

A HIGHER DIMENSIONAL FOLIATED DONALDSON THEORY, I*

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Abstract. We introduce a foliated anti-self dual equation for higher dimensional smooth manifolds with codimension-4 Riemannian foliations. Several fundamental results including a compactification theorem for the moduli space are formulated and proved, towards the defining of a possible Donaldson type invariant for such foliated manifolds.

Key words. Riemannian foliation, foliated anti-self duality, transverse anti-self duality, taut foliation, pseudogroup, foliation cycle.

AMS subject classifications. Primary 57R57; Secondary 53C12, 58H10.

1. Introduction. Higher dimensional instanton equations were first proposed by Physicists in the 1980s, see Corrigan et al [6]. Mathematically a program of study was initiated by Donaldson and Thomas [11] in 1998, where several related theories were presented, with possible approaches carefully outlined and analyzed. Since then there has been a considerable amount of follow-up works by Thomas [33], Donaldson-Segal [10], Haydys [20], Sá Earp [30], among others. Independently Tian [34] laid down the analytical foundation by establishing the first important compactification theorem for the instanton moduli space. A common key feature in [11, 34, 10] is the natural occurrences of various “calibrated” submanifolds.

It is well known that gauge theoretic invariants have provided us new and necessary tools in the understanding of three and four dimensional smooth manifolds. The situation is rather different for higher dimensional manifolds in the sense that, during the 1970s, R.C. Kirby and L.C. Siebenmann had already classified smooth structures on topological manifolds by utilizing Milnor’s microbundle theory. Therefore it would be more interesting and indeed make more sense to place additional geometric structures on the manifolds and seek possible classifications in the higher dimension case. Thus [11], [10], [34], etc. focus on special geometric structures such as Calabi-Yau, G_2 and Spin(7) structures.

In this paper we consider foliation structures and develop an instanton gauge theory for smooth manifolds with foliations. By working on this relatively simple situation, we are able to resolve many of the outstanding issues, as suggested in the general discussions from [11, 10, 34], in order to define the appropriate invariants. Moreover our foliation case provides a good prototype to test and confirm a number of new characteristics about the higher dimensional Donaldson theory.

Specifically, let us take an n -dimensional compact smooth manifold M , $n \geq 4$, together with a transversely oriented Riemannian foliation \mathcal{F} of codimension 4. Given a bundle-like metric g , we introduce the foliated anti-self dual equation for a connection A ,

$$*(F_A \wedge \chi) = -F_A,$$

by invoking the Hodge star $*$ and the \mathcal{F} -characteristic form $\chi = \chi_{\mathcal{F}}$ with respect to g . Such an equation has in fact appeared in [6] and [34], where χ was taken to be any differential form of degree $(n-4)$. However in our special case with foliations, a crucial

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point made in the paper is that we apply the equation for basic connections A only. As a consequence of this election, the above equation is equivalent to the transverse anti-self dual equation $\overline{\ast}F_A = -F_A$, which is in terms of the Hodge star $\overline{\ast}$ defined on the normal bundle $Q = TM/T\mathcal{F}$ of the foliation. In particular being a foliated anti-self dual connection depends only on the component metric g_Q rather than the whole metric g that is involved in the original equation. This new equation is very much in the spirit of transverse foliation geometry, which was initiated historically by Reinhart [27] and has been under extensive research by A. Haefliger, P. Molino, F.W. Kamber, Ph. Tondue, to just name a few. (The other important way to study foliations is to utilize A. Connes' non-commutative geometry, which is more general and most suitable for the leafwise geometry.)

Regarding the equation above, other than the characteristic form $\chi_{\mathcal{F}}$, one could also take χ to be the Kaehler 2-form, the calibration 3-form or the Cayley 4-form respectively, when M happens to be a complex 3-fold, a G_2 -manifold or a Spin(7)-manifold. This is indeed what were explored originally in [11, 10]. Since the equation is under-determined, they need to impose additional conditions to secure the ellipticity. This is comparable to our insistence that the connection A be basic, in case χ is the characteristic form of a foliation.

Following Haefliger [17, 18], we further make the transition from the equation $\overline{\ast}F_A = -F_A$ to a third equation $\ast'F_{A'} = -F_{A'}$, which is now defined on a transversal Y based on the choice of a foliation atlas. Here the 4-dimensional (non-compact, disconnected) smooth manifold Y comes with a pseudogroup action and the last equation is the standard four-dimensional anti-self dual equation that was employed by S.K. Donaldson in his pioneering work during the 1980s. The issues that Y is non-compact and disconnected can be overcome since the connection A' on Y is invariant under the pseudogroup action.

In summary, we are guided throughout the paper by the following equivalences:

$$\begin{aligned} \text{Foliated } \chi\text{-theory on } M &\Leftrightarrow \text{Transverse theory through } Q \\ &\Leftrightarrow \text{Equivariant standard theory on a transversal } Y. \end{aligned}$$

By using the above equivalences and adapting properly the original Donaldson gauge theory to Y , we prove a compactification theorem for the foliated instanton moduli space, as a major improvement of the compactification theorem of Tian [34] for the more general moduli space. In particular certain compact leaves of \mathcal{F} will naturally arise as the branched locus of the limiting singular connections. With the strategy at hand, we also obtain complete results for the linear theory and appropriate perturbations of the foliated moduli space. The solutions to such issues were only outlined and partially addressed in [11, 10]. One key technical assumption required [11, 34, 10] is that the degree $(n-4)$ form χ be closed. This is weakened in our foliation case to that the characteristic form $\chi_{\mathcal{F}} = \chi$ just needs to be \mathcal{F} -relatively closed, namely the foliation \mathcal{F} is to be taut.

The traditional transverse foliation geometry on (M, \mathcal{F}) developed by Reinhart and others should be viewed as a sort of geometry on the leaf space M/\mathcal{F} . The point is that M/\mathcal{F} itself is a highly pathological topological space – for example not even Hausdorff in general. Hence it does not allow a direct approach as far as geometry is concerned. Likewise our foliated Donaldson theory defined on M is really meant to be a sort of Donaldson theory on the 4-dimensional leaf space M/\mathcal{F} . Clearly then, when $\dim M = 4$, the foliation \mathcal{F} is trivial and our foliated instanton theory reduces to the original Donaldson theory.

In a sequel, we will consider the orientability issue of the moduli space and further

define a Donaldson type invariant for (M, \mathcal{F}) as a certain multi-linear function on a subgroup of the foliation homology $H_2^f(M)$.

For a brief description of the paper, Section 2 sets up notations and basic results on foliations. Section 3 introduces a foliated anti-self dual equation together with preliminary discussions. Section 4 concerns transverse ellipticity/index theory and their applications, followed by Section 5 where a perturbation theory of foliated anti-self dual moduli space is presented. Finally Section 6 contains the compactification theorems of the moduli space as the main results of the paper.

2. Generalities on foliations. Some of the standard references include Tondeur [35], Molino [24], and Candel-Conlon [5]. Our purpose is to introduce necessary notations and results scattering in the literature, and along the way, to make several new observations, which will all be utilized throughout the paper.

To begin, let \mathcal{F} denote a p -dimensional foliation defined on an n -dimensional compact oriented smooth manifold M . Leaves are then p -dimensional immersed submanifolds of M . For the current section, the codimension $q := n - p$ is arbitrary.

2.1. Transverse/basic geometry. In this subsection, no metric is required. A foliation atlas $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ yields local foliation coordinates (x, y) with $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_q)$. Thus the last q components of the coordinate transformation $\gamma_{\alpha\beta}$ satisfy

$$\frac{\partial \gamma_{\alpha\beta}^{p+j}}{\partial x_i} = 0, \quad i = 1, \dots, p, \quad j = 1, \dots, q.$$

On each chart, the submanifold defined by the equation $y = \text{const}$ is called a *plaque*; plaques can be joined together to form a leaf. In terms of the tangent bundle or distribution $L := T\mathcal{F}$, the Frobenius theorem says that L is involutive, i.e. tangential fields in ΓL are closed under Lie bracket. One sets $Q = TM/L$ to be the normal bundle of \mathcal{F} , yielding the exact sequence

$$0 \longrightarrow L \longrightarrow TM \longrightarrow Q \longrightarrow 0$$

and its dual $L^* \leftarrow T^*M \leftarrow Q^* \leftarrow 0$. In the transverse foliation theory, Q plays a distinguished role as the “tangent bundle” of the leaf space M/\mathcal{F} , which has a highly pathological topology.

By definition, a *foliated principal bundle* $\pi : P \rightarrow M$ is a principal bundle together with a *lifted foliation* \mathcal{F}^P of \mathcal{F} . More precisely at each point $p \in P$ and its image $z = \pi(p)$, π_* maps $T_p\mathcal{F}^P$ isomorphically onto $T_z\mathcal{F}$, $T_p\mathcal{F}^P \cap T_pP_z = \{0\}$, and $T\mathcal{F}^P$ is invariant under the structure group G -action on P . The last two conditions mean that $T\mathcal{F}^P$ can be viewed as the horizontal distribution of a *partial connection* on P . Since $T\mathcal{F}^P$ is involutive, the partial connection is flat. Note also $\dim \mathcal{F}^P = \dim \mathcal{F}$. Thus the lifted foliation \mathcal{F}^P is such that its leaves are G -invariant and are mapped via π diffeomorphically onto leaves of \mathcal{F} .

A *foliated vector bundle* $E \rightarrow M$ is such that its frame bundle P_E is foliated. Alternatively and more directly a foliated vector bundle E is given by a flat *partial connection* $\overset{\bullet}{\nabla}$. Namely $\overset{\bullet}{\nabla}_X s$ is defined for all sections $s \in \Gamma E$ but for tangential fields $X \in \Gamma L$ only, and it satisfies the usual kind of Leibnitz rule

$$(1) \quad \overset{\bullet}{\nabla}_X(fs) = (Xf)s + f\overset{\bullet}{\nabla}_X s.$$

Likewise the curvature $\overset{\bullet}{R}(X, X') = \overset{\bullet}{\nabla}_X \circ \overset{\bullet}{\nabla}_{X'} - \overset{\bullet}{\nabla}_{X'} \circ \overset{\bullet}{\nabla}_X - \overset{\bullet}{\nabla}_{[X, X']}$ is valid for tangential fields X, X' only, and is trivial for a flat $\overset{\bullet}{\nabla}$. A *basic section* $s \in \Gamma E$ is such that $\overset{\bullet}{\nabla}_X s = 0, \forall X \in \Gamma L$. Because of (1), the concept can be localized: s is basic iff its restriction $s|_U$ to any open set is basic with respect to the partial connection $\overset{\bullet}{\nabla}|_U$. One uses $\Gamma_b(E)$ for the set of all basic sections.

Since the lifted foliation on P_E is $GL(r)$ -invariant ($r = \text{Rank}E$), it descends to a foliation \mathcal{F}^E on E , with leaves mapped diffeomorphically onto the leaves of \mathcal{F} through the projection $E \rightarrow M$. Then a section $s \in \Gamma E$ is basic iff s maps each leaf of \mathcal{F} diffeomorphically onto a leaf of \mathcal{F}^E .

If E, E' are foliated vector bundles, then the naturally related bundles $E^*, \wedge^r E, \otimes^r E, E \oplus E', \text{Hom}(E, E')$ etc. all have the induced foliated bundle structures. One can see this easily by using the corresponding operations on partial connections. In particular a *basic fiber metric* (Riemannian or Hermitian) h on E is one such that $h \in \Gamma_b(E^* \otimes E^*)$. However such a basic fiber metric does not always exist in every foliated vector bundle E . For example the existence of a basic metric on Q would mean that \mathcal{F} is a Riemannian foliation (see the next subsection).

EXAMPLE 2.1. The quotient bundle Q is canonically foliated by the Bott (partial) connection $\overset{\circ}{\nabla}$. This is defined to be $\overset{\circ}{\nabla}_X s = \pi[X, Y]$, where $X \in \Gamma L$ and $Y \in \Gamma(TM)$ is a (local) lifting of $s \in \Gamma Q$ via the projection $\pi : TM \rightarrow Q$. The Jacobi identity for vector fields implies the flatness of $\overset{\circ}{\nabla}$. In contrast, neither TM nor L is a foliated bundle.

The Bott partial connection induces one, still denoted by $\overset{\circ}{\nabla}$, on $\wedge^r Q^*$. This gives rise to basic sections

$$\Gamma_b(\wedge^r Q^*) = \{\alpha \in \Gamma(\wedge^r Q^*) \mid \overset{\circ}{\nabla}_X \alpha = 0, \forall X \in \Gamma L\}.$$

These can be viewed as differential forms on M because of the natural inclusion $\Gamma(\wedge^r Q^*) \subset \Gamma(\wedge^r T^*M) = \Omega^r(M)$. Indeed one can identify $\Gamma_b(\wedge^r Q^*) = \Omega_b^r(M)$, the latter consisting of *basic differential forms* $\alpha \in \Omega^r(M)$, that is to say

$$(2) \quad \iota_X \alpha = 0, \iota_X d\alpha = 0, \forall X \in \Gamma L$$

where ι_X is the contraction operator. The first condition means that α is reducible to $\Gamma(\wedge^r Q^*)$ while the second means α is a basic section on $\wedge^r Q^*$. Locally in a foliation chart (x, y) , a basic form has the expression

$$(3) \quad \alpha = \sum_{|J|=r} f_J(y) dy_J$$

where $f_J(y)$ is independent of x . In particular *basic functions* are exactly leafwise constant functions.

By the description (2), the exterior differential d preserves basic forms, giving rise to the basic form complex $\{\Omega_b^*(M), d\}$. The associated De Rham cohomology $H_b^r(M)$, $0 \leq r \leq q$, is called the *basic cohomology* of \mathcal{F} . One checks easily that $H_b^0(M) = \mathbf{R}$ and the natural map $H_b^1(M) \rightarrow H^1(M)$ is injective. In general the natural map $H_b^r(M) \rightarrow H^r(M)$ is neither injective nor surjective.

Return to the general foliated vector bundle $(E, \overset{\bullet}{\nabla})$ together with the flat partial connection. The basic section set $\Gamma_b(E)$ is a module over basic functions. Under

a foliation atlas, transition functions of E are all basic and vice versa. The bundle $\wedge^r Q^* \otimes E$ is foliated by $\overset{\circ}{\nabla} \otimes \overset{\bullet}{\nabla}$. The set of *basic forms with values in E* ,

$$\Omega_b^r(E) = \Gamma_b(\wedge^r Q^* \otimes E) \subset \Omega^r(E),$$

is also a module over the space $C_b^\infty(M)$ of basic functions. Each $\xi \in \Omega_b^r(E)$ can be locally written as $\xi = \sum \alpha_i \otimes s_i$ for some basic forms α_i and basic sections s_i of E .

In the case of foliated principal bundle P with a lifted foliation \mathcal{F}^P , a *basic connection* on P is a regular connection whose connection form on P is a basic 1-form with respect to \mathcal{F}^P . (This implies the the horizontal distribution of the connection contains $T\mathcal{F}^P$, i.e. the connection is *adapted to \mathcal{F}^P* .) Then a *basic connection* ∇ on E is associated with a basic connection in the frame bundle of E . Directly on E , a connection ∇ is basic iff it is adapted to \mathcal{F} , $\nabla_X = \overset{\bullet}{\nabla}_X$, and its curvature satisfies $\iota_X R = 0$ for all $X \in \Gamma L$.

REMARK 2.2. Although the flat partial connection on E can always be extended to an adapted connection, the adapted connection may not be a basic connection. In fact E may not admit any basic connection at all. The obstruction to the existence is the Atiyah-Molino class (a secondary characteristic class of the foliated bundle), see Molino [24] and Kamber-Tondeur [22]. Fundamentally this is due to the lack of a partition of unity $\{\lambda_\alpha\}$, consisting of basic functions, subordinated to any foliation cover \mathcal{U} of M . (Being basic, λ_α is constant along each plaque, hence $\text{Supp}\lambda_\alpha \subset U_\alpha$ can not hold.) It is for the same reason that a foliated vector bundle does not always admit a basic fiber metric.

A basic frame s of E is of course a basic section of the foliated frame bundle P_E , i.e. $s \in \Gamma_b(P_E)$. Alternatively $s = (s_1, \dots, s_m)$ where each s_i is a basic section of E . By restricting the foliation structure to an open set of M , one can also introduce the concept of local basic frames, which always exist when the open set is small enough.

LEMMA 2.3. *For a basic connection ∇ in a foliated vector bundle E , the following hold true:*

- 1) *Locally under a basic frame of E , one can write $\nabla = d + A$ with $A \in \Omega_b^1(\text{End}E)$. Consequently the curvature R is a global basic form: $R \in \Omega_b^2(\text{End}E)$.*
- 2) *∇ preserves basic sections, i.e. $\nabla : \Gamma_b(E) \rightarrow \Omega_b^1(E)$. More generally the same is true for the extended differential d_∇ , namely $d_\nabla : \Omega_b^r(E) \rightarrow \Omega_b^{r+1}(E)$.*

Proof. Let P be the foliated frame bundle of E . Then ∇ lifts to a unique basic connection on P ; thus the connection 1-form ω on P is basic.

1) A local basic frame s of E comes from a local basic section t of P ; t being basic means that it maps local leaves of M to local leaves of P diffeomorphically. Then the connection matrix A is the pull-back of ω via t , hence A consists of basic form entries since ω is basic. As for the curvature R , locally $R = dA + A \wedge A$ hence is basic also.

2) Under the local basic frame s , each basic section ξ of E is given a vector valued function f consisting of basic function components. Hence $\nabla \xi = (df + Af)s$ is basic. Thus $\nabla : \Gamma_b(E) \rightarrow \Omega_b^1(E)$. In general d_∇ is extended through the Leibnitz rule, and $d_\nabla : \Omega_b^r(E) \rightarrow \Omega_b^{r+1}(E)$ with $r > 0$ follows from the initial case $r = 0$ just shown. \square

Note that the lemma is not true if ∇ is only an adapted connection. Also we do not have the operator $\nabla : \Gamma E \rightarrow \Gamma(Q^* \otimes E)$, despite having $\nabla : \Gamma_b(E) \rightarrow \Gamma_b(Q^* \otimes E)$ ($= \Omega_b^1(E)$). For instance we have $d : C_b^\infty(M) \rightarrow \Gamma_b(Q^*)$ but not $d : C^\infty(M) \rightarrow \Gamma(Q^*)$,

since $f_y\eta_y dy \neq f_x\xi_y dy + f_y\eta_y dy$ unless $f_x = 0$ i.e. f is basic, where $z' = (\xi(x, y), \eta(y))$ is a foliation coordinate change.

It follows from Lemma 2.3 that the set $\mathcal{A}_b(E)$ of basic connections forms an affine space modeled on $\Omega_b^1(\text{End}E)$. From this and adapting the usual Chern-Weil transgression argument, one can readily prove the following result.

PROPOSITION 2.4. *Suppose a foliated complex vector bundle E admits a basic Hermitian metric h and a compatible unitary basic connection ∇ with curvature R . Then the class $[\det(1 + \frac{i}{2\pi}R)] \in H_b^*(M)$ is well-defined and is independent of the choices of ∇ and h .*

We will call the resulted individual classes $c_k^b(E) \in H_b^{2k}(M), 0 \leq k \leq \text{rank}E$, the *basic Chern classes* of E . Being basic cohomology classes, $c_k^b(E)$ is also subject to $0 \leq 2k \leq q$. The basic Pointryagin classes $p_k^b(E) \in H_b^{4k}(M)$ of a real foliated bundle E admitting a basic Riemannian fiber metric and a compatible basic connection are also defined since the complexified bundle $E \otimes \mathbf{C}$ carries an induced foliation structure.

REMARK 2.5. In the literature, the characteristic classes of a foliated bundle belong to the standard De Rham cohomology $H^*(M)$, see Kamber-Tondeur [22, 23] for example. These foliated characteristic classes are defined by using the partial flatness of the bundle, hence are secondary exotic classes. According to [23], such classes measure the incompatibility between the reducibilities of the bundle structure and the foliation structure into a smaller group. For instance, in the case of the foliated normal bundle Q , of course its structure group is always reducible to $O(q)$ but its associated Bott partial connection may not be so. The incompatibility here gives rise to the Godbillon-Vey class of a foliation. Foliated characteristic classes are typically not liftable to the basic cohomology $H_b^*(M)$; in fact some of them belong to $H^r(M)$ with $r > q$. Basic characteristic classes and foliated characteristic classes are best treated as complementary classes, since the existence of a compatible basic connection as required in Proposition 2.4 forces most foliated characteristic classes to vanish.

2.2. Taut, tight and Riemannian Foliations. In this subsection, we will bring metrics to the geometric discussions.

Any metric g on M gives an orthogonal splitting $TM = L \oplus L^\perp$ with $g = g_L \oplus g_{L^\perp}$, and an isomorphism $L^\perp = Q$. In turn Q inherits a metric $g_Q = g_{L^\perp}$. In fact one has the following equivalence

$$(4) \quad g \iff (\pi_L, g_L, g_Q)$$

where $\pi_L : TM \rightarrow L$ is the projection that characterizes the splitting $TM = L \oplus Q$. Of course π_L is subject to $\pi_L^2 = \pi_L$. And π_L is equivalent to an injection $i : Q \rightarrow TM$ such that $\pi \circ i = \text{Id}$, where $\pi : TM \rightarrow Q$ is the quotient projection as before.

Fix a metric g and suppose \mathcal{F} is tangentially hence transversally oriented. Choose an oriented orthonormal basis

$$(e_1, \dots, e_p, \dots, e_n) \subset L \oplus L^\perp$$

at any point. The *characteristic form* $\chi = \chi_{\mathcal{F}} \in \Omega^p(M)$ of \mathcal{F} is defined by

$$\chi(Z_1, \dots, Z_p) = \det(g(e_i, Z_j)_{1 \leq i, j \leq p})$$

for any $Z_i \in \Gamma(TM)$. While the *transverse volume form*

$$\nu \in \Gamma(\wedge^q Q^*) \subset \Omega^q(M)$$

is the volume form of g_Q but viewed as a q -form on M through the natural inclusion.

REMARK 2.6. It will be useful to note that χ is the trivial extension of the tangential volume form of \mathcal{F} to TM via the splitting $TM = L \oplus Q$ given by π_L . In terms of the equivalence (4) above, χ depends on g_L and the projection π_L alone so is independent of g_Q . On the other hand, ν depends on g_Q alone.

Let ∇^M denote the Levi-Civita connection on (TM, g) . The mean curvature field (of the leaf submanifolds) $\tau \in \Gamma Q = \Gamma L^\perp$ is defined by $\tau = \sum_{i=1}^n (\nabla_{e_i}^M e_i)^\perp$. The mean curvature form is the dual 1-form $\kappa \in \Gamma Q^* \subset \Omega^1(M)$.

An \mathcal{F} -trivial form φ on M is of degree $\geq p$ and such that

$$(5) \quad \iota_{X_1} \circ \dots \circ \iota_{X_p} \varphi = 0$$

for all tangential fields $X_1, \dots, X_p \in \Gamma L$. A useful fact is the following Rummler formula [29]:

$$(6) \quad d\chi = -\kappa \wedge \chi + \varphi$$

for some \mathcal{F} -trivial $(p + 1)$ -form φ . This formula is the main reason why κ features prominently in transverse foliation geometry.

Call \mathcal{F} is a *taut foliation* if there is a metric on M such that $\kappa = 0$, i.e. leaves of \mathcal{F} are minimal submanifolds. (Equivalently \mathcal{F} is *harmonic* with respect to this metric, namely the projection $\pi : TM \rightarrow Q$, viewed as Q -valued 1-form in $\Omega^1(Q)$, is harmonic.) As an application, Rummler [29, 32, 17] showed that \mathcal{F} is taut iff there is a p -form ξ on M such that ξ restricts non-trivially to L_z at each point $z \in M$ and ξ is \mathcal{F} -closed (i.e. $d\xi$ is \mathcal{F} -trivial). When \mathcal{F} is g -harmonic, one can take $\xi = \chi$ to be the g -characteristic form.

In a related situation, \mathcal{F} is called a *tight foliation* if there is a p -dimensional calibration on M such that all leaves are calibrated submanifolds. Then Harvey and Lawson [19] showed that \mathcal{F} is tight iff there is a closed p -form ξ on M such that ξ restricts non-trivially to L_z at each point $z \in M$. They also showed that \mathcal{F} is tight iff \mathcal{F} is taut and has a complementary distribution that is integrable.

Taut and tight are for leaf directions. Going transversally, a metric g on M is called *bundle-like* if the restriction g_{L^\perp} satisfies $\mathcal{L}_X g_{L^\perp} = 0$ for all $X \in \Gamma L$. Call \mathcal{F} a *Riemannian foliation* if it admits a bundle-like metric.

More to the point, \mathcal{F} being Riemannian can be characterized through a metric g_Q on Q alone without invoking any full metric on M . For this one defines the Lie derivative

$$(7) \quad \mathcal{L}_X g_Q(s, s') = X g_Q(s, s') - g_Q(\overset{\circ}{\nabla}_X s, s') - g_Q(s, \overset{\circ}{\nabla}_X s')$$

for $s, s' \in \Gamma Q$ and $X \in \Gamma L$, utilizing the Bott partial connection $\overset{\circ}{\nabla}$ on Q . Then \mathcal{F} is Riemannian iff Q admits a *holonomy invariant metric* g_Q , i.e. $\mathcal{L}_X g_Q = 0$ for all $X \in \Gamma L$. The equivalence follows from the fact that any holonomy invariant metric g_Q on Q extends to a bundle-like metric. (Indeed, using (4), g_Q together with each metric on L and projection $TM \rightarrow L$ determines a unique bundle-like metric.)

For another characterization, a metric g_Q on Q is holonomy invariant iff g_Q is a basic metric on Q i.e. $g_Q \in \Gamma_b(Q^* \otimes Q^*)$. The latter is in turn equivalent to that $g_Q(s, t)$ is a basic function for any basic sections $s, t \in \Gamma_b(Q)$.

The last definition is suitable for a foliated vector bundle E : one says E is a *Riemannian foliated bundle* (or *Hermitian foliated bundle*) if E admits a basic Riemannian (or Hermitian resp.) fiber metric h , i.e. h is a basic section of $E^* \otimes E^*$.

EXAMPLE 2.7. For a Riemannian foliation \mathcal{F} , the foliated bundle Q admits a canonical basic connection of some sort. Indeed given the Levi-Civita connection ∇^M of a bundle-like metric g , the following defines a basic connection ∇' :

$$(8) \quad \nabla'_Z s = \begin{cases} \overset{\circ}{\nabla}_Z s & \text{for } Z \in \Gamma L \\ \pi(\nabla_Z^M Y) & \text{for } Z \in \Gamma L^\perp \end{cases}$$

where Y is a local lifting of s . In fact ∇' is completely determined by g_Q and is the only g_Q -compatible and torsion free basic connection on Q , see [35].

In general, however, a Riemannian (or Hermitian) foliated bundle E does not necessarily admit any basic connection, whether compatible with the metric or not. This is again because of the non-existence of basic partition of unity as pointed out in Remark 2.2. Nonetheless, a Hermitian foliated holomorphic bundle over a complex foliated manifold does admit a basic connection, as one can check easily the unique connection compatible to both structures is in fact basic.

Being Riemannian is definitely a restriction for any foliation \mathcal{F} . One necessary condition is that all basic cohomology groups $H_b^r(M)$ of \mathcal{F} must be finite dimensional. This is a consequence of Molino's structure theorem for Riemannian foliations [24]. In the case of flows (1-dimensional foliations), any nonsingular Killing field gives a Riemannian flow (which is also taut). On the other hand the famous Anosov flows are all non-Riemannian.

Suppose now \mathcal{F} is Riemannian and fix a bundle-like metric $g = g_L \oplus g_{L^\perp}$ on $TM = L \oplus L^\perp$. Identify $Q = L^\perp, g_Q = g_{L^\perp}$ as before. Because g_Q is holonomy invariant, the *transverse Hodge star*, $\bar{*} : \wedge^r Q^* \rightarrow \wedge^{q-r} Q^*$, preserves the basic sections, that is, $\bar{*} : \Gamma_b(\wedge^r Q^*) \rightarrow \Gamma_b(\wedge^{q-r} Q^*)$. Hence it passes on to yield

$$\bar{*} : \Omega_b^r(M) \rightarrow \Omega_b^{q-r}(M) \quad \text{for } r = 0, 1, \dots, q$$

in view of (2). (Note that $\bar{*}$ will not preserve basic forms if g_Q is not holonomy invariant. Also $\bar{*}$ is not defined in the complement of $\Omega_b^r(M) \subset \Omega^r(M)$.) It relates the usual Hodge star $* : \Omega^r(M) \rightarrow \Omega^{n-r}(M)$ via

$$(9) \quad *\alpha = \bar{*}\alpha \wedge \chi, \quad \bar{*}\alpha = (-1)^{p(q-r)} *(\alpha \wedge \chi)$$

for $\alpha \in \Omega_b^r(M)$. (The orientations in L^\perp, L are combined in such an order to orient TM .) In particular $\chi = *\nu$ but $\nu = (-1)^{pq} *\chi$. Also ν is a basic form, hence closed. It generates a top degree basic cohomology class $[\nu] \in H_b^q(M)$. (Again these use the assumption that g_Q is holonomy invariant.)

Alvarez [1] gave an orthogonal decomposition

$$\Omega^r(M) = \Omega_b^r(M) \oplus \Omega_b^r(M)^\perp$$

and showed the basic component $\kappa_b \in \Omega_b^1(M)$ of κ is a closed 1-form. Hence the twisted exterior differential $d_\kappa = d - \kappa_b \wedge$ forms a complex, resulting the *twisted basic cohomology* $\tilde{H}_b^r(M)$. By using the Hodge decompositions for the basic Laplacians

$$\Delta_b = d_b \delta_b + \delta_b d_b, \quad \Delta_{b,\kappa} = d_\kappa \delta_\kappa + \delta_\kappa d_\kappa,$$

Kamber-Tondue [22] and Park-Richardson [26] proved the following twisted Poincare duality:

THEOREM 2.8. *Under a bundle-like metric g , the twisted paring*

$$\begin{aligned} \tilde{H}_b^r(M) \times H_b^{q-r}(M) &\longrightarrow \mathbf{R} \\ ([\alpha], [\beta]) &\mapsto \int_M \alpha \wedge \beta \wedge \chi \end{aligned}$$

is well-defined and non-degenerate.

Unlike the classical Poincare duality, the paring here depends on the metric g through χ .

Dominguez [8] has shown that any Riemannian foliation \mathcal{F} carries *tense* bundle-like metrics, i.e. having basic mean curvature form: $\kappa = \kappa_b$. For such metrics the twisted cohomology $\tilde{H}_b^r(M)$ is simply from the twisted differential $d_\kappa = d - \kappa \wedge$. To avoid excessive notations with κ_b , we will usually use tense bundle-like metrics in the paper.

COROLLARY 2.9. *Suppose \mathcal{F} is a taut Riemannian foliation so we can choose a bundle-like metric g such that $\kappa = 0$. Then the map*

$$\begin{aligned} H_b^q(M) &\longrightarrow \mathbf{R} \\ [\alpha] &\mapsto \int_M \alpha \wedge \chi \end{aligned}$$

is a well-defined and is an isomorphism. Moreover $[\nu] \in H_b^q(M)$ is a generator.

Note that $\kappa = 0$ is already needed so that the integral is well-defined on the cohomology classes. If the Riemannian foliation \mathcal{F} is a non-taut, the integral will depend on the representatives of each class and $H_b^q(M)$ is trivial, see [34].

REMARK 2.10. The main results in Alvarez [1] show that for a Riemannian foliation \mathcal{F} , the class $[\kappa_b] \in H_b^1(M)$ is independent of the choice of bundle-like metrics. Namely $[\kappa_b]$ is a topological invariant for a Riemannian foliation \mathcal{F} . The class $[\kappa_b]$ vanishes iff \mathcal{F} is taut. In particular any Riemannian foliation on a manifold with $H^1(M) = 0$ is taut. On the other hand Anosov flows are taut but not Riemannian as mentioned above, so Alvarez’s theorems do not apply.

3. Foliated/transverse anti-self dual equations. We now take M to be a compact oriented smooth manifold of dimension $n \geq 4$ and \mathcal{F} a transversely oriented Riemannian foliation of codimension $q = 4$. For convenience we usually work with bundle-like metrics that are tense so that the mean curvature form κ is basic hence closed as well.

Consider a foliated $SU(2)$ -principal bundle $P \rightarrow M$ and its associated foliated bundles, $\text{Ad}P = P \times_{\text{Ad}} SU(2)$, $\text{ad}P = P \times_{\text{ad}} \mathfrak{su}(2)$, $E = P \times_{\rho} \mathbf{C}^2$, through the standard (adjoint) representations of $SU(2)$. The foliated structures on $\text{Ad}P$ and $\text{ad}P$ can be best characterized through the induced flat partial connections.

Let $\mathcal{A}_b = \mathcal{A}_b(P)$ denote the set of basic connections on P . For $A \in \mathcal{A}_b$, consider the usual Yang-Mills functional

$$YM(A) = \int_M |F_A|^2 * 1 = - \int_M \text{Tr}(F_A \wedge *F_A),$$

where F_A is the curvature of A . (To be more in tune with gauge theory, we use different connection and curvature notations from the last section.) Since the curvature F_A is

a basic form, $F_A \in \Omega_b^2(\text{ad}P)$, one can apply (9) and rewrite the functional in terms of the transverse Hodge star

$$(10) \quad YM(A) = - \int_M \text{Tr}(F_A \wedge \overline{*}F_A) \wedge \chi$$

where χ is the characteristic $(n - 4)$ -form of \mathcal{F} . More generally for $\alpha, \beta \in \Omega_b^r(\text{ad}P)$,

$$(11) \quad (\alpha, \beta) = - \int_M \text{Tr}(\alpha \wedge \overline{*}\beta) \wedge \chi$$

defines the inner product on $\Omega_b^r(\text{ad}P)$.

Since A is a basic connection, its extended differential preserves basic forms, $d_A : \Omega_b^r(\text{ad}P) \rightarrow \Omega_b^{r+1}(\text{ad}P)$, by Lemma 2.3.

PROPOSITION 3.1. *The Euler-Lagrange equation of the Yang-Mills functional (10) on \mathcal{A}_b is $d_{A,\kappa}^{\overline{*}}F_A = 0$, where*

$$(12) \quad d_{A,\kappa}^{\overline{*}} = -\overline{*}(d_A - \kappa \wedge)\overline{*}$$

is the formal basic adjoint of $d_A : \Omega_b^1(\text{ad}P) \rightarrow \Omega_b^2(\text{ad}P)$.

Proof. Let $A \in \mathcal{A}_b, \alpha \in \Omega_b^1(\text{ad}P)$. Then $F_{A+t\alpha} = F_A + td_A\alpha + t^2\alpha \wedge \alpha$ as usual. Insert this in the first variation of (10) and apply Rummmler formula (6) for the term $d\chi$ below:

$$\begin{aligned} & \frac{d}{dt}YM(A+t\alpha)|_{t=0} \\ &= \frac{d}{dt}|_{t=0} \int_M \text{Tr}(F_{A+t\alpha} \wedge \overline{*}F_{A+t\alpha}) \wedge \chi \\ &= 2 \int_M \text{Tr}(d_A\alpha \wedge \overline{*}F_A) \wedge \chi \\ &= -2 \int_M \text{Tr}[\alpha \wedge (d_A(\overline{*}F_A) \wedge \chi + \overline{*}F_A \wedge d\chi)] \\ &= -2 \int_M \text{Tr}[\alpha \wedge (d_A(\overline{*}F_A) \wedge \chi + \overline{*}F_A \wedge (-\kappa \wedge \chi + \varphi))] \\ &= -2 \int_M \text{Tr}[\alpha \wedge (d_A(\overline{*}F_A) \wedge \chi - \kappa \wedge \overline{*}F_A \wedge \chi)] \\ &= 2 \int_M \text{Tr}[\alpha \wedge \overline{*}(-\overline{*}d_A(\overline{*}F_A) + \overline{*}(\kappa \wedge \overline{*}F_A))] \wedge \chi \\ &= 2 \int_M \text{Tr}[\alpha \wedge \overline{*}(d_{A,\kappa}^{\overline{*}}F_A) \wedge \chi] \\ &= 2(\alpha, d_{A,\kappa}^{\overline{*}}F_A). \end{aligned}$$

Here we drop the term $\alpha \wedge \overline{*}F_A \wedge \varphi$ in the integrand, as it vanishes because φ is \mathcal{F} -trivial and $\alpha \wedge \overline{*}F_A$ is basic. (In a local foliation coordinates $(x_1, \dots, x_p, y_1, \dots, y_q)$, $\alpha \wedge \overline{*}F_A$ contains dy_j only, while φ contains no top degree term $dx_1 \wedge \dots \wedge dx_p$ in the leafwise x_i -portion.)

Set $\frac{d}{dt}YM(A+t\alpha)|_{t=0} = 0$ for every $\alpha \in \Omega_b^1(\text{ad}P)$. This yields the Euler-Lagrange equation $d_{A,\kappa}^{\overline{*}}F_A = 0$. \square

Because of the κ -twisting in $d_{A,\kappa}^{\bar{*}}$, we will call $d_{A,\kappa}^{\bar{*}}F_A = 0$ or explicitly

$$(13) \quad d_A \bar{*}F_A - \kappa \wedge \bar{*}F_A = 0$$

the *twisted transverse Yang-Mills equation*. (Both the usual adjoint $*d_A*$ and the untwisted transverse adjoint $\bar{*}d_A\bar{*}$ are not suitable here. The first operator does not even preserve the basic forms.)

If the metric g is not tense, then one needs to replace κ with κ_b throughout. In [16], Glazebrook and Kamber discussed the family index theory of Dirac operators coupled with basic connections in an auxiliary foliated bundle, generalizing partially one of Atiyah-Singer earlier works without the presence of foliations.

Note for our codimension-4 case, the transverse Hodge star

$$\bar{*} : \Omega_b^2(\text{ad}P) \rightarrow \Omega_b^2(\text{ad}P)$$

satisfies $\bar{*}^2 = \text{Id}$. So its (± 1) -eigenspaces yields the decomposition

$$(14) \quad \Omega_b^2(\text{ad}P) = \Omega_b^+(\text{ad}P) \oplus \Omega_b^-(\text{ad}P)$$

which is orthogonal under (11). (One emphasizes that this is a global decomposition, not pointwise or fiberwise.) These may be called transverse self-dual and anti-self dual forms respectively. One can then write $F_A = F_A^+ + F_A^-$ for $A \in \mathcal{A}_b$. By definition, A is *transverse anti-self dual* (TASD) if $F_A^+ = 0$, i.e.

$$(15) \quad \bar{*}F_A = -F_A.$$

In view of (9), as a crucial fact, the above is equivalent to

$$(16) \quad *(F_A \wedge \chi) = -F_A$$

utilizing the Hodge star $*$ on M and the characteristic form χ of \mathcal{F} . We will call (16) the *foliated anti-self dual* (FASD) equation, which is actually defined for all connections A . However when A is basic, we have the equivalence

$$\text{TASD} \iff \text{FASD}$$

under the bundle-like metric g ; in particular in this case whether A is FASD depends on the holonomy invariant metric g_Q alone, which is perhaps not so obvious at first glance, since equation (16) certainly involves the whole metric g .

REMARK 3.2. Equation (16) resembles more the higher dimensional ASD equations presented in the literature [6, 34]. But one of the main points of our paper is that by viewing FASD connections as TASD connections, we will be able to apply directly many results from the classical 4-dimensional gauge theory, particularly when the results are local in nature.

Similarly one can apply (9) and rewrite (13) so one has the *twisted foliated Yang-Mills equation*

$$(17) \quad d*(F_A \wedge \chi) - \kappa \wedge *(F_A \wedge \chi) = 0.$$

In general for an arbitrary Riemannian foliation \mathcal{F} , an FASD connection A may not be a minimum for the Yang-Mills functional YM .

PROPOSITION 3.3. *Suppose \mathcal{F} is a taut Riemannian foliation and choose a bundle-like metric g with $\kappa = 0$. Then the Yang-Mills functional YM on \mathcal{A}_b has a lower bound $8\pi^2\tilde{k}$, where \tilde{k} denotes the value $\int_M c_2^b(P) \wedge \chi$ and $c_2^b(P)$ is the basic Chern class of P as defined in Proposition 2.4. The lower bound is realized precisely by FASD connections. Furthermore \tilde{k} depends on the metric g , more precisely it depends on the tangential metric g_L and the projection $\pi_L : TM \rightarrow L$ but is independent of g_Q .*

Proof. As earlier on, consider the foliated vector bundle $E = P \times_\rho \mathbf{C}^2$ using the standard representation $\rho : SU(2) \rightarrow \text{Aut}(\mathbf{C}^2)$. Then from Proposition 2.4, the basic Chern class is

$$c_2^b(E) = \frac{1}{8\pi^2} [\text{Tr}(F_A \wedge F_A)] \in H_b^4(M),$$

using any basic connection A on P . (For the induced basic connection on E , the curvature remains the same.) By Corollary 2.9, the integral $\int_M c_2^b(E) \wedge \chi = \tilde{k}$ is well-defined on the class; in particular the value $8\pi^2\tilde{k}$ is independent of A .

Recall $F_A^+ \perp F_A^-$ and compute formally in the inner product (11):

$$\begin{aligned} 8\pi^2\tilde{k} &= \int_M \text{Tr}(F_A \wedge F_A) \wedge \chi \\ &= -(F_A, \overline{*}F_A) \\ &= -(F_A^+ + F_A^-, F_A^+ - F_A^-) \\ &= -(F_A^+, F_A^+) + (F_A^-, F_A^-) \\ &= YM(A) - 2\|F_A^+\|^2. \end{aligned}$$

This shows that $YM(A) \geq 8\pi^2\tilde{k}$ on \mathcal{A}_b and the lower bound is realized by any FASD connection A .

From Remark 2.6, χ depends on g_L and π_L alone; the same is true for the mean curvature form κ , see [1] for example. It follows that \tilde{k} depends on g_L and π_L alone as well. \square

Under the tautness condition, we call the geometry constant

$$(18) \quad \tilde{k} = \tilde{k}(P) = \int_M c_2^b(P) \wedge \chi$$

the *foliation charge* of P , in analogy with the classical gauge theory. However here \tilde{k} is not quite a topological invariant due to its dependence on g_L . Nonetheless we will only perturb the metric g_Q later in Section 5, so the charge \tilde{k} will still remain the same.

REMARK 3.4. Without the tautness assumption, the integral

$$\int_M \text{Tr}(F_A \wedge F_A) \wedge \chi$$

will depend on the basic connection A . (In fact since $H_b^4(M) = \{0\}$ for any non-taut Riemannian foliation \mathcal{F} , it seems likely that there is always a sequence of basic connections A_i such that the integral above converges to zero.) Moreover, an FASD connection A is not necessarily twisted transverse Yang-Mills/twisted foliated Yang-Mills, as defined in (13), (17). Instead A is transverse Yang-Mills without twisting:

$d_A \bar{*}F_A = 0$ (viewing A as T ASD in (15)). Applying d_A to (16) together with the Bianchi identity, one sees that A also satisfies

$$(19) \quad d_A * F_A + d\chi \wedge F_A = 0$$

which might be called the foliated Yang-Mills equation. Thus an F ASD connection A is also always foliated Yang-Mills. Regardless of foliations, equations similar to (19) have been proposed in the physics literature [6]. However a common drawback for all such untwisted higher dimensional Yang-Mills equations is that they do not arise as the Euler-Lagrange equations of any type of energy functionals. Our twisted Yang-Mills equation (17) is advantageous in this regard, although it is not satisfied by F ASD connections.

4. Transverse ellipticity, index and the virtual dimension. Next we take up the linear theory of F ASD/T ASD, but beginning first with the general theory on transverse ellipticity. Suppose E, E' are foliated vector bundles on a manifold M with respect to a foliation \mathcal{F} .

DEFINITION 4.1. A *basic differential operator* is a linear operator $D : \Gamma_b(E) \rightarrow \Gamma_b(E')$ such that locally in a foliation coordinate $z = (x, y)$ and under basic frames of E, E' ,

$$(20) \quad D = \sum_{|\gamma| \leq m} C_\gamma(y) \frac{\partial^{|\gamma|}}{\partial y_1^{\gamma_1} \dots \partial y_q^{\gamma_q}}$$

where the matrix-valued functions $C_\gamma(y)$ all depend on y only. The concept is taken from [2] and [25], which provide some of the background material for this section.

We emphasize that D does not give rise or extend naturally to a differential operator $\tilde{D} : \Gamma E \rightarrow \Gamma E'$. For example a basic connection ∇ on E induces a first order basic differential operator $\nabla : \Gamma_b(E) \rightarrow \Gamma_b(Q^* \otimes E)$ but no natural extension $\tilde{\nabla} : \Gamma(E) \rightarrow \Gamma(Q^* \otimes E)$ exists; compare with Lemma 2.3 and the remark following it.

Given a basic differential operator $D : \Gamma_b(E) \rightarrow \Gamma_b(E')$, one can define its (transverse) symbol homomorphism $\sigma : \tilde{\pi}^*E \rightarrow \tilde{\pi}^*E'$ in the standard way, where $\tilde{\pi} : Q^* \rightarrow M$ is the projection. More precisely the homomorphism $\sigma(z, \xi) : E_z \rightarrow E'_z$ is given by $\sum_{|\gamma|=m} C_\gamma(y)\xi^\gamma$ for any covector $\xi \in Q_z^* \subset T_z^*M$ in the local expression (20).

DEFINITION 4.2. D is *transverse elliptic* if its transverse symbol is an isomorphism away from the 0-section, namely $\sigma(z, \xi)$ is an isomorphism for any $z \in M$ and non-zero $\xi \in Q_z^*$.

Likewise one defines in the usual way a *transverse elliptic complex* of basic differential operators. One such example is the basic De Rham complex

$$\dots \rightarrow \Gamma_b(\wedge^r Q^*) \xrightarrow{d} \Gamma_b(\wedge^{r+1} Q^*) \xrightarrow{d} \dots,$$

whose cohomology yields of course the basic cohomology $H_b^\bullet(M)$.

Suppose E, E' are Riemannian (or Hermitian) foliated vector bundles, so that they admit basic fiber metrics h, h' . Assume \mathcal{F} is a Riemannian foliation associated with a bundle-like metric g . Working locally one checks easily that for a basic differential

operator $D : \Gamma_b(E) \rightarrow \Gamma_b(E')$, the formal adjoint $D^* : \Gamma_b(E') \rightarrow \Gamma_b(E)$ is defined and basic as well. (Choose a foliated chart $U \subset M$ with coordinates (x, y) and basic frames for E, E' . Then h, h', g, D are all represented by (matrix) functions and derivatives in y -variables only. Consider $(D\xi, \eta) = \int_U \eta^T \cdot h \cdot D\xi \cdot \rho dx dy$, where the basic sections ξ, η , fiber metric h , the derivatives in D and the density ρ of g are all in y only. Applying the integration by parts gives $(D\xi, \eta) = (\xi, D^*\eta)$ explicitly, in which D^* surely contains coefficients and differentials in y only. This verifies D^* is a basic differential operator.) If D is transverse elliptic then so is D^* .

There is no issue to construct various appropriate Sobolev spaces for basic sections $\Gamma_b(E)$. For example one can restrict the usual Sobolev norms from ΓE to the subspace $\Gamma_b(E)$; more directly one can use a basic connection globally or basic functions locally under basic frames. We will suppress any explicit Sobolev norms unless it is necessary to specify them otherwise.

By using Molino's structure theorem [24] and a version of equivariant index, El Kacimi [21] proved the following fundamental result.

THEOREM 4.3. *Make the afore-stated assumptions. In addition, assume D is transverse elliptic. Then $D : \Gamma_b(E) \rightarrow \Gamma_b(E')$ is Fredholm with a finite index $\text{Ind} D = \dim \text{Ker} D - \dim \text{Ker} D^*$.*

Proof. We sketch another proof here by working on a transversal manifold Y with a holonomy pseudogroup \mathbf{H} action. The idea is to interpret the transverse ellipticity of D as the usual ellipticity on Y and apply a modified standard pseudodifferential operator theory to find a parametrix on Y . (It will be crucial for us to utilize the transversal Y in the following sections.)

Choose a (finite) foliation atlas $\mathcal{U} = \{U_i, \varphi_i\}$, where $\varphi_i : U_i \rightarrow \mathbf{R}^p \times \mathbf{R}^q$ is a plaque-preserving diffeomorphism, so that E, E', D, g, h, h' etc are all trivialized appropriately on each chart U_i . Assume also each plaque in U_i intersects at most one plaque in U_j for any $i \neq j$ (shrink the atlas if necessary). Then the foliation cocycles $\tau_{ij} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ can be introduced by matching the plaques, where $\phi_i : U_i \rightarrow \mathbf{R}^q$ is the composition of φ_i with the projection $\mathbf{R}^p \times \mathbf{R}^q \rightarrow \mathbf{R}^q$. Denote by \mathbf{H} the holonomy pseudogroup generated by $\{\tau_{ij}\}$; each element in \mathbf{H} is a finite chain of cocycles. For each i , fix a q -dimensional submanifold $Y_i \subset M$ so that ϕ_i restricts to a diffeomorphism $Y_i \cong \mathbf{R}^q$. (It will be convenient to simply treat $Y_i = \mathbf{R}^q$ under this diffeomorphism.) The disjoint union $Y = \coprod Y_i \subset M$ is a complete transversal of the foliation and is acted upon by \mathbf{H} in the sense of a pseudogroup.

By restriction, each \mathcal{F} -basic object in M is in one-to-one correspondence with an ordinary \mathbf{H} -invariant object on Y . For example, a basic differential form on M corresponds to an \mathbf{H} -invariant differential form of the same degree on Y . (To be precise a form α on Y is \mathbf{H} -invariant if $\rho^*(\alpha|_{R(\rho)}) = \alpha|_{D(\rho)}$ for any $\rho \in \mathbf{H}$, where $R(\rho), D(\rho)$ are respectively the range and domain of ρ .) The holonomy invariant metric g_Q on Q associates with an ordinary metric \tilde{g} on Y such that \mathbf{H} consists of \tilde{g} -isometries, and vice versa. (Hence \mathcal{F} is Riemannian iff Y carries an \mathbf{H} -invariant metric.) Likewise the foliated bundle E corresponds uniquely to a usual bundle $\tilde{E} \rightarrow Y$ with a lifted \mathbf{H} -action and the basic sections $\Gamma_b(E)$ correspond to usual \mathbf{H} -equivariant sections $\Gamma_{\mathbf{H}}(\tilde{E})$. The basic differential operator $D : \Gamma_b(E) \rightarrow \Gamma_b(E')$ of course gives rise to an \mathbf{H} -equivariant differential operator $\tilde{D} : \Gamma(\tilde{E}) \rightarrow \Gamma(\tilde{E}')$. The transverse ellipticity of D means exactly that \tilde{D} is elliptic. The main idea in our proof is to modify the usual pseudodifferential operator theory to construct a parametrix for \tilde{D} . Along the process one must deal with the issue that Y is non-compact, and this is overcome

by utilizing the \mathbf{H} -invariance. (Implicitly one pretends to work in the compact but otherwise pathological quotient space Y/\mathbf{H} .)

Choose a partition of unity $\lambda = \{\lambda_i\}$ subordinated to \mathcal{U} . Define a function $\theta = (\theta_i)$ on Y by

$$(21) \quad \theta_i(y) = \int_P \lambda_i \cdot \chi$$

where $P = P_y$ is the plaque through $y \in Y_i$ and χ is the characteristic form of g . Clearly θ has a compact support on Y and is independent of g_Q and the projection $TM \rightarrow L$.

For any \mathbf{H} -invariant function $f = (f_i)$ on Y , introduce its modified (“ θ -weighted”) integral

$$(22) \quad \int'_Y f d\mu_Y = \sum \int_{Y_i} \theta_i(y) f_i(y) d\mu_{Y_i}$$

with the \tilde{g} -measure $d\mu_Y$. One can use the \mathbf{H} -invariance of f to check easily that the integral is actually independent of the choice of λ , though still dependent upon g_L . In the same spirit the modified Fourier transform $\hat{f} = (\hat{f}_i)$ is defined via

$$\hat{f}_i(\eta) = \int_{Y_i} \theta_i(y) e^{-i\langle y, \eta \rangle} f_i(y) d\mu_Y.$$

With the θ -factor, one further defines the modified pseudodifferential operators for \mathbf{H} -invariant symbol classes from \tilde{E} to \tilde{E}' as well as the modified Sobolev spaces. (Recall everything is trivialized on $Y_i = \mathbf{R}^q$.) Adjusting the standard proof from Wells [36] for example, one obtains a parametrix on \mathbf{H} -invariant sections for any \mathbf{H} -invariant elliptic operator on Y . Consequently and in particular one has $\dim \text{Ker} \tilde{D} < \infty$ for our $\tilde{D} : \Gamma_{\mathbf{H}}(\tilde{E}) \rightarrow \Gamma_{\mathbf{H}}(\tilde{E}')$. Since the fiber metrics h, h' on E, E' are basic, replacing D with its formal adjoint D^* , we have $\dim \text{Ker} \tilde{D}^* < \infty$ for the descendant $\tilde{D}^* : \Gamma_{\mathbf{H}}(\tilde{E}') \rightarrow \Gamma_{\mathbf{H}}(\tilde{E})$.

Pulling back to M , we have then that $\dim \text{Ker} D = \dim \text{Ker} \tilde{D}$ and $\dim \text{Ker} D^* = \dim \text{Ker} \tilde{D}^*$ are both finite hence the finiteness of $\text{Ind} D$ as well. \square

Expanding (22), we define the modified integral for any \mathbf{H} -invariant top form $\alpha = (\alpha_i)$ on Y :

$$\int'_Y \alpha = \sum \int_{Y_i} \theta_i(y) \alpha_i(y).$$

In particular $\int'_Y \tilde{\nu} = \text{vol}_g(M)$ where $\tilde{\nu}$ corresponds to the transverse volume form ν on M .

REMARK 4.4. The function θ defined in (21) plays a vital role here and in later sections; a few comments are in order.

1) θ is positive on a compact subset of Y that intersects all \mathbf{H} orbits. If each θ_i is extended trivially over other components of Y , then the collection $\{\theta_i\}$ can be viewed as a partition of unity on Y subordinated to the trivial cover $\{Y_i\}$, making the integral definition (22) more natural. It is also worth noting that θ is defined whether or not \mathcal{F} is Riemannian; indeed it is simply independent of g_Q as stated above and can be viewed as a \mathcal{F} -leafwise datum. Said differently, the integral (22) on Y incorporates the \mathcal{F} -leaf geometry through θ .

2) Haefliger [17] defined a surjective linear map (“integration along leaves”)

$$(23) \quad \int_{\mathcal{F}} : \Omega^{p+r}(M) \rightarrow \Omega_c^r(Y)/\Xi$$

where Ξ is the subspace generated by elements of the form $\beta - h^*\beta$ with $\beta \in \Omega_c^r(Y)$, $h \in \mathbf{H}$ such that $\text{supp}\beta \subset \text{dom}h$. As a special case, one computes $\int_{\mathcal{F}} \chi = [\theta]$ in $\Omega_c^0(Y)/\Xi$; thus the class $[\theta]$ is independent of λ just like the integral (22). It is shown in [17] that \mathcal{F} is taut iff $d[\theta] = [d\theta] = 0$ in $\Omega_c^1(Y)/\Xi$, i.e. $[\theta]$ might be viewed as a constant function in this case.

For another relevant situation, given a leaf ℓ of \mathcal{F} , set

$$\text{vol}(\ell) = \sum_i \sum_{y \in \ell \cap Y_i} \theta_i(y).$$

This defines a leaf volume function, $\text{vol} : M \rightarrow (0, +\infty]$, as studied in the literature by D.B.A. Epstein, R.D. Edwards, K.C. Millet and D. Sullivan [12].

3) By definition, θ is not \mathbf{H} -invariant. To compensate for it one can turn to the “average” of λ_i along leaves, $\bar{\lambda}_i : M \rightarrow [0, 1]$, where

$$\bar{\lambda}_i(z) = \frac{\int_{\bar{\ell}} \lambda_i d\mu_g}{\int_{\bar{\ell}} d\mu_g} = \frac{\int_{\bar{\ell}} \lambda_i d\mu_g}{\text{vol}(\bar{\ell})}$$

in which $\bar{\ell}$ is the closure of the leaf ℓ through z . Because \mathcal{F} is Riemannian, $\bar{\ell} \subset M$ is a closed embedded submanifold, see Chapter 5 of Molino [24]. Each $\bar{\lambda}_i$ is a basic function as it is already constant along leaf closures. Thus $\bar{\lambda}_i$ restricts to an \mathbf{H} -invariant function $\tilde{\lambda}_i$ on Y . However no $\tilde{\lambda}_i$ is compactly supported, unlike θ . Note as well $\sum \tilde{\lambda}_i = 1$ on Y .

Now resume our gauge theory set-up in Section 3. The *basic gauge group* $\mathcal{G}_b = \Gamma_b(\text{Ad}P)$ acts on the set \mathcal{A}_b , preserving FASD basic connections. Differentiating the action and linearizing the equation (15) at a FASD basic connection A lead to the fundamental complex

$$(24) \quad \Omega_b^0(\text{ad}P) \xrightarrow{d_A} \Omega_b^1(\text{ad}P) \xrightarrow{d_A^+} \Omega_b^+(\text{ad}P)$$

where d_A^+ is the composition of d_A with the projection onto the basic self-dual 2-forms. It is straight forward check that (24) is a transverse elliptic complex. From Theorem 4.3, we have the following corollary.

COROLLARY 4.5. *Under a bundle-like metric, the cohomology groups of (24) have finite ranks so the Euler characteristic χ^* is well defined.*

As usual $-\chi^*$ is equal to the index of the “roll-over” operator $d_A^* + d_A^+ : \Omega_b^1(\text{ad}P) \rightarrow \Omega_b^0(\text{ad}P) \oplus \Omega_b^+(\text{ad}P)$. Write $\tilde{d}_P = -\chi^*$ and call it the virtual dimension of the FASD moduli space. Certainly \tilde{d}_P is independent of the choice of A .

In general it is rather difficult to compute $\text{Ind}D$ for an arbitrary transverse elliptic basic operator D . Brüning et al [4] give an integral formula for $\text{Ind}D$ which contains some type of eta invariants. In the special case of gauge theory, it is tempting to guess that

$$(25) \quad \tilde{d}_P = 8\tilde{k} - 3(1 - \tilde{b}_1 + \tilde{b}_2^+) + \zeta$$

in terms of the foliation charge from (18) and the basic Betti numbers, where ζ is some geometric term depending on the tangential metric g_L only. Then one must impose that \mathcal{F} be taut as well.

REMARK 4.6. In [13], H. Fan considers a somewhat related case where the foliation \mathcal{F} has one dimension and the compact manifold M has non-empty boundary such that ∂M is transverse to the flows of \mathcal{F} . A key difference is that his so-called self-duality operator is fully elliptic on M , while our fundamental complex (24) is only *transverse* elliptic. To compute the index, he applies a classical index formula of P. Gilkey and evaluates the eta invariant term of ∂M in the formula. For our more general foliation, the relevant eta invariants of [4] are defined on some stratifications of singular foliations which are not directly tied to a non-compact transversal Y of the foliation.

In the special case of Sasakian manifolds (among Riemannian taut flows), Biswas and Schumacher [3] introduced and discussed the moduli spaces of Sasakian stable bundles.

5. Perturbation of the FASD moduli space. Still use the set up in Section 3. Call a basic connection $A \in \mathcal{A}_b$ *reducible* if $\text{ad}P = L' \oplus L''$ for some foliated line bundles L', L'' and $A = A' \oplus A''$ correspondingly for some basic connections A', A'' .

PROPOSITION 5.1. *Suppose $A \in \mathcal{A}_b$ is not flat. Then the following are equivalent:*

- 1) $d_A : \Omega_b^0(\text{ad}P) \rightarrow \Omega_b^1(\text{ad}P)$ has a non-trivial kernel;
- 2) A is reducible;
- 3) $\mathcal{G}_{b,A}/\pm 1$ is non-trivial, where $\mathcal{G}_{b,A}$ is the stabilizer of the \mathcal{G}_b action at A .

Proof. This models verbatim the original proof in Freed-Uhlenbeck [15].

1) \Rightarrow 2). Take any non-trivial $u \in \text{Ker}d_A$. Then as in [15], u has a constant \mathbf{i} -eigenvalue λ and eigenvector e , $ue = \lambda e$. Because u is basic, e is basic as well. Set $L' = \langle e \rangle, L'' = (L')^{-1}$. Since $d_A e = 0$ and A is basic, $A = A' \oplus A''$ splits correspondingly into basic connections.

2) \Rightarrow 3). Under $\text{ad}P = L' \oplus L''$, the acting group $\mathcal{G}_{b,A}/\pm 1$ contains a subgroup $\text{diag}(e^{i\theta}, e^{-i\theta})$.

3) \Rightarrow 1). Take any non-trivial element u in the Lie algebra of $\mathcal{G}_{b,A}/\pm 1$. Then $d_A u = 0$ and of course u is basic. \square

Thus $\mathcal{G}_b/\pm 1$ acts freely on the set of irreducible basic connections \mathcal{A}_b^* .

As a base metric, let us fix a bundle-like but not necessarily tense metric $g = g_L + g_{L^\perp} = g_L + g_Q$, with the induced identifications $TM = L \oplus L^\perp = L \oplus Q$. Unlike the usual 4-dimensional gauge theory, we can not take all perturbations in $\mathcal{N} = \Gamma(\text{GL}(TM))$ since they do not preserve bundle-like metrics. Instead we use only those in $\mathcal{N}_b = \Gamma_b(\text{GL}(Q)) \subset \mathcal{N}$. Clearly for each $\varphi \in \mathcal{N}_b$, φ^*g is also bundle-like and at the same time has the same tangential component g_L . Plainly put, the basic automorphisms $\mathcal{N}_b = \Gamma_b(\text{GL}(Q))$ parameterize the set of all holonomy invariant metrics on Q while fixing the tangential metric. We will use \mathcal{N}_b as the perturbation space for our moduli space of FASD connections.

THEOREM 5.2. *Let $P_+ : \Omega_b^2(\text{ad}P) \rightarrow \Omega_b^+(\text{ad}P)$ be the projection onto the g_Q -self dual basic forms. Then the map*

$$\begin{aligned} \mathcal{P} = \mathcal{P}_b : \mathcal{A}_b^* \times \mathcal{N}_b &\longrightarrow \Omega_b^+(\text{ad}P) \\ (A, \varphi) &\longmapsto P_+((\varphi^{-1})^*F_A) \end{aligned}$$

has 0 as a regular value. For ease on the notations, we have suppressed the C^k and Sobolev L^2_1 norms that must be imposed on the spaces above.

Proof. As alluded in the proof of Theorem 4.3, the key will be to use the transversal manifold Y introduced in that proof. Since Y is 4-dimensional, we have the usual pointwise self-duality on Y . Note that in the original proof of Theorem 3.4 in Freed-Uhlenbeck [15], most of the arguments are pointwise in nature. One can modify and adapt their proof for Y , keeping in mind that our Y is non-compact, disconnected as well as with a pseudogroup action.

Let (A, φ) be a zero of \mathcal{P} and we need to show that the differential

$$\delta\mathcal{P} = \delta\mathcal{P}_{A, \varphi} : T_A(\mathcal{A}_b^*) \oplus T_\varphi(\mathcal{N}_b) \longrightarrow \Omega_b^+(\text{ad}P)$$

is surjective at (A, φ) , namely $\text{Coker}(\delta\mathcal{P}) = 0$. The bundle-like metric g defines the L^2 -inner product (\cdot, \cdot) on $\Omega_b^+(\text{ad}P)$. In turn

$$(26) \quad \Phi \in \text{Coker}(\delta\mathcal{P}) \Leftrightarrow (\delta\mathcal{P}(\xi), \Phi) = 0$$

for all $\xi \in \text{Dom}(\delta\mathcal{P}) = \Omega_b^1(\text{ad}P) \oplus \Gamma_b(\mathfrak{gl}(Q))$.

Of course the foliated bundle P corresponds uniquely to an $SU(2)$ -bundle P' on Y with a lifted \mathbf{H} -action, where \mathbf{H} is the holonomy pseudogroup defined in the proof of Theorem 4.3. In the same spirit, g_Q corresponds uniquely to an \mathbf{H} -invariant metric g'_Q on Y and basic sections in $\Omega_b^2(\text{ad}P)$ to \mathbf{H} -invariant sections in $\Omega_{\mathbf{H}}^2(\text{ad}P')$. From $\mathcal{P}(A, \varphi) = 0$, A corresponds to an \mathbf{H} -invariant ASD connection A' on P' with respect to $\varphi'^*(g'_Q)$, where $\varphi' \in \Gamma(\text{GL}(TY))$ corresponds to φ . Moreover \mathcal{P} corresponds to the obviously defined \mathcal{P}' .

To show $\delta\mathcal{P}$ is surjective, it is equivalent to showing that

$$\delta\mathcal{P}' = \mathcal{P}'_{A', \varphi'} : T_{A'}(\mathcal{A}_{\mathbf{H}}^*) \oplus T_{\varphi'}(\mathcal{N}'_{\mathbf{H}}) \longrightarrow \Omega_{\mathbf{H}}^+(\text{ad}P')$$

is surjective, where $\mathcal{N}'_{\mathbf{H}} = \Gamma_{\mathbf{H}}(\text{GL}(TY))$. To have the appropriate regularity property, one must use suitable Sobolev spaces for the above spaces, which are constructed in the proof of Theorem 4.3. Pick any $\Phi' \in \text{Coker}(\delta\mathcal{P}')$ and we need to show $\Phi' = 0$. From the transverse elliptic complex (24) and its corresponding *elliptic* complex on Y , and proceeding as in [15], one shows that Φ' is a function of sufficient smoothness rather than just a distribution.

Let $\Phi \in \text{Coker}(\delta\mathcal{P})$ correspond to Φ' ; Φ is a function also. From (26) and using (12), one has the pointwise equation

$$(27) \quad \bar{*}d_A \bar{*}\tilde{\Phi} - \bar{*}(\kappa_b \wedge \tilde{\Phi}) = 0,$$

where $\tilde{\Phi} = \varphi^*(\Phi)$, $\bar{*}$ is with respect to φ^*g_Q and κ is the mean curvature form of φ^*g . The basic component κ_b is used since φ^*g is not necessarily tense. Varying the second component of $\delta\mathcal{P}$ one also has the pointwise equation

$$(28) \quad (r^*F_A, \tilde{\Phi})_{\varphi^*g} = 0$$

for all $r \in \Gamma_b(\mathfrak{gl}(Q))$. Because everything is basic, equations (27) and (28) correspond respectively to the pointwise equations on Y

$$(29) \quad *'d_{A'}*\tilde{\Phi}' - *'(\kappa'_b \wedge \tilde{\Phi}') = 0, \quad (r'^*F_{A'}, \tilde{\Phi}')_{\varphi'^*g'_Q} = 0$$

where $*'$ is the $\varphi'^*(g'_Q)$ -Hodge star and $r' \in \Gamma_{\mathbf{H}}(\mathfrak{g}(TY))$. Applying Lemma 3.7 of [15] to the second equation of (29), $\text{Im}(F_{A'})$ and $\text{Im}(\tilde{\Phi}')$ are pointwise perpendicular; hence one of $F_{A'}, \tilde{\Phi}'$ has rank 1 whenever they are both non-zero.

We now move entirely to the 4-dimensional manifold Y with the metric $\varphi'^*(g'_Q)$. Since A' is ASD, together with the usual Bianchi identity, $F_{A'}$ satisfies the elliptic linear equation

$$\Delta_{A'} F_{A'} = 0,$$

where $\Delta_{A'} = (d_{A'}^* d_{A'} + d_{A'} d_{A'}^*)$ and $d_{A'}^* = - *' d_{A'} *'$. By definition, $\tilde{\Phi}'$ is self-dual. Together with the first equation in (29), this implies that $\tilde{\Phi}'$ also satisfies an elliptic linear equation:

$$\Delta_{A'} \tilde{\Phi}' - d_{A'}^*(\kappa'_b \wedge \tilde{\Phi}') + d_{A'} *'(\kappa'_b \wedge \tilde{\Phi}') = 0.$$

As a consequence of these elliptic equations, a slight generalization of the unique continuation theorem says that if $F_{A'}$ or $\tilde{\Phi}'$ vanishes in some open set of Y then it vanishes identically. (Here is a quick proof the generalization. From the construction in the proof of 4.3, $Y = \coprod_{i=1}^m Y_i$, each Y_i being an open and connected. Suppose $F_{A'}$ vanishes on some open set $U \subset Y$ and $U \cap Y_1 \neq \emptyset$. By the original Aronszajn's unique continuation theorem, $F_{A'}$ vanishes entirely on Y_1 . By the \mathbf{H} -invariance, $F_{A'}$ vanishes on the open set $\text{Im}(\tau_{12}) \subset Y_2$, where τ_{12} is the foliation cocycle used in the proof of 4.3. By Aronszajn's theorem again, $F_{A'}$ vanishes on Y_2 . Continue the same argument, then $F_{A'}$ vanishes on the whole Y .)

Following [15] roughly, the rest of the proof proceeds like this. Suppose on the contrary that $\Phi' \neq 0$ so $\tilde{\Phi}' = \varphi'^*(\Phi') \neq 0$ at some point. From the unique continuation theorem above, $\tilde{\Phi}'$ does not vanish in some open and dense subset of Y . We will derive a contradiction in both of the following cases:

Case 1. $\text{Rk}(F_{A'}) = 2$ at some point in Y hence in an open neighborhood. From the previous and an earlier paragraphs, $\tilde{\Phi}'$ has rank 1 in a possibly smaller neighborhood Z . Set $d_{A',\kappa} = d_{A'} - \kappa'_b \wedge$. Equation (29) and the self-duality of $\tilde{\Phi}'$ implies $d_{A',\kappa} \tilde{\Phi}' = 0$. Write $\tilde{\Phi}' = \alpha \otimes u$ with α a self-dual 2-form and u a unit section of $\text{ad}P$ in some open set of Z . Because κ'_b is of length 1, $d_{A',\kappa} \alpha \otimes u = 0$ implies $d_{A',\kappa} \alpha = 0$ similar to [15]. On the other hand, since κ'_b is 1-form, $d_{A',\kappa}^2 u = d_{A'}^2 u = [F_{A'}, u]$, which is non-zero by Lemma 3.12 of [15]. This contradicts the just derived $d_{A',\kappa} u = 0$.

Case 2. $\text{Rk}(F_{A'}) \leq 1$ on Y . Since A' is not flat, the above unique continuation theorem implies that $F_{A'}$ has rank 1 in some open and dense subset $V = \{F_{A'} = 0\}^c$ of Y . Locally in V , write $F_{A'} = \sigma \otimes u$ as before, with $\|u\| = 1$. From $d_{A'} F_{A'} = 0$ (Bianchi identity, no κ -twist anymore), one gets $d_{A'} u = 0$. Using the same argument as [15], one shows that $V \cap Y_i$ is connected for each i . Since $d_{A'} u = 0$, u extends over $V^c = \{F_{A'} = 0\}$ to \hat{u} on each Y_i . Therefore $d_{A'} \hat{u} = 0$ and $\hat{u} \neq 0$, contradicting that A namely A' is irreducible.

Hence we must have $\Phi' = 0$ which means $\Phi = 0$ and consequently $\text{Coker} \delta \mathcal{P} = 0$. That is to say, 0 is a regular value of \mathcal{P} . \square

The standard argument brings us the following perturbation theorem.

THEOREM 5.3. *The parameterized moduli space $\overline{\mathcal{M}}_b^* := \mathcal{P}_b^{-1}(0)/\mathcal{G}_b \subset (\mathcal{A}_b^*/\mathcal{G}_b) \times \mathcal{N}_b$ is a smooth Banach manifold. The projection map $\overline{\pi}_b : \overline{\mathcal{M}}_b^* \rightarrow \mathcal{N}_b$ is Fredholm of index \hat{d}_P . Consequently there is a Baire subset $\{\varphi\}$ of \mathcal{N}_b , whose FASD moduli spaces $\mathcal{M}_b^*(\varphi^*g)$ are all smooth of dimension \hat{d}_P .*

Proof. Here the \mathcal{G}_b action on \mathcal{A}_b^* has a slice given by $\text{Ker}d_{A,\kappa_b}^*$, in terms of the twisted formal adjoint as in Proposition 3.1. Together with 5.2, the tangent space of $\overline{\mathcal{M}}_b^*$ can then be represented by $\text{Ker}(\delta\mathcal{P}) \cap \text{Ker}d_{A,\kappa_b}^*$. That $\overline{\pi}_b$ is Fredholm follows from

$$\text{Ker}(\delta\overline{\pi}_b) \cong \mathbf{H}_b^1, \text{Coker}(\delta\overline{\pi}_b) \cong \mathbf{H}_b^+$$

where $\mathbf{H}_b^1, \mathbf{H}_b^+$ are the cohomology groups of the complex (24). This also identifies the index with that of (24), since \mathbf{H}_b^0 is trivial because of irreducibility and Proposition 5.1. Finally the standard Sard-Smale theorem yields the Baire set as the regular value set of $\overline{\pi}_b$. \square

REMARK 5.4. The perturbation theorem can be adapted easily to taut foliations. In this case, choose the base metric g so that \mathcal{F} is a harmonic Riemannian foliation. Recall the mean curvature form κ depends only on the tangential metric component g_L and the projection $\pi : TM \rightarrow L$ (a consequence of Rummmler’s formula (6), see also [1]), hence it will remain to be trivial for all the perturbed metrics $\varphi^*g, \varphi \in \mathcal{N}_b$. In particular for those generic φ the FASD moduli spaces $\mathcal{M}_b^*(\varphi^*g)$ are all smooth and come with the same foliation charge \tilde{k} .

6. Foliation cycles and compactification of the moduli space. Denote by $\mathcal{T}_r = \mathcal{T}_r(M)$ the set of all r -dimensional currents on M , namely the set of topological linear functionals on $\Omega_c^r(M) = \Omega^r(M)$. Let \mathcal{F}^r denote the set of \mathcal{F} -trivial r -forms, $r \geq p$, as defined in (5). For convenience, also set $\mathcal{F}^r = \Omega^r(M)$ when $r < p$. The annihilator set is $\text{Ann}(\mathcal{F}^r) = \{T \in \mathcal{T}_r : T \text{ vanishes on } \mathcal{F}^r\}$. Inspired by Haefliger [17], we propose the following definition.

DEFINITION 6.1. $T \in \mathcal{T}_r$ is called a *foliation current* if $T \in \text{Ann}(\mathcal{F}^r)$ and $\partial T \in \text{Ann}(\mathcal{F}^{r-1})$.

Use \mathcal{T}_r^f for the set of all r -dimensional foliation currents. Note that $\mathcal{T}_0^f = \dots = \mathcal{T}_{p-1}^f = \{0\}$ and any p -dimensional foliation current is automatically closed, since

$$\text{Ann}(\mathcal{F}^0) = \dots = \text{Ann}(\mathcal{F}^{p-1}) = \{0\}.$$

The resulted *foliation homology*

$$H_r^f(M) = \text{Ker}\{\partial : \mathcal{T}_r^f \rightarrow \mathcal{T}_{r-1}^f\} / \text{Im}\{\partial : \mathcal{T}_{r+1}^f \rightarrow \mathcal{T}_r^f\}$$

satisfies $H_0^f(M) = \dots = H_{p-1}^f(M) = \{0\}$. These trivial groups present a valid reason to truncate and re-set $H_r^f(M) := H_{p+r}^f(M)$, where $r = 0, \dots, q$.

Recall each differential form $\tau \in \Omega^r(M)$ defines a diffuse current $\tau^\wedge \in \mathcal{T}_{n-r}$ through $\tau^\wedge(\omega) = \int_M \tau \wedge \omega$. If τ is a basic form, it is not hard to confirm that τ^\wedge will be a foliation current. Furthermore Sergiescu [31] proves the following De Rham duality theorem for the basic cohomology.

THEOREM 6.2. *For any Riemannian foliation \mathcal{F} , there is a canonical isomorphism*

$$H_b^r(M) \xrightarrow{\cong} H_{q-r}^f(M)$$

induced by the diffuse map $[\tau] \mapsto [\tau^\wedge]$.

Now specialize to p -dimensional currents. Even though a foliation p -current T is always closed, T is not quite a foliation cycle according to our next definition.

DEFINITION 6.3. A *foliation cycle* is a foliation p -current $T \in \mathcal{T}_p^f$ that is represented by integration (i.e. a current of integration).

This means that T can be extended to the set of continuous p -forms on M as a continuous functional. By adding this condition, one can characterize foliation cycles in several other ways, which appear quite distinctive at first glance.

PROPOSITION 6.4. *The following are equivalent statements for any p -dimensional current T :*

- 1) T is a foliation cycle.
- 2) T is closed and lies in the topological closure of the set of finite linear combinations of Dirac foliation currents.
- 3) T is a closed current of integration whose tangent simple p -vector field is linearly dependent with that of \mathcal{F} , namely $\vec{T} = \lambda \vec{\mathcal{F}} \in \Gamma(\wedge^p(T\mathcal{F}))$ a.e. for some scalar function λ .
- 4) T corresponds to a holonomy invariant signed (Borel) measure μ on a transversal Y via “integration along leaves”.

In part 2), a Dirac foliation current is one given by a p -vector in the leaf tangent space $T_x\mathcal{F}$ at a point $x \in M$. The various equivalences in Proposition 6.4 are proved respectively by Ruelle-Sullivan, Harvey-Lawson and Haefliger: 2) \Leftrightarrow 4) in [28] (see also Theorem 10.2.12 of [5]), 2) \Leftrightarrow 3) in [19], and 1) \Leftrightarrow 4) in [17]. One can also impose that the foliation current T in 1) be *positive*, which corresponds to $\lambda > 0$ in 2), positive coefficients in the linear combinations in 3), and a positive measure μ in 4).

The correspondence 1) \Leftrightarrow 4) is particularly relevant for us, for it serves as a prelude to the proof of our convergence theorem 6.5 below. Explicitly, given a holonomy invariant (i.e. \mathbf{H} -invariant) signed measure μ on Y , the corresponding foliation cycle $T_\mu : \Omega^p(M) \rightarrow \mathbf{R}$ is given by

$$(30) \quad T_\mu(\omega) = \sum_i \int_{Y_i} \left(\int_{P_y} \lambda_i \omega \right) d\mu$$

where $\{\lambda_i\}$ is a partition of unity subordinated to the foliation cover $\{U_i\}$ and P_y is the plaque through y . More generally, given a 0-dimensional \mathbf{H} -invariant current T' on Y (not necessarily a measure), the analogous formula

$$(31) \quad T(\omega) = \sum_i T' \left(\int_{P_y} \lambda_i \omega \right)$$

establishes a one-to-one correspondence between \mathbf{H} -invariant 0-currents on Y and foliation p -currents on M . In fact for \mathbf{H} -invariant currents T' of arbitrary dimension r on Y , the same formula (31) still makes sense and gives again an isomorphism with $(p+r)$ -dimensional foliation currents T on M . (Both (30) and (31) are independent of the choice of $\{\lambda_i\}$.) The last isomorphism further appears as the bottom map in the following commutative diagram:

$$(32) \quad \begin{array}{ccc} \Omega_b^{q-r}(M) & \xrightarrow{\cong} & \Omega_{\mathbf{H}}^{q-r}(Y) \\ \downarrow & & \downarrow \\ \mathcal{T}_{p+r}^f(M) & \xleftarrow{\cong} & \mathcal{T}_r^{\mathbf{H}}(Y). \end{array}$$

Here $\Omega_{\mathbf{H}}^{q-r}(Y), \mathcal{T}_r^{\mathbf{H}}(Y)$ contain \mathbf{H} -invariant forms and currents respectively on Y . The top horizontal map is the restriction map which is also an isomorphism. The vertical maps are the injective diffuse homomorphisms (the left one induces the isomorphism in Theorem 6.2).

Moreover it is shown in [17] that $\mathcal{T}_r^{\mathbf{H}}(Y) = (\Omega_c^r(Y)/\Xi)^*$ and the bottom isomorphism is the dual of the linear map $\int_{\mathcal{F}}$ stated in (23). (Namely $\mathcal{T}_{p+r}^{\mathbf{f}}(M) \subset (\Omega^{p+r}(M))^*$ is the image set of the dual map $\int_{\mathcal{F}}^*$, which is of course injective.) Our definition 6.1 for foliation currents is motivated by this isomorphism and Theorem 6.2. Originally foliation currents were defined differently in [28, 19] to be those currents, but not necessarily closed, from parts 2), 3) of Proposition 6.4. Their definitions are valid for dimension p currents only, while 6.1 in our case is applicable for all dimensions.

Recall a rectifiable current T is the limit, under the mass norm, of Lipschitz images of finite polyhedral chains with integer coefficients, see [14]. Equivalently T is given by its the support S (a rectifiable subset of M) and integer multiplicity Θ through the formula $T(\omega) = \int_S \omega(\vec{S})\Theta d\mathcal{H}^r$, where \vec{S} is the tangent simple p -vector field of S and \mathcal{H}^r is the Hausdorff measure. Here is our main convergence theorem on the FASD moduli space $\mathcal{M}_b(E)$.

THEOREM 6.5. *Suppose \mathcal{F} is a codimension-4 taut transversely oriented Riemannian foliation on a compact oriented smooth manifold M of dimension ≥ 4 . Let $\{A_\alpha\}$ be a sequence of FASD basic connections defined on a foliated $SU(2)$ -bundle $E \rightarrow M$. Then there exists a rectifiable foliation cycle $T = (S, \Theta)$ and an FASD connection \tilde{A} defined in another foliated $SU(2)$ -bundle \tilde{E} such that the following hold:*

- 1) *There is a subsequence $\{A_{\beta}\}$ converging to \tilde{A} on $M \setminus S$ after applying basic bundle isomorphisms $\rho_\beta : E|_{M \setminus S} \rightarrow \tilde{E}|_{M \setminus S}$.*
- 2) *As foliation cycles, the diffuse currents $\frac{1}{8\pi^2}(\text{Tr}(F_{A_\beta}^2))^\wedge$ converge weakly to $\frac{1}{8\pi^2}(\text{Tr}(F_{\tilde{A}}^2))^\wedge + T$.*
- 3) *The foliation charges of E, \tilde{E} are constrained by*

$$\tilde{k}(E) = \tilde{k}(\tilde{E}) + \mathbf{M}(T)$$

where \mathbf{M} is the mass norm for currents.

- 4) *The support S of T consists of a finite number of compact leaves, $S = \{L_1, \dots, L_w\}$. With the multiplicity $\Theta = (\theta_1, \dots, \theta_w)$, the mass norm $\mathbf{M}(T) = \theta_1 \text{vol}(L_1) + \dots + \theta_w \text{vol}(L_w)$, thus depending on the tangential metric g_L only.*

Proof. We continue working with the transversal $Y = \coprod_{i=1}^m Y_i$ and holonomy pseudogroup \mathbf{H} . Unless otherwise said, notations will be carried over from the proofs of Theorems 4.3 and 5.2. We further choose the foliation atlas $\mathcal{U} = \{U_i, \varphi_i\}_{i=1}^m$ to be *regular* in the sense of Definition 1.2.11 of Candel-Conlon [5] (see their Lemma 1.2.17 for the existence). Essentially this means that the closure \overline{U}_i is compact and is contained in another foliation chart W_i as foliated sets. Hence the plaque P_y of U_i through any $y \in Y_i$ has the closure \overline{P}_y contained in a plaque of W_i and the extended plaque \tilde{P}_y of U_i is defined even if $y \in \tilde{Y}_i$.

The foliated $SU(2)$ -bundle E associates a unique $SU(2)$ -bundle $E' \rightarrow Y$ with a lifted \mathbf{H} -action. Since \mathcal{F} is Riemannian, the transverse volume form ν is basic so corresponds to an \mathbf{H} -invariant top form ν' on Y . Of course ν' is simply the volume form of g'_Q .

By Proposition 3.3 and since A_α is FASD, i.e. TASD, one has for each foliation chart U_i ,

$$\begin{aligned} 8\pi^2\tilde{k} &= \int_M \text{Tr}(F_{A_\alpha} \wedge F_{A_\alpha}) \wedge \chi = \int_M |F_{A_\alpha}|^2 \nu \wedge \chi \\ &\geq \int_{\overline{U}_i} |F_{A_\alpha}|^2 \nu \wedge \chi = \int_{\overline{Y}_i} (\int_{\overline{P}_y} \chi) |F_{A_\alpha}|^2 \nu', \end{aligned}$$

where A'_α , corresponding to A_α , is \mathbf{H} -invariant ASD on $E' \rightarrow Y$. In the above we have applied the Fubini theorem for the iterated integral and the fact that $|F_{A_\alpha}|^2$ is a basic function. Set

$$K_i = \min_{y \in \overline{Y}_i} \int_{\overline{P}_y} \chi.$$

Since the restricted χ is the g_L -volume form of \overline{P}_y and each \overline{P}_y is contained in a plaque of W_i , one sees that K_i is strictly positive (and finite). It follows that

$$\int_{Y_i} |F_{A'_\alpha}|^2 \nu' \leq 8\pi^2\tilde{k}/K_i$$

or more uniformly

$$(33) \quad \int_Y |F_{A'_\alpha}|^2 \nu' \leq 8\pi^2\tilde{k}/K$$

if one sets $K = \min\{K_1, \dots, K_m\}$. From its definition, K depends on the tangential metric g_L .

By (33), the measures $\frac{1}{8\pi^2}|F_{A'_\alpha}|^2 \nu'$ are bounded, hence weakly converge to a measure: $\frac{1}{8\pi^2}|F_{A'_\alpha}|^2 \nu' \rightarrow \mu$ on Y (taking a subsequence if necessary). Since $\int_Y d\mu \leq \tilde{k}/K$, for any $\epsilon > 0$, there are at most $\tilde{k}/(\epsilon^2 K)$ “blow-up” points $\{y_1, \dots, y_l\}$ in Y which do not lie in any g'_Q -geodesic ball of μ -measure less than ϵ^2 . Hence around each point $y \in \Omega := Y \setminus \{y_1, \dots, y_l\}$, there is a small geodesic ball $D_y \subset Y$ such that $|F_{A'_\alpha}|^2 \nu' \leq \epsilon^2$ for large α . Choose ϵ small enough so that a version of Uhlenbeck’s theorem, Proposition (4.4.9) of Donaldson-Kronheimer [9], applies to give us a (sub)sequence of gauge transformations u'_α on $E'|_\Omega$ with $u'_\alpha(A'_\alpha)$ converging to A' on Ω . Of course $\int_Y |F_{A'}|^2 \nu' \leq 8\pi^2\tilde{k}/K$. From the 4-dimensional Uhlenbeck removable singularity theorem (Theorem 4.4.12 of [9]), one extends A' smoothly to another g'_Q -ASD connection \tilde{A}' on $\tilde{E}' \rightarrow Y$, where the restriction $\tilde{E}'|_\Omega$ is isomorphic to E' . By checking the proof of Uhlenbeck’s theorem, one sees that \tilde{E}' carries a lifted \mathbf{H} -action and \tilde{A}' is \mathbf{H} -invariant. Thus they correspond to a foliated bundle \tilde{E} and an FASD-basic connection \tilde{A} respectively on M .

From $\frac{1}{8\pi^2}|F_{A'_\alpha}|^2 \nu' \rightarrow \mu$ on Y and $|F_{A'_\alpha}|^2 = |F_{u'_\alpha(A'_\alpha)}|^2 \rightarrow |F_{A'}|^2$ on Ω , one can write

$$(34) \quad \mu = \frac{1}{8\pi^2}|F_{A'}|^2 \nu' + \sum_{r=1}^l n_r \delta_{y_r}$$

where $n_r = \mu(f_r) - \frac{1}{8\pi^2} \int_Y f_r |F_{A'}|^2 \nu'$ and f_r is any compact-supported function that equals 1 near y_r and 0 near each $y_j \neq y_r$. Choose a small ball B centered at y_r such

that $f_r|_B \equiv 1$. Then

$$\begin{aligned}
 8\pi^2 n_r &= \lim_{\alpha \rightarrow \infty} \int_B f_r |F_{A_\alpha}|^2 \nu' - \int_B f_r |F_{A'}|^2 \nu' \\
 &= \lim_{\alpha \rightarrow \infty} \int_B \text{Tr}(F_{A_\alpha} \wedge F_{A_\alpha}) - \int_B \text{Tr}(F_{A'} \wedge F_{A'}) \\
 &= \lim_{\alpha \rightarrow \infty} \int_B (\text{Tr}(F_{u_\alpha(A_\alpha)} \wedge F_{u_\alpha(A_\alpha)}) - \text{Tr}(F_{A'} \wedge F_{A'})) \\
 &\equiv \lim_{\alpha \rightarrow \infty} \int_{\partial B} (\text{CS}(u_\alpha(A_\alpha)) - \text{CS}(A')) \pmod{8\pi^2 \mathbb{Z}} \\
 &= 0
 \end{aligned}$$

where CS stands for the Chern-Simons form. Thus n_r is an integer, as in the case of the classical 4-dimensional gauge theory.

Let \hat{S} be the set of leaves each of which passes through at least one point in $W := \{y_1, \dots, y_l\}$. Define $S = \cup\{L : L \in \hat{S}\}$ to be the underlying set of points in \hat{S} , i.e. S is the saturation of W . Because $|F_{A_\alpha}|^2$ are basic functions on M , each element in $S \cap Y$ is a blow-up point for the sequence $\{|F_{A_\alpha}|^2\}$. Thus $S \cap Y = W$ hence a finite set. Consequently each leaf L in \hat{S} must be compact; otherwise, in view that Y is a complete transversal, L being non-compact would imply that L intersects Y infinitely many times. (More precisely note that each foliation chart U_i has a compact closure and $L \cap U_i$ contains $|L \cap Y_i|$ plaques, which is a finite number less than $|S \cap Y|$. It follows from Theorem 5 on page 51 of [7] that L must be closed.)

Since the measure μ is \mathbf{H} -invariant, the set W is also \mathbf{H} -invariant. Write $\hat{S} = \{L_1, \dots, L_w\}$ explicitly. Then y_r is \mathbf{H} -equivariant to y_s iff y_r and y_s belong to a common leaf $L_j \in \hat{S}$. Said differently each leaf $L_j \in \hat{S}$ represents an \mathbf{H} -equivalence class in W and w is then the number of elements in the quotient set W/\mathbf{H} .

From (34), the set $\{n_1, \dots, n_l\}$ is also \mathbf{H} -invariant, since the other two terms in the equation are so. This means that $n_r = n_s$ if y_r and y_s are \mathbf{H} -equivariant. Thus the common integer $n_r = n_s$ can be associated with the leaf $L_j \in \hat{S}$ and will be denoted by θ_j , regarded as the multiplicity of L_j . Pooling together we have a p -dimensional current T with support S and multiplicity $\Theta = (\theta_1, \dots, \theta_w)$,

$$T = L_1 + \dots + L_1 + \dots + L_w + \dots + L_w,$$

where each L_j is repeated θ_j times.

Certainly T is a rectifiable current. By condition 3) in Proposition 6.4, T is a foliation cycle. Alternatively we check that T is associated with the \mathbf{H} -invariant Dirac

measure $\bar{\delta} = \sum_{r=1}^l n_r \delta_{y_r}$ through the formula (30): for any $\omega \in \Omega^p(M)$,

$$\begin{aligned} T(\omega) &= \sum_{j=1}^w \theta_j \int_{L_j} \omega \\ &= \sum_{j=1}^w \theta_j \int_{L_j} \sum_{i=1}^m \lambda_i \omega \\ &= \sum_i \sum_j \theta_j \int_{L_j} \lambda_i \omega \\ &= \sum_i \sum_j \theta_j \int_{L_j \cap Y_i} \lambda_i \omega \quad (\text{since } \text{supp} \lambda_i \subset U_i) \\ &= \sum_i \sum_j \theta_j \sum_{y_r \in L_j \cap Y_i} \int_{P_{y_r}} \lambda_i \omega \\ &= \sum_i \sum_j n_r \sum_{y_r \in L_j \cap Y_i} \int_{P_{y_r}} \lambda_i \omega \quad (\text{from the definition of } \theta_j) \\ &= \sum_r n_r \int_{P_{y_r}} \lambda_i \omega = \sum_r n_r \delta_{y_r} \left(\int_{P_y} \lambda_i \omega \right). \end{aligned}$$

That is to say, $T(\omega) = \sum_i \int_{Y_i} (\int_{P_y} \lambda_i \omega) d\bar{\delta}$. (Note the function $\int_{P_y} \lambda_i \omega$ does have compact support in y .) By 4) of Proposition 6.4, T is a foliation cycle again. (To be logically correct, (30) also requires that the foliation cover $\{U_i\}$ be regular.)

Now turn to calculate

$$\begin{aligned} \int_M \frac{1}{8\pi^2} \text{Tr}(F_{A_\alpha}^2) \wedge \omega &= \frac{1}{8\pi^2} \int_M |F_{A_\alpha}|^2 \nu \wedge \omega \\ &= \frac{1}{8\pi^2} \int_M |F_{A_\alpha}|^2 \nu \wedge \left(\sum_i \lambda_i \omega \right) \\ &= \frac{1}{8\pi^2} \sum_i \int_{Y_i} \left(\int_{P_y} \lambda_i \omega \right) |F_{A_\alpha}|^2 \nu' \quad (\text{Fubini's theorem}). \end{aligned}$$

From the weak convergence $\frac{1}{8\pi^2} |F_{A'_\alpha}|^2 \nu' \rightarrow \mu$ and (34), as $\alpha \rightarrow \infty$, the last term converges to

$$\frac{1}{8\pi^2} \sum_i \int_{Y_i} \left(\int_{P_y} \lambda_i \omega \right) |F_{A'}|^2 \nu' + \sum_r n_r \delta_{y_r} \left(\int_{P_y} \lambda_i \omega \right)$$

which equals $\int_M \frac{1}{8\pi^2} \text{Tr}(F_{\bar{A}}^2) \wedge \omega + T(\omega)$ as we have just exhibited. Thus we have shown the diffuse current sequence $\frac{1}{8\pi^2} (\text{Tr}(F_{A_\alpha}^2))^\wedge$ converges weakly to $\frac{1}{8\pi^2} (\text{Tr}(F_{\bar{A}}^2))^\wedge + T$. Since $\text{Tr} F_{A_\alpha}^2$ is a basic form, $\frac{1}{8\pi^2} (\text{Tr}(F_{A_\alpha}^2))^\wedge$ is a foliation current hence a foliation cycle as well (it is a current of integration for sure). The same is true for $\frac{1}{8\pi^2} (\text{Tr}(F_{\bar{A}}^2))^\wedge$. This completes the proof of part 2) of the theorem.

From the weak convergence in 2), we have, as $\alpha \rightarrow \infty$,

$$\int_M \frac{1}{8\pi^2} \text{Tr}(F_{A_\alpha}^2) \wedge \chi \rightarrow \int_M \frac{1}{8\pi^2} \text{Tr}(F_{\bar{A}}^2) \wedge \chi + T(\chi),$$

which leads to the charge equation in 3).

Part 4) and the rest of the theorem are pretty clear. \square

REMARK 6.6. 1) In essence, Theorem 4.2.3 of Tian [34] implies that the current T appearing in our Theorem 6.5 is a rectifiable foliation cycle. Then Corollary 1.16 of [19] shows also that the support S of T consists of a finite number of compact leaves. Our proof here is much shorter and Theorem 6.5 contains a lot stronger results than [34]. For example the limit connection \tilde{A} extends smoothly on the entire manifold M , meaning the singular set $S([\tilde{A}])$ is empty. In the general case of [34], it is only conjectured that $S([\tilde{A}])$ has Hausdorff codimension ≥ 6 (see p. 263 therein).

2) The tautness assumption in 6.5 is weaker than χ being closed, which is as assumed in [34] (cf. the discussion on tightness in Subsection 2.2). Even without the tautness assumption, our arguments above show that parts 1), 2), 4) of Theorem 6.5 continue to hold, as long as the sequence $\int_M \frac{1}{8\pi^2} \text{Tr}(F_{A_\alpha}^2) \wedge \chi$ is bounded for all α . The essential fact needed here is that a basic connection A on E is FASD iff the corresponding \mathbf{H} -invariant connection A' on E' is ASD, regardless whether \mathcal{F} is taut. (In contrast, A is *twisted* foliated Yang-Mills (17) iff A' is Yang-Mills without the tautness.)

The existence of a compact leaf is a fundamental problem in foliation theory. On one extreme, a theorem of Molino [24] says that for a Riemannian foliation \mathcal{F} with all leaves compact, the leaf space M/\mathcal{F} is an orbifold. The converse is evidently true as well.

To state the compactification theorem, introduce the set $\Upsilon_E = \{(\hat{E}, T)\}$ where \hat{E} is a foliated $SU(2)$ -vector bundle and $T = \{L_1, \dots, L_n\}$ consists of compact leaves, repeated according to multiplicity. The pair (\hat{E}, S) is subject to the condition

$$(35) \quad \tilde{k}(\hat{E}) + \text{vol}(L) = \tilde{k}(E)$$

where $\text{vol}(T) = \text{vol}(L_1) + \dots + \text{vol}(L_n)$. Define the ideal FASD moduli space

$$\mathcal{IM}_b(E) = \{([A], T) \mid [A] \in \mathcal{M}_b(\hat{E}), (\hat{E}, T) \in \Upsilon_E\},$$

whose topology is characterized through the sequence convergence: a sequence $([A_\alpha], T_\alpha)$ in $\mathcal{IM}_b(E)$ is convergent to $([A], T)$ if after suitable basic gauge transformations $A_\alpha \rightarrow A$ on $M \setminus (\cup_\alpha \text{supp} T_\alpha \cup \text{Supp} T)$ and $T_\alpha \rightarrow T$ weakly as currents.

THEOREM 6.7. *Suppose \mathcal{F} is a taut foliation and g is a bundle-like metric with mean curvature form $\kappa = 0$. Then the ideal moduli space $\mathcal{IM}_b(E)$ is compact. Moreover $\mathcal{IM}_b(E)$ depends on the leafwise metric component of g alone.*

Proof. The compactness is a consequence of Theorem 6.5. The sole dependence on g_L is due to that Equation (35) involves g_L only in the foliation charge and leaf volumes. \square

In a future paper, we plan to address the orientability issue of the moduli space $\mathcal{M}_b = \mathcal{M}_b(E)$. We will use a natural foliation structure on \mathcal{B} to define a slant product $H_b^4(\mathcal{B} \times M) \times H_2^f(M) \rightarrow H_b^2(\mathcal{B})$. Then we define a Donaldson invariant as a multilinear functional on $H_2^f(M)$ by making a suitable pairing with the fundamental class $[\mathcal{M}_b]$. In order to get an integer-valued invariant we restrict to a subgroup of $H_2^f(M)$, whose classes are generated by rectifiable currents. In the case of all leaves being compact, we would also like to compare our foliated theory with the existing orbifold Donaldson theory, which is defined on the orbifold leaf space M/\mathcal{F} .

A foiliated Seiberg-Witten theory is contained in Andrew Renner's University of Missouri doctoral thesis (expected in 2015).

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