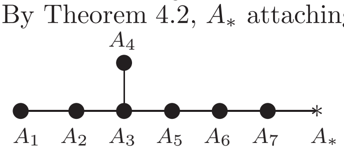
$$Z\Big|_{A_* \cup \Gamma_1} = 2\ n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6$$

Since $A_i \cdot Z = A_i(-K) = 0$ for $1 \le i \le 6$, we have the following system of equations

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 = 0 \end{cases}$$

which imply $n_6 = \frac{4}{3}$, which contradicts the fact that n_6 is an integer.

By Theorem 4.2, A_* attaching on E_7 must be of the following form.



$$Z\Big|_{A_* \cup \Gamma_1} = n_1\ n_2\ \overset{n_4}{n_3}\ n_5\ n_6\ n_7\ 2$$

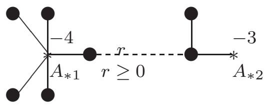
Since $A_i \cdot Z = A_i \cdot (-K) = 0$ for $1 \le i \le 7$, we have the following system of equations.

$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 2 = 0 \end{cases}$$

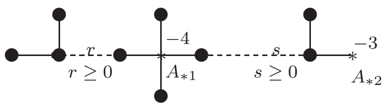
We get $n_1 = 2$, then we are in case (13). By Theorem 4.2, A_* cannot attach on E_8 because $A_* \cdot Z_1 \ge 2$. ◻

THEOREM 6.28. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (7) of Proposition 6.2 holds,*

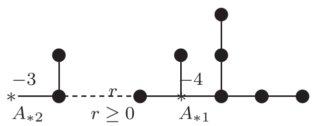
i.e., there exists two effective components A_{*1} and A_{*2} with $A_{*1}^2 = -4, A_{*2}^2 = -3$ and $z_{*1} = 2, z_{*2} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.

(1a) 

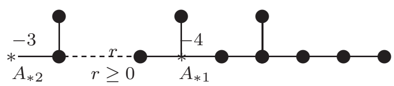
$$Z = \begin{array}{ccccc} 1\,1 & & & 1 & \\ \underline{2} & 2 & \dots & 2 & \underline{1} \\ 1\,1 & & & & \end{array}$$

(1b) 

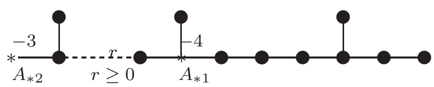
$$Z = \begin{array}{ccccccccc} & 1 & & & 1 & & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & 2 & \dots & 2 & \underline{1} \\ & & & & 1 & & & & \end{array}$$

(1c$_1$) 

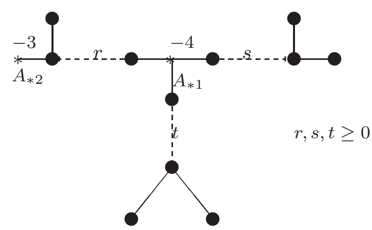
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & 1 & 2 & & \\ \underline{1} & 2 & \dots & 2 & \underline{2} & 3 & 2 & 1 \end{array}$$

(1c$_2$) 

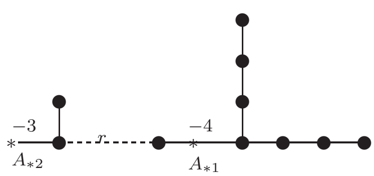
$$Z = \begin{array}{cccccccccc} & 1 & & & 1 & & 2 & & & \\ \underline{1} & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$$

(1c$_3$) 

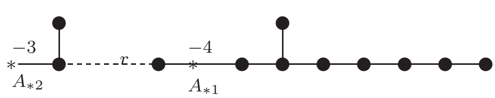
$$Z = \begin{array}{ccccccccccc} & 1 & & & 1 & & & & 3 & & \\ \underline{1} & 2 & \dots & 2 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(1d) 

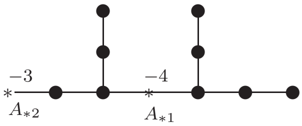
$$Z = \begin{array}{ccccccccc} & 1 & & & & & & 1 & \\ \underline{1} & 2 & \cdots & 2 & \underline{2} & 2 & \cdots & 2 & 1 \\ & & & & 2 & & & & \\ & & & & \vdots & & & & \\ & & & & 2 & & & & \\ & & & & 1\quad 1 & & & & \end{array}$$

(1e$_1$) 

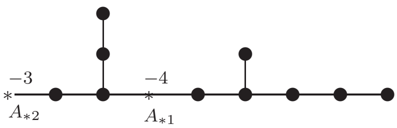
$$Z = \begin{array}{ccccccccc} & & & & & 1 & & & \\ & & & & & 2 & & & \\ & 1 & & & & 3 & & & \\ \underline{1} & 2 & \cdots & 2 & \underline{2} & 4 & 3 & 2 & 1 \end{array}$$

(1e$_2$) 

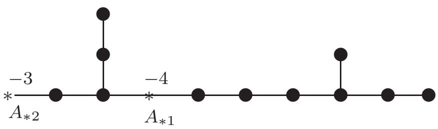
$$Z = \begin{array}{ccccccccccccc} & 1 & & & & & 3 & & & & & & \\ \underline{1} & 2 & \cdots & 2 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

(2a$_1$) 

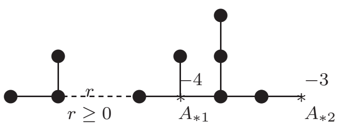
$$\begin{array}{lccccccc} & & & 1 & & 1 & & \\ & & & 2 & & 2 & & \\ Z = & \underline{1} & 2 & 3 & \underline{2} & 3 & 2 & 1\end{array}$$

(2a$_2$) 

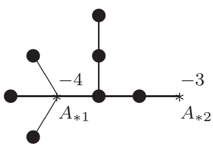
$$\begin{array}{lccccccccc} & & & 1 & & & & & & \\ & & & 2 & & & 2 & & & \\ Z = & \underline{1} & 2 & 3 & \underline{2} & 3 & 4 & 3 & 2 & 1\end{array}$$

(2a$_3$) 

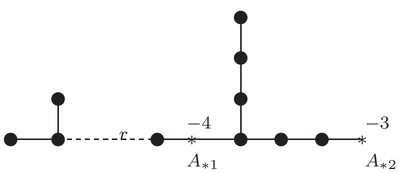
$$\begin{array}{lcccccccccc} & & & 1 & & & & & & & \\ & & & 2 & & & & & 3 & & \\ Z = & \underline{1} & 2 & 3 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2\end{array}$$

(2b) 

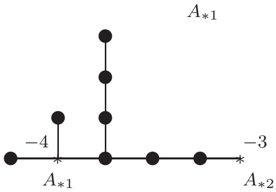
$$\begin{array}{lcccccccc} & & & & & & 1 & & \\ & & 1 & & & 1 & 2 & & \\ Z = & 1 & 2 & \ldots & 2 & \underline{2} & 3 & 2 & \underline{1}\end{array}$$

(2c) 

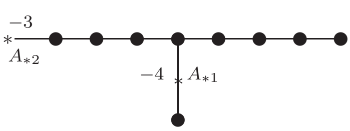
$$\begin{array}{lcccccc} & & & & 1 & & \\ & & 1 & & 2 & & \\ Z = & 1 & & \underline{2} & 3 & 2 & \underline{1} \\ & & 1 & & & & \end{array}$$

(3a) 

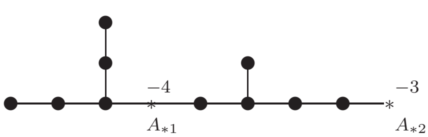
$$\begin{array}{lccccccccc} & & & & & & 1 & & & \\ & & & & & & 2 & & & \\ & & 1 & & & & 3 & & & \\ Z = & 1 & 2 & \cdots & 2 & \underline{2} & 4 & 3 & 2 & \underline{1}\end{array}$$

(3b) 

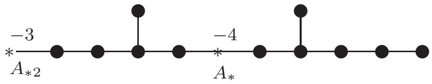
$$\begin{array}{lcccccc} & & & 1 & & & \\ & & & 2 & & & \\ & & 1 & 3 & & & \\ Z = & 1 & \underline{2} & 4 & 3 & 2 & \underline{1}\end{array}$$

(4) 

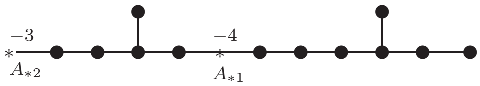
$$\begin{array}{lccccccccc} Z = & \underline{1} & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ & & & & & \underline{2} & & & & \\ & & & & & 1 & & & & \end{array}$$

(5a$_1$) 

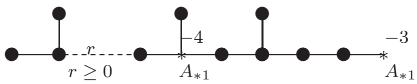
$$\begin{array}{lccccccccc} & & & 1 & & & & & & \\ & & & 2 & & & 2 & & & \\ Z = & 1 & 2 & 3 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1}\end{array}$$

(5a$_2$) 

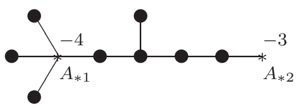
$$Z = \begin{array}{ccccccccccc} & & & 2 & & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$$

(5a$_3$) 

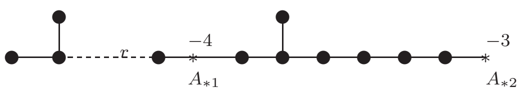
$$Z = \begin{array}{cccccccccccc} & & & 2 & & & & & & 3 & & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(5b) 

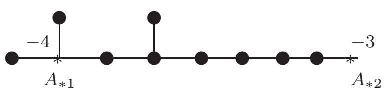
$$Z = \begin{array}{ccccccccccc} & 1 & & & 1 & & 2 & & & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(5c) 

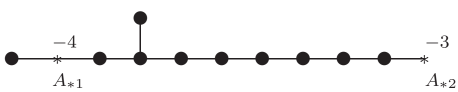
$$Z = \begin{array}{cccccccc} & 1 & & & 2 & & & \\ 1 & & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ & 1 & & & & & & \end{array}$$

(6a) 

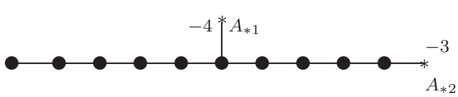
$$Z = \begin{array}{ccccccccccccc} & 1 & & & & & 3 & & & & & \\ 1 & 2 & \cdots & 2 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(6b) 

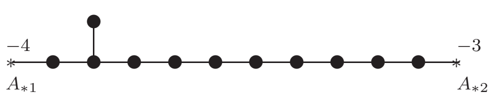
$$Z = \begin{array}{ccccccccc} & 1 & & 3 & & & & & \\ 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(7) 

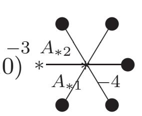
$$Z = \begin{array}{ccccccccccc} & & & 4 & & & & & & & \\ 1 & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(8) 

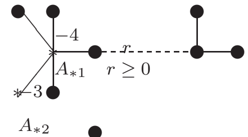
$$Z = \begin{array}{ccccccccccc} & & & & & \underline{2} & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(9) 

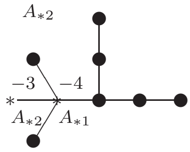
$$Z = \begin{array}{cccccccccccc} & & 5 & & & & & & & & & \\ \underline{2} & 6 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(10) 

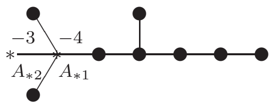
$$Z = \begin{array}{ccccc} & 1 & & 1 & \\ \underline{1} & & \underline{2} & & 1 \\ & 1 & & 1 & \end{array}$$

(11) 

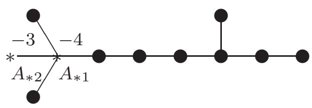
$$Z = \begin{array}{cccccc} 1 & 1 & & & 1 & \\ & \underline{2} & 2 & \ldots & 2 & 1 \\ \underline{1} & 1 & & & & \end{array}$$

(12) 

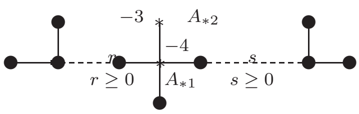
$$Z = \begin{array}{cccccc} & & & 1 & & \\ & 1 & & 2 & & \\ \underline{1} & & \underline{2} & 3 & 2 & 1 \\ & 1 & & & & \end{array}$$

(13) 

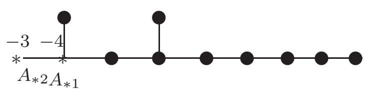
$$Z = \begin{array}{cccccccc} & 1 & & & 2 & & & \\ \underline{1} & & \underline{2} & 3 & 4 & 3 & 2 & 1 \\ & 1 & & & & & & \end{array}$$

(14) 

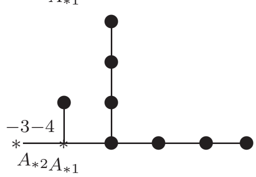
$$Z = \begin{array}{ccccccccc} & 1 & & & & & 3 & & \\ \underline{1} & & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \\ & 1 & & & & & & & \end{array}$$

(15) 

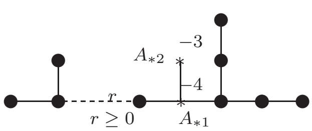
$$Z = \begin{array}{cccccccc} & 1 & & & \underline{1} & & 1 & \\ 1 & 2 & \ldots & 2 & \underline{2} & 2 \ldots & 2 & 1 \\ & & & & 1 & & & \end{array}$$

(16) 

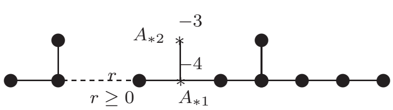
$$Z = \begin{array}{ccccccccc} & 1 & & 3 & & & & & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

(17) 

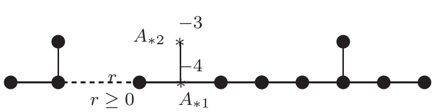
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & & \\ & 1 & 3 & & & \\ \underline{1} & \underline{2} & 4 & 3 & 2 & 1 \end{array}$$

(18) 

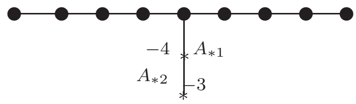
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & \underline{1} & 2 & & \\ 1 & 2 & \ldots & 2 & \underline{2} & 3 & 2 & 1 \end{array}$$

(19) 

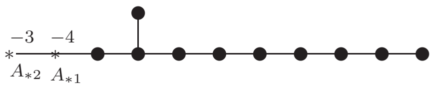
$$Z = \begin{array}{cccccccccc} & 1 & & & \underline{1} & & 2 & & & \\ 1 & 2 & \ldots & 2 & \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$$

(20) 

$$Z = \begin{array}{ccccccccccc} & 1 & & & \underline{1} & & & & 3 & & \\ 1 & 2 & \ldots & 2 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(21) 

$$\begin{array}{l} Z = 1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\ 1 \\ \underline{2} \\ \underline{1} \end{array}$$

(22) 

$$\begin{array}{l} 4 \\ Z = \underline{1}\ \underline{2}\ 5\ \ 8\ \ 7\ 6\ 5\ 4\ 3\ 2\ 1 \end{array}$$

Proof. Since the singularity is minimally elliptic, $A_{*1}^2 = -4, A_{*2}^2 = -3$, $z_{*1} = 2, z_{*2} = 1$, we have

$$A_{*1} \cdot (Z - 2A_{*1}) = -A_{*1} \cdot (K + 2A_{*1}) = A_{*1}^2 + 2 - 2A_{*1}^2 = 6. \tag{6.165}$$

$$A_{*2} \cdot (Z - A_{*2}) = -A_{*2} \cdot (K + A_{*2}) = A_{*2}^2 + 2 - A_{*2}^2 = 2. \tag{6.166}$$

Let Γ' be the graph obtained by deleting A_{*1} and A_{*2} from Γ. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \ldots, Z_m$ respectively. (6.165) implies that

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 6, \tag{6.167}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z\Big|_{\Gamma_j} = 2. \tag{6.168}$$

Since we have two effective component, by Corrollay 6.4 we have

$$A_{*i} \cdot Z_j = 1 \qquad \text{for } i = 1, 2 \text{ and } 1 \le j \le m. \tag{6.169}$$

Consider first that A_{*1} and A_{*2} do not meet. Then Proposition 6.26 applies.

In case (1) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 2+1+1+1+1$, according to Proposition 6.27, then we are in case ($1a$).

In case (1) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 2+2+1+1$, according to Proposition 6.27, then we are in case ($1b$).

In case (1) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 2+3+1$, according to Proposition 6.27, then we are in case ($1c_1$), case($1c_2$), case($1c_3$).

In case (1) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 2+2+2$, according to Proposition 6.27, then we are in case ($1d$).

In case (1) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 2+4$, according to Proposition 6.27, then we are in case ($1e_1$), case($1e_2$).

In case (2) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3+3$, according to Proposition 6.27, then we are in case ($2a_1$), case($2a_2$), case($2a_3$).

In case (2) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3+2+1$, according to Proposition 6.27, then we are in case ($2b$).

In case (2) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3+1+1+1$, according to Proposition 6.27, then we are in case ($2c$).

In case (3) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 4 + 2$, according to Proposition 6.27, then we are in case ($3a$).

In case (3) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 4+1+1$, according to Proposition 6.27, then we are in case ($3b$).

In case (4) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 5 + 1$, according to Proposition 6.27, then we are in case (4).

In case (5) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3 + 3$, according to Proposition 6.27, then we are in case ($5a_1$), case($5a_2$), case($5a_3$).

In case (5) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3+2+1$, according to Proposition 6.27, then we are in case ($5b$).

In case (5) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 3 + 1 + 1 + 1$, according to Proposition 6.27, then we are in case ($5c$).

In case (6) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 4 + 2$, according to Proposition 6.27, then we are in case ($6a$).

In case (6) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 4+1+1$, according to Proposition 6.27, then we are in case ($6b$).

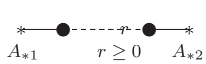
In case (7) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 5 + 1$, according to Proposition 6.27, then we are in case (7).

In case (8) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 6 + 0$, according to Proposition 6.27, then we are in case (8).

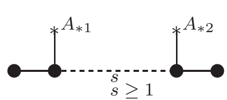
In case (9) of Proposition 6.26, if the decomposition (6.155) at A_{*1} is $6 = 6 + 0$, according to Proposition 6.16, then we are in case (9).

If $A_{*1} \cdot A_{*2} \neq 0$, then $A_{*1} \cdot A_{*2} = 1$. It follows that $A_{*1} \cdot (Z - 2A_{*1} - A_{*2}) = -A_{*1} \cdot (K + 2A_{*1} + A_{*2}) = -A_{*1}^2 + 1 = 5$. For $5 = 1+1+1+1+1$, we are in case (10). For $5 = 1 + 1 + 1 + 2$, we are in case (11). For $5 = 1 + 1 + 3$, we are in case (12), case (13) and case (14). For $5 = 2 + 2 + 1$, we are in case (15). For $5 = 4 + 1$, we are in case (16) and case (17). For $5 = 3 + 2$, we are in case-(18), case (19), case (20). For $5 = 5 + 0$, we are in case (21), case (22). $\square$

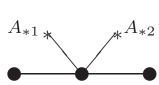
Proposition 6.29. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective compon Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients z_{*1} of A_{*1} and z_{*2} of A_{*2} in Z are one and $A_{*1}^2 = -3$, $A_{*2}^2 = -6$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and the restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following form.*

(1) 

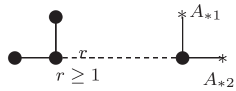
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{1}\ 1\ \ldots\ 1\ \underline{1}$$

(2) 

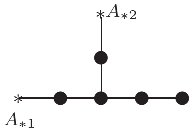
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \ldots & 2 & 1 \end{array}$$

(3) 

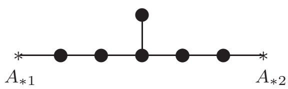
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & & 2 & & 1 \end{array}$$

(4) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & 1 & & \underline{1} & \\ 1 & 2 & \dots & 2 & \underline{1}\end{array}$$

(5) 

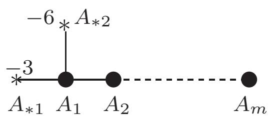
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1\end{array}$$

(6) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1}\end{array}$$

Proof. (I) Assume that Γ_1 is of the form of case (1) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



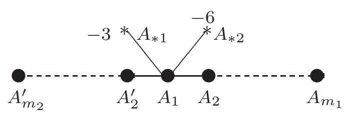
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & & & \\ 1 & n_1 & n_2 & \dots\dots & n_m. \end{array}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \leq i \leq m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \leq i \leq m \\ n_m = \frac{2}{m+1} \end{cases}.$$

Therefore $m = 1$ and it is a special case in case (1).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & 1 & & 1 & \\ n'_{m_2} \dots n'_2 & & n_1 & & n_2 \dots n_{m_1}. \end{array}$$

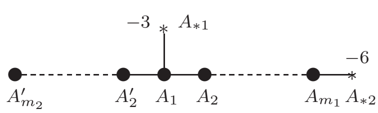
As in the proof of Proposition 6.9, we have $n_{m_1} = 1$, $n'_{m_2} = 1$ and $m_1 = m_2 = n_1 = 2$. We are in case (3).

Consider A_{*1} and A_{*2} attaching an Γ_1 in the following form.

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ n_1\ n_2 \dots n_m\ 1.$$

As in the proof of Proposition 6.9, we have $n_1 = n_2 = \cdots = n_m = 1$ and we are in case (1).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & 1 & & \\ n'_{m_2} \dots n'_2 & n_1 & n_2 \dots n_{m_1} & 1. \end{array}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$ and $A'_j \cdot Z = A'_j \cdot (-K) = A_j'^2 + 2 = 0$, $2 \le j \le m_2$, we have

$$(6.170)\qquad \begin{cases} -2n_{m_1} + n_{m_1-1} + 1 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases}$$

$$(6.171)\qquad \begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases}$$

$$(6.172)\qquad 1 - 2n_1 + n_2 + n'_2 = 0$$

(6.170) implies

$$(6.173)\qquad n_j = (m_1 - j + 1)n_{m_1} - (m_1 - j), \qquad 1 \le j \le m_1 - 1$$

(6.173) implies

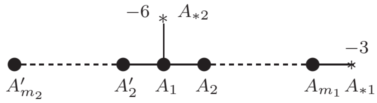
$$(6.174)\qquad n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \le j \le m_2 - 1$$

(6.173) and (6.174) imply

$$(6.175)\qquad m_1 n_{m_1} - (m_1 - 1) = m_2 n'_{m_2} = n_1$$

(6.172), (6.173) and (6.174) imply $n_{m_1} + n'_{m_2} = 2$. So we have $n_{m_1} = n'_{m_2} = 1$. Then $n_1 = 1$ and $n_2 + n'_2 = 1$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2 \ \ \overset{1}{n_1} \ \ n_2\dots n_{m_1}\ 1.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$ and $A'_j \cdot Z = A'_j \cdot (-K) = A_j'^2 + 2 = 0$, $2 \le j \le m_2$, we have

$$(6.176)\qquad \begin{cases} -2n_{m_1} + n_{m_1-1} + 1 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases}$$

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\qquad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.177}$$

$$1 - 2n_1 + n_2 + n'_2 = 0 \tag{6.178}$$

(6.176) implies

$$n_j = (m_1 - j + 1)n_{m_1} - 4(m_1 - j), \qquad 1 \le j \le m_1 - 1 \tag{6.179}$$

(6.177) implies

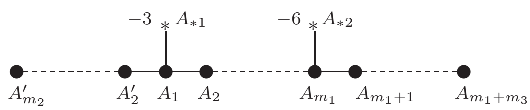
$$n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \le j \le m_2 - 1 \tag{6.180}$$

(6.179) and (6.180) imply

$$m_1 n_{m_1} - 4(m_1 - 1) = m_2 n'_{m_2} = n_1 \tag{6.181}$$

(6.178), (6.179) and (6.180) imply $n_{m_1} + n'_{m_2} = 2$. So we have $n_{m_1} = n'_{m_2} = 1$. Then $n_1 = 1$ and $n_2 + n'_2 = 1$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2} \ldots n'_2 \ \overset{1}{n_1} \ \ n_2 \cdots \overset{1}{n_{m_1}} \ \ n_{m_1+1} \ldots n_{m_1+m_3}.$$

And $m_1 \ge 2$, $m_3 \ge 1$, $m_2 \ge 2$ By the same argument as before, we have the following equations

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\qquad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.182}$$

$$-2n_1 + n'_2 + n_2 + 1 = 0 \tag{6.183}$$

$$\begin{cases} -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 = 0 \\ \qquad\qquad \vdots \\ -2n_{m_1-2} + n_{m_1-3} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \end{cases} \tag{6.184}$$

$$-2n_{m_1}+n_{m_1-1}+n_{m_1+1}+1=0 \tag{6.185}$$

$$\begin{cases} -2n_{m_1+1}+n_{m_1}+n_{m_1+2}=0 \\ -2n_{m_1+2}+n_{m_1+1}+n_{m_1+3}=0 \\ \qquad\vdots \\ -2n_{m_1+m_3-1}+n_{m_1+m_3-2}+n_{m_1+m_3}=0 \\ -2n_{m_1+m_3}+n_{m_1+m_3-1}=0 \end{cases} \tag{6.186}$$

(6.182) implies

$$n'_j=(m_2-j+1)n'_{m_2} \qquad 1\le j\le m_2 \tag{6.187}$$

(6.183) and (6.187) imply

$$n_2=(m_2+1)n'_{m_2}-1 \tag{6.188}$$

(6.188) and (6.184) imply

$$n_j=(m_2+j-1)n'_{m_2}-(j-1), \qquad 2\le j\le m_1, \tag{6.189}$$

(6.186) implies

$$n_{m_1+j}=(m_3-j+1)n_{m_1+m_3}, \qquad 0\le j\le m_3, \tag{6.190}$$

(6.189) and (6.190) imply

$$n_{m_1}=(m_2+m_1-1)n'_{m_2}-(m_1-1)=(m_3+1)n_{m_1+m_3} \tag{6.191}$$

(6.189), (6.190) and (6.185) imply

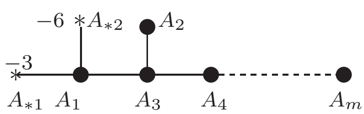
$$(m_2+m_1)n'_{m_2}-m_1=m_3n_{m_1+m_3}+1 \tag{6.192}$$

(6.191) and (6.192) imply $n'_{m_2}+n_{m_1+m_3}=2$. So $n'_{m_2}=n_{m_1+m_3}=1$ and $m_2=m_3+1$. And $n_1=n_2=\cdots=n_{m_1}=m_2$.

Observe that $-1=A_{*1}^2+2=-3+n_1$, so $m_2=n_{m_1}\le 2$. Because $m_2\ge 2$, we have $m_2=2$ and $m_3=1$.Therefore we are in case (2).

(II) Assume that Γ_1 is of the form $D_m(m\ge 4)$of case (2) in Theorem 4.2.

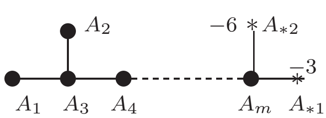
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1}=\begin{array}{ccccc} & 1 & n_2 & & \\ 1 & n_1 & n_3 & n_4\dots n_m. \end{array}$$

As in the proof of Proposition 6.10, we have $n_m=1$, $n_1=\frac{m}{2}$. Since$-1=A_{*1}^2+2=A_{*1}\cdot Z\ge -3+n_1$, so $m\le 4$. And $m\ge 4$, so $m=4$ and we are in a special case in case (4).

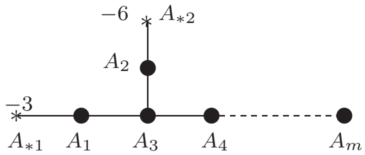
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1}=\begin{array}{cccccc} & n_2 & & & 1 & \\ n_1 & n_3 & n_4\dots & & n_m & 1. \end{array}$$

As in the proof of Proposition 6.10, we have $n_1 = n_2 = 1$ and $n_3 = n_4 = \cdots = n_m = 2$. So we are in case (4).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{matrix} & & 1 & \\ & & n_2 & \\ 1 & n_1 & n_3 & n_4\ldots n_m. \end{matrix}$$

By the same argument as before, we have the following equations.

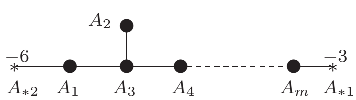
$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + 1 + n_3 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.193}$$

$$-2n_3 + n_1 + n_2 + n_4 = 0 \tag{6.194}$$

From (6.193) we get $n_1 = n_2 = \frac{1}{2} + \frac{n_3}{2}$. And associated with (6.194), we have $n_4 = n_3 - 1$. By (6.194), $n_5 = 2n_4 - n_3 = 2(n_3 - 1) - n_3 = n_3 - 2$. Then $n_6 = 2n_5 - n_4 = 2(n_3 - 2) - (n_3 - 1) = n_3 - 3$.

By induction we get $n_k = n_{k-1} - 1$ for $k \geq 4$. By (6.194) we have $n_{m-1} = 2n_m$. So $n_m = 1$, $n_{m-1} = 2$ and so on. So $n_4 = m - 3$, $n_3 = m - 2$. Since $-1 = A_{*1}^2 + 2 \geq -3 + n_1$, $n_1 = \frac{1}{2} + \frac{m-2}{2} \leq 2$. Then $m \leq 5$ and n_1 is an integer. So $m = 5$. We are in case (5).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{matrix} & & n_2 & & \\ 1 & n_1 & n_3 & n_4\ldots n_m & 1. \end{matrix}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.195}$$

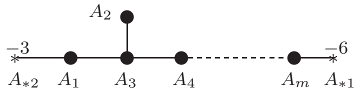
$$-2n_m + n_{m-1} + 1 = 0 \tag{6.196}$$

(6.195) implies

$$n_1 = \frac{1}{2} + n_2. \tag{6.197}$$

Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{array}{cccccc} & & n_2 & & & \\ 1 & n_1 & n_3 & n_4 \ldots n_m & 1. \end{array}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.198}$$

$$-2n_m + n_{m-1} + 1 = 0 \tag{6.199}$$

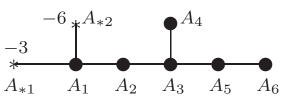
(6.198) implies

$$n_1 = \frac{1}{2} + n_2. \tag{6.200}$$

Contradiction!

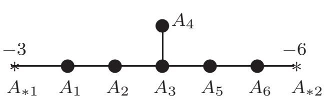
(III) Assume that Γ_1 is of the form E_6 of case (3) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & n_4 & & \\ 1 & n_1 & n_2 & n_3 & n_5 & n_6. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible. Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A*1\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & n_4 & & & \\ 1 & n_1 & n_2 & n_3 & n_5 & n_6 & 1. \end{array}$$

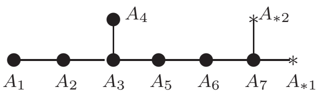
By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + 1 = 0 \end{cases}$$

Since $-1 = A_{*1}^2 + 2 \geq -3 + n_1$, we have $n_1 \leq 2$. Then $n_1 = 2$. We have $n_2 = 3, n_3 = 4, n_4 = 2, n_5 = 3, n_6 = 2$. We are in case (6).

(IV) Assume that Γ_1 is of the form E_7 of case (4) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_7 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & n_4 & & & 1 & \\ n_1 & n_2 & n_3 & n_5 & n_6 & n_7 & 1. \end{array}$$

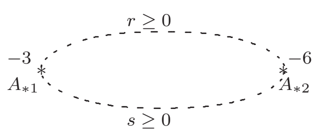
By the same argument as before, we have the following equations

$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 2 = 0 \end{cases} \tag{6.201}$$

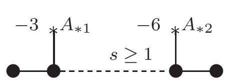
We get $n_7 = 3$. But $A_{*1} \cdot Z = -3 + n_7 = 0$. Contradiction!

(V) Assume that Γ_1 is of the form E_8 of case (5) in Theorem 4.2. This case cannot happen because $A_{*1} \cdot Z_1 \geq 2$. ∎

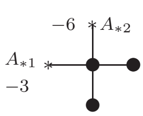
THEOREM 6.30. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (8) of Proposition 6.2 holds, i.e., there exist two effective components A_{*1} and A_{*2} with $A_{*1}^2 = -3, A_{*2}^2 = -6$ and $z_{*1} = 1 = z_{*2}$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

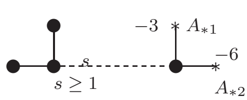
$Z = \underline{1}$ (ellipse) $\underline{1}$

(2) 

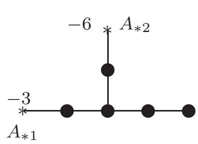
$$Z = \begin{array}{cccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \ldots & 2 & 1 \end{array}$$

(3) 

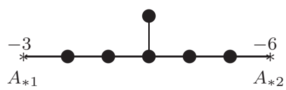
$$Z = \begin{array}{ccc} & \underline{1} & \\ \underline{1} & 2 & 1 \\ & 1 & \end{array}$$

(4) 

$$Z = \begin{array}{ccccc} & 1 & & \underline{1} & \\ 1 & 2 & \ldots & 2 & \underline{1} \end{array}$$

(5) 

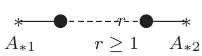
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1 \end{array}$$

(6) 

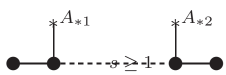
$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. This follows from Proposition 6.3 , Proposition 6.13 and Proposition 6.29. ∎

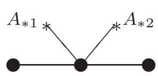
Proposition 6.31. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients z_{*1} of A_{*1} and z_{*2} of A_{*2} in Z are one and $A_{*1}^2 = -4$, $A_{*2}^2 = -5$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and the restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following form.*

(1) 

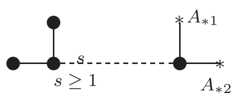
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{1}\ 1\ \ldots\ 1\ \underline{1}$$

(2) 

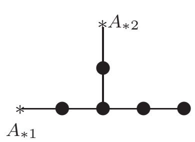
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \ldots\ldots & 2 & 1 \end{array}$$

(3) 

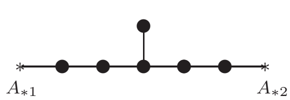
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & & 2 & & 1 \end{array}$$

(4) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & 1 & & \underline{1} & \\ 1 & 2 & \ldots & 2 & \underline{1} \end{array}$$

(5) 

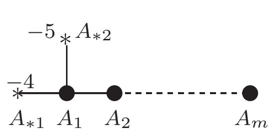
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1 \end{array}$$

(6) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. (I) Assume that Γ_1 is of the form of case (1) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



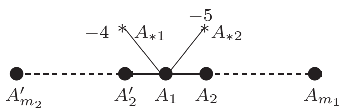
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & & & \\ 1 & n_1 & n_2 & \ldots\ldots & n_m. \end{array}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 2 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{2}{m+1} \end{cases}.$$

Therefore $m = 1$ and it is a special case in case (1).

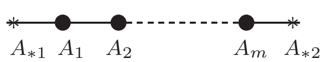
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2 \quad \stackrel{1}{} \quad n_1 \quad \stackrel{1}{} \quad n_2\dots n_{m_1}.$$

As in the proof of Proposition 6.9, we have $n_{m_1} = 1$, $n'_{m_2} = 1$ and $m_1 = m_2 = n_1 = 2$. We are in case (3).

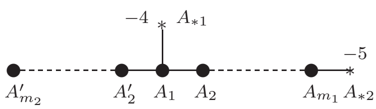
Consider A_{*1} and A_{*2} attaching an Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = 1\ n_1\ n_2\dots n_m\ 1.$$

As in the proof of Proposition 6.9, we have $n_1 = n_2 = \cdots = n_m = 1$ and we are in case (1).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2\ \stackrel{1}{n_1}\ n_2\dots n_{m_1}\ 1.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$ and $A'_j \cdot Z = A'_j \cdot (-K) = A_j'^2 + 2 = 0$, $2 \le j \le m_2$, we have

$$\begin{cases} -2n_{m_1} + n_{m_1-1} + 1 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad \vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \tag{6.202}$$

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.203}$$

$$1 - 2n_1 + n_2 + n'_2 = 0 \tag{6.204}$$

(6.202) implies

$$n_j = (m_1 - j + 1)n_{m_1} - (m_1 - j), \qquad 1 \le j \le m_1 - 1 \tag{6.205}$$

(6.203) implies

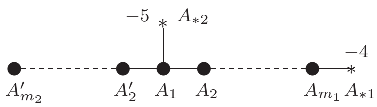
$$n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \le j \le m_2 - 1 \tag{6.206}$$

(6.205) and (6.206) imply

$$m_1 n_{m_1} - (m_1 - 1) = m_2 n'_{m_2} = n_1 \tag{6.207}$$

(6.204), (6.205) and (6.206) imply $n_{m_1} + n'_{m_2} = 2$. So we have $n_{m_1} = n'_{m_2} = 1$. Then $n_1 = 1$ and $n_2 + n'_2 = 1$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2 \;\; \overset{1}{n_1} \;\; n_2\dots n_{m_1}\; 1.$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$ and $A'_j \cdot Z = A'_j \cdot (-K) = A_j'^2 + 2 = 0$, $2 \le j \le m_2$, we have

$$\begin{cases} -2n_{m_1} + n_{m_1-1} + 1 = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \\ \qquad\vdots \\ -2n_3 + n_2 + n_4 = 0 \\ -2n_2 + n_1 + n_3 = 0 \end{cases} \tag{6.208}$$

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.209}$$

$$1 - 2n_1 + n_2 + n'_2 = 0 \tag{6.210}$$

(6.208) implies

$$n_j = (m_1 - j + 1)n_{m_1} - 4(m_1 - j), \qquad 1 \le j \le m_1 - 1 \tag{6.211}$$

(6.209) implies

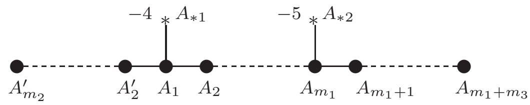
$$n'_j = (m_2 - j + 1)n'_{m_2}, \qquad 1 \le j \le m_2 - 1 \tag{6.212}$$

(6.211) and (6.212) imply

$$m_1 n_{m_1} - 4(m_1 - 1) = m_2 n'_{m_2} = n_1 \tag{6.213}$$

(6.210), (6.211) and (6.212) imply $n_{m_1} + n'_{m_2} = 2$. So we have $n_{m_1} = n'_{m_2} = 1$. Then $n_1 = 1$ and $n_2 + n'_2 = 1$. Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = n'_{m_2}\dots n'_2\ \overset{1}{n_1}\ \ n_2\cdots \overset{1}{n_{m_1}}\ \ n_{m_1+1}\dots n_{m_1+m_3}.$$

And $m_1 \geq 2$, $m_3 \geq 1$, $m_2 \geq 2$ By the same argument as before, we have the following equations

$$\begin{cases} -2n'_{m_2} + n'_{m_2-1} = 0 \\ -2n'_{m_2-1} + n'_{m_2-2} + n'_{m_2} = 0 \\ \qquad\qquad \vdots \\ -2n'_3 + n'_2 + n'_4 = 0 \\ -2n'_2 + n_1 + n'_3 = 0 \end{cases} \tag{6.214}$$

$$-2n_1 + n'_2 + n_2 + 1 = 0 \tag{6.215}$$

$$\begin{cases} -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 = 0 \\ \qquad\qquad \vdots \\ -2n_{m_1-2} + n_{m_1-3} + n_{m_1-1} = 0 \\ -2n_{m_1-1} + n_{m_1-2} + n_{m_1} = 0 \end{cases} \tag{6.216}$$

$$-2n_{m_1} + n_{m_1-1} + n_{m_1+1} + 1 = 0 \tag{6.217}$$

$$\begin{cases} -2n_{m_1+1} + n_{m_1} + n_{m_1+2} = 0 \\ -2n_{m_1+2} + n_{m_1+1} + n_{m_1+3} = 0 \\ \qquad\qquad \vdots \\ -2n_{m_1+m_3-1} + n_{m_1+m_3-2} + n_{m_1+m_3} = 0 \\ -2n_{m_1+m_3} + n_{m_1+m_3-1} = 0 \end{cases} \tag{6.218}$$

(6.214) implies

$$n'_j = (m_2 - j + 1)n'_{m_2} \qquad 1 \leq j \leq m_2 \tag{6.219}$$

(6.215) and (6.219) imply

$$n_2 = (m_2 + 1)n'_{m_2} - 1 \tag{6.220}$$

(6.220) and (6.216) imply

$$n_j = (m_2 + j - 1)n'_{m_2} - (j - 1), \qquad 2 \leq j \leq m_1, \tag{6.221}$$

(6.218) implies

$$n_{m_1+j} = (m_3 - j + 1)n_{m_1+m_3}, \qquad 0 \leq j \leq m_3, \tag{6.222}$$

(6.221) and (6.222) imply

$$n_{m_1} = (m_2 + m_1 - 1)n'_{m_2} - (m_1 - 1) = (m_3 + 1)n_{m_1+m_3} \tag{6.223}$$

(6.221), (6.222) and (6.217) imply

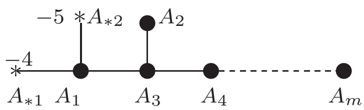
$$(m_2 + m_1)n'_{m_2} - m_1 = m_3 n_{m_1+m_3} + 1 \tag{6.224}$$

(6.223) and (6.224) imply $n'_{m_2} + n_{m_1+m_3} = 2$. So $n'_{m_2} = n_{m_1+m_3} = 1$ and $m_2 = m_3 + 1$. And $n_1 = n_2 = \cdots = n_{m_1} = m_2$.

Observe that $A_{*1} \cdot (Z - A_{*1}) = A_{*1}^2 + 2 - A_{*1}^2 = 2$, so $m_2 = n_{m_1} \leq 2$. Because $m_2 \geq 2$, we have $m_2 = 2$ and $m_3 = 1$.Therefore we are in case (2).

(II) Assume that Γ_1 is of the form $D_m (m \geq 4)$of case (2) in Theorem 4.2.

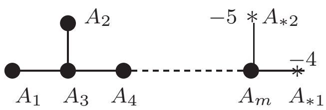
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} 1 & n_2 & & \\ 1\ n_1 & n_3 & n_4 \ldots n_m. \end{array}$$

As in the proof of Proposition 6.10, we have $n_m = 1$, $n_1 = \frac{m}{2}$. Since$A_{*1} \cdot (Z - A_{*1}) = A_{*1}^2 + 2 - A_{*1}^2 = 2 \geq n_1$, so $m \leq 4$. And $m \geq 4$, so $m = 4$ and we are in a special case in case (4).

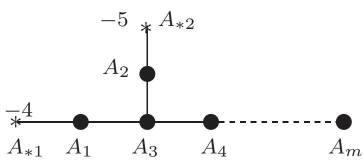
Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & n_2 & & & 1 & \\ n_1 & n_3 & n_4 & \ldots & n_m & 1. \end{array}$$

As in the proof of Proposition 6.10, we have $n_1 = n_2 = 1$ and $n_3 = n_4 = \cdots = n_m = 2$. So we are in case (4).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & 1 & & \\ & n_2 & & \\ 1\ n_1 & n_3 & n_4 \ldots n_m. \end{array}$$

By the same argument as before, we have the following equations.

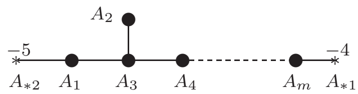
$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + 1 + n_3 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad \vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \tag{6.225}$$

$$-2n_3 + n_1 + n_2 + n_4 = 0 \tag{6.226}$$

From (6.225) we get $n_1 = n_2 = \frac{1}{2} + \frac{n_3}{2}$. And associated with (6.226), we have $n_4 = n_3 - 1$. By (6.225), we have $n_4 = n_3 - 1$. By (6.194), $n_5 = 2n_4 - n_3 = 2(n_3 - 1) - n_3 = n_3 - 2$. Then $n_6 = 2n_5 - n_4 = 2(n_3 - 2) - (n_3 - 1) = n_3 - 3$.

By induction we get $n_k = n_{k-1} - 1$ for $k \geq 4$. By (6.194) we have $n_{m-1} = 2n_m$. So $n_m = 1$, $n_{m-1} = 2$ and so on. So $n_4 = m-3$, $n_3 = m-2$. Since $-1 = A_{*1}^2 + 2 \geq -3 + n_1$, $n_1 = \frac{1}{2} + \frac{m-2}{2} \leq 2$. Then $m \leq 5$ and n_1 is an integer. So $m = 5$. We are in case (5).

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{matrix} & & n_2 & & \\ 1 & n_1 & n_3 & n_4 \ldots n_m & 1. \end{matrix}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.227}$$

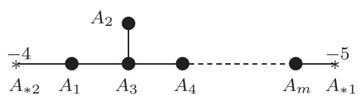
$$-2n_m + n_{m-1} + 1 = 0 \tag{6.228}$$

(6.227) implies

$$n_1 = \frac{1}{2} + n_2. \tag{6.229}$$

Contradiction!

Consider A_{*1} and A_{*2} attaching on Γ_1 in the following form



$$Z\Big|_{A_{1*}\cup A_{2*}\cup\Gamma_1} = \begin{matrix} & & n_2 & & \\ 1 & n_1 & n_3 & n_4 \ldots n_m & 1. \end{matrix}$$

By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_3 = 0 \\ -2n_2 + n_3 = 0 \\ -2n_3 + n_1 + n_2 + n_4 = 0 \\ -2n_4 + n_3 + n_5 = 0 \\ \qquad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \end{cases} \tag{6.230}$$

$$-2n_m + n_{m-1} + 1 = 0 \tag{6.231}$$

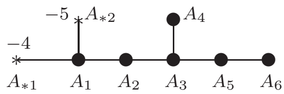
(6.230) implies

$$n_1 = \frac{1}{2} + n_2. \tag{6.232}$$

Contradiction!

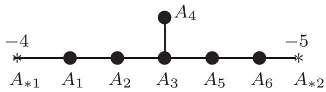
(III) Assume that Γ_1 is of the form E_6 of case (3) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & & n_4 & & \\ 1 & n_1 & n_2 & n_3 & n_5 & n_6. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible. Consider A_{*1} and A_{*2} attaching on E_6 in the following form



$$Z\Big|_{A*1\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & n_4 & & & \\ 1 & n_1 & n_2 & n_3 & n_5 & n_6 & 1. \end{array}$$

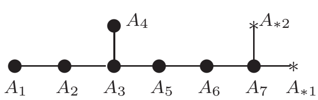
By the same argument as before, we have the following equations.

$$\begin{cases} -2n_1 + 1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + 1 = 0 \end{cases}$$

Since $A_{*1} \cdot (Z - A_{*1}) = A_{*1}^2 + 2 - A_{*1}^2 = 2 \geq n_1$, we have $n_1 \leq 2$. Then $n_1 = 2$. We have $n_2 = 3, n_3 = 4, n_4 = 2, n_5 = 3, n_6 = 2$. We are in case (6).

(IV) Assume that Γ_1 is of the form E_7 of case (4) in Theorem 4.2.

Consider A_{*1} and A_{*2} attaching on E_7 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & n_4 & & & 1 & \\ n_1 & n_2 & n_3 & n_5 & n_6 & n_7 & 1. \end{array}$$

By the same argument as before, we have the following equations

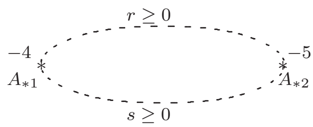
$$\begin{cases} -2n_1 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ -2n_3 + n_2 + n_4 + n_5 = 0 \\ -2n_4 + n_3 = 0 \\ -2n_5 + n_3 + n_6 = 0 \\ -2n_6 + n_5 + n_7 = 0 \\ -2n_7 + n_6 + 2 = 0 \end{cases} \tag{6.233}$$

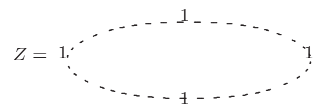
We get $n_7 = 3$. But $A_{*1} \cdot Z = -3 + n_7 = 0$. Contradiction!

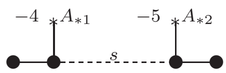
(V) Assume that Γ_1 is of the form E_8 of case (5) in Theorem 4.2. This case cannot happen because $A_{*1} \cdot Z_1 \geq 2$. □

THEOREM 6.32. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (9) of Proposition 6.2 holds,*

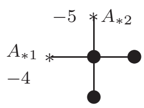
i.e., there exist two effective components A_{*1} *and* A_{*2} *with* $A_{*1}^2 = -4, A_{*2}^2 = -5$ *and* $z_{*1} = 1 = z_{*2}$*, then the weighted dual graph* Γ *of the exceptional set is one of the following forms.*

(1)


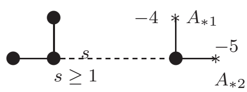


(2)


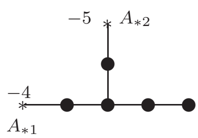
$$Z = \begin{array}{cccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \dots & 2 & 1 \end{array}$$

(3)


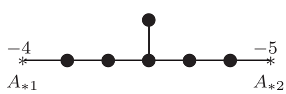
$$Z = \begin{array}{ccc} & \underline{1} & \\ \underline{1} & 2 & 1 \\ & 1 & \end{array}$$

(4)


$$Z = \begin{array}{cccccc} & 1 & & \underline{1} & \\ 1 & 2 & \dots & 2 & \underline{1} \end{array}$$

(5)


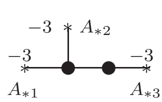
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1 \end{array}$$

(6)


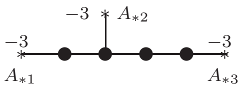
$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. This follows from Proposition 6.3, Proposition 6.13 and Proposition 6.31. ☐

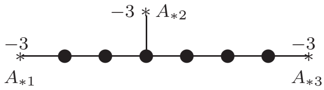
Proposition 6.33. *Let* Γ *be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle* Z*. Let* Γ' *be the subgraph of* Γ *by removing all the effective components of* Γ*. Let* A_{*1}*,* A_{*2} *and* A_{*3} *be three effective components of* Γ*. Suppose that* Γ_1 *is a connected component of* Γ' *which corresponds to a rational double point graph in Theorem 4.2. Suppose also that* Γ_1 *intersects with* A_{*1}*,* A_{*2} *and* A_{*3}*, but disjoint from other effective component. Let* Z_1 *be the fundamental cycle on* Γ_1*. Suppose that* $A_{*1} \cdot Z_1 = 1 = A_{*2} \cdot Z_1 = A_{*3} \cdot Z_1$*. If* A_{*1}*,* A_{*2} *and* A_{*3} *are mutually disjoint,* $z_{*1} = 2 = z_{*2}$*,* $z_{*3} = 1$*, and* $A_{*1}^2 = -3 = A_{*2}^2 = A_{*3}^2$*, then* $A_{*1} \cup A_{*2} \cup A_{*3} \cup \Gamma_1$ *and restriction of* Z *on* $A_{*1} \cup A_{*2} \cup A_{*3} \cup \Gamma_1$ *must be one of the following form.*

(1)


$$Z\Big|_{A_{*1} \cup A_{*2} \cup A_3 \cup \Gamma_1} = \begin{array}{cccc} & \underline{2} & & \\ \underline{2} & 3 & 2 & \underline{1} \end{array}$$

(2) 

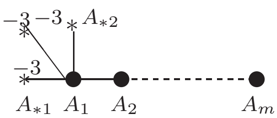
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_3\cup\Gamma_1} = \begin{array}{cccccc} & & \underline{2} & & & \\ \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(3) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_3\cup\Gamma_1} = \begin{array}{cccccccc} & & & \underline{2} & & & & \\ \underline{2} & 3 & 4 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. (I) Assume that Γ_1 is of the form of case (1) in Theorem 4.2.

Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



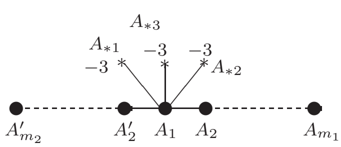
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & 1\ 2 & & & & \\ 2 & n_1 & n_2 & \ldots\ldots & n_m. \end{array}$$

Since $A_i \cdot Z = A_i \cdot (-K) = A_i^2 + 2 = 0$, $1 \le i \le m$, we have the following system of equations.

$$\begin{cases} -2n_1 + 5 + n_2 = 0 \\ -2n_2 + n_1 + n_3 = 0 \\ \quad\vdots \\ -2n_{m-1} + n_{m-2} + n_m = 0 \\ -2n_m + n_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} n_i = (m-i+1)n_m \quad 1 \le i \le m \\ n_m = \frac{5}{m+1} \end{cases}.$$

Therefore $m = 4$, then $n_m = 1$, $n_1 = mn_m = 4$. Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \ge n_1 = 4$. Contradicton!

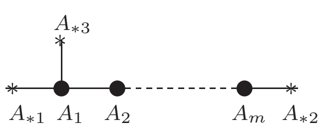
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccc} & 2\ 1\ 2 & \\ n'_{m_2}\ldots n'_2 & n_1 & n_2\ldots n_{m_1}. \end{array}$$

Since $A_{*3} \cdot (Z - A_{*3}) = A_{*3}^2 + 2 - A_{*3}^2 = 2 \ge n_1$, then $-2n_1 + 5 + n_2 + n'_2 > 0$. Contradiction!

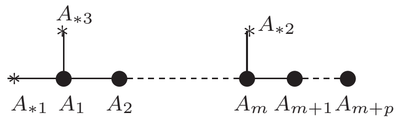
Consider A_{*1}, A_{*2} and A_{*3} attaching an Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccc} & 1 & & \\ 2 & n_1 & n_2\ldots n_m & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \ge n_1$, and $n_2 \ge 2$, then $-2n_1+2+1+n_2 > 0$. Contradiction!

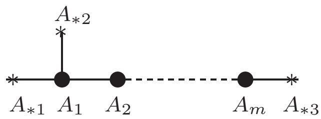
Consider A_{*1}, A_{*2} and A_{*3} attaching an Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = 2\ \ \overset{1}{n_1}\ \ n_2\cdots \overset{2}{n_m}\ \ n_{m+1}\cdots\ n_{m+p}.$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, and $n_2 \geq 2$, then $-2n_1+2+1+n_2 > 0$. Contradiction!

Consider A_{*1}, A_{*2} and A_{*3} attaching an Γ_1 in the following form.

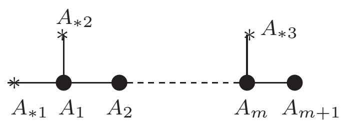


$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = 2\ \ \overset{2}{n_1}\ \ n_2\ldots n_m\ 1.$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_m$. If $n_m = 1$, then $n_{m-1} = n_{m_2} = \cdots = n_1 = 1$, $-2n_1+2+2+n_2 > 0$, contradiction. So $n_m = 2$ and $n_1 = m+1$. Then $-2n_1+2+2+n_2 = -2m-2+2+2+m = 0$. We have $m = 2$. Then We are in case (1).

Consider A_{*1}, A_{*2} and A_{*3} attaching an Γ_1 in the following form.

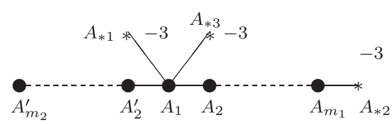
Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_m$. So $n_m = 2$ and $n_{m+1}+n_{m-1} = 3$. So n_{m+1} has to be 1 and Γ_1 can only have $m+1$ components.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = 2\ \ n_1\ \ n_2\cdots \overset{2}{n_m}\ \ \overset{1}{n_{m+1}}.$$

And $n_{m-1} = 2$. Thus $n_{m-1} = n_{m-2} = \cdots = n_2 = n_1 = 2$. But $-2n_1+2+2+n_2 = -4+6 = 2 = 0$. Contradiction!

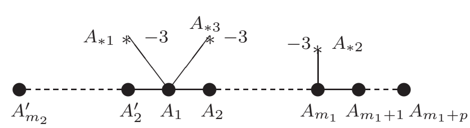
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = n'_{m_2}\ldots n'_2\quad \overset{2}{n_1}\quad \overset{1}{n_2}\ldots n_{m_1}\ 2.$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$. Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2+2 = 0$, $1 \leq i \leq m$ and $A'_j\cdot Z = A'_j\cdot(-K) = A_j'^2+2 = 0$, $2 \leq j \leq m_2$, we have $1+2-2n_1+n_2+n'_2 = 0$. Thus $n_2+n'_2 = 1$. Contradiction!

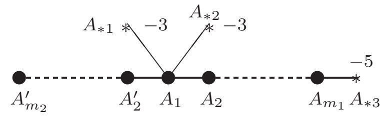
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = n'_{m_2}\ldots \overset{2}{n'_2}\quad n_1\quad \overset{1}{n_2}\cdots \overset{2}{n_{m_1}}\ \ n_{m_1+1}\ \cdots\ n_{m_1+p}.$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$. Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2+2 = 0$, $1 \leq i \leq m$ and $A'_j \cdot Z = A'_j\cdot(-K) = A_j^{'2}+2 = 0$, $2 \leq j \leq m_2$, we have $1+2-2n_1+n_2+n'_2 = 0$. Thus $n_2+n'_2 = 1$. Contradiction!

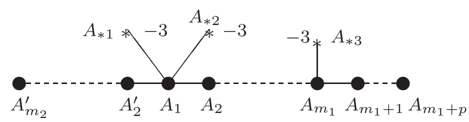
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & 2 & 2 & & \\ n'_{m_2}\dots n'_2 & n_1 & & n_2\dots n_{m_1} & 1. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_{m_1}$, $n_{m_1} = 2$. Since $A_i\cdot Z = A_i\cdot(-K) = A_i^2+2 = 0$, $1 \leq i \leq m$ and $A'_j\cdot Z = A'_j\cdot(-K) = A_j^{'2}+2 = 0$, $2 \leq j \leq m_2$, we have $n_1 = m_1+1, n_2 = m_1$. Thus $n'_2 = 2n_1-2-2-n_2 = m_1-2$. Since $A_{*1}\cdot(Z-2A_{*1}) = A_{*1}^2+2-2A_{*1}^2 = 5 \geq n_1$, $m_1+1 \leq 5$. So $n'_2 = m_1-2 \leq 2$ and $n'_3 = m_1-5 < 0$. So m_2 cannot be larger than 2. But $2n'_2 = n_1$ and then $2m_1-4 = m_1+1$. We have $m_1 = 5$. This contradicts to the fact that $m_1 \leq 4$.

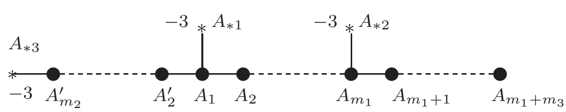
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & 2 & 2 & 1 & & \\ n'_{m_2}\dots n'_2 & n_1 & n_2\cdots n_{m_1} & & n_{m_1+1} & \cdots\ n_{m_1+p}. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_{m_1}$, $n_{m_1} = 2$. So $n_{m_1+1} = 1$ and $p = 1$. Thus $n_{m_1-1} = 2 = n_{m_1-2} = \dots = n_1$. But $-2n_1+2+2+n_2+n'_2 = 2+n'_2 = 0$. This contradicts with the fact that $n'_2 > 0$.

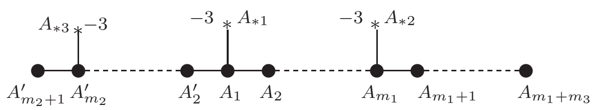
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & & 2 & 2 & \\ 1 & n'_{m_2}\dots n'_2 & n_1 & n_2\cdots n_{m_1} & n_{m_1+1}\dots n_{m_1+m_3}. \end{array}$$

And $m_1 \geq 2$, $m_3 \geq 1$, $m_2 \geq 2$ By the same argument as before, we have $n'_{m_2} = 2$, $n_1 = m_2+1 \leq 5$, $n_{m_1} = m_2-m_1+2 \leq 5$ and $n_{m_1+1} = m_2-m_1-1 \geq 1$. So $n_{m_1+1} = 1$, $n_{m_1} = 4$ or $n_{m_1+1} = 2$, $n_{m_1} = 5$. Both are impossible because $-2n_{m_1+1}+n_{m_1}+n_{m_1+2} = 0$. Contradiction!

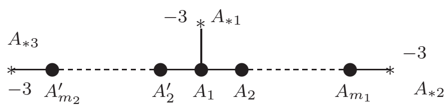
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form. By the same argument as before, we have $n'_{m_2} = 2$ and $n'_{m_2-1} \geq 2$. So $n'_{m_2+1} = 1$ and the graph has to be like the following form:



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & 1 & 2 & 2 & \\ n'_{m_2+1} & n'_{m_2}\dots n'_2 & n_1 & n_2\cdots n_{m_1} & n_{m_1+1}\dots n_{m_1+m_3}. \end{array}$$

We have $n'_{m_2-1} = 2 = n'_{m_2-2} = n'_2 = n_1$. But $-2n_1 + 2 + n_2 + n'_2 = n_2 = 0$ Contradiction!

Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.

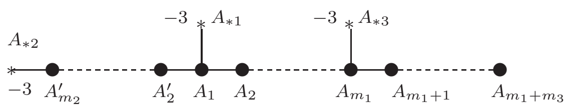


$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = 1\ n'_{m_2}\dots n'_2\ \overset{2}{n_1}\ \ n_2\dots n_{m_1}\ 2.$$

And $m_1 \geq 2$, $m_3 \geq 1$, $m_2 \geq 2$ By the same argument as before, we have $n'_{m_2} = 2$, $3 \leq n_1 = m_2 + 1 \leq 5$, $n_{m_1-1} = m_2 - m_1 + 3$, $n_{m_1} = m_2 - m_1 + 2 \leq 5$ and $-2n_{m_1} + 2 + n_{m_1-1} = 0$. So $n_{m_1} = 3$.

Thus $m_2 - m_1 = 1$. Because $2 \leq m_2 \leq 4$ and $m_1 \geq 2$, we have $m_1 = 2$, $m_2 = 3$ or $m_1 = 3$, $m_2 = 4$. Then we are in case (2) and case (3).

Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



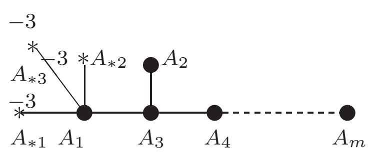
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = 2\ n'_{m_2}\dots n'_2\ \overset{2}{n_1}\ \ n_2\cdots \overset{1}{n_{m_1}}\ \ n_{m_1+1}\dots n_{m_1+m_3}.$$

And $m_1 \geq 2$, $m_3 \geq 1$, $m_2 \geq 2$ By the same argument as before, we have $n'_{m_2} = 2$, $n_{m_1+1} = 1$. So $n_{m_1-1} = 2 = n_{m_1-2} = \dots = n_2 = n_1$. But $-2n_1 + 2 + n_2 + n'_2 = 0$. Then $n'_2 = 0$. Contradiction!

And A_{*3} can not intersect in the middle of the Γ_1 and between the components intersected by A_{*1} and A_{*2}. Because the component intersects with A_{*3} must have factor 2 in Z_1 and can only connect with one component with factor 3 or two components with factors 2 and 1.

(II) Assume that Γ_1 is of the form $D_m (m \geq 4)$of case (2) in Theorem 4.2.

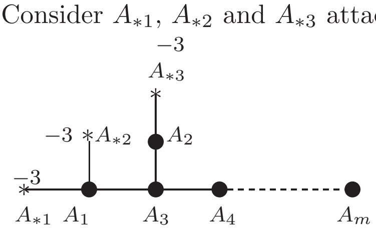
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} \begin{array}{l} \ \ \ \ 1\ \ 2\ \ n_2 \\ =2\ n_1\ n_3\ \ n_4\dots n_m. \end{array}$$

As in the proof of Proposition 6.10, we have $n_m = \frac{5}{2}$. This contradicts the fact that n_m is an integer.

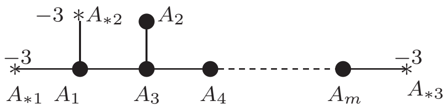
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} \begin{array}{l} \ \ \ \ \ \ \ \ \ \ \ 1 \\ \ \ \ \ \ \ 2\ \ n_2 \\ = 2\ \ n_1\ n_3\ \ n_4\dots n_m. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_2$, $n_2 = 2$. So $n_3 = 3$ and $-2n_1+2+2+3 = 0$. We get $n_1 = \frac{7}{2}$. This contradicts the fact that n_1 is an integer.

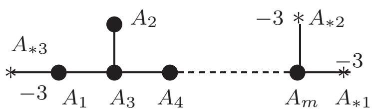
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & 2 & n_2 & & & \\ 2 & n_1 & n_3 & n_4\dots n_m & 1. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_m$, $n_m = 2$. So $n_3 = m+1$, $n_2 = \frac{m+1}{2}$ and $-2n_1+2+2+m+1 = 0$. We get $n_1 = \frac{m+5}{2}$. And $-2n_3+n_2+n_1+n_4 = 0$, we get $1 = 0$. Contradiction!

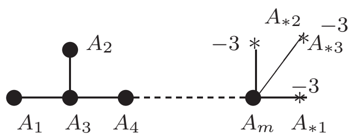
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & & n_2 & & 2 & \\ 1 & n_1 & n_3 & n_4\dots & n_m & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$ and $n_3 = 3$. But $n_2 = \frac{n_3}{2}$. This contradicts with the fact that n_2 is an integer.

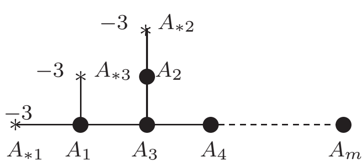
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & n_2 & & 2 & 1 \\ n_1 & n_3 & n_4\dots & n_m & 2. \end{array}$$

As in the proof of Proposition 6.10, we have $n_1 = \frac{5}{2}$. This contradicts the fact that n_1 is an integer.

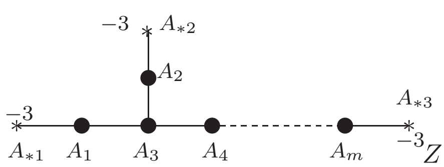
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccc} & & 2 & \\ & 1 & n_2 & \\ 2 & n_1 & n_3 & n_4\dots n_m. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$ and $n_3 = 1$. But $n_2 \geq 2$ and $n_4 \geq 1$. This contradicts with the fact that $-2n_3+n_2+n_1+n+4 = 0$.

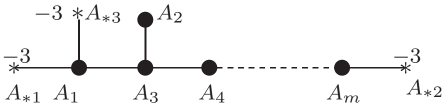
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & & 2 & & \\ & & n_2 & & \\ 2 & n_1 & n_3 & n_4\dots n_m & 1. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$ and $n_3 = m-1$. So $n_1 = n_2 = \frac{m+1}{2}$. Then $-2n_3+n_1+n_2+n_4 = -2m+2+m+1+m-2 = 1$. This contradicts with the fact that $A_3\cdot Z = 0$.

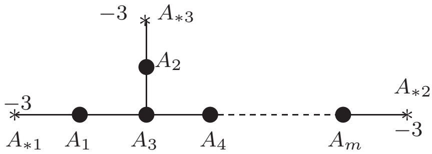
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & n_2 & & & \\ 2 & n_1 & n_3 & n_4\ldots n_m & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, $n_1 = 2$. So $n_3 = 1$, $n_2 = \frac{1}{2}$. This contradicts with the fact that n_2 is an integer.

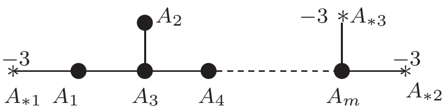
Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccc} & & 1 & & \\ & & n_2 & & \\ 2 & n_1 & n_3 & n_4\ldots n_m & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_2$, $n_2 = 2$ and $n_3 = 3$. So $-2n_1+2+3 = 0$ and $n_1 = \frac{5}{2}$. This contradicts with the fact that n_1 is an integer.

Consider A_{*1}, A_{*2} and A_{*3} attaching on Γ_1 in the following form.

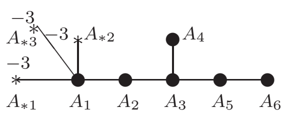


$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & & n_2 & & 1 & \\ 2 & n_1 & n_3 & n_4\cdots & n_m & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_m$, $n_m = 2$. So $n_{m-1} = 1$. But $-2n_{m-1}+n_m+n_{m-2} = 0$. Contradiction!

(III) Assume that Γ_1 is of the form E_6 of case (3) in Theorem 4.2.

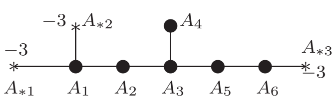
Consider A_{*1}, A_{*2} and A_{*3} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & 1 & 2 & n_4 & & \\ 2 & n_1 & n_2 & n_3 & n_5 & n_6. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible.

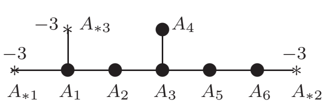
Consider A_{*1}, A_{*2} and A_{*3} attaching on E_6 in the following form



$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccccc} & 2 & & n_4 & & & \\ 2 & n_1 & n_2 & n_3 & n_5 & n_6 & 1. \end{array}$$

As in the proof of Proposition 6.11, we find out that this case is not possible.

Consider A_{*1}, A_{*2} and A_{*3} attaching on E_6 in the following form

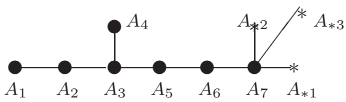


$$Z\Big|_{A*1\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{ccccccc} & 1 & & n_4 & & & \\ 2 & n_1 & n_2 & n_3 & n_5 & n_6 & 2. \end{array}$$

Since $A_{*3}\cdot(Z-A_{*3}) = A_{*3}^2+2-A_{*3}^2 = 2 \geq n_1$, we have $n_1 = 2$. We have $n_2 = 1$. But n_2 should be larger than 1. So this is not possible.

(IV) Assume that Γ_1 is of the form E_7 of case (4) in Theorem 4.2.

Consider A_{*1}, A_{*2} and A_{*3} attaching on E_7 in the following form

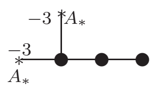


$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup\Gamma_1} = \begin{array}{cccccc} & & n_4 & & & 2\ 1 \\ n_1 & n_2 & n_3 & n_5 & n_6 & n_7 2. \end{array}$$

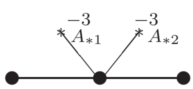
As in the proof of Proposition 6.11, this is not possible.

(V) Assume that Γ_1 is of the form E_8 of case (5) in Theorem 4.2. This case cannot happen because $A_{*1} \cdot Z_1 \geq 2$. $\square$

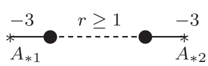
Proposition 6.34. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with both A_{*1} and A_{*2}, but no other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose $A_{*1} \cdot Z_1 = A_{*2} \cdot Z_1 = 1$. If $A_{*1} \cdot A_{*2} = 0$ and the coefficients z_{*1} of A_{*1} and z_{*2} of A_{*2} in Z are 2 and $A_{*1}^2 = A_{*2}^2 = -3$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following forms.*

(1) 

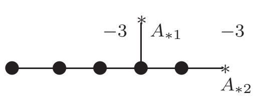
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & \underline{2} & & \\ \underline{2} & 3 & 2 & 1 \end{array}$$

(2) 

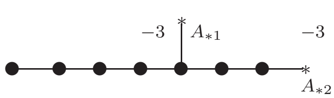
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{2} & & \underline{2} & \\ 2 & & 4 & & 2 \end{array}$$

(3) 

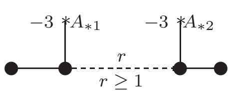
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \underline{2}\ 2\ \ldots\ 2\ \underline{2}$$

(4) 

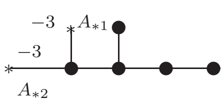
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & & \underline{2} & & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} \end{array}$$

(5) 

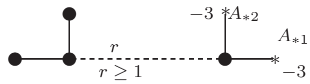
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccc} & & & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & \underline{2} \end{array}$$

(6) 

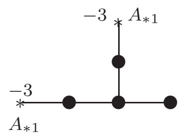
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{2} & & \underline{2} & \\ 2 & 4 & \ldots & 4 & 2 \end{array}$$

(7) 

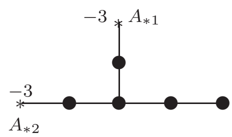
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{2} & 3 & & \\ \underline{2} & 5 & 6 & 4 & 2 \end{array}$$

(8) 

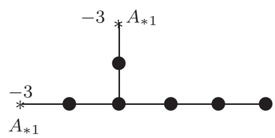
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & 2 & & \underline{2} & \\ 2 & 4 & \dots & 4 & \underline{2}\end{array}$$

(9) 

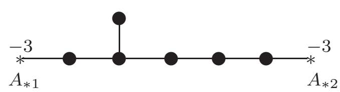
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & & \underline{2} & \\ & & 3 & \\ \underline{2} & 3 & 4 & 2\end{array}$$

(10) 

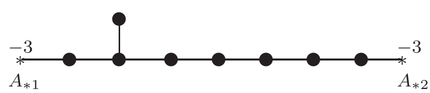
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccc} & & \underline{2} & & \\ & & 4 & & \\ \underline{2} & 4 & 6 & 4 & 2\end{array}$$

(11) 

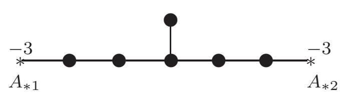
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & \underline{2} & & & \\ & & 5 & & & \\ \underline{2} & 5 & 8 & 6 & 4 & 2\end{array}$$

(12) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & 3 & & & & \\ \underline{2} & 4 & 6 & 5 & 4 & 3 & \underline{2}\end{array}$$

(13) 

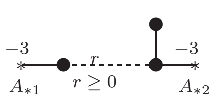
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccccc} & & 4 & & & & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2}\end{array}$$

(14) 

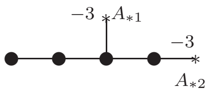
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & 4 & & & \\ \underline{2} & 4 & 6 & 8 & 6 & 4 & \underline{2}\end{array}$$

Proof. The proof is the same as Proposition 6.18 in [C-X-Y]'s paper. ∎

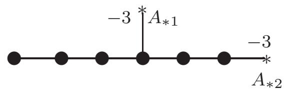
Proposition 6.35. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1} and A_{*2} be two effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1} and A_{*2}, but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose that $A_{*1} \cdot Z_1 = 1 = A_{*2} \cdot Z_1$. If $A_{*1} \cdot A_{*2} = 0$, $z_{*1} = 2$, $z_{*2} = 1$ (coefficient of A_{*1} and A_{*2} in Z respectively), and $A_{*1}^2 = -3 = A_{*2}^3$, then $A_{*1} \cup A_{*2} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup \Gamma_1$ must be one of the following form.*

(1) 

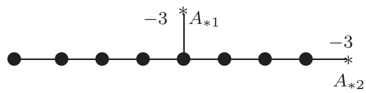
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccc} & & & 1 \\ \underline{2} & 2\dots & 2 & \underline{1}\end{array}$$

(2) 

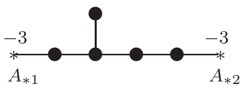
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & \underline{2} & & \\ 1 & 2 & 3 & 2 & \underline{1} \end{array}$$

(3) 

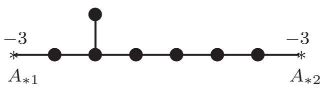
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccc} & & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(4) 

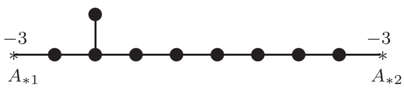
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{ccccccccc} & & & & \underline{2} & & & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

(5) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccc} & & 2 & & & \\ \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

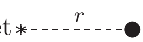
(6) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccc} & & 3 & & & & & \\ \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

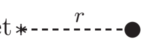
(7) 

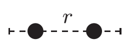
$$Z\Big|_{A_{*1}\cup A_{*2}\cup\Gamma_1} = \begin{array}{cccccccccc} & & 4 & & & & & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. The proof is similar as Proposition 6.33. ∎

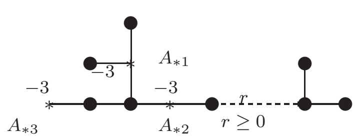
Remark. In the following Theorem, when $r \geq 1$, let  denote

with one effective component, r vertices ● which represent nonsingular rational curves with weight -2. and r edges;

when $r = 0$, let  denote $*$, only one effective component.

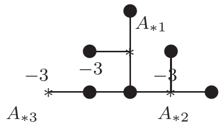
And  denote  with r vertices and $r+1$ edges. ● is a nonsingular rational curve with weight -2.

Theorem 6.36. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (10) of Proposition 6.2 holds, i.e., there exist three effective components A_{*1}, A_{*2} and A_{*3} with $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = -3$ and $z_{*1} = z_{*2} = 2$, $z_{*3} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following form.*

(1) 

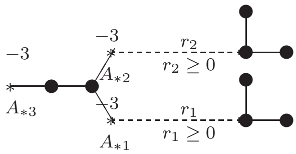
$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & 1 & \underline{2} & & & & 1 & \\ \underline{1} & 2 & 3 & \underline{2} & 2\ldots & & 2 & 1 \end{array}$$

(2)



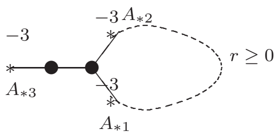
$$Z=\begin{array}{ccccc} & & 1 & & \\ & 1 & \underline{2} & 1 & \\ \underline{1} & 2 & 3 & \underline{2} & 1\end{array}$$

(3)



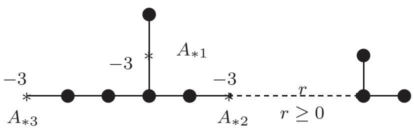
$$Z=\begin{array}{cccccc} & & & & 1 & \\ & & & \underline{2}\ldots & 2 & 1\\ \underline{1} & 2 & 3 & & & \\ & & & \underline{2}\ldots & 2 & 1 \\ & & & & 1 & \end{array}$$

(4)



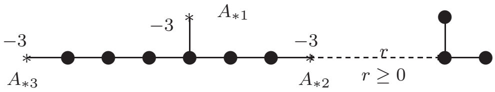
$$Z=\begin{array}{ccc} & \underline{2} & \\ \underline{1} & 2 & 3 \\ & \underline{2} & \end{array}$$

(5)



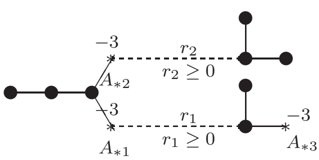
$$Z=\begin{array}{ccccccccc} & & & 1 & & & & & \\ & & & \underline{2} & & & & 1 & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & \ldots & 2 & 1\end{array}$$

(6)



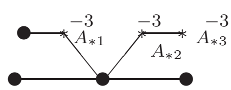
$$Z=\begin{array}{ccccccccccc} & & & & \underline{2} & & & & & 1 & \\ \underline{1} & 2 & 3 & 4 & 5 & 4 & 3 & \underline{2} & \ldots & 2 & 1\end{array}$$

(7)



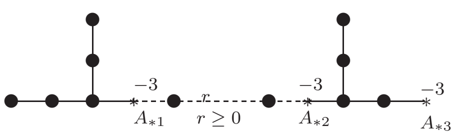
$$Z=\begin{array}{cccccc} & & & & 1 & \\ & & & \multicolumn{2}{c}{\underbrace{\underline{2}\ldots 2}} & 1\\ 1 & 2 & 3 & & & \\ & & & \multicolumn{2}{c}{\underbrace{\underline{2}\ldots 2}} & \underline{1} \\ & & & & 1 & \end{array}$$

(8)



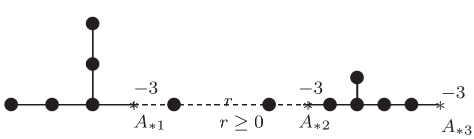
$$Z=\begin{array}{ccccc}1&\underline{2}& &\underline{2}&\underline{1}\\ 2& &4& &2\end{array}$$

(9)



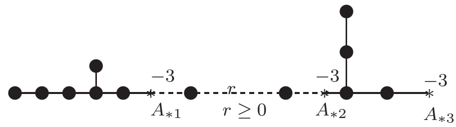
$$Z=\begin{array}{cccccccccc} & & 1 & & & & & 1 & & \\ & & 2 & & & & & 2 & & \\ 1 & 2 & 3 & \underline{2} & 2\ldots & 2 & \underline{2} & 3 & 2 & \underline{1}\end{array}$$

(10)



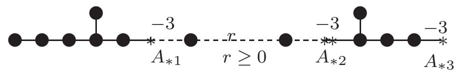
$$Z=\begin{array}{cccccccccccc} & & 1 & & & & & & & & & \\ & & 2 & & & & & & 2 & & & \\ 1 & 2 & 3 & \underline{2} & 2\ldots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1}\end{array}$$

(11)



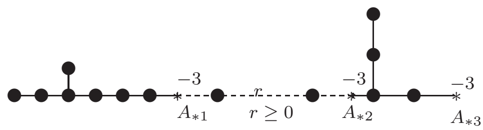
$$Z = \begin{array}{ccccccccccccc} & & & & & & & & & & 1 & & \\ & & & 2 & & & & & & & 2 & & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & 3 & 2 & \underline{1} \end{array}$$

(12)



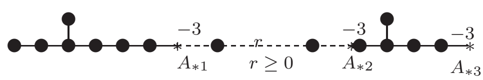
$$Z = \begin{array}{ccccccccccccccc} & & & 2 & & & & & & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(13)



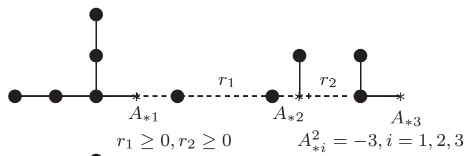
$$Z = \begin{array}{ccccccccccccc} & & & & & & & & & & 1 & & \\ & & 3 & & & & & & & & 2 & & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \dots & 2 \; \underline{2} & 3 & 2 & \underline{1} \end{array}$$

(14)



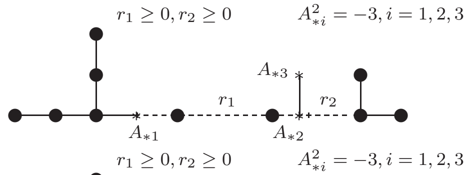
$$Z = \begin{array}{ccccccccccccccc} & & 3 & & & & & & & & & 2 & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} \; 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(15)



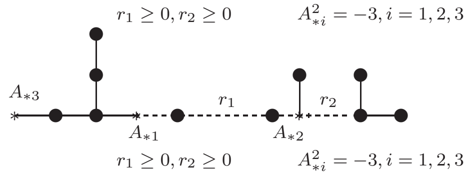
$$Z = \begin{array}{ccccccccccc} & & 1 & & & & & & & & \\ & & 2 & & & & & 1 & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & \underline{1} \end{array}$$

(16)



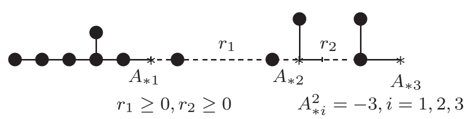
$$Z = \begin{array}{ccccccccccc} & & 1 & & & & & & & & \\ & & 2 & & & & & \underline{1} & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

(17)



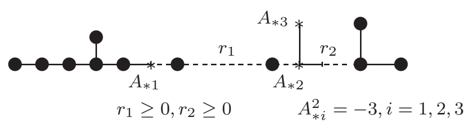
$$Z = \begin{array}{ccccccccccc} & & 1 & & & & & & & & \\ & & 2 & & & & & 1 & & 1 & \\ \underline{1} & 2 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

(18)



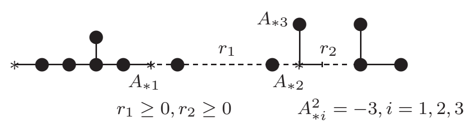
$$Z = \begin{array}{ccccccccccccc} & & & 2 & & & & & & 1 & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & \underline{1} \end{array}$$

(19)

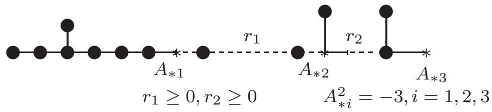


$$Z = \begin{array}{ccccccccccccc} & & & 2 & & & & & & \underline{1} & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

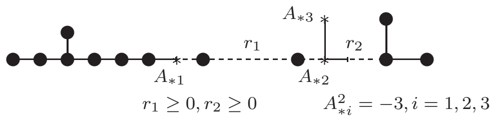
(20)



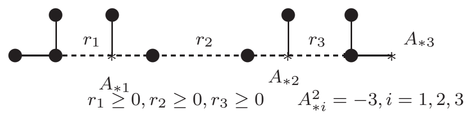
$$Z = \begin{array}{ccccccccccccc} & & & 2 & & & & & & 1 & & 1 & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

(21) 

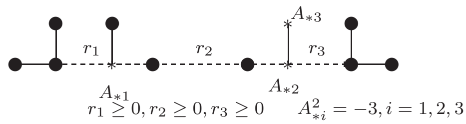
$$Z=\begin{array}{cccccccccccccc} & & 3 & & & & & & & & 1 & & 1 & \\ 2&4&6&5&4&3&\underline{2}&2&\dots&2&\underline{2}&\cdots&2&\underline{1}\end{array}$$

(22) 

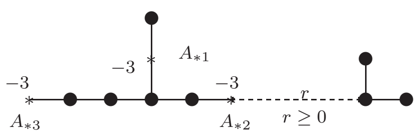
$$Z=\begin{array}{cccccccccccccc} & & 3 & & & & & & & & \underline{1} & & 1 & \\ 2&4&6&5&4&3&\underline{2}&2&\dots&2&\underline{2}&\cdots&2&1\end{array}$$

(23) 

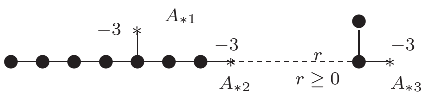
$$Z=\begin{array}{ccccccccccc} & 1 & & 1 & & & & 1 & & 1 & \\ 1&2&\cdots&\underline{2}&2&\dots&2&\underline{2}&\dots&2&\underline{1}\end{array}$$

(24) 

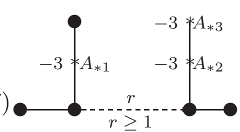
$$Z=\begin{array}{ccccccccccc} & 1 & & 1 & & & & \underline{1} & & 1 & \\ 1&2&\cdots&\underline{2}&2&\dots&2&\underline{2}&\dots&2&1\end{array}$$

(25) 

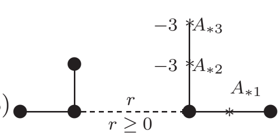
$$Z=\begin{array}{ccccccccc} & & & 1 & & & & & \\ & & & \underline{2} & & & & 1 & \\ 1&2&3&4&3&\underline{2}&\dots&2&1\end{array}$$

(26) 

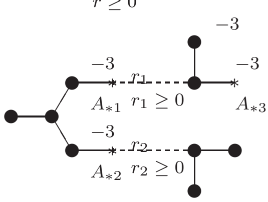
$$Z=\begin{array}{ccccccccccc} & & & & \underline{2} & & & & & 1 & \\ 1&2&3&4&5&4&3&\underline{2}&\dots&2&\underline{1}\end{array}$$

(27) 

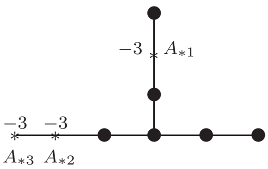
$$Z=\begin{array}{ccccc} & 1 & & \underline{1} & \\ & \underline{2} & & \underline{2} & \\ 2&4&\dots&4&2\end{array}$$

(28) 

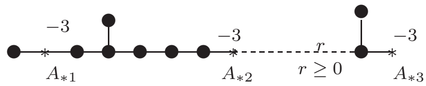
$$Z=\begin{array}{cccccc} & & & \underline{1} & & \\ & 2 & & \underline{2} & & \\ 2&4&\dots&4&\underline{2}&1\end{array}$$

(29) 

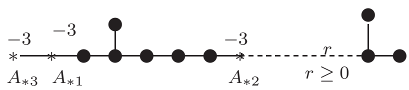
$$Z=\begin{array}{ccccc} & & & 1 & \\ & & 3 & \underbrace{\underline{2}\dots 2} & \underline{1} \\ 2 & 4 & & & \\ & & 3 & \underbrace{\underline{2}\dots 2} & 1 \\ & & & 1 & \end{array}$$

(30) 

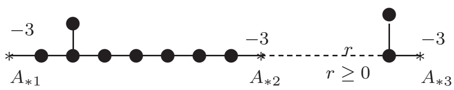
$$Z=\begin{array}{cccccc} & & & 1 & & \\ & & & \underline{2} & & \\ & & & 4 & & \\ \underline{1} & \underline{2} & 4 & 6 & 4 & 2 \end{array}$$

(31) 

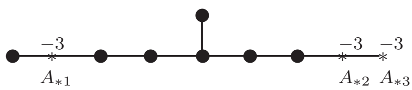
$$Z=\begin{array}{ccccccccccc} & & & 3 & & & & & & 1 & \\ 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & \underline{2} & \dots & 2 & \underline{1} \end{array}$$

(32) 

$$Z=\begin{array}{ccccccccccc} & & & 3 & & & & & & 1 & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & \underline{2} & \dots & 2 & 1 \end{array}$$

(33) 

$$Z=\begin{array}{ccccccccccccc} & & 4 & & & & & & & & & 1 & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2} & \dots & & 2 & \underline{1} \end{array}$$

(34) 

$$Z=\begin{array}{ccccccccc} & & & & 4 & & & & \\ 1 & \underline{2} & 4 & 6 & 8 & 6 & 4 & \underline{2} & \underline{1} \end{array}$$

Proof. Since the singularity is minimally elliptic, $A_{*1}^2 = A_{*2}^2 = A_{*3}^2 = -3$, $z_{*1} = z_{*2} = 2$, $z_{*3} = 1$, we have

$$A_{*1} \cdot (Z - 2A_{*1}) = -A_{*1} \cdot (K + 2A_{*1}) = A_{*1}^2 + 2 - 2A_{*1}^2 = 5. \tag{6.234}$$

$$A_{*2} \cdot (Z - 2A_{*1}) = -A_{*1} \cdot (K + 2A_{*1}) = A_{*1}^2 + 2 - 2A_{*1}^2 = 5. \tag{6.235}$$

$$A_{*3} \cdot (Z - A_{*2}) = -A_{*2} \cdot (K + A_{*2}) = A_{*2}^2 + 2 - A_{*2}^2 = 2. \tag{6.236}$$

Let Γ' be the graph obtained by deleting A_{*1}, A_{*2} and A_{*3} from Γ. Let $\Gamma_1, \dots, \Gamma_m$ be the connected components of Γ' with fundamental cycles $Z_1, \dots, Z_m$ respectively. (6.234) implies that

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 5, \tag{6.237}$$

$$\sum_{j=1}^{m} A_{*1} \cdot Z\Big|_{\Gamma_j} = 5, \tag{6.238}$$

$$\sum_{j=1}^{m} A_{*2} \cdot Z\Big|_{\Gamma_j} = 2. \tag{6.239}$$

Consider first that A_{*1}, A_{*2} and A_{*3} do not meet and they connect with the same connected components of Γ. Then Proposition 6.33 applies.

In case (1) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 1 + 1$ and $5 = 3 + 2$, according to Proposition 6.16, then we are in case (1).

In case (1) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 1 + 1$ and $5 = 3 + 1 + 1$, according to Proposition 6.16, then we are in case (2).

In case (1) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 2$ and $5 = 3 + 2$, according to Proposition 6.16, then we are in case (3) and case (4).

In case (2) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 3 + 2$, according to Proposition 6.16, then we are in case (5)($r \geq 1$).

In case (2) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 3 + 1 + 1$, according to Proposition 6.16, then we are in case (5)($r = 0$).

In case (3) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 5 + 0$ and $5 = 3 + 2$, according to Proposition 6.16, then we are in case (6)($r \geq 1$).

In case (3) of Proposition 6.33, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 5 + 0$ and $5 = 3 + 1 + 1$, according to Proposition 6.16, then we are in case (6)($r = 0$).

Consider secondly that A_{*1}, A_{*2} and A_{*3} do not meet and each two connect with the same connected components of Γ. And $A_{*2} \cdot A_{*3}$ may not equal to zero. Then Proposition 6.34 and Proposition 6.35 applies.

In case (1) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 2$ and $5 = 3 + 2$, according to Proposition 6.16, then we are in case (7)($r_1 \geq 1, r_2 \geq 1$). In case (1) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 2$ and $5 = 3 + 1 + 1$, according to Proposition 6.16, then we are in case (7)($r_1 \geq 1, r_2 = 0$).

In case (1) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 1 + 1$ and $5 = 3 + 1 + 1$, according to Proposition 6.16, then we are in case (7)($r_1 = r_2 = 0$).

If $A_{*2} \cdot A_{*3} \neq 0$. In case (2) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 4 + 1$, according to Proposition 6.16, then we are in case (8).

In case (3) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 2 + 3$ and $5 = 2 + 3$, according to Proposition 6.16's case (3), case (7), case (11) and Proposition 6.35 's case (2), case (5), then we are in case (9) to case (14).

In case (3) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 2 + 3$ and $5 = 2 + 2 + 1$, according to Proposition 6.16's case (3), case (7), case (11) and Proposition 6.35's case (1), then we are in case (15) to case (20).

In case (3) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 2 + 2 + 1$ and $5 = 2 + 2 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (21)($r_1 \geq 1$, $r_3 \geq 1$) and case (22)($r_1 \geq 1$, $r_3 \geq 1$).

In case (3) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 2 + 2 + 1$ and $5 = 2 + 1 + 1 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (21)($r_1 \geq 1$, $r_3 = 0$) and case (22)($r_1 \geq 1$, $r_3 = 0$).

In case (3) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 2+1+1+1$ and $5 = 2+1+1+1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (21)($r_1 = r_3 = 0$).

In case (4) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 3 + 2$, according to Proposition 6.16 and Propsition 6.35, then we are in case (23)$(r \geq 1)$.

In case (4) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 3 + 1 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (23)$(r = 0)$.

In case (5) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 5 + 0$ and $5 = 3 + 2$, according to Proposition 6.16 and Propsition 6.35, then we are in case (24)$(r \geq 1)$.

In case (5) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 5 + 0$ and $5 = 3 + 1 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (24)$(r = 0)$.

In case (6) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 4 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (25)$(r \geq 1)$.

In case (7) it is not possible because there is no place for A_{*3}.

In case (8) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 4 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (26)$(m - 4 \geq 1)$.

In case (9) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 2$ and $5 = 3 + 2$, according to Proposition 6.16 and Propsition 6.35, then we are in case (27)$(r_1 \geq 1, r_2 \geq 1)$.

In case (9) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 1 + 1$ and $5 = 3 + 1 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (27)$(r_1 = r_2 = 0)$.

In case (9) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 3 + 2$ and $5 = 3 + 1 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (27)$(r_1 \geq 1, r_2 = 0)$.

In case (10) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 4 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (28).

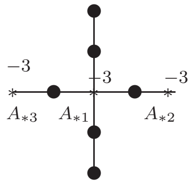
In case (11) it is not possible because there is no place for A_{*3}.

In case (12) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 3 + 2$, according to Proposition 6.16 and Propsition 6.35, then we are in case (29) and case (30).

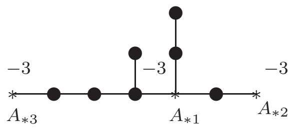
In case (13) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 5 + 0$ and $5 = 3 + 2$, according to Proposition 6.16 and Propsition 6.35, then we are in case (31).

In case (14) of Proposition 6.34, if the decomposition (6.237) at A_{*1} and A_{*2} are $5 = 4 + 1$ and $5 = 4 + 1$, according to Proposition 6.16 and Propsition 6.35, then we are in case (32). ∎

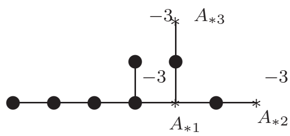
THEOREM 6.37. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi \colon M \to V$ be the minimal resolution of p. If case (11) of Proposition 6.2 holds, i.e., there exists three effective components A_{*1} , A_{*2} andA_{*3} with $A_{*1}^2 = -3 = A_{*2}^2 = A_{*3}^2$ and $z_{*1} = 3, z_{*2} = 1 = z_{*3}$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

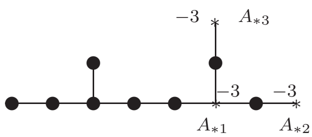
$$\begin{array}{cccccc} & & & 1 & & \\ & & & 2 & & \\ Z = & \underline{1} & 2 & \underline{3} & 2 & \underline{1} \\ & & & 2 & & \\ & & & 1 & & \end{array}$$

(2) 

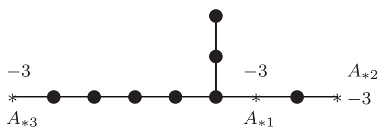
$$\begin{array}{cccccccc} & & & & & 1 & & \\ & & & & 2 & 2 & & \\ Z = & \underline{1} & 2 & 3 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

(3) 

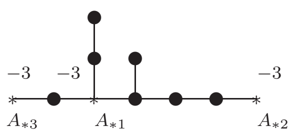
$$\begin{array}{cccccccc} & & & & & \underline{1} & & \\ & & & & 2 & 2 & & \\ Z = & 1 & 2 & 3 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

(4) 

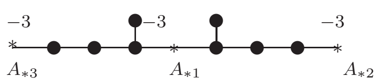
$$\begin{array}{ccccccccc} & & & & & & \underline{1} & & \\ & & & 3 & & & 2 & & \\ Z = & 2 & 4 & 6 & 5 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

(5) 

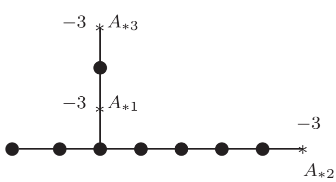
$$\begin{array}{cccccccccc} & & & & & & 2 & & & \\ & & & & & & 4 & & & \\ Z = & \underline{1} & 2 & 3 & 4 & 5 & 6 & \underline{3} & 2 & \underline{1} \end{array}$$

(6) 

$$\begin{array}{cccccccc} & & & 1 & & & & \\ & & & 2 & 2 & & & \\ Z = & \underline{1} & 2 & \underline{3} & 4 & 3 & 2 & \underline{1} \end{array}$$

(7) 

$$\begin{array}{cccccccccc} & & & & 2 & & 2 & & & \\ Z = & \underline{1} & 2 & 3 & 4 & \underline{3} & 4 & 3 & 2 & \underline{1} \end{array}$$

(8) 

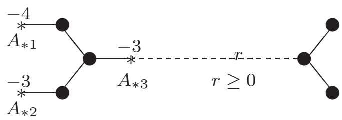
$$\begin{array}{ccccccccc} & & & \underline{1} & & & & & \\ & & & 2 & & & & & \\ & & & \underline{3} & & & & & \\ Z = & 2 & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. The proof is similar to those of Theorem 6.36, followed from Proposition 6.3, Proposition 6.15 and Proposition 6.22.

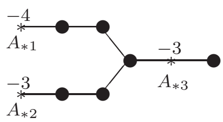
And its result is almost the same as Theorem 6.23, unless the case (4) in Theorem 6.23. It is because there is only one component in it that has weight 1 but

Theorem 6.36 must have two. ☐

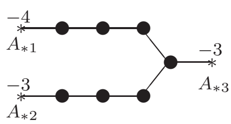
THEOREM 6.38. *Let (V,p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (12) of Proposition 6.2 holds, i.e., there exist three effective components A_{*1}, A_{*2} and A_{*3} with $A_{*1}^2 = -4, A_{*2}^2 = -3 = A_{*3}^2$ and $z_{*1} = z_{*2} = 1$, $z_{*3} = 2$, then the weighted dual graph Γ of the exceptional set is one of the following form.*

(1) 

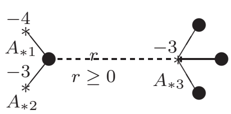
$$Z = \begin{array}{cccccc} \underline{1} & 2 & & & & 1 \\ & & 3 & \underline{2} & \ldots 2 & \\ \underline{1} & 2 & & & & 1 \end{array}$$

(2) 

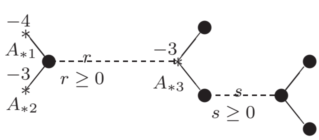
$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & 1 \\ \underline{1} & 2 & 3 & & & \end{array}$$

(3) 

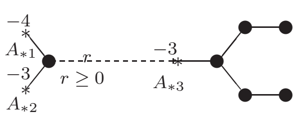
$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & 4 & & \\ & & & & 5 & \underline{2} \\ \underline{1} & 2 & 3 & 4 & & \end{array}$$

(4) 

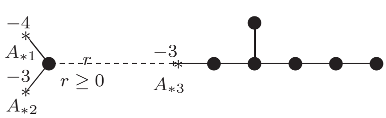
$$Z = \begin{array}{ccc} \underline{1} & & 1 \\ & 2 \ldots \underline{2} & 1 \\ \underline{1} & & 1 \end{array}$$

(5) 

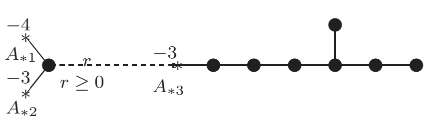
$$Z = \begin{array}{ccccc} \underline{1} & & & 1 & \\ & 2 \ldots & \underline{2} & & 1 \\ \underline{1} & & & 2 \ldots 2 & \\ & & & & 1 \end{array}$$

(6) 

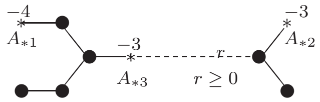
$$Z = \begin{array}{cccccc} \underline{1} & & & & 2 & 1 \\ & 2 \ldots & \underline{2} & 3 & & \\ \underline{1} & & & & 2 & 1 \end{array}$$

(7) 

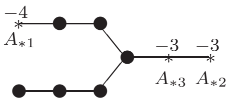
$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2 \ldots & \underline{2} & 3 & 4 & 3 & 2 & 1 \\ \underline{1} & & & & & & & \end{array}$$

(8) 

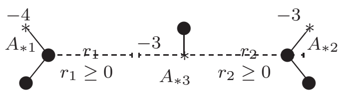
$$Z = \begin{array}{cccccccc} \underline{1} & & & & & 3 & & \\ & 2 \ldots & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \\ \underline{1} & & & & & & & \end{array}$$

(9) 

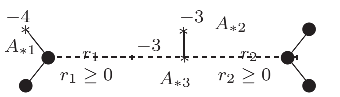
$$Z = \begin{array}{cccccc} \underline{1} & 2 & & & & \underline{1} \\ & & 3 & \underline{2} & \ldots 2 & \\ 1 & 2 & & & & 1 \end{array}$$

(10) 

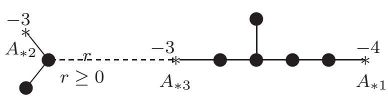
$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & \underline{1} \\ 1 & 2 & 3 & & & \end{array}$$

(11) 

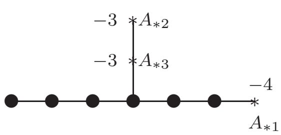
$$Z = \begin{array}{ccccc} \underline{1} & & 1 & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

(12) 

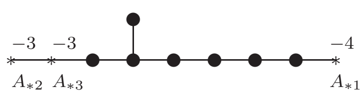
$$Z = \begin{array}{ccccc} \underline{1} & & \underline{1} & & 1 \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

(13) 

$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2\ldots & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ 1 & & & & & & & \end{array}$$

(14) 

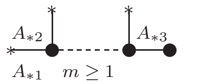
$$Z = \begin{array}{ccccccc} & & & \underline{1} & & & \\ & & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(15) 

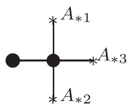
$$Z = \begin{array}{ccccccccc} & & & 3 & & & & & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

Proof. The proof is similar to those of Theorem 6.36, followed from Proposition 6.3, Proposition 6.16. □

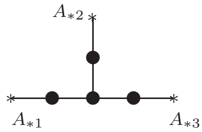
PROPOSITION 6.39. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cuylce Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1}, A_{*2} and A_{*3} be three effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1}, A_{*2}, A_{*3}, but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose that $A_{*1} \cdot Z_1 = 1 = A_{*2} \cdot Z_1 = A_{*3} \cdot Z_1$. If A_{*1}, A_{*2}, and A_{*3} are mutually disjoint, $z_{*1} = 1 = z_{*2} = z_{*3}$, and $A_{*1}^2 \leq -3$, $A_{*2}^2 \leq -3$, $A_{*3}^2 \leq -3$, then $A_{*1} \cup A_{*2} \cup A_{*3} \cup \Gamma_1$ and restriction of Z on $A_{*1} \cup A_{*2} \cup A_{*3} \cup \Gamma_1$ must be one of the following form.*

(1) 

$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup \Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \dots & 2 & 1 \end{array}$$

(2) 

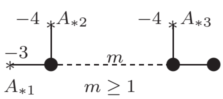
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup \Gamma_1} = \begin{array}{ccc} & \underline{1} & \\ 1 & 2 & \underline{1} \\ & \underline{1} & \end{array}$$

(3) 

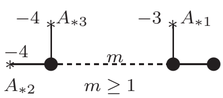
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup \Gamma_1} = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & \underline{1} \end{array}$$

Proof. The proof is the same as these in Proposition 6.33. ☐

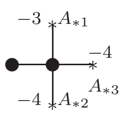
THEOREM 6.40. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi \colon M \to V$ be the minimal resolution of p. If case (13) of Proposition 6.2 holds, i.e., there exists three effective compounds A_{*1}, A_{*2} and A_{*3} with $A_{*1}^2 = -3$, $A_{*2}^2 = A_{*3}^2 = -4$ and $z_{*1} = z_{*2} = z_{*3} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

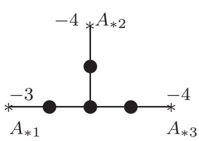
$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \dots\dots & 2 & 1 \end{array}$$

(2) 

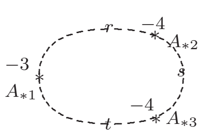
$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \dots\dots & 2 & 1 \end{array}$$

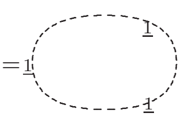
(3) 

$$Z = \begin{array}{ccc} & \underline{1} & \\ 1 & 2 & \underline{1} \\ & \underline{1} & \end{array}$$

(4) 

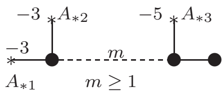
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & \underline{1} \end{array}$$

(5) 

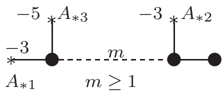
$Z =$ 

Proof. This follows from Proposition 6.3, Proposition 6.13, Proposition 6.29 and Proposition 6.39. ☐

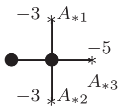
THEOREM 6.41. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi \colon M \to V$ be the minimal resolution of p. If case (14) of Proposition 6.2 holds,* *i.e., there exists three effective compounds A_{*1}, A_{*2} and A_{*3} with $A_{*1}^2 = -3 = A_{*2}^2$, $A_{*3}^2 = -5$ and $z_{*1} = z_{*2} = z_{*3} = 1$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

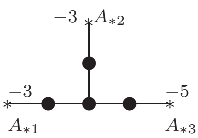
$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \cdots\cdots & 2 & 1 \end{array}$$

(2) 

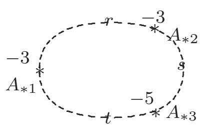
$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \cdots\cdots & 2 & 1 \end{array}$$

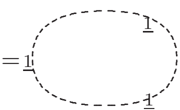
(3) 

$$Z = \begin{array}{ccc} & \underline{1} & \\ 1 & 2 & \underline{1} \\ & \underline{1} & \end{array}$$

(4) 

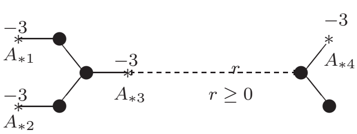
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & \underline{1} \end{array}$$

(5) 

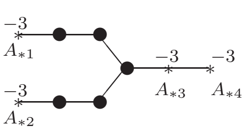
$Z =$ 

Proof. This is similar to the proof of Theorem 6.40. ∎

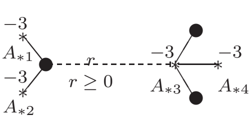
Theorem 6.42. *Let (V, p) be a germ of minimally elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution of p. If case (15) of Proposition 6.2 holds, i.e., there exist three effective components A_{*1}, A_{*2} and A_{*3} with $A_{*1}^2 = -3 = A_{*2}^2 = A_{*3}^2 = A_{*4}^2$ and $z_{*1} = z_{*2} = 1 = z_{*4}$, $z_{*3} = 2$, then the weighted dual graph Γ of the exceptional set is one of the following form.*

(1) 

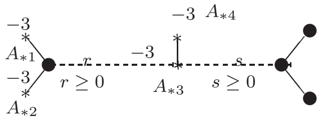
$$Z = \begin{array}{cccccc} \underline{1} & 2 & & & & \underline{1} \\ & & 3 & \underline{2} & \ldots 2 & \\ \underline{1} & 2 & & & & 1 \end{array}$$

(2) 

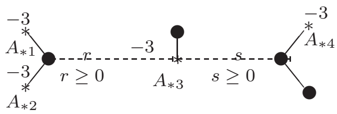
$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & \underline{1} \\ \underline{1} & 2 & 3 & & & \end{array}$$

(3) 

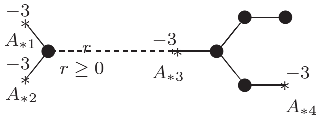
$$Z = \begin{array}{cccc} \underline{1} & & & 1 \\ & 2\ldots & \underline{2} & \underline{1} \\ \underline{1} & & & 1 \end{array}$$

(4) 

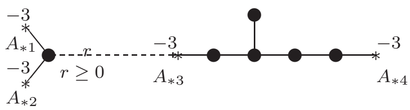
$$Z = \begin{array}{ccccc} \underline{1} & & \underline{1} & & 1 \\ & 2\ldots & \underline{2} & \ldots 2 & \\ \underline{1} & & & & 1 \end{array}$$

(5) 

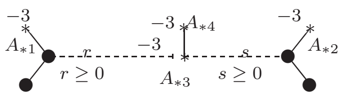
$$Z = \begin{array}{ccccc} \underline{1} & & 1 & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ \underline{1} & & & & 1 \end{array}$$

(6) 

$$Z = \begin{array}{cccccc} \underline{1} & & & & 2 & \underline{1} \\ & 2\ldots & \underline{2} & 3 & & \\ \underline{1} & & & & 2 & 1 \end{array}$$

(7) 

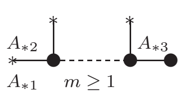
$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2\ldots & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ \underline{1} & & & & & & & \end{array}$$

(8) 

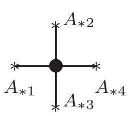
$$Z = \begin{array}{ccccc} \underline{1} & & \underline{1} & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

Proof. This follows from Proposition 6.3, Proposition 6.13, Proposition 6.16 and Proposition 6.35. ∎

PROPOSITION 6.43. *Let Γ be the minimal resolution graph of a minimally elliptic singularity with fundamental cycle Z. Let Γ' be the subgraph of Γ by removing all the effective components of Γ. Let A_{*1}, A_{*2}, A_{*3} and A_{*4} be four effective components of Γ. Suppose that Γ_1 is a connected component of Γ' which corresponds to a rational double point graph in Theorem 4.2. Suppose also that Γ_1 intersects with A_{*1}, A_{*2}, A_{*3} and A_{*4}, but disjoint from other effective component. Let Z_1 be the fundamental cycle on Γ_1. Suppose that $A_{*1} \cdot Z_1 = 1 = A_{*2} \cdot Z_1 = A_{*3} \cdot Z_1 = A_{*4} \cdot Z_1$. If A_{*1}, A_{*2}, A_{*3} and A_{*4} are mutually disjoint, $z_{*1} = z_{*2} = z_{*3} = z_{*4} = 1$, and $A_{*1}^2 \leq -3$, $A_{*2}^2 \leq -3$, $A_{*3}^2 \leq -3$, $A_{*4}^2 \leq -3$, then $A_{*1} \cup A_{*2} \cup A_{*3} \cup A_{*4} \cup \Gamma_1$ and the restriction of Z on $A_{*1} \cup A_{*2} \cup A_{*3} \cup A_{*4} \cup \Gamma_1$ must be one of the following form.*

(1) 

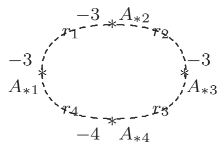
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup A_{*4}\cup\Gamma_1} = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ \underline{1} & 2 & \ldots & 2 & \underline{1} \end{array}$$

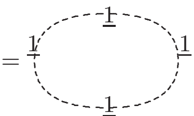
(2) 

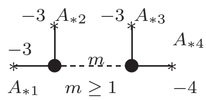
$$Z\Big|_{A_{*1}\cup A_{*2}\cup A_{*3}\cup A_{*4}\cup\Gamma_1} = \begin{array}{ccc} & \underline{1} & \\ \underline{1} & 2 & \underline{1} \\ & \underline{1} & \end{array}$$

Proof. The proof is the same as these in Proposition 6.35. ◻

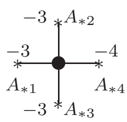
Theorem 6.44. *Let (V,p) be a germ of minimally elliptic singularity. Let $\pi\colon M \longrightarrow V$ be the minimal resolution of p. If case (16) of Proposition 6.2 holds, i.e., there exists four effective components A_{*1}, A_{*2}, A_{*3} and A_{*4} with $A_{*1}^2 = -3 = A_{*2}^2 = A_{*3}^2, A_{*4}^2 = -4$ and $z_{*1} = 1 = z_{*2} = z_{*3} = z_{*4}$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

$Z =$ 

(2) 

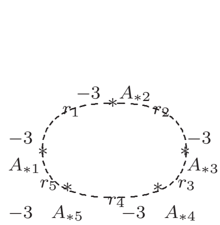
$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \cdots\cdots\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \underline{1}$$

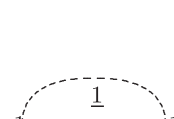
(3) 

$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\\ \underline{1}\end{array}\ \underline{1}$$

Proof. This follows from Proposition 6.3 and Proposition 6.43. ◻

Theorem 6.45. *Let (V,p) be a germ of minimally elliptic singularity. Let $\pi\colon M \longrightarrow V$ be the minimal resolution of p. If case (17) of Proposition 6.2 holds, i.e., there exists five effective components A_{*1}, A_{*2}, A_{*3} , A_{*4} and A_{*5} with $A_{*1}^2 = -3 = A_{*2}^2 = A_{*3}^2 = A_{*4}^2 = A_{*5}^2$ and $z_{*1} = 1 = z_{*2} = z_{*3} = z_{*4} = z_{*5}$, then the weighted dual graph Γ of the exceptional set is one of the following forms.*

(1) 

$Z =$ 

Proof. This follows from Proposition 6.3, Proposition 6.29, Proposition 6.35 and Proposition 6.43. ◻

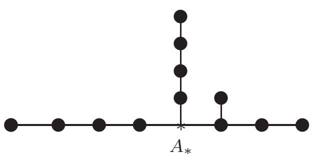
7. Complete list of weighted dual graphs of minimally elliptic singularities with $Z^2 = -5$.

The following graphs correspond to those exceptional cases in Proposition 3.7.

	$A_* \cdot A_*$	
1. E_ℓ	-5	$det(A_i \cdot A_j) = -5$
2. N_0	-5	$det(A_i \cdot A_j) = -5$
3. C_u	-5	$det(A_i \cdot A_j) = -5$
4. T_a	$-2, -7$	$det(A_i \cdot A_j) = 10$
5. T_r	$-2, -2, -7$	$det(A_i \cdot A_j) = -15$
6. T_a	$-3, -6$	$det(A_i \cdot A_j) = 14$
7. T_r	$-2, -3, -6$	$det(A_i \cdot A_j) = -23$
8. T_a	$-4, -5$	$det(A_i \cdot A_j) = 16$
9. T_r	$-2, -4, -5$	$det(A_i \cdot A_j) = -27$
10. T_r	$-3, -3, -5$	$det(A_i \cdot A_j) = -32$
11. T_r	$-3, -4, -4$	$det(A_i \cdot A_j) = -35$

I. The following graphs correspond to those exceptional cases in Theorem 6.12.

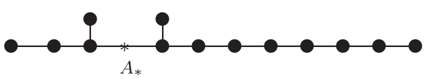
(1)



$\det(A_i \cdot A_j) = -25$

1
2
3
4 3
$Z = 1\ 2\ 3\ 4\ \underline{5}\ 6\ \ 4\ 2$

(2)

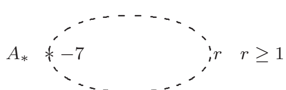


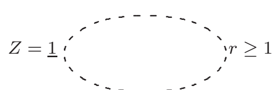
$\det(A_i \cdot A_j) = 52$

3 4
$Z = 2\ 4\ \ 6\ \ \underline{5}\ \ 8\ \ 7\ 6\ 5\ 4\ 3\ 2\ 1$

II. The following graphs correspond to those exceptional cases in Theorem 6.14.

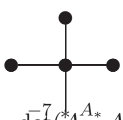
(1)





$\det(A_i \cdot A_j) = 13, r = 1$
$\det(A_i \cdot A_j) = (-1)^{r+1} 5(r+1), r > 1$

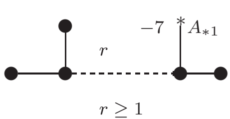
(2)



$\det(A_i \cdot A_j) = -20$

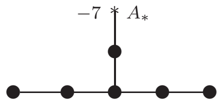
1
$Z = 1\ \ 2\ \ 1$
$\underline{1}$

(3)



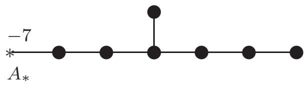
1 $\underline{1}$
$Z = 1\ \ 2\ \ \ldots\ \ 2\ \ 1$

$\det(A_i \cdot A_j) = (-1)^{r+1}20, r \geq 1$

(4) 

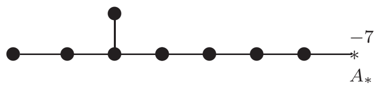
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = -15$

(5) 

$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & 1 \end{array}$$

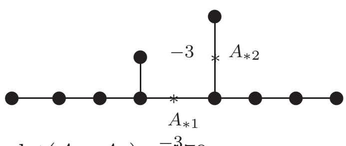
$\det(A_i \cdot A_j) = 10$

(6) 

$$Z = \begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

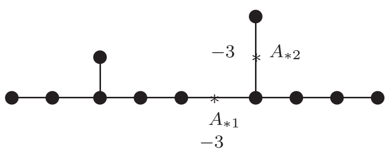
$\det(A_i \cdot A_j) = -5$

III. The following graphs correspond to those exceptional cases in Theorem 6.18.

(1) 

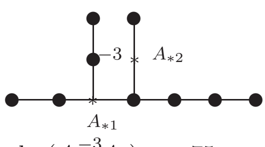
$$Z = \begin{array}{ccccccccc} & & & & & 1 & & & \\ & & & 2 & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = 170$

(2) 

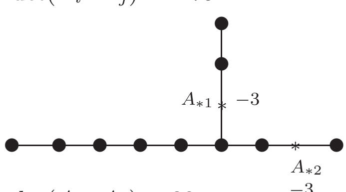
$$Z = \begin{array}{cccccccccc} & & & & & & 1 & & & \\ & & 3 & & & & \underline{2} & & & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = -93$

(3) 

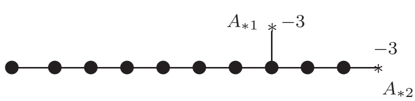
$$Z = \begin{array}{ccccccc} & & 1 & 1 & & & \\ & & 2 & \underline{2} & & & \\ 1 & 2 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = -75$

(4) 

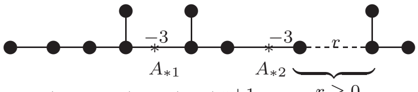
$$Z = \begin{array}{ccccccccc} & & & & & 1 & & & \\ & & & & & 2 & & & \\ & & & & & \underline{3} & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & \underline{2} & 1 \end{array}$$

$\det(A_i \cdot A_j) = 38$

(5) 

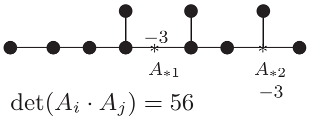
$$Z = \begin{array}{ccccccccccc} & & & & & & & \underline{3} & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 6 & 4 & \underline{2} \end{array}$$

$\det(A_i \cdot A_j) = 13$

(6) 

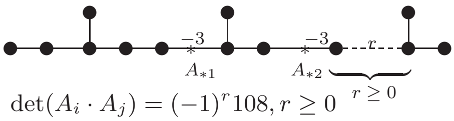
$$\begin{array}{ccccccccccccc} & & & & 2 & & 2 & & & & & 1 & \\ Z= & 1 & 2 & 3 & 4 & \underline{3} & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}56, r \geq 0$

(7) 

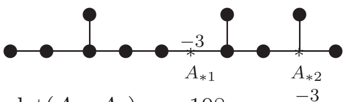
$$\begin{array}{cccccccccc} & & & & 2 & & 2 & & 1 & \\ Z= & 1 & 2 & 3 & 4 & \underline{3} & 4 & 3 & \underline{2} & 1\end{array}$$

$\det(A_i \cdot A_j) = 56$

(8) 

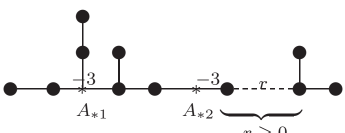
$$\begin{array}{ccccccccccccccc} & & & 3 & & & & 2 & & & & & 1 & \\ Z= & 2 & 4 & 6 & 5 & 4 & \underline{3} & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}108, r \geq 0$

(9) 

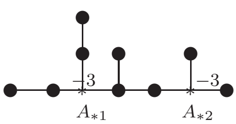
$$\begin{array}{ccccccccccc} & & & 3 & & & & 2 & & 1 & \\ Z= & 2 & 4 & 6 & 5 & 4 & \underline{3} & 4 & 3 & \underline{2} & 1\end{array}$$

$\det(A_i \cdot A_j) = -108$

(10) 

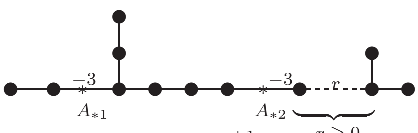
$$\begin{array}{cccccccccc} & & & 1 & & & & & & \\ & & & 2 & 2 & & & & 1 & \\ Z= & 1 & 2 & \underline{3} & 4 & 3 & \underline{2} & 2 \ \ldots & 2 & 1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}84, r \geq 0$

(11) 

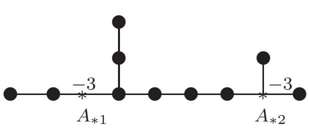
$$\begin{array}{cccccccc} & & & 1 & & & & \\ & & & 2 & 2 & & 1 & \\ Z= & 1 & 2 & \underline{3} & 4 & 3 & \underline{2} & 1\end{array}$$

$\det(A_i \cdot A_j) = -84$

(12) 

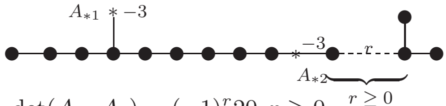
$$\begin{array}{cccccccccccccc} & & & & 2 & & & & & & & & & \\ & & & & 4 & & & & & & & 1 & & \\ Z= & 1 & 2 & \underline{3} & 6 & 5 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & 1 & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}44, r \geq 0$

(13) 

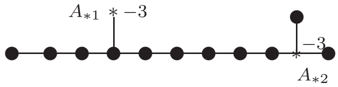
$$\begin{array}{cccccccccc} & & & & 2 & & & & & \\ & & & & 4 & & & & 1 & \\ Z= & 1 & 2 & \underline{3} & 6 & 5 & 4 & 3 & \underline{2} & 1\end{array}$$

$\det(A_i \cdot A_j) = 44$

(14) 

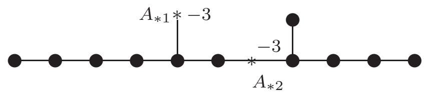
$Z = 2\;4\;6\;\overset{\underline{3}}{8}\;\;7\;6\;5\;4\;3\;\underline{2}\;\;2\;\ldots\;\overset{1}{2}\;\;1$

$\det(A_i \cdot A_j) = (-1)^r 20, r \geq 0$

(15) 

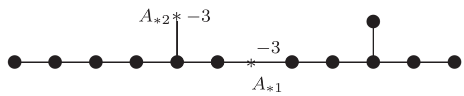
$Z = 2\;4\;6\;\overset{\underline{3}}{8}\;\;7\;6\;5\;4\;3\;\overset{1}{\underline{2}}\;\;1$

$\det(A_i \cdot A_j) = -20$

(16) 

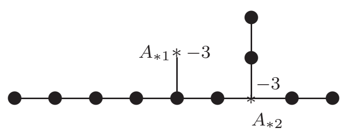
$Z = 1\;2\;3\;4\;\overset{\underline{2}}{5}\;\;4\;\underline{3}\;\overset{2}{4}\;\;3\;2\;1$

$\det(A_i \cdot A_j) = -32$

(17) 

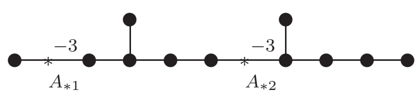
$Z = 1\;2\;3\;4\;\overset{\underline{2}}{5}\;\;4\;\underline{3}\;4\;5\;\overset{3}{6}\;\;4\;2$

$\det(A_i \cdot A_j) = 16$

(18) 

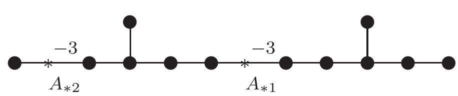
$Z = 1\;2\;3\;4\;\overset{\underline{2}}{5}\;\;4\;\;\overset{\overset{1}{2}}{\underline{3}}\;\;2\;1$

$\det(A_i \cdot A_j) = -38$

(19) 

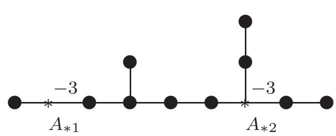
$Z = 1\;\underline{2}\;4\;\overset{3}{6}\;\;5\;4\;\underline{3}\;\overset{2}{4}\;\;3\;2\;1$

$\det(A_i \cdot A_j) = -28$

(20) 

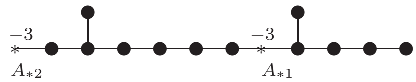
$Z = 1\;\underline{2}\;4\;\overset{3}{6}\;\;5\;4\;\underline{3}\;4\;5\;\overset{3}{6}\;\;4\;2$

$\det(A_i \cdot A_j) = 14$

(21) 

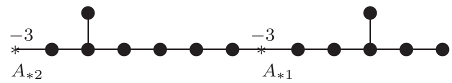
$Z = 1\;\underline{2}\;4\;\overset{3}{6}\;\;5\;4\;\overset{\overset{1}{2}}{\underline{3}}\;\;2\;1$

$\det(A_i \cdot A_j) = 42$

(22) 

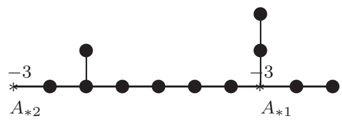
$\det(A_i \cdot A_j) = 14$

$$Z = \begin{array}{cccccccccccc} & & 4 & & & & & & 2 & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 4 & 3 & 2 & 1 \end{array}$$

(23) 

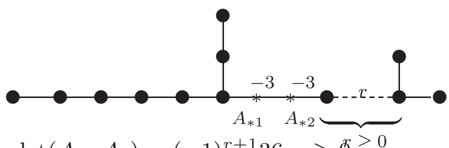
$\det(A_i \cdot A_j) = -7$

$$Z = \begin{array}{ccccccccccccc} & & 4 & & & & & & & & 3 & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 4 & 5 & 6 & 4 & 2 \end{array}$$

(24) 

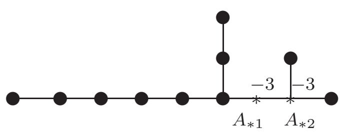
$\det(A_i \cdot A_j) = 17$

$$Z = \begin{array}{cccccccccc} & & & & & & & 1 & & \\ & & 4 & & & & & 2 & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & \underline{3} & 2 & 1 \end{array}$$

(25) 

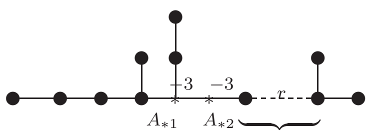
$\det(A_i \cdot A_j) = (-1)^{r+1}36, r \geq 0$

$$Z = \begin{array}{ccccccccccc} & & & & & 2 & & & & & \\ & & & & & 4 & & & & 1 & \\ 1 & 2 & 3 & 4 & 5 & 6 & \underline{3} & \underline{2} & 2 \ \ldots & 2 & 1 \end{array}$$

(26) 

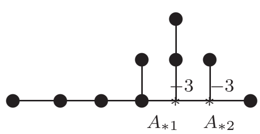
$\det(A_i \cdot A_j) = 36$

$$Z = \begin{array}{ccccccccc} & & & & & 2 & & & \\ & & & & & 4 & & 1 & \\ 1 & 2 & 3 & 4 & 5 & 6 & \underline{3} & \underline{2} & 1 \end{array}$$

(27) 

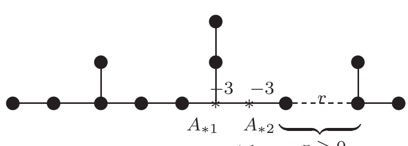
$\det(A_i \cdot A_j) = (-1)^r 72, r \geq 0$

$$Z = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & 2 & & & 1 & \\ 1 & 2 & 3 & 4 & \underline{3} & \underline{2} & 2 \ \ldots & 2 & 1 \end{array}$$

(28) 

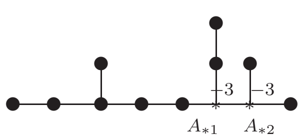
$\det(A_i \cdot A_j) = -72$

$$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & & & 2 & 2 & 1 & \\ 1 & 2 & 3 & 4 & \underline{3} & \underline{2} & 1 \end{array}$$

(29) 

$\det(A_i \cdot A_j) = (-1)^{r+1}132, r \geq 0$

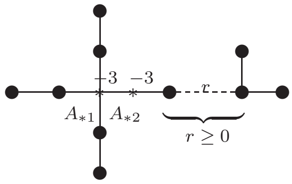
$$Z = \begin{array}{cccccccccc} & & & & & 1 & & & & \\ & & 3 & & & 2 & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & \underline{2} & 2 \ \ldots & 2 & 1 \end{array}$$

(30) 

$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & & 3 & & & 2 & 1 & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & \underline{2} & 1 \end{array}$$

$\det(A_i \cdot A_j) = 132$

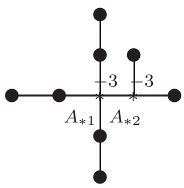
(31)



$$\begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & 1 & \\ Z = 1 & 2 & \underline{3} & \underline{2} & 2 & \dots & 2 & 1 \\ & & 2 & & & & & \\ & & 1 & & & & & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}108, r \geq 0$

(32)

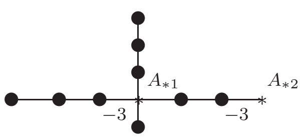


$$\begin{array}{ccccc} & & 1 & & \\ & & 2 & 1 & \\ Z = 1 & 2 & \underline{3} & \underline{2} & 1 \\ & & 2 & & \\ & & 1 & & \end{array}$$

$\det(A_i \cdot A_j) = 108$

IV. The following graphs correspond to those exceptional cases in Theorem 6.21.

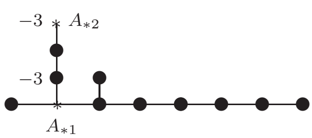
(1)



$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 2 & & & \\ & & & 3 & & & \\ Z = 1 & 2 & 3 & \underline{4} & 3 & 2 & \underline{1} \\ & & & 2 & & & \end{array}$$

$\det(A_i \cdot A_j) = -144$

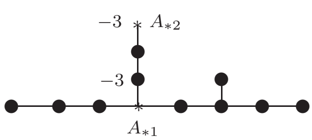
(2)



$$\begin{array}{cccccccc} & \underline{1} & & & & & & \\ & 2 & & & & & & \\ & 3 & 3 & & & & & \\ Z = 2 & \underline{4} & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

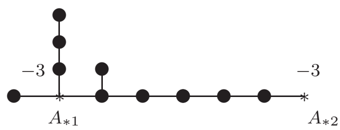
$\det(A_i \cdot A_j) = 32$

(3)



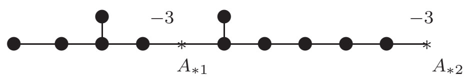
$$\begin{array}{cccccccc} & & & \underline{1} & & & & \\ & & & 2 & & & & \\ & & & 3 & & 3 & & \\ Z = 1 & 2 & 3 & \underline{4} & 5 & 6 & 4 & 2 \end{array}$$

$\det(A_i \cdot A_j) = 32$

(4) 

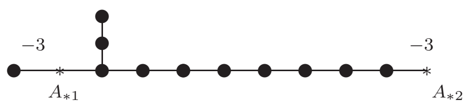
$$Z=\begin{array}{cccccccc} & 1 & & & & & & \\ & 2 & & & & & & \\ & 3 & 3 & & & & & \\ 2 & \underline{4} & 6 & 5 & 4 & 3 & 2 & \underline{1}\end{array}$$

$\det(A_i \cdot A_j) = 54$

(5) 

$$Z=\begin{array}{ccccccccccc} & & 3 & & & 3 & & & & & \\ 2 & 4 & 6 & 5 & \underline{4} & 6 & 5 & 4 & 3 & 2 & \underline{1}\end{array}$$

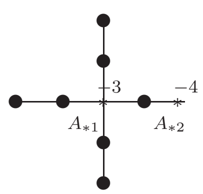
$\det(A_i \cdot A_j) = -17$

(6) 

$$Z=\begin{array}{ccccccccccc} & & 3 & & & & & & & & \\ & & 6 & & & & & & & & \\ 2 & \underline{4} & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1}\end{array}$$

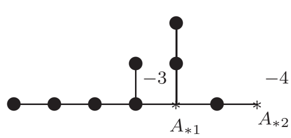
$\det(A_i \cdot A_j) = -22$

V. The following graphs correspond to those exceptional cases in Theorem 6.23.

(1) 

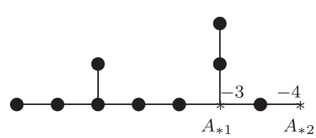
$$Z=\begin{array}{ccccc} & & 1 & & \\ & & 2 & & \\ 1 & 2 & \underline{3} & 2 & \underline{1} \\ & & 2 & & \\ & & 1 & & \end{array}$$

$\det(A_i \cdot A_j) = -81$

(2) 

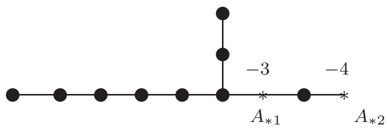
$$Z=\begin{array}{ccccccc} & & & & 1 & & \\ & & & 2 & 2 & & \\ 1 & 2 & 3 & 4 & \underline{3} & 2 & \underline{1}\end{array}$$

$\det(A_i \cdot A_j) = 54$

(3) 

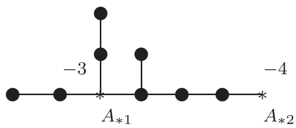
$$Z=\begin{array}{cccccccc} & & & & & 1 & & \\ & & 3 & & & 2 & & \\ 2 & 4 & 6 & 5 & 4 & \underline{3} & 2 & \underline{1}\end{array}$$

$\det(A_i \cdot A_j) = -27$

(4) 

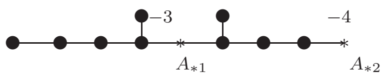
$\det(A_i \cdot A_j) = -27$

$$\begin{array}{cccccccccc} &&&&&2&&&\\ &&&&&4&&&\\ Z= 1&2&3&4&5&6&\underline{3}&2&\underline{1} \end{array}$$

(5) 

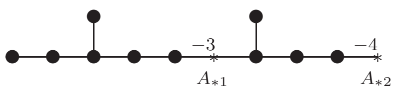
$\det(A_i \cdot A_j) = 60$

$$\begin{array}{ccccccc} &&1&&&&\\ &&2&2&&&\\ Z= 1&2&\underline{3}&4&3&2&\underline{1} \end{array}$$

(6) 

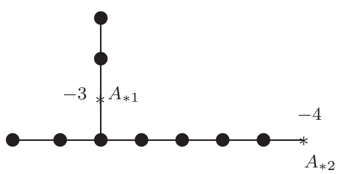
$\det(A_i \cdot A_j) = -40$

$$\begin{array}{ccccccccc} &&&2&&2&&&\\ Z= 1&2&3&4&\underline{3}&4&3&2&\underline{1} \end{array}$$

(7) 

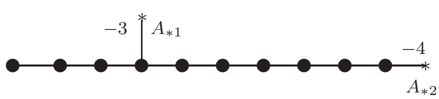
$\det(A_i \cdot A_j) = 20$

$$\begin{array}{ccccccccc} &&3&&&2&&&\\ Z = 2&4&6&5&4\,\underline{3}&4&3&2&\underline{1} \end{array}$$

(8) 

$\det(A_i \cdot A_j) = -31$

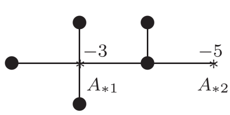
$$\begin{array}{ccccccccc} &&1&&&&&&\\ &&2&&&&&&\\ &&\underline{3}&&&&&&\\ Z= 2&4&6&5&4&3&2&\underline{1} \end{array}$$

(9) 

$\det(A_i \cdot A_j) = 242$

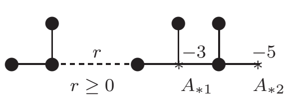
$$\begin{array}{ccccccccccc} &&&\underline{3}&&&&&&&\\ Z = 2&4&6&8&7&6&5&4&3&2&\underline{1} \end{array}$$

VI. The following graphs correspond to those exceptional cases in Theorem 6.25.

(1) 

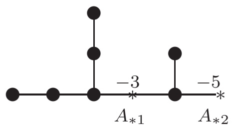
$\det(A_i \cdot A_j) = -76$

$$\begin{array}{cccc} &1&1&\\ Z = 1&\underline{2}&2&\underline{1}\\ &1&& \end{array}$$

(2) 

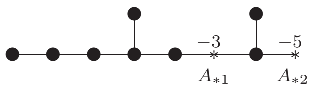
$\det(A_i \cdot A_j) = (-1)^r 76, r \geq 0$

$$\begin{array}{ccccccc} &1&&&1&1&\\ Z = 1&2&\ldots&2&\underline{2}&2&\underline{1} \end{array}$$

(3) 

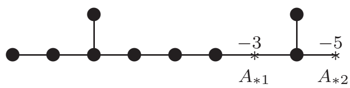
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -57$

(4) 

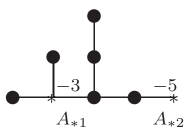
$$Z = \begin{array}{cccccccc} & & & 2 & & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 38$

(5) 

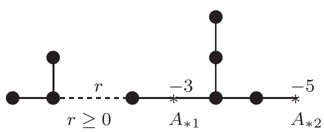
$$Z = \begin{array}{ccccccccc} & & 3 & & & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -19$

(6) 

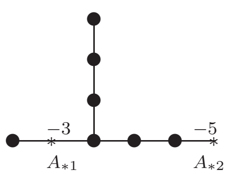
$$Z = \begin{array}{ccccc} & & 1 & & \\ & 1 & 2 & & \\ 1 & \underline{2} & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 60$

(7) 

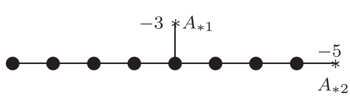
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & & 2 & & \\ 1 & 2 & \ldots & 2 & \underline{2} & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}60, r \geq 0$

(8) 

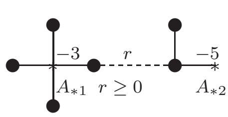
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & & \\ & & 3 & & & \\ 1 & \underline{2} & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -41$

(9) 

$$Z = \begin{array}{ccccccccc} & & & & \underline{2} & & & & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

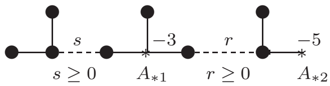
$\det(A_i \cdot A_j) = 26$

(10) 

$$Z = \begin{array}{cccccc} & 1 & & & 1 & \\ 1 & \underline{2} & 2 & \ldots & 2 & \underline{1} \\ & 1 & & & & \end{array}$$

$\det(A_i \cdot A_j) = -76, r = 0$

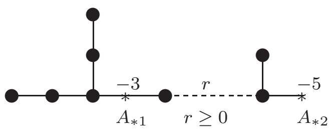
(11)



$$Z = \begin{array}{ccccccccc} & 1 & & & 1 & & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+s}(76 + 12s)$
$r \geq 0, s \geq 0$

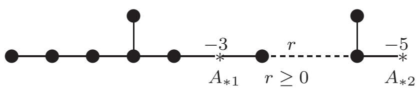
(12)



$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & 1 & \\ 1 & 2 & 3 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}(57 + 9r), r \geq 0$

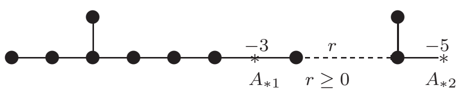
(13)



$$Z = \begin{array}{ccccccccc} & & & 2 & & & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 \; \dots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}(38 + 6r), r \geq 0$

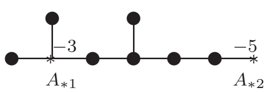
(14)



$$Z = \begin{array}{ccccccccccc} & & 3 & & & & & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \dots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}(19 + 3r), r \geq 0$

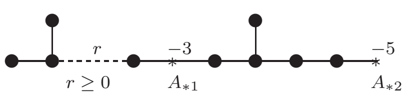
(15)



$$Z = \begin{array}{ccccccc} & 1 & & 2 & & & \\ 1 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -44$

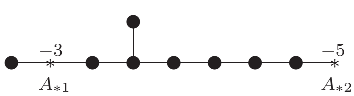
(16)



$$Z = \begin{array}{cccccccccc} & 1 & & & & & 2 & & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}44, r \geq 0$

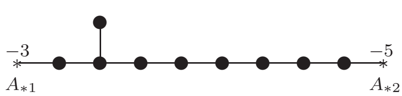
(17)



$$Z = \begin{array}{ccccccccc} & & & 3 & & & & & \\ 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

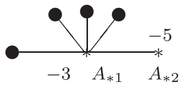
$\det(A_i \cdot A_j) = 22$

(18)



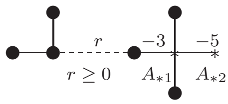
$$Z = \begin{array}{cccccccccc} & & 4 & & & & & & & \\ \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -200$

(19)


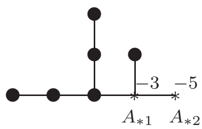
$\det(A_i \cdot A_j) = 64$

$$Z = \begin{array}{ccc} & 1\ 1\ 1 & \\ 1 & \underline{2} & \underline{1} \end{array}$$

(20)


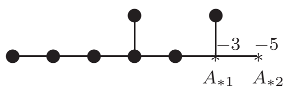
$\det(A_i \cdot A_j) = (-1)^{r+1}64, r \geq 0$

$$Z = \begin{array}{cccccc} & 1 & & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & \underline{1} \\ & & & & 1 & \end{array}$$

(21)


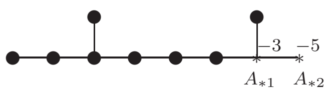
$\det(A_i \cdot A_j) = 48$

$$Z = \begin{array}{ccccc} & & 1 & & \\ & & 2 & 1 & \\ 1 & 2 & 3 & \underline{2} & \underline{1} \end{array}$$

(22)


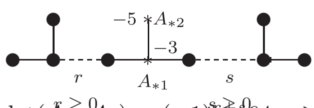
$\det(A_i \cdot A_j) = -32$

$$Z = \begin{array}{ccccccc} & & & 2 & & 1 & \\ 1 & 2 & 3 & 4 & 3 & \underline{2} & \underline{1} \end{array}$$

(23)


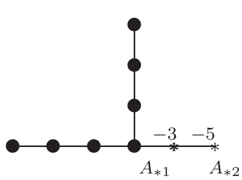
$\det(A_i \cdot A_j) = 16$

$$Z = \begin{array}{cccccccc} & & 3 & & & & 1 & \\ 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & \underline{1} \end{array}$$

(24)


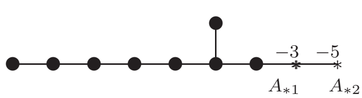
$\det(A_i \cdot A_j) = (-1)^{r+s}64, r \geq 0, s \geq 0$

$$Z = \begin{array}{ccccccccc} & 1 & & & \underline{1} & & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & 2 & \dots & 2 & 1 \end{array}$$

(25)


$\det(A_i \cdot A_j) = -32$

$$Z = \begin{array}{cccccc} & & & 1 & & \\ & & & 2 & & \\ & & & 3 & & \\ 1 & 2 & 3 & 4 & \underline{2} & \underline{1} \end{array}$$

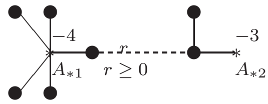
(26)


$\det(A_i \cdot A_j) = 16$

$$Z = \begin{array}{ccccccccc} & & & & & 3 & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & \underline{2} & \underline{1} \end{array}$$

VII. The following graphs correspond to those exceptional cases in Theorem 6.28.

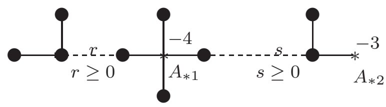
(1)



$\det(A_i \cdot A_j) = (-1)^r(128+16r), r \geq 0$

$$Z = \begin{array}{ccccc} 1\,1 & & & 1 & \\ \underline{2} & 2 & \ldots & 2 & \underline{1} \\ 1\,1 & & & & \end{array}$$

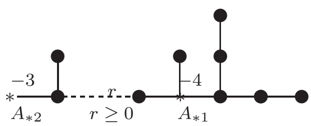
(2)



$\det(A_i \cdot A_j) = (-1)^{r+s+1}(128+16s)$
$r \geq 0, s \geq 0$

$$Z = \begin{array}{cccccccccc} & 1 & & & 1 & & & & 1 & \\ 1 & 2 & \ldots & 2 & \underline{2} & 2 & \ldots & & 2 & \underline{1} \\ & & & & 1 & & & & & \end{array}$$

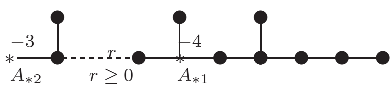
(3)



$\det(A_i \cdot A_j) = (-1)^r(96+12r), r \geq 0$

$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & 1 & 2 & & \\ \underline{1} & 2 & \ldots & 2 & \underline{2} & 3 & 2 & 1 \end{array}$$

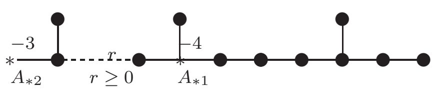
(4)



$\det(A_i \cdot A_j) = (-1)^{r+1}(64+8r), r \geq 0$

$$Z = \begin{array}{cccccccccc} & 1 & & & 1 & & 2 & & & \\ \underline{1} & 2 & \ldots & 2 & \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$$

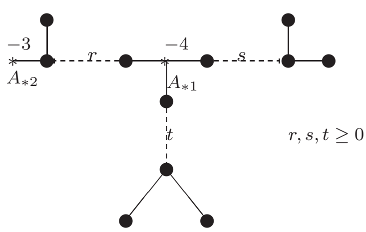
(5)



$\det(A_i \cdot A_j) = (-1)^r(32+4r), r \geq 0$

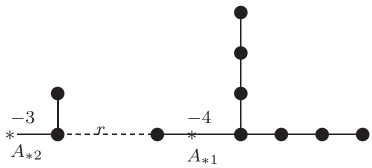
$$Z = \begin{array}{ccccccccccc} & 1 & & & 1 & & & & 3 & & \\ \underline{1} & 2 & \ldots & 2 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

(6)



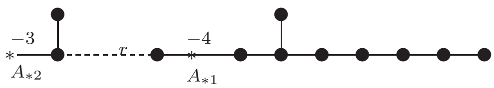
$\det(A_i \cdot A_j) = (-1)^{r+s+t}(128+16r)$
$r \geq 0, s \geq 0, t \geq 0$

$$Z = \begin{array}{ccccccccc} & 1 & & & & & & 1 & \\ \underline{1} & 2 & \cdots & 2 & \underline{2} & 2 & \cdots & 2 & 1 \\ & & & & 2 & & & & \\ & & & & \vdots & & & & \\ & & & & 2 & & & & \\ & & & 1 & & 1 & & & \end{array}$$

(7) 

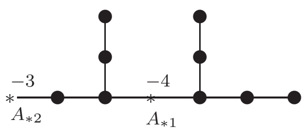
$$Z=\begin{array}{ccccccccc} &&&&&1&&&\\ &&&&&2&&&\\ &1&&&&3&&&\\ \underline{1}&2&\cdots&2&\underline{2}&4&3&2&1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}(64+8r), r \geq 0$

(8) 

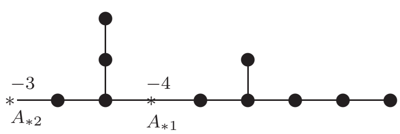
$$Z=\begin{array}{ccccccccccc} &1&&&&&3&&&&\\ \underline{1}&2&\cdots&2&\underline{2}&4&6&5&4&3&2\;1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}(32+4r), r \geq 0$

(9) 

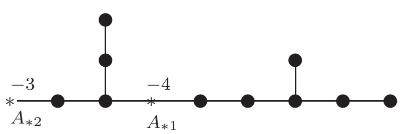
$$Z=\begin{array}{ccccccc} &&1&&1&&\\ &&2&&2&&\\ 1&2&3&\underline{2}&3&2&1\end{array}$$

$\det(A_i \cdot A_j) = -75$

(10) 

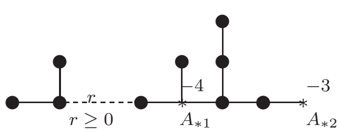
$$Z=\begin{array}{ccccccccc} &&1&&&&&&\\ &&2&&&2&&&\\ 1&2&3&\underline{2}&3&4&3&2&1\end{array}$$

$\det(A_i \cdot A_j) = 50$

(11) 

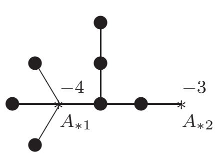
$$Z=\begin{array}{cccccccccc} &&1&&&&&&&\\ &&2&&&&&3&&\\ 1&2&3&\underline{2}&3&4&5&6&4&2\end{array}$$

$\det(A_i \cdot A_j) = -25$

(12) 

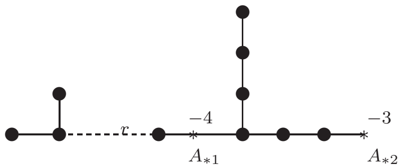
$$Z=\begin{array}{cccccccc} &&&&&1&&\\ &1&&&1&2&&\\ 1&2&\ldots&2&\underline{2}&3&2&\underline{1}\end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r}100, r \geq 0$

(13) 

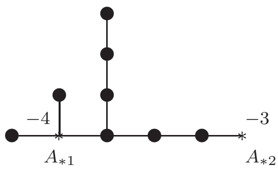
$$Z=\begin{array}{cccccc} &&&1&&\\ &1&&2&&\\ 1&&\underline{2}&3&2&\underline{1}\\ &1&&&&\end{array}$$

$\det(A_i \cdot A_j) = -84$

(14) 

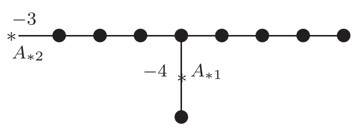
$$Z = \begin{array}{cccccccc} & & & & 1 & & & \\ & & & & 2 & & & \\ & 1 & & & 3 & & & \\ 1 & 2 \cdots & 2 & \underline{2} & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}68, r \geq 0$

(15) 

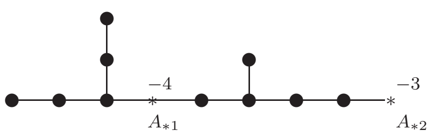
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & & \\ & 1 & 3 & & & \\ 1 & \underline{2} & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 68$

(16) 

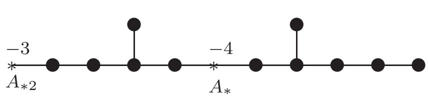
$$Z = \begin{array}{ccccccccc} \underline{1} & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ & & & & \underline{2} & & & & \\ & & & & 1 & & & & \end{array}$$

$\det(A_i \cdot A_j) = -43$

(17) 

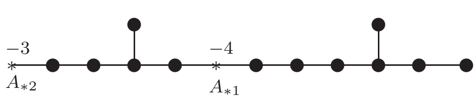
$$Z = \begin{array}{ccccccccc} & & 1 & & & & & & \\ & & 2 & & & 2 & & & \\ 1 & 2 & 3 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 54$

(18) 

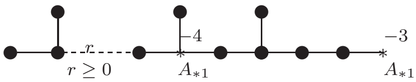
$$Z = \begin{array}{ccccccccccc} & & & 2 & & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 3 & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = -36$

(19) 

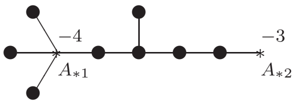
$$Z = \begin{array}{cccccccccccc} & & & 2 & & & & & & 3 & & \\ \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \end{array}$$

$\det(A_i \cdot A_j) = 18$

(20) 

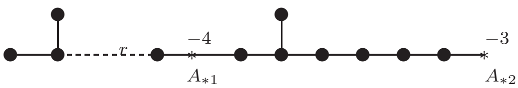
$$\begin{array}{cccccccccccc} & 1 & & & 1 & & 2 & & & \\ Z = 1 & 2 & \ldots & 2 & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}72, r \geq 0$

(21) 

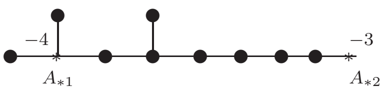
$$\begin{array}{cccccccc} & 1 & & & 2 & & & \\ Z = 1 & & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ & 1 & & & & & & \end{array}$$

$\det(A_i \cdot A_j) = 72$

(22) 

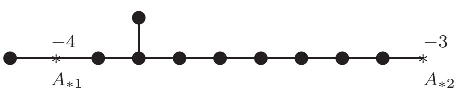
$$\begin{array}{ccccccccccccc} & 1 & & & & & 3 & & & & & \\ Z = 1 & 2 & \cdots & 2 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 36, r \geq 0$

(23) 

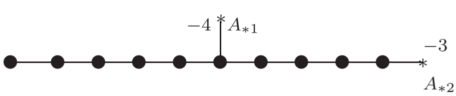
$$\begin{array}{ccccccccc} & 1 & & 3 & & & & & \\ Z = 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -36$

(24) 

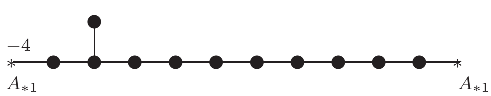
$$\begin{array}{cccccccccccc} & & & 4 & & & & & & & & \\ Z = 1 & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 18$

(25) 

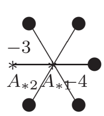
$$\begin{array}{ccccccccccc} & & & & & \underline{2} & & & & & \\ Z = 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 26$

(26) 

$$\begin{array}{ccccccccccccc} & & 5 & & & & & & & & & \\ Z = \underline{2} & 6 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

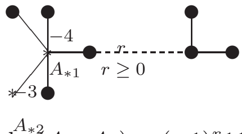
$\det(A_i \cdot A_j) = -9$

(27) 

$$\begin{array}{ccccc} & 1 & & 1 & \\ Z = \underline{1} & & \underline{2} & & 1 \\ & 1 & & 1 & \end{array}$$

$\det(A_i \cdot A_j) = -112$

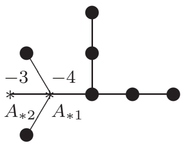
(28)



$$Z = \begin{array}{cccccc} 1 & 1 & & & 1 & \\ & \underline{2} & 2 & \dots & 2 & 1 \\ \underline{1} & 1 & & & & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 112, r \geq 0$

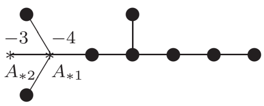
(29)



$$Z = \begin{array}{cccccc} & & & 1 & & \\ & 1 & & 2 & & \\ \underline{1} & & \underline{2} & 3 & 2 & 1 \\ & 1 & & & & \end{array}$$

$\det(A_i \cdot A_j) = -84$

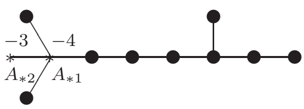
(30)



$$Z = \begin{array}{cccccccc} & 1 & & & 2 & & & \\ \underline{1} & & \underline{2} & 3 & 4 & 3 & 2 & 1 \\ & 1 & & & & & & \end{array}$$

$\det(A_i \cdot A_j) = 56$

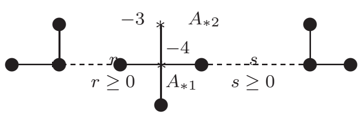
(31)



$$Z = \begin{array}{ccccccccc} & 1 & & & & & 3 & & \\ \underline{1} & & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \\ & 1 & & & & & & & \end{array}$$

$\det(A_i \cdot A_j) = -28$

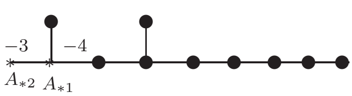
(32)



$$Z = \begin{array}{cccccccc} & 1 & & & \underline{1} & & 1 & \\ 1 & 2 & \dots & 2 & \underline{2} & 2\dots & 2 & 1 \\ & & & & 1 & & & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+s+1} 112, r = 0, s = 0$

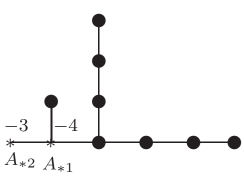
(33)



$$Z = \begin{array}{ccccccccc} & 1 & & 3 & & & & & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

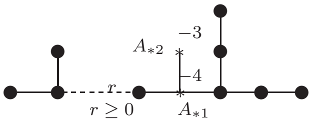
$\det(A_i \cdot A_j) = -28$

(34)



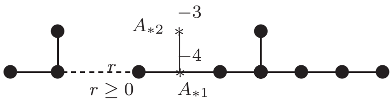
$$Z = \begin{array}{cccccc} & & 1 & & & \\ & & 2 & & & \\ & 1 & 3 & & & \\ \underline{1} & \underline{2} & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = 56$

(35) 

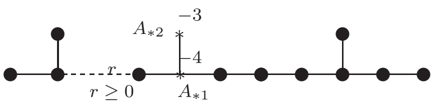
$$Z = \begin{array}{cccccccc} & & & & & 1 & & \\ & 1 & & & \underline{1} & 2 & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 84, r \geq 0$

(36) 

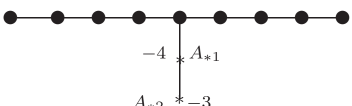
$$Z = \begin{array}{ccccccccc} & 1 & & & \underline{1} & & 2 & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3 & 4 & 3\ 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1} 56, r \geq 0$

(37) 

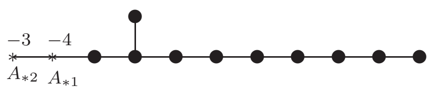
$$Z = \begin{array}{ccccccccc} & 1 & & & \underline{1} & & 3 & & \\ 1 & 2 & \dots & 2 & \underline{2} & 3\ 4\ 5 & 6 & 4 & 2 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 28, r \geq 0$

(38) 

$$Z = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ & & & & \underline{2} & & & & \\ & & & & \underline{1} & & & & \end{array}$$

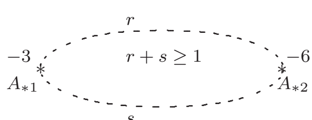
$\det(A_i \cdot A_j) = -35$

(39) 

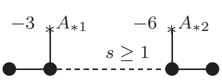
$$Z = \begin{array}{ccccccccccc} & & & 4 & & & & & & & \\ \underline{1} & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = 14$

VIII. The following graphs correspond to those exceptional cases in Theorem 6.30.

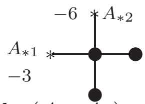
(1) 

$\det(A_i \cdot A_j) = (-1)^{r+s}[9(r+s) + 4rs + 14], r + s \geq 1$

(2) 

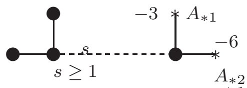
$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \dots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{s+1}(36 + 4s), s \geq 1$

(3) 

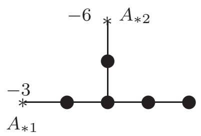
$\det(A_i \cdot A_j) = -36$

$$Z = \begin{array}{ccc} & \underline{1} & \\ \underline{1} & 2 & 1 \\ & 1 & \end{array}$$

(4) 

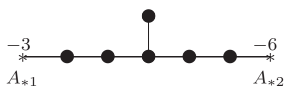
$\det(A_i \cdot A_j) = (-1)^{s+1}36, s \geq 1$

$$Z = \begin{array}{ccccc} & 1 & & \underline{1} & \\ 1 & 2 & \dots & 2 & \underline{1} \end{array}$$

(5) 

$\det(A_i \cdot A_j) = -31$

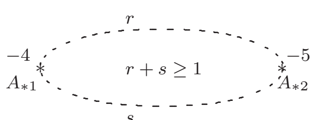
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1 \end{array}$$

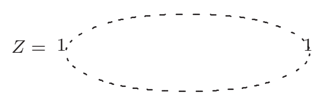
(6) 

$\det(A_i \cdot A_j) = 22$

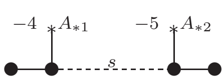
$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

IX. The following graphs correspond to those exceptional cases in Theorem 6.32.

(1) 

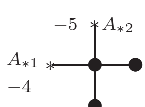


$\det(A_i \cdot A_j) = (-1)^{r+s}[11(r+s) + 6rs + 16], r + s \geq 1$

(2) 

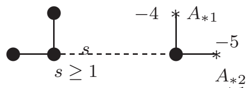
$\det(A_i \cdot A_j) = (-1)^{s+1}(44 + 6s), s \geq 1$

$$Z = \begin{array}{ccccc} & \underline{1} & & \underline{1} & \\ 1 & 2 & \dots & 2 & 1 \end{array}$$

(3) 

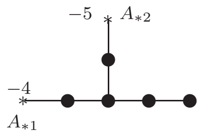
$\det(A_i \cdot A_j) = -44$

$$Z = \begin{array}{ccc} & \underline{1} & \\ \underline{1} & 2 & 1 \\ & 1 & \end{array}$$

(4) 

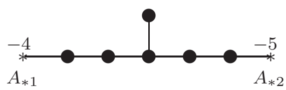
$$Z = \begin{array}{cccccc} & 1 & & & \underline{1} & \\ 1 & 2 & \ldots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}44, s \geq 1$

(5) 

$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & 1 \end{array}$$

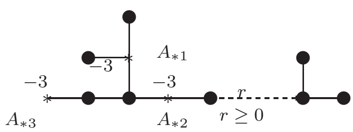
$\det(A_i \cdot A_j) = -39$

(6) 

$$Z = \begin{array}{ccccccc} & & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

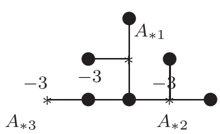
$\det(A_i \cdot A_j) = 28$

X. The following graphs correspond to those exceptional cases in Theorem 6.36.

(1) 

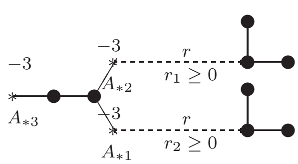
$$Z = \begin{array}{cccccccc} & & 1 & & & & & \\ & 1 & \underline{2} & & & & 1 & \\ \underline{1} & 2 & 3 & \underline{2} & 2 & \ldots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 128, r \geq 0$

(2) 

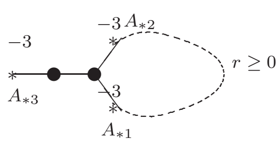
$$Z = \begin{array}{ccccc} & & 1 & & \\ & 1 & \underline{2} & 1 & \\ \underline{1} & 2 & 3 & \underline{2} & 1 \end{array}$$

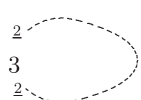
$\det(A_i \cdot A_j) = -128$

(3) 

$$Z = \begin{array}{cccccc} & & & & 1 & \\ & & & \underline{2} \ldots & 2 & 1 \\ \underline{1} & 2 & 3 & & & \\ & & & \underline{2} \ldots & 2 & 1 \\ & & & & 1 & \end{array}$$

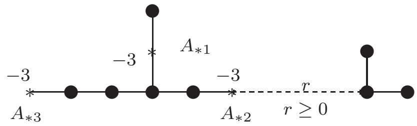
$\det(A_i \cdot A_j) = (-1)^{r+s+1}128, r_1 \geq 0, r_2 \geq 0$

(4) 

$Z = \underline{1}\ 2\ 3$ 

$\det(A_i \cdot A_j) = (-1)^{r+1}(16 + 8r), r \geq 0$

(5)



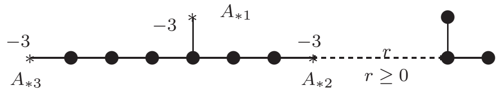
$\det(A_i \cdot A_j) = (-1)^r 92, r \geq 0$

1

$\underline{2} \qquad 1$

$Z = \underline{1}\ 2\ 3\ 4\ \ 3\ \underline{2}\ \dots\ 2\ \ 1$

(6)

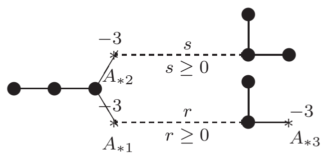


$\det(A_i \cdot A_j) = (-1)^{r+1} 60, r \geq 0$

$\underline{2} \qquad 1$

$Z = \underline{1}\ 2\ 3\ 4\ 5\ \ 4\ 3\ \underline{2}\ \dots\ 2\ \ 1$

(7)



$\det(A_i \cdot A_j) = (-1)^{r+s+1}(116 + 16s), r \geq 0, s \geq 0$

1

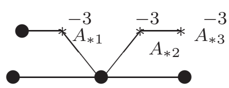
$\underline{2} \dots 2 \quad 1$

$Z = 1 \quad 2 \quad 3$

$\underline{2} \dots 2 \quad \underline{1}$

1

(8)

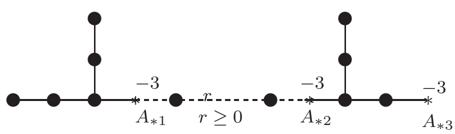


$1\ \underline{2} \quad \underline{2}\ \underline{1}$

$Z = 2 \quad 4 \quad 2$

$\det(A_i \cdot A_j) = -36$

(9)



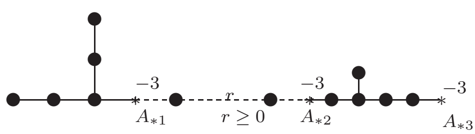
$\det(A_i \cdot A_j) = (-1)^r (96 + 21r), r \geq 0$

$1 \qquad 1$

$2 \qquad 2$

$Z = 1\ 2\ \ 3\ \underline{2}\ 2 \dots\ 2\ \underline{2}\ \ 3\ \ 2\ \underline{1}$

(10)



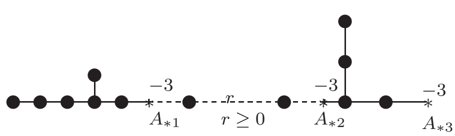
$\det(A_i \cdot A_j) = (-1)^{r+1}(69 + 15r), r \geq 0$

1

$2 \qquad 2$

$Z = 1\ 2\ \ 3\ \ \underline{2}\ \ 2 \dots\ 2\ \underline{2}\ 3\ \ 4\ \ 3\ 2\ \underline{1}$

(11)

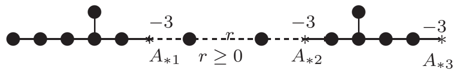


$\det(A_i \cdot A_j) = (-1)^{r+1}(64 + 14r), r \geq 0$

1

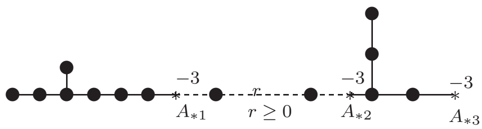
$2 \qquad 2$

$Z = 1\ 2\ 3\ 4\ \ 3\ \underline{2}\ \ 2 \dots\ 2\ \underline{2}\ \ 3\ \ 2\ \underline{1}$

(12) 

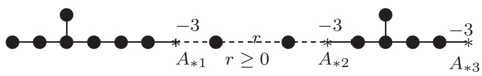
$$Z = 1\ 2\ 3\ \overset{2}{4}\ \ 3\ \underline{2}\ 2 \ldots\ 2\ \underline{2}\ 3\ \ \overset{2}{4}\ \ 3\ 2\ \underline{1}$$

$\det(A_i \cdot A_j) = (-1)^r(46 + 10r), r \geq 0$

(13) 

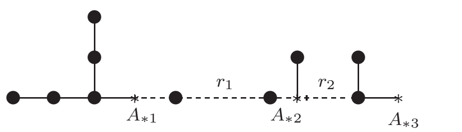
$$Z = 2\ 4\ \overset{3}{6}\ \ 5\ 4\ 3\ \underline{2}\ 2 \ldots\ 2\ \underline{2}\ \ \overset{\overset{1}{2}}{3}\ \ 2\ \underline{1}$$

$\det(A_i \cdot A_j) = (-1)^r(32 + 7r), r \geq 0$

(14) 

$$Z = 2\ 4\ \overset{3}{6}\ \ 5\ 4\ 3\ \underline{2}\ 2\ \ldots\ 2\ \underline{2}\ 3\ \ \overset{2}{4}\ \ 3\ 2\ \underline{1}$$

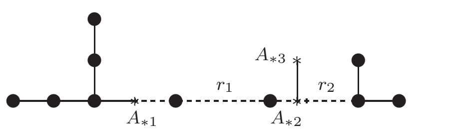
$\det(A_i \cdot A_j) = (-1)^{r+1}(23 + 5r), r \geq 0$

(15) 

$r_1 \geq 0, r_2 \geq 0$ $A_{*i}^2 = -3, i = 1, 2, 3$

$$Z = 1\ 2\ \ \overset{\overset{1}{2}}{3}\ \underline{2}\ 2\ \ldots\ 2\ \ \overset{1}{\underline{2}}\ \ \cdots \overset{1}{2}\ \ \underline{1}$$

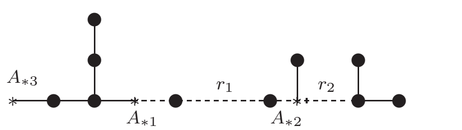
$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2}(24r_1 + 15r_2 + 3r_1r_2 + 108), r_1, r_2 \geq 0$

(16) 

$r_1 \geq 0, r_2 \geq 0$ $A_{*i}^2 = -3, i = 1, 2, 3$

$$Z = 1\ 2\ \ \overset{\overset{1}{2}}{3}\ \underline{2}\ 2\ \ldots\ 2\ \ \overset{\underline{1}}{\underline{2}}\ \ \cdots \overset{1}{2}\ \ 1$$

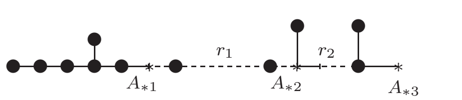
$\det(A_i \cdot A_j) = (-1)^{r_1+r_2}(108 + 24r_1), r_1, r_2 \geq 0$

(17) 

$r_1 \geq 0, r_2 \geq 0$ $A_{*i}^2 = -3, i = 1, 2, 3$

$$Z = \underline{1}\ 2\ \ \overset{\overset{1}{2}}{3}\ \underline{2}\ 2\ \ldots\ 2\ \ \overset{1}{\underline{2}}\ \ \cdots \overset{1}{2}\ \ 1$$

$\det(A_i \cdot A_j) = (-1)^{r_1+r_2}(128 + 28r_1), r_1, r_2 \geq 0$

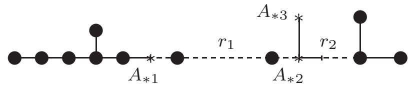
(18) 

$r_1 \geq 0, r_2 \geq 0$ $A_{*i}^2 = -3, i = 1, 2, 3$

$$Z = 1\ 2\ 3\ \overset{2}{4}\ \ 3\ \underline{2}\ 2\ \ldots\ 2\ \ \overset{1}{\underline{2}}\ \ \cdots \overset{1}{2}\ \ \underline{1}$$

$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2+1}(16r_1 + 10r_2 + 2r_1r_2 + 72), r_1, r_2 \geq 0$

(19)

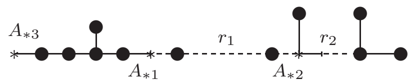


$r_1 \geq 0, r_2 \geq 0$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{cccccccccccccc} & & & & 2 & & & & & & \underline{1} & & 1 & \\ Z = & 1 & 2 & 3 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r_1+r_2+1}(72 + 16r_1), r_1, r_2 \geq 0$

(20)

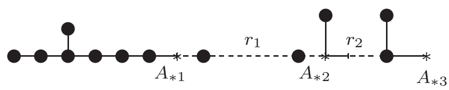


$r_1 \geq 0, r_2 \geq 0$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{cccccccccccccc} & & & & 2 & & & & & & 1 & & 1 & \\ Z = & \underline{1} & 2 & 3 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r_1+r_2+1}(92 + 20r_1), r_1, r_2 \geq 0$

(21)

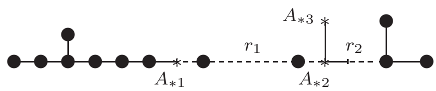


$r_1 \geq 0, r_2 \geq 0$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{ccccccccccccccc} & & & 3 & & & & & & & & 1 & & 1 & \\ Z = & 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & \underline{2} & \cdots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2}(8r_1 + 5r_2 + r_1r_2 + 36), r_1, r_2 \geq 0$

(22)

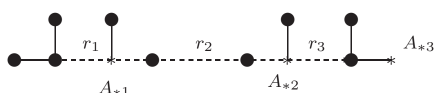


$r_1 \geq 0, r_2 \geq 0$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{ccccccccccccccc} & & & 3 & & & & & & & & \underline{1} & & 1 & \\ Z = & 2 & 4 & 6 & 5 & 4 & 3 & \underline{2} & 2 & \ldots & 2 & \underline{2} & \cdots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r_1+r_2}(36 + 8r_1), r_1, r_2 \geq 0$

(23)

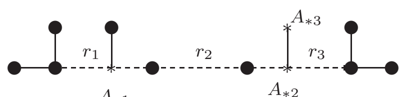


$r_i \geq 0, i = 1, 2, 3$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{cccccccccccc} & & 1 & & 1 & & & & 1 & & 1 & \\ Z = & 1 & 2 & \cdots & \underline{2} & 2 & \ldots & 2 & \underline{2} & \ldots & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2+r_3}(32r_1 + 20r_2 + 4r_1r_2 + 144), r_1, r_2, r_3 \geq 0$

(24)

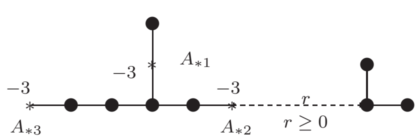


$r_i \geq 0, i = 1, 2, 3$ $\quad A_{*i}^2 = -3, i = 1, 2, 3$

$$\begin{array}{cccccccccccc} & & 1 & & 1 & & & & \underline{1} & & 1 & \\ Z = & 1 & 2 & \cdots & \underline{2} & 2 & \ldots & 2 & \underline{2} & \ldots & 2 & 1 \end{array}$$

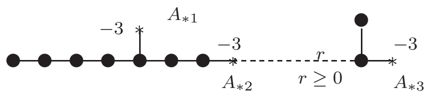
$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2+r_3}(32r_1 + 144), r_1, r_2, r_3 \geq 0$

(25)



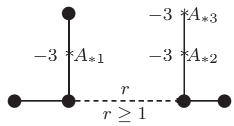
$$\begin{array}{cccccccccc} & & & & 1 & & & & & \\ & & & & \underline{2} & & & & 1 & \\ Z = & 1 & 2 & 3 & 4 & 3 & \underline{2} & \ldots & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}92, r \geq 0$

(26) 

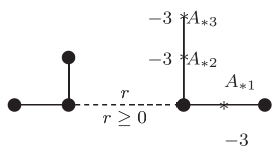
$$Z=\begin{array}{ccccccccccc} & & & & \underline{2} & & & & & 1 & \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & \underline{2} & \dots & 2 & \underline{1}\end{array}$$

$\det(A_i\cdot A_j)=(-1)^{r+1}(51+7r), r\geq 0$

(27) 

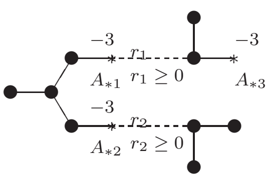
$$Z=\begin{array}{ccccc} & 1 & & \underline{1} & \\ & \underline{2} & & \underline{2} & \\ 2 & 4 & \dots & 4 & 2\end{array}$$

$\det(A_i\cdot A_j)=(-1)^{r+1}(36+2r), r\geq 1$

(28) 

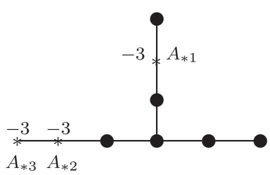
$$Z=\begin{array}{cccccc} & & & \underline{1} & & \\ & 2 & & \underline{2} & & \\ 2 & 4 & \dots & 4 & \underline{2} & 1\end{array}$$

$\det(A_i\cdot A_j)=(-1)^{r+1}36, r\geq 0$

(29) 

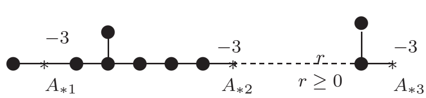
$$Z=2\quad 4\quad \begin{array}{ccc} & & \\ 3 & \underbrace{\underline{2}\dots \overset{1}{2}} & \underline{1} \\ & & \\ 3 & \underbrace{\underline{2}\dots 2}_{1} & 1\end{array}$$

$\det(A_i\cdot A_j)=(-1)^{r_1+r_2}(88+12r_1), r_1\geq 0, r_2\geq 0$

(30) 

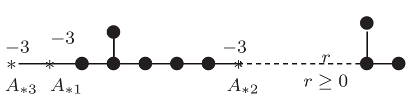
$$Z=\begin{array}{cccccc} & & & 1 & & \\ & & & \underline{2} & & \\ & & & 4 & & \\ \underline{1} & \underline{2} & 4 & 6 & 4 & 2\end{array}$$

$\det(A_i\cdot A_j)=-29$

(31) 

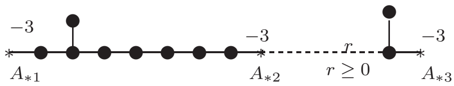
$$Z=\begin{array}{ccccccccccc} & & & 3 & & & & & & 1 & \\ 1 & \underline{2} & 4 & 6 & 5 & 4 & 3 & \underline{2} & \dots & 2 & \underline{1}\end{array}$$

$\det(A_i\cdot A_j)=(-1)^{r+1}(44+6r), r\geq 0$

(32) 

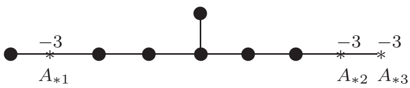
$$Z=\begin{array}{ccccccccccc} & & & 3 & & & & & & 1 & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & \underline{2} & \dots & 2 & 1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 12, r \geq 0$

(33) 

$$\begin{array}{ccccccccccccc} & & 4 & & & & & & & & & 1 & \\ Z = & \underline{2} & 5 & 8 & 7 & 6 & 5 & 4 & 3 & \underline{2} & \ldots & 2 & \underline{1} \end{array}$$

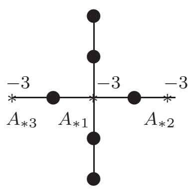
$\det(A_i \cdot A_j) = (-1)^r (22 + 3r), r \geq 0$

(34) 

$$\begin{array}{cccccccccc} & & & & & 4 & & & & \\ Z = & 1 & \underline{2} & 4 & 6 & 8 & 6 & 4 & \underline{2} & \underline{1} \end{array}$$

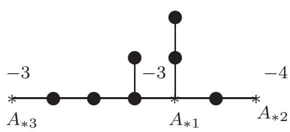
$\det(A_i \cdot A_j) = 20$

XI. The following graphs correspond to those exceptional cases in Theorem 6.37.

(1) 

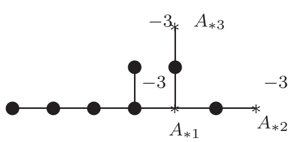
$$\begin{array}{cccccc} & & & 1 & & \\ & & & 2 & & \\ Z = & \underline{1} & 2 & \underline{3} & 2 & \underline{1} \\ & & & 2 & & \\ & & & 1 & & \end{array}$$

$\det(A_i \cdot A_j) = -105$

(2) 

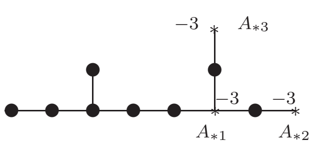
$$\begin{array}{cccccccc} & & & & & 1 & & \\ & & & & 2 & 2 & & \\ Z = & \underline{1} & 2 & 3 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 76$

(3) 

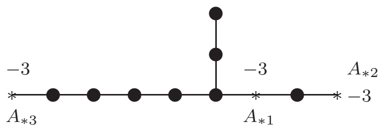
$$\begin{array}{cccccccc} & & & & & \underline{1} & & \\ & & & & 2 & 2 & & \\ Z = & 1 & 2 & 3 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = 70$

(4) 

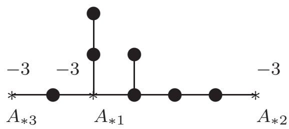
$$\begin{array}{ccccccccc} & & & & & & \underline{1} & & \\ & & & 3 & & & 2 & & \\ Z = & 2 & 4 & 6 & 5 & 4 & \underline{3} & 2 & \underline{1} \end{array}$$

$\det(A_i \cdot A_j) = -35$

(5) 

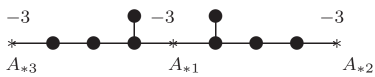
$\det(A_i \cdot A_j) = -39$

$$Z=\begin{array}{ccccccccc} & & & & & 2 & & & \\ & & & & & 4 & & & \\ \underline{1} & 2 & 3 & 4 & 5 & 6 & \underline{3} & 2 & \underline{1}\end{array}$$

(6) 

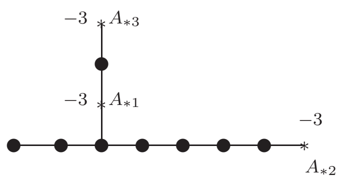
$\det(A_i \cdot A_j) = 76$

$$Z=\begin{array}{ccccccc} & & 1 & & & & \\ & & 2 & 2 & & & \\ \underline{1} & 2 & \underline{3} & 4 & 3 & 2 & \underline{1}\end{array}$$

(7) 

$\det(A_i \cdot A_j) = -55$

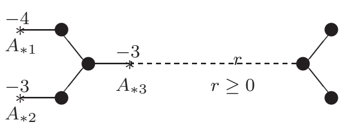
$$Z=\begin{array}{ccccccccc} & & & 2 & & 2 & & & \\ \underline{1} & 2 & 3 & 4 & \underline{3} & 4 & 3 & 2 & \underline{1}\end{array}$$

(8) 

$\det(A_i \cdot A_j) = -39$

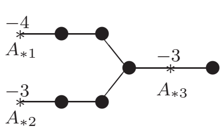
$$Z=\begin{array}{cccccccc} & & 1 & & & & & \\ & & 2 & & & & & \\ & & \underline{3} & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1}\end{array}$$

XII. The following graphs correspond to those exceptional cases in Theorem 6.38.

(1) 

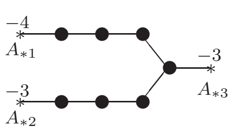
$$Z=\begin{array}{cccccc} \underline{1} & 2 & & & & 1 \\ & & 3 & \underline{2} & \ldots 2 & \\ \underline{1} & 2 & & & & 1\end{array}$$

$\det(A_i \cdot A_j) = (-1)^r 92, r \geq 0$

(2) 

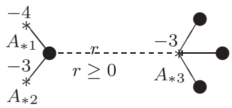
$$Z=\begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & 1 \\ \underline{1} & 2 & 3 & & & \end{array}$$

$\det(A_i \cdot A_j) = -65$

(3) 

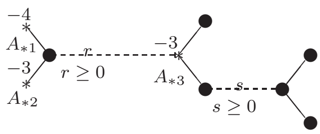
$$Z=\begin{array}{cccccc} \underline{1} & 2 & 3 & 4 & & \\ & & & & 5 & \underline{2} \\ \underline{1} & 2 & 3 & 4 & & \end{array}$$

$\det(A_i \cdot A_j) = 159$

(4) 

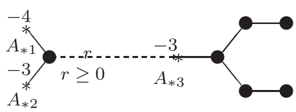
$$Z = \begin{array}{ccc} \underline{1} & & 1 \\ & 2\dots\underline{2} & 1 \\ \underline{1} & & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r(88 + 20r), r \geq 0$

(5) 

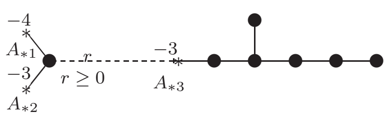
$$Z = \begin{array}{cccccc} \underline{1} & & & 1 & & \\ & 2\dots & \underline{2} & & & 1 \\ \underline{1} & & & & 2\dots2 & \\ & & & & & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+s+1}(88 + 20r), r \geq 0, s \geq 0$

(6) 

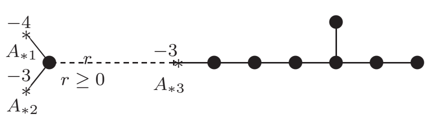
$$Z = \begin{array}{cccccc} \underline{1} & & & & 2 & 1 \\ & 2\dots & \underline{2} & 3 & & \\ \underline{1} & & & & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r(66 + 15r), r \geq 0$

(7) 

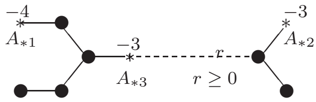
$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2\dots & \underline{2} & 3 & 4 & 3 & 2 & 1 \\ \underline{1} & & & & & & & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+1}(44 + 10r), r \geq 0$

(8) 

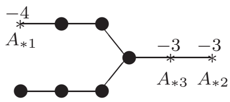
$$Z = \begin{array}{ccccccccc} \underline{1} & & & & & & 3 & & \\ & 2\dots & \underline{2} & 3 & 4 & 5 & 6 & 4 & 2 \\ \underline{1} & & & & & & & & \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r(22 + 5r), r \geq 0$

(9) 

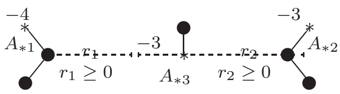
$\det(A_i \cdot A_j) = (-1)^r(88 + 11r), r \geq 0$

$$Z = \begin{array}{cccccc} \underline{1} & 2 & & & & \underline{1} \\ & & 3 & \underline{2} & \ldots 2 & \\ 1 & 2 & & & & 1 \end{array}$$

(10) 

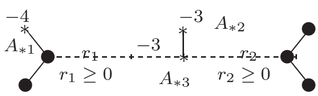
$\det(A_i \cdot A_j) = -56$

$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & \underline{1} \\ 1 & 2 & 3 & & & \end{array}$$

(11) 

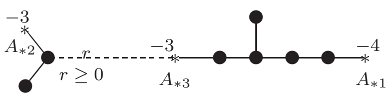
$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2}(16r_1 + 12r_2 + 2r_1r_2 + 88), r_1, r_2 \geq 0$

$$Z = \begin{array}{ccccc} \underline{1} & & 1 & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

(12) 

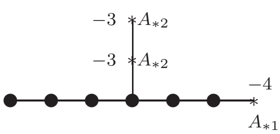
$\det(A_i \cdot A_j) = (-1)^{r_1+r_2}(88 + 16r_1), r_1, r_2 \geq 0$

$$Z = \begin{array}{ccccc} \underline{1} & & \underline{1} & & 1 \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

(13) 

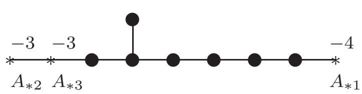
$\det(A_i \cdot A_j) = (-1)^{r+1}(60 + 8r), r \geq 0$

$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2\ldots & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ 1 & & & & & & & \end{array}$$

(14) 

$\det(A_i \cdot A_j) = -56$

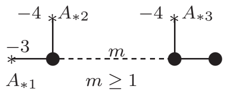
$$Z = \begin{array}{ccccccc} & & & \underline{1} & & & \\ & & & \underline{2} & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{array}$$

(15) 

$\det(A_i \cdot A_j) = 30$

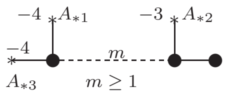
$$Z = \begin{array}{ccccccccc} & & & 3 & & & & & \\ \underline{1} & \underline{2} & 4 & 6 & 5 & 4 & 3 & 2 & \underline{1} \end{array}$$

XIII. The following graphs correspond to those exceptional cases in Theorem 6.40.

(1) 

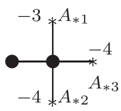
$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \ldots\ldots\ \begin{array}{c}\underline{1}\\ 2\end{array}\ 1$$

$\det(A_i \cdot A_j) = (-1)^{m+1}(64 + 10m), m \geq 1$

(2) 

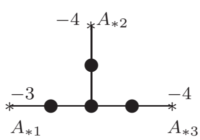
$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \ldots\ldots\ \begin{array}{c}\underline{1}\\ 2\end{array}\ 1$$

$\det(A_i \cdot A_j) = (-1)^{m+1}(64 + 8m), m \geq 1$

(3) 

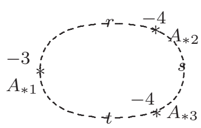
$$Z = 1\ \begin{array}{c}\underline{1}\\ 2\\ \underline{1}\end{array}\ \underline{1}$$

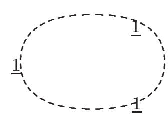
$\det(A_i \cdot A_j) = -64$

(4) 

$$Z = \underline{1}\ 2\ \begin{array}{c}\underline{1}\\ 2\\ 3\end{array}\ 2\ \underline{1}$$

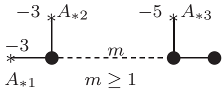
$\det(A_i \cdot A_j) = -63$

(5) 

$Z =$ 

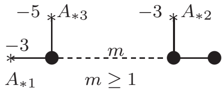
$\det(A_i \cdot A_j) =$
$(-1)^{r+s+t+1}(35 + 19r + 21s$
$+19t + 10rs + 8rt + 10st + 4rst), r, s, t \geq 0$

XIV. The following graphs correspond to those exceptional cases in Theorem 6.41.

(1) 

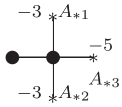
$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \ldots\ldots\ \begin{array}{c}\underline{1}\\ 2\end{array}\ 1$$

$\det(A_i \cdot A_j) = (-1)^{m+1}(57 + 9m), m = 1$

(2) 

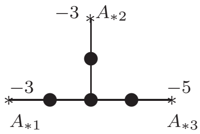
$$Z = \underline{1}\ \begin{array}{c}\underline{1}\\ 2\end{array}\ \ldots\ldots\ \begin{array}{c}\underline{1}\\ 2\end{array}\ 1$$

$\det(A_i \cdot A_j) = (-1)^{m+1}(57 + 7m), m \geq 1$

(3) 

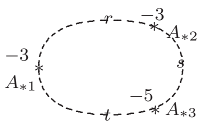
$\det(A_i \cdot A_j) = -57$

$$Z = \begin{array}{ccc} & \underline{1} & \\ 1 & 2 & \underline{1} \\ & \underline{1} & \end{array}$$

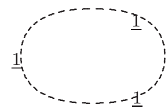
(4) 

$\det(A_i \cdot A_j) = -55$

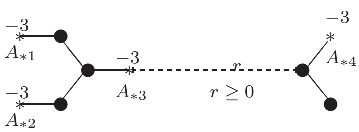
$$Z = \begin{array}{ccccc} & & \underline{1} & & \\ & & 2 & & \\ \underline{1} & 2 & 3 & 2 & \underline{1} \end{array}$$

(5) 

$\det(A_i \cdot A_j) =$
$(-1)^{r+s+t+1}(32 + 16r + 18s+$
$18t + 7rs + 9st + 7rt + 3rst), r, s, t \geq 0$

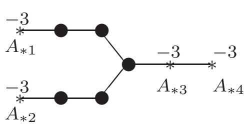
$Z =$ 

XV. The following graphs correspond to those exceptional cases in Theorem 6.42.

(1) 

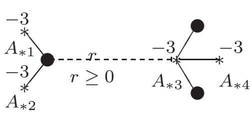
$\det(A_i \cdot A_j) = (-1)^r(110 + 15r), r \geq 0$

$$Z = \begin{array}{cccccc} \underline{1} & 2 & & & & \underline{1} \\ & & 3 & \underline{2} & \dots 2 & \\ \underline{1} & 2 & & & & 1 \end{array}$$

(2) 

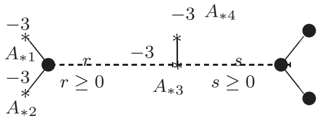
$\det(A_i \cdot A_j) = -77$

$$Z = \begin{array}{cccccc} \underline{1} & 2 & 3 & & & \\ & & & 4 & \underline{2} & \underline{1} \\ \underline{1} & 2 & 3 & & & \end{array}$$

(3) 

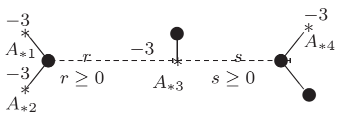
$\det(A_i \cdot A_j) = (-1)^r(108 + 24r), r \geq 0$

$$Z = \begin{array}{cccc} \underline{1} & & & 1 \\ & 2 \dots & \underline{2} & \underline{1} \\ \underline{1} & & & 1 \end{array}$$

(4) 

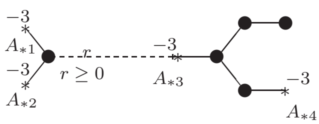
$$Z = \begin{array}{cccccc} \underline{1} & & \underline{1} & & 1 \\ & 2\ldots & \underline{2} & \ldots 2 & \\ \underline{1} & & & & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+s+1}(108 + 24r), r, s \geq 0$

(5) 

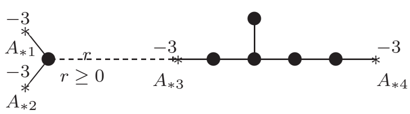
$$Z = \begin{array}{ccccc} \underline{1} & & 1 & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ \underline{1} & & & & 1 \end{array}$$

$\det(A_i \cdot A_j) =$
$(-1)^{r+s}(108 + 24r + 15s + 3rs), r, s \geq 0$

(6) 

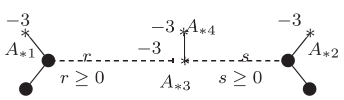
$$Z = \begin{array}{cccccc} \underline{1} & & & & 2 & \underline{1} \\ & 2\ldots & \underline{2} & 3 & & \\ \underline{1} & & & & 2 & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^r(96 + 21r), r \geq 0$

(7) 

$$Z = \begin{array}{cccccccc} \underline{1} & & & & 2 & & & \\ & 2\ldots & \underline{2} & 3 & 4 & 3 & 2 & \underline{1} \\ \underline{1} & & & & & & & \end{array}$$

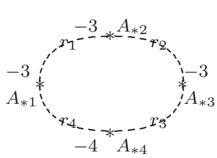
$\det(A_i \cdot A_j) = (-1)^{r+1}(69 + 15r), r \geq 0$

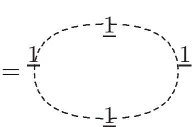
(8) 

$$Z = \begin{array}{ccccc} \underline{1} & & \underline{1} & & \underline{1} \\ & 2\ldots & \underline{2} & \ldots 2 & \\ 1 & & & & 1 \end{array}$$

$\det(A_i \cdot A_j) = (-1)^{r+s}[108 + 15(r + s) + 2rs], r, s \geq 0$

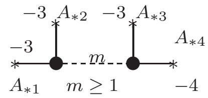
XVI. The following graphs correspond to those exceptional cases in Theorem 6.44.

(1) 

$Z =$ 

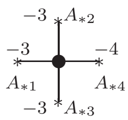
$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2+r_3+r_4}(66 + 31r_1 + 31r_2$
$+34r_3 + 34r_4 + 12r_1r_2 + 15r_1r_3+$
$13r_1r_4 + 15r_2r_4 + 16r_3r_4 + 13r_2r_3+$

$5r_1r_2r_3 + 5r_1r_2r_4 + 6r_1r_3r_4 + 6r_2r_3r_4$
$+2r_1r_2r_3r_4), r_1, r_2, r_3, r_4 \geq 0$

(2) 

$Z = \underline{1}\ \begin{matrix}\underline{1}\\2\end{matrix}\ \ldots\ \begin{matrix}\underline{1}\\2\end{matrix}\ \underline{1}$

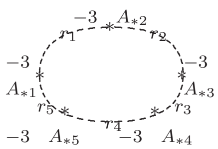
$\det(A_i \cdot A_j) = (-1)^{m+1}(81 + 15m), m \geq 1$

(3) 

$Z = \underline{1}\ \begin{matrix}\underline{1}\\2\\\underline{1}\end{matrix}\ \underline{1}$

$\det(A_i \cdot A_j) = -81$

XVII. The following graphs correspond to those exceptional cases in Theorem 6.45.

(1) 

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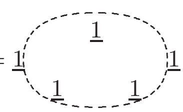
$Z =$ 

$\det(A_i \cdot A_j) =$
$(-1)^{r_1+r_2+r_3+r_4+r_5+1}$
$[121 + 55(r_1 + r_2 + r_3 + r_4 + r_5)$
$+21(r_1r_2 + r_2r_3 + r_3r_4 + r_4r_5$
$+r_5r_1) + 24(r_1r_3+$
$r_2r_4 + r_3r_5 + r_4r_1 + r_5r_2)$
$+8(r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_5+$
$r_4r_5r_1 + r_5r_1r_2) + 9(r_1r_2r_4+$
$rr_2r_3r_5 + r_3r_4r_1 +_4 r_5r_2 + r_5r_1r_3)$
$+3(r_1r_2r_3r_4 + r_2r_3r_4r_5 + r_3r_4r_5r_1 + r_4r_5r_1r_2+$
$r_5r_1r_2r_3) + r_1r_2r_3r_4r_5]$
$r_i \geq 0, i = 1, 2, 3, 4, 5$

8. Proof of the main theorem. Recall that the link L of a normal singularity is called a rational homology sphere (RHS) if $H_1(L, \mathbb{Q}) = 0$. L is called an integral homology sphere (IHS) if $H_1(L, \mathbb{Z}) = 0$. It is well known that L is an RHS if and only if the weighted dual graph Γ is a tree and the genus of each vertex equals to zero. L is IHS if additionally the determinant of the intersection matrix $(A_i \cdot A_j)$ is $\pm$ 1. From the list of graphs in Section 7, we know that there are exactly 222 weighted dual graphs of minimally elliptic singularities with $Z^2 = -5$ where Z is the fundamental cycle. And for each graph, it is easy to check that $\det(A_i \cdot A_j) \neq \pm 1$. i.e., there is no integral homology sphere link. Thus the main theorem is proved.

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