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005

ON THE SIEGEL-WEIL FORMULA OVER FUNCTION FIELDS*

FU-TSUN WEI†

Abstract. The aim of this article is to prove the Siegel-Weil formula over function fields for the dual reductive pair $(\mathrm{Sp}_n, \mathrm{O}(V))$, where Sp_n is the symplectic group of degree $2n$ and (V, Q_V) is an anisotropic quadratic space with even dimension. This is a function field analogue of Kudla and Rallis' result. By this formula, the theta series is identified with the special value of the Siegel-Eisenstein series on Sp_n at a critical point.

Key words. Function field, theta series, Eisenstein series, automorphic form.

AMS subject classifications. 11F27, 11F55, 11M36, 11R58.

Introduction. In the number field case, the Siegel-Weil formula, discovered by Siegel [16] and considerably extended by Weil ([18] and [19]) in the representation theoretical language, connects special values of Eisenstein series with theta series from quadratic spaces. In [19], Weil assumed the critical point in question lies in the absolute convergence of the Eisenstein series. After Weil, Kudla and Rallis ([9], [10], and [15]) explored the analytic behavior of the meromorphic continuation of the Eisenstein series, and extended Weil's result to the case beyond the convergence range for the dual reductive pair $(\mathrm{Sp}_n, \mathrm{O}(V))$. Here (V, Q_V) is a non-degenerate quadratic space with even dimension, $\mathrm{O}(V)$ is the associated orthogonal group, and Sp_n is the symplectic group of degree $2n$. Many extensions and variations of this formula are studied in the number field case, and have applications to the special values of automorphic L-functions. However, there is a lack of knowledge about this formula in the function field context, except, Harris [4] dealt with $(\mathrm{SL}_2, \mathrm{O}(V))$ (resp. $(\widetilde{\mathrm{SL}_2}, \mathrm{O}(V))$, where $\widetilde{\mathrm{SL}_2}$ is the metaplectic cover of SL_2) when the dimension of V is even (resp. odd) and larger than 4. Our purpose in this article is to show a function field analogue of the Siegel-Weil formula for the dual reductive pair $(\mathrm{Sp}_n, \mathrm{O}(V))$ where V is an anisotropic quadratic space with even dimension, in order to complete the author's work ([2], [3], and [17]) on the central critical values of L-functions coming from "Drinfeld type" autumorphic forms.

Let k be a global function field with odd characteristic. Let (V, Q_V) be an anisotropic quadratic space over k with even dimension. Take a Schwartz function $\varphi \in S(V(\mathbb{A}_k)^n)$, where $\mathbb{A}_k$ is the adele ring of k, the associated theta series on $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$ is defined by

$$\theta(g, h, \varphi) := \sum_{x \in V(k)} \big(\omega(g, h)\varphi\big)(x), \quad \forall (g, h) \in \mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k).$$

Here ω is the Weil representation of $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$ on the Schwartz space $S(V(\mathbb{A}_k)^n)$ (cf. Section 1.3). Since V is anisotropic, the following integral is well-defined:

$$I^n(g, \varphi) := \int_{\mathrm{O}(V)(k) \backslash \mathrm{O}(V)(\mathbb{A}_k)} \theta(g, h, \varphi) dh.$$

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†Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan (ftwei@math.sinica.edu.tw).

The measure dh is induced from the Haar measure on $\mathrm{O}(V)(\mathbb{A}_k)$ normalized so that the volume of $\mathrm{O}(V)(k)\backslash \mathrm{O}(V)(\mathbb{A}_k)$ is 1. Then the main theorem of this article is:

THEOREM 0.1. *For $\varphi \in S(V(\mathbb{A}_k)^n)$ and $g \in \mathrm{Sp}_n(\mathbb{A}_k)$,*

$$E(g, s_{(n)}, \Phi_\varphi) = \epsilon(n) \cdot I^n(g, \varphi).$$

Here:

(i) *$E(g, s, \Phi_\varphi)$ is the Siegel-Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_k)$ associated to the Siegel section Φ_φ* (cf. Section 3)*;*

(ii) *$s_{(n)} = \dim(V)/2 - (n+1)/2$; and*

(iii)

$$\epsilon(n) = \begin{cases} 1 & \text{if } \dim(V) > n+1, \\ 2 & \text{if } \dim(V) \le n+1. \end{cases}$$

Note that $s_{(n)}$ could be out of the convergence region of the Siegel-Eisenstein series: $\mathrm{Re}(s) > (n+1)/2$. In particular, the functional equation of the Eisenstein series relates the values at s and $-s$, which means that $s_{(n)} = 0$ is the central critical point of the Eisenstein series when $\dim(V) = n+1$. This gives an application to the central critical values of automorphic L-functions. In concrete terms, we express the L-function in which we are interested by an integral representation involving the Siegel-Eisenstein series. Then at $s = s_{(n)}$, the theta series $I^n(g, \varphi)$ provides a plenty of arithmetic information and leads us to explicit formulas for the special values of L-functions in question.

The strategy of the proof for Theorem 0.1 follows from [9]. The first step is to show that $I^n(g, \varphi)$ and $E(g, s, \Phi_\varphi)$ are both concentrated on the standard Borel subgroup of Sp_n (Proposition 2.3 and 3.2). Then we review the meromorphic continuation of the Eisenstein series, and make sure that $E(g, s, \Phi_\varphi)$ is holomorphic at $s_{(n)}$. Finally, we prove the equality of the constant term of $I^n(g, \varphi)$ and $E(g, s_{(n)}, \Phi_\varphi)$ along the Siegel-parabolic subgroup. Therefore $E(g, s_{(n)}, \Phi_\varphi) - \epsilon(n) I^n(g, \varphi)$ must be a cusp form on $\mathrm{Sp}_n(\mathbb{A}_k)$ which is orthogonal to all the cusp forms on $\mathrm{Sp}_n(\mathbb{A}_k)$. This assures the result. It is worth pointing out that the dimension of V must be 2 or 4 by Hasse-Minkowski principle. In particular:

- when $\dim(V) = 2$, $(V, Q_V) \cong (F, \alpha \cdot \mathrm{Nr}_{F/k})$ for $\alpha \in k^\times$ where F/k is a quadratic extension and $\mathrm{Nr}_{F/k}$ is the norm form on F/k;
- when $\dim(V) = 4$ and the discriminant of V is a square in k, $(V, Q_V) \cong (D, \mathrm{Nr}_{D/k})$ where D is a division quaternion algebra over k and $\mathrm{Nr}_{D/k}$ is the reduced norm form on D/k;
- when $\dim(V) = 4$ and the discriminant d of V is a non-square in k,

$$(V, Q_V) \cong \left(F_d^o \oplus D^o, \alpha \cdot (\mathrm{Nr}_{F^o/k} \oplus \mathrm{Nr}_{D^o/k})\right)$$

  for $\alpha \in k^\times$ where $F_d = k(\sqrt{d})$, D is a division quaternion algebra over k such that F_d cannot embed into D, and

$$F_d^o \text{ (resp. } D^o) = \{b \in F_d \text{ (resp. } D) : \mathrm{Tr}(b) = 0\}.$$

This observation simplifies the proof. However, there are several techniques used in [9] which were not verified in the function field case. Therefore for the sake of completeness, some further discussions are sought in the appendices, including Fourier coefficients of theta series, the Jacquet module of the Schwartz space, and Maass-Jacquet-Shalika Eisenstein series on GL_n. We point out that in the number field case, Kudla-Rallis use a differential operator (introduced by Maass) at one archimedean place to obtain the continuation and also the functional equation of the Maass-Jacquet-Shalika Eisenstein series. Our approach is to write down directly the explicit form of the meromorphic continuation, and the functional equation shows up accordingly.

The structure of this article is organized as follows. We set up basic notations in Section 1.1 and 1.2, and recall the Weil representation of $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$ on the Schwartz space $S(V(\mathbb{A}_k)^n)$ in Section 1.3. In Section 2, we introduce the theta series $I^n(\cdot, \varphi)$, and show that it is concentrated on the standard Borel subgroup of Sp_n. In Section 3, we investigate the analytic behavior of Siegel-Eisenstein series by studying the constant terms along standard parabolic subgroups. Also shown in Section 3 is that the Eisenstein series are concentrated on the Borel subgroup of Sp_n. We recall in Section 4 the meromorphic continuation of the intertwining operators on Siegel sections, and deduce in Section 5 the holomorphic property of Siegel-Eisenstein series at $s_{(n)}$ by using Maass-Jacquet-Shalika Eisenstein series on GL_n. Finally, we show the Siegel-Weil formula for the case when $\dim(V) = n+1$ in Section 6.1, and prove the other cases by a finite induction process in Section 6.2 and 6.3. In Appendix A, we follow Rallis [15] to study the non-singular Fourier coefficients of $I^n(g, \varphi)$ and $E(g, s, \Phi_\varphi)$. Appendix B is a review of the Jordan structure of the Jacquet module of the Schwartz space $S(V(k_v)^n)$ (where k_v is the completion of k at a place v). Finally, we show the meromorphic continuation and functional equation of the Maass-Jacquet-Shalika Eisenstein series on GL_n in Appendix C.

1. Preliminary.

1.1. Basic setting. Let k be a global function field with finite constant field $\mathbb{F}_q$, i.e. k is a finitely generated field extension of transcendence degree one over $\mathbb{F}_q$ and $\mathbb{F}_q$ is algebraically closed in k. In this article we always assume that q is **odd**. For each place v of k, the completion of k at v is denoted by k_v, and O_v is the valuation ring in k_v. Take a uniformizer π_v in O_v. We set $\mathbb{F}_v := O_v/\pi_v O_v$, the residue field at v. The cardinality of $\mathbb{F}_v$ is denoted by q_v. For each $\alpha \in k_v$,

$$|\alpha|_v := q_v^{-\operatorname{ord}_v(\alpha)}.$$

The adele ring of k is denoted by $\mathbb{A}_k$. We let $O_{\mathbb{A}_k} := \prod_v O_v$, the maximal compact subring of $\mathbb{A}_k$. For any element $\alpha = (\alpha_v)_v \in \mathbb{A}_k^\times$, the norm $|\alpha|_{\mathbb{A}_k}$ is defined to be

$$|\alpha|_{\mathbb{A}_k} := \prod_v |\alpha_v|_v.$$

We fix a non-trivial additive character $\psi = \otimes_v \psi_v : \mathbb{A}_k \to \mathbb{C}^\times$ which is trivial on k (here $\psi_v(x_v) := \psi(0, ..., 0, x_v, 0, ...)$, for all x_v in k_v).

1.2. The symplectic group Sp_n. For a positive integer n, let

$$\mathrm{Sp}_n = \left\{ g \in \mathrm{GL}_{2n} \;\middle|\; {}^t g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

We view Sp_n as an affine algebraic group over k. Let $B_n = T_n \cdot U_n$ be the standard Borel subgroup of Sp_n where

$$T_n = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \;\middle|\; a = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} (=: \mathrm{diag}(t_1, ..., t_n)), t_i \in \mathbb{G}_m \right\}$$

and

$$U_n = \left\{ \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \in \mathrm{Sp}_n \;\middle|\; a = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_n \right\}.$$

A parabolic subgroup P of Sp_n is called *standard* if B_n is a subgroup of P. The *Siegel-parabolic subgroup* P_n is equal to $M_n \cdot N_n$, where

$$M_n := \left\{ \mathfrak{m}(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_n \right\}$$

and

$$N_n := \left\{ \mathfrak{n}(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \;\middle|\; b = {}^t b \in \mathrm{Mat}_n \right\}.$$

Let

$$\mathrm{Sym}_n := \{ b = {}^t b \in \mathrm{Mat}_n \}.$$

Note that GL_n and Sym_n are isomorphic to M_n and N_n by the map $\mathfrak{m}$ and $\mathfrak{n}$ respectively. For $0 < r < n$ we let $P_r = M_r N_r$ be the maximal proper parabolic subgroup of Sp_n where

$$N_r = \left\{ \begin{pmatrix} I_r & x & y & z \\ 0 & I_{n-r} & {}^t z & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & -{}^t x & I_{n-r} \end{pmatrix} \in \mathrm{Sp}_n \right\},$$

$$M_r = \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & {}^t \alpha^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} \;\middle|\; \alpha \in \mathrm{GL}_r, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{n-r} \right\}.$$

The above description of M_r gives us a natural isomorphism between M_r and $\mathrm{GL}_r \times \mathrm{Sp}_{n-r}$.

Let $N_{\mathrm{Sp}_n}(T_n)$ be the normalizer of T_n in Sp_n. The *Wyel group* $N_{\mathrm{Sp}_n}(T_n)/T_n$ is denoted by W_{Sp_n}. We shall use the same symbol for an element of $N_{\mathrm{Sp}_n}(T_n)$ and its image in W_{Sp_n}. The Bruhat decomposition says that

$$\mathrm{Sp}_n = \coprod_{w \in W_{\mathrm{Sp}_n}} B_n w B_n.$$

Let $X(T_n)$ be the group of (algebraic) characters on T_n. Then $X(T_n) = \oplus_{i=1}^n \mathbb{Z}x_i$, where x_i is the character on T_n satisfying

$$x_i(\mathrm{diag}(t_1, ..., t_n)) = t_i.$$

We define an $\mathbb{R}$-bilinear form $< \cdot, \cdot >$ on $X(T_n) \otimes_{\mathbb{Z}} \mathbb{R}$ by setting

$$< x_i, x_j >= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and extending bilinearly. The left action of W_{Sp_n} on $X(T_n) \otimes_{\mathbb{Z}} \mathbb{R}$ (induced by conjugation on T_n) is orthogonal with respect to $< \cdot, \cdot >$, i.e.,

$$< wx, wy >=< x, y > \text{ for all } w \in W_{\mathrm{Sp}_n} \text{ and } x, y \in X(T_n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The set of roots of Sp_n with respect to T_n is denoted by Δ_n. For each root $\alpha \in \Delta_n$, let N_α be the unipotent subgroup of Sp_n associated to α, and the reflection associated to α is denoted by w_α. We let Δ_n^+ be the set of positive roots (with respect to B_n), and the simple roots are

$$\begin{aligned} \alpha_i &= x_i - x_{i+1}, \ 1 \leq i < n, \\ \alpha_n &= 2x_n. \end{aligned}$$

It is known that the Wyel group W_{Sp_n} is generated by w_{α_i}.

1.3. The Weil representation of $\mathrm{Sp}_n \times \mathrm{O}(V)$**.** Let (V, Q_V) be an anisotropic quadratic space over k which has even dimension. By Hasse-Minkowski principle (cf. Theorem 2.12 and Section 3.1 in [11]), the dimension of V must be 2 or 4. Set

$$< x, y >_V := Q_V(x + y) - Q_V(x) - Q_V(y), \quad \forall x, y \in V,$$

the bilinear form associated to Q_V. The orthogonal group of V is denoted by $\mathrm{O}(V)$, i.e.

$$\mathrm{O}(V) = \{h \in \mathrm{GL}(V) \mid Q_V(hx) = Q_V(x), \quad \forall x \in V\}.$$

Here we view $\mathrm{O}(V)$ as an affine algebraic group over k.

For each place v of k, we have fixed an additive character ψ_v on k_v in Section 1.1. Let $V(k_v) := V \otimes_k k_v$ and let $S(V(k_v)^n)$ be the space of *Schwartz functions on* $V(k_v)^n$, i.e. the space of functions on $V(k_v)$ which are locally constant and compactly supported. The (*local*) *Weil representation* $\omega_v(= \omega_{v,\psi_v})$ of $\mathrm{Sp}_n(k_v) \times O(V)(k_v)$ on $S(V(k_v)^n)$ is determined by the following: for every $\varphi_v \in S(V(k_v)^n)$ and $x \in V(k_v)^n$,

$$(\omega_v(h)\varphi_v)(x) := \varphi_v(h^{-1}x_1, ..., h^{-1}x_n), \ \forall h \in \mathrm{O}(V)(k_v), ;$$

$$\left(\omega_v \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \varphi_v\right)(x) := \chi_{V,v}(\det a) |\det a|_v^{\frac{\dim(V)}{2}} \cdot \varphi_v(x \cdot a), \ \forall a \in \mathrm{GL}_n(k_v);$$

$$\left(\omega_v \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \varphi_v\right)(x) := \psi_v\Big(\mathrm{Trace}\big(b \cdot Q_V^{(n)}(x)\big)\Big) \cdot \varphi_v(x), \ \forall b \in \mathrm{Sym}_n(k_v);$$

$$\left(\omega_v \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \varphi_v\right)(x) := \varepsilon_v(V)^n \cdot \widehat{\varphi}_v(x).$$

Here:

- $\chi_{V,v} : k_v^\times \to \{\pm 1\}$ is the quadratic character associated to V at v, i.e.

$$\chi_{V,v}(\alpha) := \big(\alpha, (-1)^{\frac{\dim(V)}{2}} \det(V)\big)_v,$$

  where $(\cdot,\cdot)_v : k_v^\times \times k_v^\times \to \{\pm 1\}$ is the Hilbert quadratic symbol and

$$\det(V) := \det\Big((\frac{1}{2} < x_i, x_j >_V)_{1 \leq i,j \leq \dim(V)} \Big) \in k^\times/(k^\times)^2$$

  for any k-basis $\{x_1, ..., x_{\dim(V)}\}$ of V.
- $Q_V^{(n)} : V^n \to \mathrm{Sym}_n$ is the *moment map,* i.e. for any $x = (x_1, ..., x_n) \in V^n$,

$$Q_V^{(n)}(x) = \left(\frac{1}{2} < x_i, x_j >_V \right)_{1 \leq i,j \leq n}.$$

- $\widehat{\varphi}_v$ is the Fourier transform of φ_v (with respect to ψ_v):

$$\widehat{\varphi}_v(x) := \int_{V(k_v)^n} \varphi_v(y) \cdot \psi_v(\sum_{i=1}^n < x_i, y_i >_V) dy, \quad \forall x = (x_1, ..., x_n) \in V(k_v)^n.$$

  The Haar measure $dy = dy_1 \cdots dy_n$ is chosen to be *self dual*, i.e.

$$\widehat{\widehat{\varphi}}_v(x) = \varphi_v(-x), \quad \forall x \in V(k_v)^n.$$

- $\varepsilon_v(V)$ is the *Weil index of* V *at* v, i.e.

$$\varepsilon_v(V) := \int_{L_v} \psi_v\big(Q_V(x)\big) dx$$

  for any sufficiently large O_v-lattice L_v in $V(k_v)$. The Haar measure dx is also chosen to be self dual.

We denote χ_V to be the character $\otimes_v \chi_{V,v} : k^\times \backslash \mathbb{A}_k^\times \to \{\pm 1\}$.

Fix an arbitrary k-basis Λ of V. For each place v of k, let φ_v^0 be the characteristic function $\Lambda_v^n \subset V(k_v)^n$, where $\Lambda_v \subset V(k_v)$ is the O_v-lattice generated by the elements in Λ. Then for almost all places v of k, it is known that

$$\omega_v(\kappa_v, \kappa_v')\varphi_v^0 = \varphi_v^0 \ \forall (\kappa_v, \kappa_v') \in \mathrm{Sp}_n(O_v) \times \mathrm{O}(V)(O_v).$$

Let $V(\mathbb{A}_k) := V \otimes_k \mathbb{A}_k$ and let $S(V(\mathbb{A}_k))$ be the space of Schwartz functions on $V(\mathbb{A}_k)$. Viewing $S(V(\mathbb{A}_k)^n)$ as the restricted tensor product $\otimes_v' S(V(k_v)^n)$ with respect to $\{\varphi_v^0\}_v$, we have the (global) Weil representation $\omega = \otimes_v \omega_v$ of $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$ on the space $S(V(\mathbb{A}_k)^n)$: for every $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}_k)^n)$ and $(g,h) = (g_v, h_v)_v$ in $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$,

$$\omega(g,h)\varphi := \otimes_v \omega_v(g_v, h_v)\varphi_v.$$

2. Theta series. Take a Schwartz function $\varphi \in S(V(\mathbb{A}_k)^n)$. For $(g,h) \in \mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$, the theta series

$$\theta(g,h,\varphi) := \sum_{x \in V(k)^n} (\omega(g)\varphi)(h^{-1}x),$$

as a function on $\mathrm{Sp}_n(\mathbb{A}_k) \times \mathrm{O}(V)(\mathbb{A}_k)$, is left $\mathrm{Sp}_n(k) \times \mathrm{O}(V)(k)$-invariant. We define

$$I^n(g,\varphi) := \int_{\mathrm{O}(V)(k)\backslash\,\mathrm{O}(V)(\mathbb{A}_k)} \theta(g,h,\varphi)dh.$$

This integral is absolutely convergent, as $\mathrm{O}(V)(k)\backslash\,\mathrm{O}(V)(\mathbb{A}_k)$ is compact. The measure dh is induced from the Haar measure on $\mathrm{O}(V)(\mathbb{A}_k)$ which is normalized so that the volume of $\mathrm{O}(V)(k)\backslash\,\mathrm{O}(V)(\mathbb{A}_k)$ is 1.

Let P be a standard parabolic subgroup of Sp_n. Write $P = M \cdot N$, where N is the unipotent radical of P and M is its standard Levi subgroup. Define

$$I^n_P(g,\varphi) := \int_{N(k)\backslash N(\mathbb{A}_k)} I^n(ng,\varphi)dn, \quad \forall g \in \mathrm{Sp}_n(\mathbb{A}_k),$$

where the measure dn on $N(k)\backslash N(\mathbb{A}_k)$ is chosen so that the total mass is 1. It is clear that

Lemma 2.1. $I^n_{P_n}(g,\varphi) = (\omega(g)\varphi)(0)$.

Let Z_M denotes the center of M, which is contained in T_n.

Lemma 2.2. *For every standard parabolic subgroup P of Sp_n, there exists a character ν_P on $Z_M(k)\backslash Z_M(\mathbb{A}_k)$ such that for every $z \in Z_M(\mathbb{A}_k)$, $g \in \mathrm{Sp}_n(\mathbb{A}_k)$, $\varphi \in S(V(\mathbb{A}_k)^n)$,*

$$I^n_P(zg,\varphi) = \nu_P(z) I^n_P(g,\varphi).$$

Proof. It is clear when $P = \mathrm{Sp}_n$, as the center of Sp_n is $\{\pm 1\}$. Next, we consider the case when $P = P_r$, $0 < r \leq n$. Then $M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{n-r}$, and

$$I^n_{P_r}(g,\varphi) = \int_{\mathrm{O}(V)(k)\backslash\,\mathrm{O}(V)(\mathbb{A}_k)} \sum_{x\in V(k)^{n-r}} \omega(g)\varphi(0,h^{-1}x)dh.$$

Therefore for $m = (a,g') \in \mathrm{GL}_r(\mathbb{A}_k) \times \mathrm{Sp}_{n-r}(\mathbb{A}_k) \cong M_r(\mathbb{A}_k)$, we get

$$I^n_P(mg,\varphi) = \chi_V(\det a) |\det a|_{\mathbb{A}_k}^{\frac{\dim V}{2}} I^{n-r}(g',\widetilde{\varphi}),$$

where $\widetilde{\varphi} \in S(V(\mathbb{A}_k)^{n-r})$ is defined by

$$\widetilde{\varphi}(x) := \omega(g)\varphi(0,x).$$

This assures the result for $P = P_r$.

In general, we can assume that P is contained in P_r, $0 < r \leq n$, and the Levi subgroup M of P is isomorphic to

$$\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_\ell} \times \mathrm{Sp}_{n-r},$$

where $r_1 + \cdots + r_\ell = r$. Note that $N = U \cdot N_r$ where U is a unipotent subgroup of M_r. Therefore

$$I^n_P(g,\varphi) = I^n_{P_r}(g,\varphi),$$

which completes the proof immediately. □

The next proposition shows that $I^n(g,\varphi)$ is concentrated on the Borel subgroup B_n:

PROPOSITION 2.3. *Take a standard parabolic subgroup* $P = M\cdot N$ *of* Sp_n *which is not equal to* B_n. *Let* ν_P *be the character of* $Z_M(k)\backslash Z_M(\mathbb{A}_k)$ *in* Lemma 2.2. *Then for every cusp form* f *on* $M(\mathbb{A}_k)$ *with central character* ν_P^{-1}, $\varphi\in S(V(\mathbb{A}_k)^n)$, *and* $g\in\mathrm{Sp}_n(\mathbb{A}_k)$, *we have*

$$\int_{Z_M(\mathbb{A}_k)M(k)\backslash M(\mathbb{A}_k)} I_P^n(mg,\varphi)f(m)dm = 0.$$

Proof. Suppose P is contained in P_r, $0<r\le n$, and $M\subset M_r\cong \mathrm{GL}_r\times\mathrm{Sp}_{n-r}$ is isomorphic to

$$\mathrm{GL}_{r_1}\times\cdots\times\mathrm{GL}_{r_\ell}\times\mathrm{Sp}_{n-r},$$

where $r_1+\cdots+r_\ell = r$. From the proof of Lemma 2.2, we have known that for every element $m=(a_1,...,a_\ell,g')\in M(\mathbb{A}_k)$, $g\in\mathrm{Sp}_n(\mathbb{A}_k)$, and $\varphi\in S(V(\mathbb{A}_k)^n)$,

$$\begin{aligned} I_P^n(mg,\varphi) &= I_{P_r}^n(mg,\varphi)\\ &= \left(\prod_{i=1}^{\ell}\chi_V(\det a_i)|\det a_i|_{\mathbb{A}_k}^{\frac{\dim V}{2}}\right) I^{n-r}(g',\widetilde{\varphi}). \end{aligned} \tag{2.1}$$

Suppose $r_i>1$ for some i. Then I_P^n is left invariant under $U_i(\mathbb{A}_k)$, where U_i is the unipotent radical of a proper parabolic subgroup P_M of M. Note that for every cusp form f on $M(\mathbb{A}_k)$,

$$\int_{U_i(k)\backslash U_i(\mathbb{A}_k)} f(ug)du = 0.$$

Therefore $(m\mapsto I_P^n(mg,\varphi))$ is orthogonal to all cusp forms on $M(\mathbb{A}_k)$.

In the case when $r_i=1$ for all $1\le i\le\ell$, we must have $r<n$ (as $P\neq B_n$). From Equation (2.1) we reduce to the case when $P=\mathrm{Sp}_n$. By Theorem A.5 in the Appendix A, we have that for every cusp form f on $\mathrm{Sp}_n(\mathbb{A}_k)$,

$$\int_{\mathrm{Sp}_n(k)\backslash\mathrm{Sp}_n(\mathbb{A}_k)} I^n(g,\varphi)\cdot f(g)\,dg = 0.$$

This completes the proof. □

3. Siegel-Eisenstein series. Let $I_{\mathbb{A}_k}(s)$ be the space of smooth functions Φ on $\mathrm{Sp}_n(\mathbb{A}_k)$ satisfying that for elements $g\in\mathrm{Sp}_n(\mathbb{A}_k)$ and $\begin{pmatrix} a & * \\ 0 & {}^t a^{-1}\end{pmatrix}\in P_n(\mathbb{A}_k)$,

$$\Phi\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1}\end{pmatrix} g\right) = \chi_V(\det a)|\det a|_{\mathbb{A}_k}^{s+\frac{n+1}{2}}\cdot\Phi(g).$$

For $g\in\mathrm{Sp}_n(\mathbb{A}_k)$ and $\Phi\in I_{\mathbb{A}_k}(s)$, it is known that the Eisenstein series

$$E(g,s,\Phi) := \sum_{\gamma\in P_n(k)\backslash\mathrm{Sp}_n(k)}\Phi(\gamma g)$$

converges absolutely for $\mathrm{Re}(s) > (n+1)/2$. From the Iwasawa decomposition

$$\mathrm{Sp}_n(\mathbb{A}_k) = P_n(\mathbb{A}_k)\cdot \mathrm{Sp}_n(O_{\mathbb{A}_k}),$$

we can extend Φ to a standard section (which is still denoted by Φ), i.e. for all $s' \in \mathbb{C}$, $a \in \mathrm{GL}_n(\mathbb{A}_k)$, $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$,

$$\Phi\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1}\end{pmatrix}\kappa, s'\right) = \chi_V(\det a)|\det a|_{\mathbb{A}_k}^{s'+\frac{n+1}{2}}\cdot \Phi(\kappa).$$

It is known that for $g \in \mathrm{Sp}_n(\mathbb{A}_k)$, $E(g,s,\Phi)$ can be extended to a meromorphic function of $s \in \mathbb{C}$ (in fact, a rational function in q^{-s}).

In this section, our aim is to show that $E(g,s,\Phi)$ is also concentrated on the Borel subgroup B_n for every section $\Phi \in I_{\mathbb{A}_k}(s)$ (in Proposition 3.2).

3.1. Constant terms of Eisenstein series. Let P be a standard parabolic subgroup of Sp_n, i.e. P is a parabolic subgroup of Sp_n containing B_n. Then $P = M\cdot N$, where M is the Levi subgroup of P and N is the unipotent radical of P. For $g \in \mathrm{Sp}_n(\mathbb{A}_k)$ and $\Phi \in I_{\mathbb{A}_k}(s)$, the *constant term of E along P* is defined by

$$E_P(g,s,\Phi) := \int_{N(k)\backslash N(\mathbb{A}_k)} E_P(ng,s,\Phi)dn.$$

Here the measure dn is induced from the Haar measure of $N(\mathbb{A}_k)$ normalized so that the volume of $N(k)\backslash N(\mathbb{A}_k)$ is 1. Let W_M be the Weyl group of M with respect to T_n, and denote by Δ_M^+ the set of positive roots with respect to the Borel subgroup $B_n \cap M$ of M. Then it is known that

$$W_{M_n,M} := \{w \in W_{\mathrm{Sp}_n} | w^{-1}\alpha \in \Delta_n^+ \text{ for all } \alpha \in \Delta_{M_n}^+, \text{ and } w\alpha \in \Delta_n^+ \text{ for all } \alpha \in \Delta_M^+\}$$

forms a set of double coset representatives of $W_{M_n}\backslash W_{\mathrm{Sp}_n}/W_M$.

LEMMA 3.1. *For each $w \in W_{M_n,M}$, let $M''_w := w^{-1}P_n w \cap M$ and $N''_w := w^{-1}P_n w \cap N$. Then*

$$E_P(g,s,\Phi) = \sum_{w\in W_{M_n,M}}\left(\sum_{\gamma'\in M''_w(k)\backslash M(k)} \Phi_w(\gamma' g)\right),$$

where

$$\Phi_w(g) := \int_{N''_w(\mathbb{A}_k)\backslash N(\mathbb{A}_k)} \Phi(wng)dn.$$

Proof. Note that $W_{M_n,M}$ also forms a set of double coset representatives of $P_n(k)\backslash \mathrm{Sp}_n(k)/P(k)$. Moreover, for each $w \in W_{M_n,M}$ we have the following bijection

$$(N''_w(k)\backslash N(k)) \times (M''_w(k)\backslash M(k)) \cong w^{-1}P(k)w \cap P(k)\backslash P(k).$$

Therefore

$$\begin{aligned} E_P(g,s,\Phi) &= \int_{N(k)\backslash N(\mathbb{A}_k)} \sum_{w\in P_n(k)\backslash \mathrm{Sp}_n(k)/P(k)} \sum_{\gamma_w \in w^{-1}P_n(k)w\cap P(k)\backslash P(k)} \Phi(w\gamma_w ng)dn \\ &= \sum_{w\in W_{M_n,M}}\left(\sum_{\gamma'\in M''_w(k)\backslash M(k)} \int_{N(k)\backslash N(\mathbb{A}_k)} \sum_{n'\in N''_w(k)\backslash N(k)} \Phi(wn'\gamma' ng)dn\right). \end{aligned}$$

It is observed that

$$\begin{aligned}
&\int_{N(k)\backslash N(\mathbb{A}_k)} \sum_{n'\in N''_w(k)\backslash N(k)} \Phi(wn'\gamma' ng)dn \\
&= \int_{N(k)\backslash N(\mathbb{A}_k)} \sum_{n'\in N''_w(k)\backslash N(k)} \Phi(w(n'n)\gamma' g)dn \\
&= \int_{N''_w(k)\backslash N(\mathbb{A}_k)} \Phi(wn\gamma' g)dn \\
&= \Phi_w(\gamma' g).
\end{aligned}$$

The last equality is from

$$\Phi(wn''g) = \Phi(wg), \quad \forall n'' \in N''_w(\mathbb{A}_k).$$

□

For $w \in W_{M_n,M}$, there exists a character μ_w on $M''_w(\mathbb{A}_k)$ trivial on $M''_w(k)$ such that for $m \in M''_w(\mathbb{A}_k)$

$$\Phi_w(mg) = \mu_w(m)\Phi_w(g).$$

Note that M''_w is a standard parabolic subgroup of M (with respect to $B_n \cap M$) for any $w \in W_{M_n,M}$. Let ν_w be the restriction of μ_w on the center $Z_M(\mathbb{A}_k)$ of $M(\mathbb{A}_k)$. Then we can write

$$E_P(g,s,\Phi) = \sum_{\text{character } \nu \text{ on } Z_M(\mathbb{A}_k)} E_{P,\nu}(g,s,\Phi),$$

where

$$E_{P,\nu}(g,s,\Phi) := \sum_{w\in W_{M_n,M}, \nu_w=\nu} \left(\sum_{\gamma\in M''_w(k)\backslash M(k)} \Phi_w(\gamma g) \right).$$

Since μ_w is trivial on $U_{M''_w}(\mathbb{A}_k)$ where $U_{M''_w}$ is the unipotent radical of M''_w when $M''_w \neq M$ and we take $U_{M''_w} = B_n \cap M$ if $M''_w = M$, we have the following result.

Proposition 3.2. *Suppose a function $\Phi \in I_{\mathbb{A}_k}(s)$ and $g \in \mathrm{Sp}_n(\mathbb{A}_k)$ are given. Let $P \supsetneq B_n$ be a standard parabolic subgroup of Sp_n. Then for any cusp form f on $M(\mathbb{A}_k)$ (where M is the Levi subgroup of P) with central character ν^{-1}, we have*

$$\int_{Z_M(\mathbb{A}_k)M(k)\backslash M(\mathbb{A}_k)} E_{P,\nu}(mg,s,\Phi)f(m)dm = 0.$$

Corollary 3.3. *For every section Φ and every standard parabolic subgroup P of Sp_n, $E(g,s,\Phi)$ and $E_P(g,s,\Phi)$ have the same set of poles. More precisely, for $s_0 \in \mathbb{C}$, let*

$$\mathrm{ord}_{s=s_0} E(\cdot,s,\Phi) = \min_{g\in \mathrm{Sp}_n(\mathbb{A}_k)} \{\mathrm{ord}_{s=s_0} E(g,s,\Phi)\}$$

and

$$\mathrm{ord}_{s=s_0} E_P(\cdot,s,\Phi) = \min_{g\in \mathrm{Sp}_n(\mathbb{A}_k)} \{\mathrm{ord}_{s=s_0} E_P(g,s,\Phi)\}.$$

Then

$$\operatorname{ord}_{s=s_0} E(\cdot, s, \Phi) = \operatorname{ord}_{s=s_0} E_P(\cdot, s, \Phi), \quad \forall s_0 \in \mathbb{C}.$$

Proof. From the definition of $E_P(g, s, \Phi)$, it is clear that

$$\operatorname{ord}_{s=s_0} E_P(\cdot, s, \Phi) \geq \operatorname{ord}_{s=s_0} E(\cdot, s, \Phi).$$

Write $P = M \cdot N$, where N is the unipotent radical of P and M is the Levi subgroup. We also have

$$\operatorname{ord}_{s=s_0} E_{B_n}(\cdot, s, \Phi) \geq \operatorname{ord}_{s=s_0} E_P(\cdot, s, \Phi),$$

as for every $g \in \operatorname{Sp}_n(\mathbb{A}_k)$

$$E_{B_n}(g, s, \Phi) = \int_{B_n(k)N(\mathbb{A}_k)\backslash B_n(\mathbb{A}_k)} E_P(ng, s, \Phi) dn.$$

It suffices to show that

$$\operatorname{ord}_{s=s_0} E(\cdot, s, \Phi) \geq \operatorname{ord}_{s=s_0} E_{B_n}(\cdot, s, \Phi).$$

Let $\ell = \operatorname{ord}_{s=s_0} E(\cdot, s, \Phi)$ and $\ell' = \operatorname{ord}_{s=s_0} E_{B_n}(\cdot, s, \Phi)$. If $\ell < \ell'$, then the function $f(g) := \lim_{s \to s_0} (s - s_0)^{-\ell} E(g, s, \Phi)$ would have

$$f_{B_n}(g) = \int_{U_n(k)\backslash U_n(\mathbb{A}_k)} f(ug) du = 0.$$

By Proposition 3.2, f is also concentrated on B_n. Therefore $f \equiv 0$ (as f is a cusp form on $\operatorname{Sp}_n(\mathbb{A}_k)$ which is also orthogonal to all cusp forms). Therefore the proof is complete. □

Take a Schwartz function $\varphi \in S(V(\mathbb{A}_k)^n)$. For any $g = \begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \cdot \kappa \in \operatorname{Sp}_n(\mathbb{A}_k)$ where $a \in \operatorname{GL}_n(\mathbb{A}_k)$ and $\kappa \in \operatorname{Sp}_n(O_{\mathbb{A}_k})$, set $s_{(n)} := \dim(V)/2 - (n+1)/2$ and

$$\Phi_\varphi(g, s) := |\det a|_{\mathbb{A}_k}^{s - s_{(n)}} \cdot (\omega(g)\varphi)(0).$$

Then Φ_φ is in $I_{\mathbb{A}_k}(s)$. We call Φ_φ the *Siegel section associated to* φ and $E(g, s, \Phi_\varphi)$ the *Siegel-Eisenstein series associated to* φ. To show that $E(g, s, \Phi_\varphi)$ is holomorphic at $s = s_{(n)}$ for every Siegel section Φ_φ, by Corollary 3.3 it suffices to get

$$\operatorname{ord}_{s=s_{(n)}} E_{P_n}(\cdot, s, \Phi_\varphi) \geq 0.$$

In the next two sections, we study the analytic behavior of $E_{P_n}(g, s, \Phi_\varphi)$ at $s = s_{(n)}$, and prove its holomorphic property.

4. The analytic behavior of $E_{P_n}(g, s, \Phi_\varphi)$ I: Intertwining operators. For $0 \leq r \leq n$, let

$$w_r := \begin{pmatrix} I_{n-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r \\ 0 & 0 & I_{n-r} & 0 \\ 0 & -I_r & 0 & 0 \end{pmatrix} \in \operatorname{Sp}_n.$$

Then $\{w_0, ..., w_n\}$ is a set of double coset representatives of $W_{M_n} \backslash W_{\mathrm{Sp}_n} / W_{M_n}$, and we have $\mathrm{Sp}_n = \coprod_{r=0}^n P_n w_r P_n$. Moreover, $M''_{w_r} = w_r^{-1} P_n w_r \cap M_n = \{\mathfrak{m}(a) : a \in Q_r\}$ where

$$Q_r := \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in \mathrm{GL}_n \;\middle|\; a \in \mathrm{GL}_{n-r},\ d \in \mathrm{GL}_r \right\},$$

and

$$N''_{w_r} = w_r^{-1} P_n w_r \cap N_n = \left\{ \mathfrak{n} \begin{pmatrix} y & z \\ {}^t z & 0 \end{pmatrix} \;\middle|\; y \in \mathrm{Sym}_{n-r},\ z \in \mathrm{Mat}_{(n-r)\times r} \right\},$$

$$N''_{w_r} \backslash N_n \cong N'_r := \left\{ \mathfrak{n} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \;\middle|\; b \in \mathrm{Sym}_r \right\}.$$

Here $\mathfrak{m}$ and $\mathfrak{n}$ are the isomorphisms introduced in Section 1.2.

Take a section $\Phi \in I_{\mathbb{A}_k}(s)$. By Lemma 3.1, we can write $E_{P_n}(g, s, \Phi)$ as

$$\sum_{r=0}^n E_{P_n}^{(r)}(g, s, \Phi),$$

where

$$E_{P_n}^{(r)}(g, s, \Phi) := \sum_{\gamma \in Q_r(k) \backslash \mathrm{GL}_n(k)} \Phi^{(r)}\big(\mathfrak{m}(\gamma) g, s\big)$$

with

$$\Phi^{(r)}(g, s) := \int_{N'_r(\mathbb{A}_k)} \Phi(w_r n g, s) dn.$$

Notice that

$$E_{P_n}^{(0)}(g, s, \Phi) = \Phi(g, s), \quad E_{P_n}^{(n)}(g, s, \Phi) = \big(M(s)\Phi\big)(g, s)$$

where $M(s) : I_{\mathbb{A}_k}(s) \to I_{\mathbb{A}_k}(-s)$ is the intertwining operator defined by

$$\big(M(s)\Phi\big)(g, s) := \int_{N_n(\mathbb{A}_k)} \Phi(w_n n g, s) dn,$$

and

$$a \mapsto E_{P_n}^{(r)}(\mathfrak{m}(a) g, s, \Phi), \quad \forall a \in \mathrm{GL}_n(\mathbb{A}_k)$$

is an Eisenstein series on $\mathrm{GL}_n(\mathbb{A}_k)$ for $0 < r < n$.

In this section, we review the meromorphic continuation of the intertwining operator $M(s)$, and show that $E_{P_n}^{(n)}(g, s, \Phi)$ is holomorphic at $s = s_{(n)}$ when $\Phi = \Phi_\varphi$ is a Siegel section.

4.1. The intertwining operator $M(s)$**.** For each place v of k, let $I_v(s)$ be the space of smooth functions Φ_v on $\mathrm{Sp}_n(k_v)$ satisfying that for $g \in \mathrm{Sp}_n(k_v)$ and $\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \in P_n(k_v)$,

$$\Phi_v\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} g\right) = \chi_{V,v}(\det a)|\det a|_v^{s+\frac{n+1}{2}}\Phi_v(g).$$

Given $\Phi_v \in I_v(s)$, we can extend Φ_v to be a standard section, i.e. for $s' \in \mathbb{C}$,

$$\Phi_v\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} \kappa_v\right) := \chi_{V,v}(\det a)|\det a|_v^{s'+\frac{n+1}{2}}\Phi_v(\kappa_v), \quad \forall a \in \mathrm{GL}_n(k_v),\ \kappa_v \in \mathrm{Sp}_n(O_v).$$

Define the intertwining operator $M_v(s) : I_v(s) \to I_v(-s)$ by

$$M_v(s)\Phi_v(g) := \int_{N_n(k_v)} \Phi_v(w_n n_v g) dn_v,$$

which converges when $s > (n-1)/2$. We state the known facts we need in the following:

Lemma 4.1. (cf. [5]) *Let*

$$a_{n,v}(s) := L_v(s + \frac{n+1}{2} - n, \chi_{V,v}) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_{k,v}(2s - n + 2i)$$

and

$$b_{n,v}(s) = L_v(s + \frac{n+1}{2}, \chi_{V,v}) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_{k,v}(2s + n - 2i + 1),$$

where

$$\zeta_{k,v}(s) := (1 - q_v^{-s})^{-1} \quad \text{and} \quad L_v(s, \chi_{V,v}) := (1 - \alpha_v(\chi_{V,v})q_v^{-s})^{-1}$$

with

$$\alpha_v(\chi_{V,v}) := \begin{cases} \chi_{V,v}(\pi_v), & \text{if } \chi_{V,v} \text{ is unramified,} \\ 0, & \text{otherwise.} \end{cases}$$

(i) For any standard section $\Phi_v \in I_v(s)$ *and* $g \in \mathrm{Sp}_n(k_v)$,

$$\frac{1}{a_{n,v}(s)} M_v(s)\Phi_v(g,s)$$

can be extended to an entire function of s.

(ii) Suppose $\chi_{V,v}$ *is unramified. Let* $\Phi_v^0(g,s) \in I_v(s)$ *be the standard section such that*

$$\Phi_v^0(\kappa_v, s) = 1, \quad \forall \kappa_v \in \mathrm{Sp}_n(O_v).$$

Then we have

$$M_v(s)\Phi_v^0(g,s) = \mathrm{vol}(N_n(O_v), dn_v) \cdot \frac{a_{n,v}(s)}{b_{n,v}(s)} \cdot \Phi_v^0(g,-s), \quad \forall g \in \mathrm{Sp}_n(k_v).$$

Recall that $I_{\mathbb{A}_k}(s)$ is the restricted tensor product of $I_v(s)$ with respect to $\{\Phi_v^0\}$. We normalize the Haar measure dn_v on $N_n(k_v)$ such that $\mathrm{vol}(N_n(O_v)) = 1$ for almosrt all places v and the Haar measure $dn = \prod_v dn_v$ on $N_n(\mathbb{A}_k)$ satisfies $\mathrm{vol}(N_n(k)\backslash N_n(\mathbb{A}_k), dn) = 1$. Then $M(s) = \otimes M_v(s)$, which converges absolutely on $\mathrm{Re}(s) > (n+1)/2$. In particular, for a factorizable section $\Phi = \otimes_v \Phi_v \in I_{\mathbb{A}_k}(s)$, let $\Sigma(\Phi, dn)$ be the finite set of places v of k such that $\Phi_v \neq \Phi_v^0$ or $\mathrm{vol}(N_n(O_v), dn_v) \neq 1$. Then by Lemma 4.1 (ii), $M(s)\Phi(g,s)$ can be expressed by

$$\frac{a_n(s)}{b_n(s)} \cdot \left[\left(\otimes_{v \notin \Sigma(\Phi,dn)} \Phi_v^0(g_v, -s) \right) \otimes \left(\otimes_{v \in \Sigma(\Phi,dn)} \frac{b_{n,v}(s)}{a_{n,v}(s)} M_v(s) \Phi_v(g_v, s) \right) \right],$$

where

$$a_n(s) := \prod_v a_{n,v}(s) \quad \text{and} \quad b_n(s) := \prod_v b_{n,v}(s).$$

By Lemma 4.1 (i), we have the meromorphic continuation of $M(s)\Phi(g,s)$ for each element $g \in \mathrm{Sp}_n(\mathbb{A}_k)$.

Lemma 4.2.

(i) Suppose χ_V *is non-trivial. Then when* $\dim(V) = 2$ *we have*

$$\mathrm{ord}_{s=s_{(n)}} \frac{a_n(s)}{b_n(s)} = \begin{cases} 0, & \text{if } n = 1, \\ +1, & \text{if } n > 1; \end{cases}$$

when $\dim(V) = 4$,

$$\mathrm{ord}_{s=s_{(n)}} \frac{a_n(s)}{b_n(s)} = \begin{cases} 0, & \text{if } n = 1 \text{ or } 3, \\ -1, & \text{if } n = 2, \\ 1, & \text{if } n > 3. \end{cases}$$

(ii) When χ_V *is trivial, we have* $\dim(V) = 4$ *and*

$$\mathrm{ord}_{s=s_{(n)}} \frac{a_n(s)}{b_n(s)} = \begin{cases} -1, & \text{if } n = 1, \\ -2, & \text{if } n = 2, \\ 0, & \text{if } n = 3, \\ +1, & \text{if } n > 3. \end{cases}$$

Recall that $S(V(\mathbb{A}_k)^n)$ can be viewed as the restricted tensor product $\otimes_v' S(V(k_v)^n)$ with respect to $\{\varphi_v^0\}_v$, where the functions φ_v^0 are chosen in Section 1.3. Suppose a factorizable Schwartz function $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}_k)^n)$ is given. Then the associated section $\Phi_\varphi = \otimes_v \Phi_{\varphi_v}$ is also factorizable.

Proposition 4.3. *For each place* v *of* k,

$$\frac{b_{n,v}(s)}{a_{n,v}(s)} M_v(s) \Phi_{\varphi_v}(g_v, s)$$

is holomorphic at $s = s_{(n)}$ *for all* $g \in \mathrm{Sp}_n(k_v)$ *and* $\varphi_v \in S(V(k_v)^n)$.

Remark. The above proposition does not hold for all sections of $I_v(s)$ in general, as

$$\mathrm{ord}_{s=s_{(n)}} b_{n,v}(s) = \begin{cases} -1, & \text{if } \dim(V) < n+1, \\ 0, & \text{if } \dim(V) \geq n+1. \end{cases}$$

Proof. We define an intertwining operator T_v from $S(V(k_v)^n)$ to $I_v(-s_{(n)})$ (as representations of $\mathrm{Sp}_n(k_v)$) by

$$\varphi_v \longmapsto \frac{1}{a_{n,v}(s_{(n)})} M_v(s_{(n)})\Phi_{\varphi_v}.$$

When $\dim(V) < n+1$, this intertwining operator T_v must be zero by the following lemma (Lemma 4.4), which tells us that $a_{n,v}(s)^{-1} M_{n,n,v}(s)\Phi_{\varphi_v}(g,s)$ has a zero at $s = s_{(n)}$ for all $\varphi \in S(V(k_v)^n)$ and $g \in \mathrm{Sp}_n(k_v)$. Therefore the result holds. □

LEMMA 4.4. *Let*

$$\ell(n,v) := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}_n(k_v)\times \mathrm{O}(V)(k_v)} \Big(S(V(k_v)^n), I_v(-s_{(n)}) \otimes \mathbf{1} \Big),$$

where $\mathbf{1}$ *is the trivial representation of* $\mathrm{O}(V)(k_v)$. *Then*

(i) *When* $\dim(V) < n+1$, *we have* $\ell(n,v) = 0$.

(ii) *When* $\dim(V) = 2$ *and* $n = 1$, $\ell(n,v) \leq 1$.

(iii) *When* $\dim(V) = 4$ *and* $n = 1$ *or* 2,

$$\ell(n,v) \leq \begin{cases} n, & \text{if } V(k_v) \text{ is isotropic,} \\ 0, & \text{if } V(k_v) \text{ is anisotropic.} \end{cases}$$

(iv) *When* $\dim(V) = 4$ *and* $n = 3$, *we have* $\ell(n,v) \leq 1$.

Proof. This is a consequence of Proposition B.1. The proof is given in Appendix B (cf. Corollary B.3 and Remark B.4). □

The above lemma implies immediately that:

COROLLARY 4.5. *If* $\dim(V) \geq n+1$ *and* $\ell(n,v) = 0$, *then the meromorphic function*

$$\frac{b_{n,v}(s)}{a_{n,v}(s)} M_{n,n,v}(s)\Phi_{\varphi}(g,s)$$

has a zero at $s = s_{(n)}$ *for all* $g \in \mathrm{Sp}_n(k_v)$ *and* $\varphi \in S(V(k_v)^n)$.

Recall that given $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}_k)^n)$, $M(s)\Phi_\varphi(g,s)$ can be expressed by

$$\frac{a_n(s)}{b_n(s)} \cdot \left[\left(\otimes_{v \notin \Sigma(\Phi_\varphi, dn)} \Phi^0_{\varphi_v}(g_v, -s)\right) \otimes \left(\otimes_{v \in \Sigma(\Phi_{\varphi_v}, dn)} \frac{b_{n,v}(s)}{a_{n,v}(s)} M_v(s)\Phi_{\varphi_v}(g_v, s) \right) \right].$$

Note that when $\dim(V) = 4$ and χ_V is non-trivial (resp. other cases), there exists at least one (resp. two) places of k such that (V, Q_V) is anisotropic over k_v. Since any Schwartz function $\varphi \in S(V(\mathbb{A}_k)^n)$ is a linear combination of factorizable functions, we finally arrive at:

PROPOSITION 4.6. *For each Siegel section* $\Phi_\varphi \in I_{\mathbb{A}_k}(s)$, $M(s)\Phi_\varphi(g,s)$ *is holomorphic at* $s = s_{(n)}$ *for all* $g \in \mathrm{Sp}_n(\mathbb{A}_k)$. *Moreover,*

$$M(s_{(n)})\Phi_\varphi(g, s_{(n)}) = 0 \quad \text{for all } g \in \mathrm{Sp}_n(\mathbb{A}_k)$$

except for the following cases:

(i) *when* $\dim(V) = 2$ *and* $n = 1$;

(ii) *when* $\dim(V) = 4$ *and* $n = 2$ *or* 3.

4.2. The intertwining operator $M_{n,r}(s)$. Fix an integer r with $0 < r < n$. We consider the standard parabolic subgroup $P_n^{(r)} := M_n^{(r)} N_n^{(r)}$ contained in P_n, where the Levi subgroup $M_n^{(r)} \subset M_n$ is equal to

$$\left\{ \mathfrak{m}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \;\middle|\; a_1 \in \mathrm{GL}_{n-r},\ a_2 \in \mathrm{GL}_r \right\}.$$

For each pair of Hecke characters μ_1 and μ_2 on $k^\times \backslash \mathbb{A}_k^\times$, Let $I_{n,r}(\mu_1, \mu_2)$ be the space of smooth functions f on $\mathrm{Sp}_n(\mathbb{A}_k)$ satisfying that for $(a_1, a_2) \in \mathrm{GL}_{n-r} \times \mathrm{GL}_r$ and $n' \in N_n^{(r)}$,

$$f\left(\mathfrak{m}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} n'g\right) = \mu_1(a_1)\mu_2(a_2) \frac{|\det a_1|_{\mathbb{A}_k}^{\frac{n+r+1}{2}}}{|\det a_2|_{\mathbb{A}_k}^{\frac{r+1}{2}}} f(g).$$

For a section $\Phi \in I_{\mathbb{A}_k}(s)$, define

$$M_{n,r}(s)\Phi(g,s) := \int_{N_r'(\mathbb{A}_k)} \Phi(w_r n g, s) dn,$$

where the Haar measure dn is normalized so that the volume of $N_r'(k)\backslash N_r'(\mathbb{A}_k)$ is 1. When $\mathrm{Re}(s) > (n+1)/2$, it is clear that $M_{n,r}(s)\Phi(\cdot, s)$ is a section in $I_{n,r}(s)$, where

$$I_{n,r}(s) := I_{n,r}(|\cdot|_{\mathbb{A}_k}^{s-\frac{r}{2}}, |\cdot|_{\mathbb{A}_k}^{-s+\frac{n-r}{2}}).$$

We also set $I_{n,0}(s) := I_{\mathbb{A}_k}(s)$ and $I_{n,n}(s) := I_{\mathbb{A}_k}(-s)$, and $M_{n,n}(s) := M(s)$ introduced in Section 4.1.

For any $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$, let $\rho(\kappa)$ denote the left action of κ by right multiplication on $I_{\mathbb{A}_k}(s)$ and $I_{n,r}(s)$. This action is independent of s. Let $\Phi \in I_{\mathbb{A}_k}(s)$ be a standard section. Given $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$, let $\Phi' := \rho(\kappa)\Phi$, which is also a standard section. Define the inclusion $i : \mathrm{Sp}_r \hookrightarrow \mathrm{Sp}_n$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} I_{n-r} & & & \\ & a & & b \\ & & I_{n-r} & \\ & c & & d \end{pmatrix}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} I_{n,0}(s) & \xrightarrow{M_{n,r}(s)} & I_{n,r}(s) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ I_{r,0}(s') & \xrightarrow[M_{r,r}(s')]{} & I_{r,r}(s') \end{array}$$

where $s' = s + \frac{n-r}{2}$. For each $g \in \mathrm{Sp}_n(\mathbb{A}_k)$, write $g = n(b)m(a)\kappa$ where $n(b) \in N_n(\mathbb{A}_k)$, $a = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in Q_r(\mathbb{A}_k)$, and $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$. Then

$$\begin{aligned} M_{n,r}(s)\Phi(g,s) &= |\det a_1|_{\mathbb{A}_k}^{s+\frac{n+1}{2}} \cdot |\det a_2|_{\mathbb{A}_k}^{-s+\frac{n+1}{2}} M_{n,r}(s)(\rho(\kappa)\Phi)(1,s) \\ &= |\det a_1|_{\mathbb{A}_k}^{s+\frac{n+1}{2}} \cdot |\det a_2|_{\mathbb{A}_k}^{-s+\frac{n+1}{2}} M_{r,r}(s')\big(i^*(\rho(\kappa)\Phi)\big)(1,s'). \end{aligned}$$

This gives the meromorphic continuation of $M_{n,r}(s)\Phi$ for each standard section Φ, and

$$\operatorname{ord}_{s=s_{(n)}} M_{n,r}(s)\Phi(g,s) = \operatorname{ord}_{s'=s_{(r)}} M_{r,r}(s')\big(i^*(\rho(\kappa)\Phi)\big)(1,s').$$

In particular, suppose Φ is factorizable. Write $\Phi = \otimes_v \Phi_v$. Recall that $\Sigma(\Phi, dn)$ is the finite set of places of k such that $\Phi_v = \Phi_v^0$ and $\operatorname{vol}(N_n(O_v)) = 1$ when $v \notin \Sigma(\Phi, dn)$. Then $M_{n,r}(s)\Phi$ is equal to

$$\frac{a_r(s')}{b_r(s')} \cdot \left[\Big(\otimes_{v\notin\Sigma(\Phi,dn)} \Phi_v^0\Big) \otimes \Big(\otimes_{v\in\Sigma(\Phi,dn)} \frac{b_{r,v}(s')}{a_{r,v}(s')} M_{n,r,v}(s)\Phi_v\Big)\right].$$

5. The analytic behavior of $E_{P_n}(g,s,\Phi_\varphi)$ II: Mass-Jacquet-Shalika Eisenstein series. Recall that when $\operatorname{Re}(s) > (n+1)/2$, for $g \in \operatorname{Sp}_n(\mathbb{A}_k)$ and a section $\Phi \in I_{\mathbb{A}_k}(s)$ we defined

$$E_{P_n}^{(r)}(g,s,\Phi) = \sum_{\gamma \in Q_r(k)\backslash \operatorname{GL}_n(k)} \Phi^{(r)}(\gamma g, s)$$

where $\Phi^{(r)}(g,s) = M_{n,r}(s)\Phi(g,s) \in I_{n,r}(s)$. The discussion in Section 4.2 gives us the meromorphic continuation of $\Phi^{(r)}$. Moreover, for each $g \in \operatorname{Sp}_n(\mathbb{A}_k)$, the function $(a \mapsto E_{P_n}^{(r)}(\mathfrak{m}(a)g,s,\Phi))$ can be viewed as a *Mass-Jacquet-Shalika Eisenstein series on* $\operatorname{GL}_n(\mathbb{A}_k)$. In this section, we recall the analytic behavior of this kind of Eisenstein series, and show that $E_{P_n}^{(r)}(\mathfrak{m}(a),s,\Phi)$ is holomorphic at $s = s_{(n)}$ for every $a \in \operatorname{GL}_n(\mathbb{A}_k)$ when $\Phi = \Phi_\varphi$ is a Siegel section.

Fix an integer r with $0 < r < n$. Set $X_r := \operatorname{Mat}_{r\times n}$ (as an affine space over k). Let μ_1, μ_2 be two Hecke characters on $k^\times\backslash\mathbb{A}_k^\times$. For any $g \in \operatorname{GL}_n(\mathbb{A}_k)$ and Schwartz function $f \in S(X_r(\mathbb{A}_k))$, define

$$F(g) = F(g,\mu_1,\mu_2,f) := \mu_1(\det g)|\det g|_{\mathbb{A}_k}^{r/2} \int_{\operatorname{GL}_r(\mathbb{A}_k)} f(h^{-1}(0,I_r)g)\mu^{-1}(\det h)d^\times h.$$

Here $\mu = \mu_1\mu_2^{-1}|\cdot|_{\mathbb{A}_k}^{n/2}$ and the Haar measure $d^\times h$ is normalized so that the volume of $\operatorname{GL}_r(O_{\mathbb{A}_k})$ is 1. This integral is absolutely convergent if $|\mu_1\mu_2^{-1}| = |\cdot|_{\mathbb{A}_k}^{\sigma}$ where $\sigma > r - n/2$.

Recall that

$$Q_r = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in \operatorname{GL}_n \,\middle|\, a_1 \in \operatorname{GL}_{n-r},\ a_2 \in \operatorname{GL}_r \right\}.$$

For $g \in \operatorname{GL}_n(\mathbb{A}_k)$ and $b = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in Q_r(\mathbb{A}_k)$, we have

$$F(bg) = \mu_1(\det a_1)\mu_2(\det a_2)\delta_{Q_r}(b)^{1/2}F(g)$$

where $\delta_{Q_r}(b) = |\det a_1|_{\mathbb{A}_k}^{r/2} \cdot |\det a_2|_{\mathbb{A}_k}^{(r-n)/2}$. The *Maass-Jacquet-Shalika Eisenstein series associated to* f, μ_1, μ_2 is defined by

$$E(g,\mu_1,\mu_2,f) = \sum_{\gamma \in Q_r(k)\backslash \operatorname{GL}_n(k)} F(\gamma g).$$

This series converges absolutely when $|\mu_1\mu_2^{-1}|_{\mathbb{A}_k} = |\cdot|_{\mathbb{A}_k}^\sigma$ with $\sigma > n/2$. The following theorem gives us the meromorphic continuation of such Eisenstein series (the proof is given in Theorem C.3):

THEOREM 5.1. *Suppose* $\mu_1 \cdot \mu_2^{-1} = |\cdot|_{\mathbb{A}_k}^\sigma$. *Then*

(1) (*Continuation*) $E(g, \mu_1, \mu_2, f)$ *can be extended to a meromorphic function in* σ (*in fact, a rational function in* $q^{-\sigma}$), *and every possible pole can only be a simple pole. Let* $P(\sigma) := P^+(\sigma) \cdot P^-(\sigma)$, *where*

$$P^{\pm}(\sigma) = \prod_{i=0}^{r-1}(1 - q^{-\sigma\pm(\frac{n}{2}-i)}).$$

Then $P(\sigma) \cdot E(g, \mu_1, \mu_2, f)$ *is entire.*

(2) (*Functional equation*) *For each* $f \in S(X_r(\mathbb{A}_k))$, *we have*

$$E(g, \mu_1, \mu_2, f) = E({}^tg^{-1}, \mu_1^{-1}, \mu_2^{-1}, \hat{f})$$

where $\hat{f}$ *is the Fourier transform of* f:

$$\hat{f}(x) := \int_{X_r(\mathbb{A}_k)} f(y)\psi(-\operatorname{Tr}(x^ty))dy.$$

The Haar measure dy *is chosen to be self-dual, i.e.* $\hat{\hat{f}}(x) = f(-x)$.

(3) Suppose that there exists a place v *of* k *such that the support of the restriction of* f *on* $X_r(k_v)$ *is contained in the set of elements with rank* r *in* $X_r(k_v)$. *Then*

$$P^+(\sigma) \cdot E(g, \mu_1, \mu_2, f) \text{ is entire.}$$

Now, we set

$$\mu_{1,s} := |\cdot|_{\mathbb{A}_k}^{s+\frac{n-r+1}{2}} \text{ and } \mu_{2,s} := |\cdot|_{\mathbb{A}_k}^{-s+\frac{r+1}{2}}.$$

Let $\widetilde{I}_{n,r}(s)$ be the space of smooth functions Ψ on $\mathrm{GL}_n(\mathbb{A}_k)$ such that for every element $p = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in Q_r(\mathbb{A}_k)$ and $g \in \mathrm{GL}_n(\mathbb{A}_k)$,

$$\Psi(pg) = \mu_{1,s}(\det a_1) \cdot \mu_{2,s}(\det a_2) \cdot \left(|\det a_1|_{\mathbb{A}_k}^{\frac{r}{2}} \cdot |\det a_2|_{\mathbb{A}_k}^{\frac{r-n}{2}}\right) \cdot \Psi(g).$$

It is clear that $\widetilde{I}_{n,r}(s) = \otimes_v' \widetilde{I}_{n,r,v}(s)$, and given $f = \otimes_v f_v \in S(X_r(\mathbb{A}_k)) = \otimes_v' S(X_r(k_v))$ we have,

$$F(g, \mu_{1,s}, \mu_{2,s}, f) = \otimes_v F_v(g_v, \mu_{1,s,v}, \mu_{2,s,v}, f_v)$$

for $g = (g_v)_v \in \mathrm{GL}_n(\mathbb{A}_k)$. Here

$$F_v(g_v, \mu_{1,s,v}, \mu_{2,s,v}, f_v) := \mu_{1,s,v}(\det g_v)|\det g_v|_v^{r/2} \int_{X_r(k_v)} f_v(h_v^{-1}(0, I_r)g_v)\mu_{s,v}^{-1}(h_v)d^\times h_v$$

and $\mu_{s,v} = \mu_{1,s,v} \cdot \mu_{2,s,v}^{-1} \cdot |\cdot|_v^{n/2}$. We have (cf. Lemma C.2)

LEMMA 5.2. (1) *For any standard section* $\Psi_v \in \widetilde{I}_{n,r,v}(s)$, *there exists a Schwartz function* f *on* $X_r(k_v)$ *supported on elements of rank* r *such that*

$$F_v(g_v, \mu_{1,v,s}, \mu_{2,v,s}, f_v) = \Psi_v(g_v, s).$$

(2) *Let f_v^0 be the charateristic function of $X_r(O_v)$. Then*

$$F_v^0(g_v) := F_v(g, \mu_{1,s,v}, \mu_{2,s,v}, f_v^0) = \prod_{i=0}^{r-1} \zeta_v(2s - 2n - r - i)\Psi_v^0$$

where $\Psi_v^0 \in \widetilde{I}_{n,r,v}(s)$ is the standard section such that

$$\Psi_v^0(\kappa_v) = 1, \ \forall \kappa_v \in \mathrm{GL}_n(O_v).$$

For any Φ (res. Φ_v) $\in I_{n,r}(s)$, we denote by $\widetilde{\Phi}$ (res. $\widetilde{\Phi}_v$) the restriction of Φ (res. Φ_v) on $M_n(\mathbb{A}_k)$ $(M_n(k_v))$. Then via the isomorphism $\mathfrak{m}$ between GL_n and M_n, $\widetilde{\Phi}$ (res. $\widetilde{\Phi}_v$) can be viewed as a function in $\widetilde{I}_{n,r}(s)$ (res. $\widetilde{I}_{n,r,v}(s)$). Let $c_r(s) := \prod_v c_{r,v}(s)$ where

$$c_{r,v}(s) := \prod_{i=0}^{r-1} \zeta_{k,v}(2s + n - r - i).$$

For each factorizable Schwartz function $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}_k)^n)$, from the discussion in Section 4.2 and Lemma 5.2 we get

$$\begin{aligned} &\widetilde{M_{n,r}(s)\Phi_\varphi} \\ &= \frac{a_r(s')}{b_r(s')} \cdot \left(\Big(\otimes_{v \notin \Sigma(\Phi_\varphi, dn)} \Psi_v^0 \Big) \otimes \Big(\otimes_{v \in \Sigma(\Phi_\varphi, dn)} \frac{b_{r,v}(s')}{a_{r,v}(s')} \widetilde{M_{n,r,v}(s)\Phi_{\varphi_v}} \Big) \right) \\ &= \frac{a_r(s')}{c_r(s) b_r(s')} \cdot \left(F^{0,\varphi} \otimes \Big(\otimes_{v \in \Sigma(\Phi_\varphi, dn)} \frac{c_{r,v}(s) b_{r,v}(s')}{a_{r,v}(s')} \widetilde{M_{n,r,v}(s)\Phi_{\varphi_v}} \Big) \right), \end{aligned}$$

where $s' = s + (r-n)/2$, $\Sigma(\Phi, dn)$ is the finite set of places v of k such that $\Phi_v \neq \Phi_v^0$ or $\mathrm{vol}(N_n(O_v)) \neq 1$, and $F^{0,\varphi} = \otimes_{v \notin \Sigma(\Phi_\varphi, dn)} F_v^0$.

Lemma 5.3. *For any $\varphi_v \in S(V(k_v)^n)$, there exists a finite collection of standard sections Φ_v^j in $I_v(s)$ such that*

$$\frac{b_{r,v}(s')}{a_{r,v}(s')} \widetilde{M_{n,r,v}(s)\Phi_{\varphi_v}}(m_v, s) = \sum_j \left(\frac{b_{r,v}(s')}{a_{r,v}(s')} \widetilde{M_{n,r,v}(s)\Phi_v^j}(1, s) \right) \Psi_v^j(m_v, s)$$

for all $m_v \in M_n(k_v)$, where $\Psi_v^j \in \widetilde{I}_{n,r,v}(s)$ is a standard section for each j.

Then Lemma 5.2 and 5.3 lead us to the following result:

Proposition 5.4. (1) *For each Siegel section $\Phi_{\varphi_v} \in I_v(s)$, there exists a finite collection of Schwartz functions $f_{v,j} \in S(X_r(k_v))$ such that*

$$\frac{c_{r,v}(s) b_{r,v}(s')}{a_{r,v}(s')} \widetilde{M_{n,r,v}(s)\Phi_{\varphi_v}} = \sum_j \beta_{v,j} F_{v,j},$$

where for each j, $\beta_{v,j}$ is a rational function in q^{-s} which is holomorphic at $s = s_{(n)}$, and

$$F_{v,j}(g_v) = F(g_v, \mu_{1,s,v}, \mu_{2,s,v}, f_{v,j}), \ \textit{for } g_v \in \mathrm{GL}_n(k_v).$$

(2) *Moreover, suppose* $a_{r,v}(s')^{-1}M_{n,r,v}(s)\Phi_{\varphi_v}(g_v, s)$ *has a zero at* $s = s_{(n)}$ *for all elements* $g_v \in \mathrm{Sp}_n(k_v)$. *Then we are able to choose suitable* $f_{v,j}$ *and* $\beta_{v,j}$ *such that either* $\beta_{v,j}$ *has a zero at* $s = s_{(n)}$ *for all* j *or the support of* $f_{v,j}$ *is contained in the set of elements of rank* r *in* $X_r(k_v)$ *for all* j.

The above proposition describes the analytic behavior of the local factors at each place $v \in \Sigma(\Phi_\varphi, dn)$. Immediately, we have

Corollary 5.5. *For any factorizable* $\varphi = \otimes_v \varphi_v \in S(V(\mathbb{A}_k)^n)$, *there exists a finite collection of Schwartz functions* $f_j \in S(X_r(\mathbb{A}_k))$ *such that*

$$\widetilde{\Phi}_{\varphi}^{(r)} = \widetilde{M_{n,r}(s)}\Phi_\varphi = \frac{a_r(s')}{c_r(s)b_r(s')} \sum_j \beta_j(s) F_j(s)$$

where for each j, β_j *is a rational function in* q^{-s} *which is holomorphic at* $s_{(n)}$, *and* F^j *is the section of* $\widetilde{I}_{n,r}(s)$ *corresponding to* f_j.

5.1. The order of $E_{P_n}^{(r)}(\mathfrak{m}(a), s, \Phi_\varphi)$ **at** $s = s_{(n)}$ **for** $0 < r < n$. By Corollary 5.5 we have that for $a \in \mathrm{GL}_2(\mathbb{A}_k)$,

$$E_{P_n}^{(r)}(\mathfrak{m}(a), s, \Phi_\varphi) = \frac{a_r(s')}{c_r(s)b_r(s')} \cdot \sum_j \beta_j(s) \cdot E(a, \mu_{1,s}, \mu_{2,s}, f_j).$$

Recall that $s_{(n)} = \dim(V)/2 - (n+1)/2$, $s' = s + (n-r)/2$, and

$$\begin{aligned}
a_r(s) &= L(s + \frac{r+1}{2} - r, \chi_V) \prod_{i=1}^{\lfloor r/2 \rfloor} \zeta_k(2s - r + 2i),\\
b_r(s) &= L(s + \frac{r+1}{2}, \chi_V) \prod_{i=1}^{\lfloor r/2 \rfloor} \zeta_k(2s + r - 2i + 1),\\
c_r(s) &= \prod_{i=0}^{r-1} \zeta_k(2s + n - r - i).
\end{aligned}$$

Therefore

$$\mathrm{ord}_{s=s_{(n)}} c_r(s)^{-1} = \begin{cases} +2, & \text{if } \dim(V) = 4 \text{ and } r = 2,\\ +1, & \text{if } (\dim(V), r) = (2,1) \text{ or } (4,3),\\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 4.2 we have

$$\mathrm{ord}_{s=s_{(n)}} \frac{a_r(s')}{c_r(s)b_r(s')} = \begin{cases} +1, & \text{if } \dim(V) = 2,\\ -1, & \text{if } (\dim(V), r) = (4,1) \text{ and } \chi_V \text{ is trivial},\\ 0, & \text{if } (\dim(V), r) = (4,1) \text{ and } \chi_V \text{ is non-trivial},\\ 0, & \text{if } (\dim(V), r) = (4,2) \text{ and } \chi_V \text{ is trivial},\\ +1, & \text{if } (\dim(V), r) = (4,2) \text{ and } \chi_V \text{ is non-trivial},\\ +1, & \text{if } \dim(V) = 4 \text{ and } r \geq 3. \end{cases}$$

Moreover, by Theorem 5.1,

$$\operatorname{ord}_{s=s_{(n)}} E(a,\mu_{1,s},\mu_{2,s},f_j) \geq \begin{cases} -1 & \text{if } (\dim(V),r)=(2,1),\ (4,2),\ \text{or } (4,3), \\ -1 & \text{if } n=2 \text{ and } (\dim(V),r)=(4,1), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\dim(V)=2$. Then $E_{P_n}^{(r)}(\mathfrak{m}(a),s,\Phi_\varphi)$ is holomorphic at $s=s_{(n)}$ and vanishes when $r>1$. Suppose $\dim(V)=4$. We have that $E_{P_n}^{(r)}(\mathfrak{m}(a),s,\Phi_\varphi)$ is holomorphic (resp. vanishes) at $s=s_{(n)}$ when $r=3$ (resp. $r>3$). Suppose $r=1$ or 2. Since (V,Q_V) is anisotropic over k, there exists at least one (resp. two) place v of k such that (V,Q_V) is still anisotropic over k_v when $\dim(V)=4$ and χ_V is trivial (resp. other cases). Therefore by Lemma 4.4 (ii) and Proposition 5.4 (ii), we always can choose β_j such that

$$\operatorname{ord}_{s=s_{(n)}} \beta_j \geq 1 \quad (\text{resp. } 2).$$

Therefore $E_{P_n}^{(r)}(\mathfrak{m}(a),s,\Phi_\varphi)$ is still holomorphic at $s_{(n)}$ and vanishes for the case when $r=2$ or $n-1>r=1$. We conclude that

PROPOSITION 5.6. *Fix an integer r with $0<r<n$. For each $a\in \mathrm{GL}_n(\mathbb{A}_k)$, $E_{P_n}^{(r)}(\mathfrak{m}(a),s,\Phi_\varphi)$ is holomorphic at $s=s_{(n)}$. Moreover, $E_{P_n}^{(r)}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi)=0$ except for the following cases:*
(i) when $\dim(V)=2$, $r=1$;
(ii) when $\dim(V)=4$, $n=2=r+1$;
(iii) when $\dim(V)=4$, $r=3$.

Together with the result in Proposition 4.6, we obtain:

COROLLARY 5.7. *For $0\leq r\leq n$, $E_{P_n}^{(r)}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi)=0$ except for the following cases:*
(i) when $r=0$;
(ii) when $\dim(V)=2$, $n\geq r=1$;
(iii) when $\dim(V)=4$, $n=2\geq r\geq 1$;
(iv) when $\dim(V)=4$, $n\geq r=3$.

6. Siegel-Weil formula. The aim of this section is to prove the Siegel-Weil formula over function fields:

THEOREM 6.1. *Let $\varphi\in S(V(\mathbb{A}_k)^n)$. Then for $g\in \mathrm{Sp}_n(\mathbb{A}_k)$,*

$$E(g,s_{(n)},\Phi_\varphi)=\epsilon(n)\cdot I(g,\varphi),$$

where $s_{(n)}=\dim(V)/2-(n+1)/2$ and

$$\epsilon(n)=\begin{cases} 1 & \text{if } \dim(V)>n+1, \\ 2 & \text{if } \dim(V)\leq n+1. \end{cases}$$

The proof is divided into three cases:
(i) $\dim(V)=n+1$;
(ii) $\dim(V)>n+1$;
(iii) $\dim(V)<n+1$.
We deal with these cases in Section 6.1, 6.2, and 6.3, separately.

6.1. Special case: $\dim(V) = n+1$**.** We first show that

Lemma 6.2. *When* $\dim(V) = n+1$,

$$M(0)\Phi_\varphi(g,0) = \Phi_\varphi(g,0)$$

for every Siegel section Φ_φ *and* $g \in \mathrm{Sp}_n(\mathbb{A}_k)$ *when* $\dim(V) = n+1$*. In particular, for every* $a \in \mathrm{GL}_n(\mathbb{A}_k)$*,*

$$E_{P_n}^{(n)}(\mathfrak{m}(a), 0, \Phi_\varphi) = (M(0)\Phi_\varphi)(\mathfrak{m}(a), 0) = \Phi_\varphi(\mathfrak{m}(a), 0) = (\omega(\mathfrak{m}(a))\varphi)(0).$$

Proof. For each place v of k, the maps

$$T_{1,v} = \big(\varphi_v \mapsto \Phi_\varphi(\cdot, 0)\big) \quad \text{and} \quad T_{2,v} = \big(\varphi_v \mapsto \frac{b_{n,v}(0)}{a_{n,v}(0)} M_v(0)\Phi_\varphi(\cdot, 0)\big)$$

are both lie in $\mathrm{Hom}_{\mathrm{Sp}_n(k_v)\times \mathrm{O}(V)(k_v)}(S(V(k_v)^n), I_{\mathbb{A}_k}(0) \otimes \mathbf{1})$. Lemma 4.4 implies that there exists $\mu_v \in \mathbb{C}$ such that $T_{2,v} = \mu_v T_{1,v}$. Recall that for every $\Phi = \otimes_v \Phi_v \in I_{\mathbb{A}_k}(s)$, $M(s)\Phi(g,s)$ can be expressed by

$$\frac{a_n(s)}{b_n(s)} \cdot \left[\big(\otimes_{v \notin \Sigma(\Phi, dn)} \Phi_v^0(g_v, -s)\big) \otimes \left(\otimes_{v \in \Sigma(\Phi, dn)} \frac{b_{n,v}(s)}{a_{n,v}(s)} M_v(s)\Phi_v(g_v, s)\right)\right].$$

Hence we can find $\mu \in \mathbb{C}$ such that $M(0)\Phi_\varphi(\cdot, 0) = \mu\Phi_\varphi(\cdot, 0)$ for every $\varphi \in S(V(\mathbb{A}_k)^n)$. It is known that $M(0) \circ M(0) : I_{\mathbb{A}_k}(0) \to I_{\mathbb{A}_k}(0)$ is the identity map. Thus $\mu = \pm 1$. By Theorem A.4, we can choose a Siegel section Φ_φ such that the Siegel-Eisenstein series $E(\cdot, 0, \Phi_\varphi)$ is not zero. Its constant term $E_{P_n}(g, s, \Phi_\varphi)$ is equal to

$$\Phi_\varphi(g, 0) + M(0)\Phi_\varphi(g, 0) = (1+\mu)\big(\omega(g)\varphi\big)(0).$$

Suppose $\mu = -1$. Then $E_{P_n}(\cdot, 0, \Phi_\varphi) \equiv 0$. By Corollary 3.3 we have $E(\cdot, 0, \Phi_\varphi) \equiv 0$, which gives us a contradiction. Therefore $\mu = 1$ and the proof is complete. □

Corollary 6.3. *The Siegel-Weil formula* (Theorem 6.1) *holds when* $\dim(V) = n+1$*.*

Proof. It suffices to show that for $\varphi \in S(V(\mathbb{A}_k)^n)$ and $g \in \mathrm{Sp}_n(\mathbb{A}_k)$,

$$E_{B_n}(g, s_{(n)}, \Phi_\varphi) = 2 \cdot I_{B_n}(g, \varphi). \tag{6.1}$$

Indeed, Proposition 2.3 and Proposition 3.2 say that $E(g, s_{(n)}, \Phi_\varphi)$ and $I(g, \varphi)$ are both concentrated on the Borel subgroup B_n. Then by (6.1), it can be shown that $E(\cdot, s_{(n)}, \Phi_\varphi) - 2I(\cdot, \varphi)$ is a cusp form on $\mathrm{Sp}_n(\mathbb{A}_k)$ which is also orthogonal to all the cusp forms on $\mathrm{Sp}_n(\mathbb{A}_k)$. Therefore the result holds.

Note that

$$E_{B_n}(g, s_{(n)}, \Phi_\varphi) = \int_{U_{M_n}(k)\backslash U_{M_n}(\mathbb{A}_k)} E_{P_n}(ug, s_{(n)}, \Phi_\varphi) du$$

and

$$I_{B_n}(g, \varphi) = \int_{U_{M_n}(k)\backslash U_{M_n}(\mathbb{A}_k)} I_{P_n}(ug, \varphi) du$$

where $U_{M_n} := U_n \cap M_n$. Write $g = n(b)m(a)\kappa$ where $\mathfrak{n}(b) \in N_n(\mathbb{A}_k)$, $\mathfrak{m}(a) \in M_n(\mathbb{A}_k)$, $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$, then

$$E_{P_n}(g, s_{(n)}, \Phi_\varphi) = E_{P_n}(\mathfrak{m}(a), s_{(n)}, \Phi_{\omega(\kappa)\varphi})$$

and

$$(\omega(g)\varphi)(0) = \omega(\mathfrak{m}(a))\big(\omega(\kappa)\varphi\big)(0).$$

To show (6.1), it is enough to prove that for every $\varphi \in S(V(\mathbb{A}_k)^n)$ and $a \in \mathrm{GL}_n(\mathbb{A}_k)$,

$$E_{P_n}(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = \Phi_\varphi(\mathfrak{m}(a), 0) + M(0)\Phi_\varphi(\mathfrak{m}(a), 0) = 2(\omega(\mathfrak{m}(a))\varphi)(0).$$

Therefore Lemma 6.2 completes the proof. □

6.2. Special case: $\dim(V) > n+1$**.** Let $\varphi_0 \in S(V(\mathbb{A}_k))$ such that $\varphi_0(0) = 1$. For every $\varphi \in S(V(\mathbb{A}_k)^n)$, take $\tilde{\varphi} := \varphi_0 \otimes \varphi \in S(V(\mathbb{A}_k)^{n+1})$. Then it is clear that for $g \in \mathrm{Sp}_n(\mathbb{A}_k)$,

$$(i^*\Phi_{\tilde{\varphi}})(g, s) = \Phi_{\tilde{\varphi}}(i(g), s - 1/2) = \Phi_\varphi(g, s),$$

where $i : \mathrm{Sp}_n \hookrightarrow \mathrm{Sp}_{n+1}$ is the embedding introduced in Section 4.2. Consider the maximal standard parabolic subgroup P_1 of Sp_{n+1} whose Levi subgroup M_1 is isomorphic to $\mathrm{GL}_1 \times \mathrm{Sp}_n$. By Lemma 3.1, we have

LEMMA 6.4. *For every standard section $\Phi \in I^{n+1}_{\mathbb{A}_k}(s)$ and $g \in \mathrm{Sp}_n(\mathbb{A}_k)$,*

$$E^{n+1}_{P_1}(i(g), s, \Phi) = E^n(g, s + \frac{1}{2}, i^*\Phi) + E^n(g, \frac{1}{2} - s, i^*M_{n+1}(s)\Phi).$$

Proof. By Lemma 3.1, we get

$$E^{n+1}_{P_1}(i(g), s, \Phi) = E^n(g, s + \frac{1}{2}, i^*\Phi) + E^n(g, s - \frac{1}{2}, i^*\Phi_w).$$

where

$$w = w_{n+1}w_n^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & I_n & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} \in \mathrm{Sp}_{n+1}(k).$$

Recall the functional equation of Siegel-Eisenstein series:

$$E^n(g, s, \Phi) = E^n(g, -s, M(s)\Phi), \quad \forall g \in \mathrm{Sp}_n(\mathbb{A}_k),\ \Phi \in I^n_{\mathbb{A}_k}(s).$$

By straightforward calculation, $M_n(s - 1/2)\big(i^*\Phi_w\big) = i^*\big(M_{n+1}(s)\Phi\big)$. Therefore the result holds. □

When $\dim(V) = 4$ and $n = 2$, by Lemma 6.2, it can be observed that for every $g \in \mathrm{Sp}_2(\mathbb{A}_k)$

$$E^2\big(g, \frac{1}{2}, i^*M_3(0)\Phi_{\tilde{\varphi}}\big) = E^2(g, \frac{1}{2}, i^*\Phi_{\tilde{\varphi}}).$$

In particular,

$$E^3_{P_1}(i(g), 0, \Phi_{\tilde{\varphi}}) = 2E^2(g, \frac{1}{2}, i^*\Phi_{\tilde{\varphi}}) = 2E^2(g, \frac{1}{2}, \Phi_\varphi).$$

On the other hand, by Corollary 6.3 we get

$$E^3_{P_1}(i(g),0,\Phi_{\tilde{\varphi}}) = 2I^3_{P_1}(i(g),\tilde{\varphi}) = 2I^2(g,\varphi).$$

Therefore

Corollary 6.5. *The Siegel-Weil formula* (Theorem 6.1) *holds when* $\dim(V) = 4$ *and* $n = 2$.

When $\dim(V) = 4$ and $n = 1$, by the same argument in Corollary 6.3, it is enough to show that for $a \in \mathrm{GL}_1(\mathbb{A}_k)$

$$E_{P_1}(\mathfrak{m}(a),0,\Phi_\varphi) = (\omega(\mathfrak{m}(a))\varphi)(0).$$

By Corollary 5.7 we have $E^{(1)}_{P_1}(\mathfrak{m}(a),0,\Phi_\varphi) = 0$. Hence

$$E_{P_1}(\mathfrak{m}(a),0,\Phi_\varphi) = E^{(0)}_{P_1}(\mathfrak{m}(a),0,\Phi_\varphi) = (\omega(\mathfrak{m}(a))\varphi)(0).$$

We conclude that

Corollary 6.6. *The Siegel-Weil formula* (Theorem 6.1) *holds when* $\dim(V) > n+1$.

6.3. Special case: $\dim(V) < n+1$**.** Set $\ell := n+1-\dim(V)$. The case when $\ell = 0$ was shown in Corollary 6.3. The remaining case is proven by an induction process on ℓ.

By the same argument in Corollary 6.3, we only need to show that for $a \in \mathrm{GL}_n(\mathbb{A}_k)$

$$E_{P_n}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi) = 2(\omega(\mathfrak{m}(a))\varphi)(0).$$

Note that when $\dim(V) \leq n+1$, by Corollary 5.7 we have that for $a \in \mathrm{GL}_n(\mathbb{A}_k)$

$$E_{P_n}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi) = \Phi_\varphi(\mathfrak{m}(a),s_{(n)}) + E^{(\dim(V)-1)}_{P_n}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi).$$

Then it is clear that $\Big(a \mapsto E_{P_n}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi)\Big)$ is concentrated on the Borel subgroup of GL_n. On the other hand, the function $\big(a \mapsto \chi_V(a)|\det(a)|_{\mathbb{A}_k}^{\frac{\dim(V)}{2}}\varphi(0)\big)$ is also concentrated on the Borel subgroup of GL_n. Therefore it is enough to show the equality for their constant terms (along Q_{n-1}):

$$\big(E_{P_n}\big)_{Q_{n-1}}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi) = 2\chi_V(a)|\det(a)|_{\mathbb{A}_k}^{\frac{\dim(V)}{2}}\varphi(0).$$

Here Q_r, $0 < r < n$, is the maximal parabolic subgroup of GL_n introduced in Section 5. Without loss of generality, we only need to consider the case when

$$a \in \mathrm{GL}_{n-1}(\mathbb{A}_k) \hookrightarrow \mathrm{GL}_1(\mathbb{A}_k) \times \mathrm{GL}_{n-1}(\mathbb{A}_k).$$

Note that $Q_{n-1} \subset P_1 \cap P_n$, and

$$\big(E_{P_n}\big)_{Q_{n-1}}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi) = \big(E_{P_1}\big)_{P^{n-1}_{n-1}}(\mathfrak{m}(a),s_{(n)},\Phi_\varphi).$$

Here P_{n-1}^{n-1} is the Siegel parabolic subgroup of Sp_{n-1}. By Lemma 6.4, we have that for $a \in \mathrm{GL}_{n-1}(\mathbb{A}_k)$.

$$E_{P_1}(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = E^{n-1}(\mathfrak{m}(a), s_{(n-1)}, i^*\Phi_\varphi) + E^{n-1}(\mathfrak{m}(a), \frac{1}{2} - s_{(n)}, i^* M_n(s_{(n)})\Phi_\varphi).$$

Since $M_n(s)\Phi_\varphi \equiv 0$ when $s = s_{(n)}$ and $\ell > 0$, it is observed that for $a \in \mathrm{GL}_{n-1}(\mathbb{A}_k)$,

$$E^{n-1}(\mathfrak{m}(a), \frac{1}{2} - s_{(n)}, i^* M_n(s_{(n)})\Phi_\varphi) = 0.$$

By induction we have that for $a \in \mathrm{GL}_{n-1}(\mathbb{A}_k)$,

$$\left(E_{P_n}\right)_{Q_{n-1}}(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = \left(E_{P_1}\right)_{P_{n-1}^{n-1}}(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = 2\chi_V(a)|\det(a)|_{\mathbb{A}_k}^{\frac{\dim(V)}{2}} \varphi(0).$$

Therefore we conclude that

Corollary 6.7. *The Siegel-Weil formula* (Theorem 6.1) *holds when* $\dim(V) < n+1$.

Appendix A. Fourier coefficients of theta series. Let f be an automorphic form on $\mathrm{Sp}_n(\mathbb{A}_k)$. For each $\beta \in \mathrm{Sym}_n(k)$, the β-th Fourier coefficient of f is

$$f_\beta(g) := \int_{\mathrm{Sym}_n(k)\backslash \mathrm{Sym}_n(\mathbb{A}_k)} f(\mathfrak{n}(b)g)\psi\big(\, Tr(-b\beta)\big)db,$$

where Tr is the trace map and the Haar measure db is normalized so that the total mass is 1. The aim of this section is to compare $E_\beta(g, s_{(n)}, \varphi)$ with $I_\beta^n(g, \varphi)$ when $\det \beta \neq 0$, and prove an analogous result (Theorem A.5) of Proposition 4.2 in [15] by the same strategy.

It is clear that

$$\begin{aligned} I_\beta^n(g,\varphi) &= \int_{\mathrm{Sym}_n(k)\backslash \mathrm{Sym}_n(\mathbb{A}_k)} I(\mathfrak{n}(b)g, \varphi)\psi(\, Tr(-b\beta))db \\ &= \int_{\mathrm{O}(V)(k)\backslash \mathrm{O}(V)(\mathbb{A}_k)} \sum_{\substack{x \in V(k)^n \\ Q_V^{(n)}(x) = \beta}} \omega(g)\varphi(h^{-1}x)dh. \end{aligned}$$

Here $Q_V^{(n)}$ is the moment map from V^n to Sym_n introduced in Section 1.3. Thus $I_\beta^n(g,\varphi) = 0$ if $(Q_V^{(n)})^{-1}(\beta)$ is empty.

Recall that a *gauge form* on a given smooth variety $\mathcal{V}$ over k is a differential ν-form over k (where $\nu = \dim(\mathcal{V})$) which is regular and non-vanishing everywhere. We refer the reader to [20] for further discussions of the gauge forms on varieties. Let dx and db be the standard gauge forms on the vector spaces V^n and Sym_n over k, respectively. The corresponding measures (i.e. Tamagawa measures) on $V^n(\mathbb{A}_k)$ and $\mathrm{Sym}_n(\mathbb{A}_k)$ (resp. $V^n(k_v)$ and $\mathrm{Sym}_n(k_v)$) are also denoted by dx and db (resp. dx_v and db_v). For every β in $\mathrm{Sym}_n(k)$ (resp. in $\mathrm{Sym}_n(k_v)$), let

$$(Q_V^{(n)})^{-1}_{\mathrm{reg}}(\beta) := \{x \in (Q_V^{(n)})^{-1}(\beta) \mid dQ_V^{(n)}(x) : V^n \to \mathrm{Sym}_n \text{ is sujective}\},$$

which is a smooth variety over k (resp. k_v), and there exists a gauge form δ_β on $(Q_V^{(n)})^{-1}_{\mathrm{reg}}(\beta)$ which is compatible with the choice of dx and db (resp. dx_v and db_v)

on V^n and Sym_n respectively. In concrete terms, δ_β induces a local measure on $(Q_V^{(n)})^{-1}(\beta)(k_v)$ so that for any L^1-function f_v on $V(k_v)^n$ (cf. [19] §6),

$$\int_{V(k_v)^n} f_v dx_v = \int_{\mathrm{Sym}_n(k_v)} \left(\int_{(Q_V^{(n)})^{-1}_{\mathrm{reg}}(b_v)(k_v)} f_v \delta_{b_v} \right) db_v.$$

In particular, suppose $\beta \in \mathrm{Sym}_n(k)$ with $\det \beta \neq 0$ and $(Q_V^{(n)})^{-1}(\beta)(k)$ is not empty, which implies that $\dim(V) \geq n$. Then

$$(Q_V^{(n)})^{-1}_{\mathrm{reg}}(\beta) = (Q_V^{(n)})^{-1}(\beta),$$

and $\mathrm{O}(V)$ acts transitively on $(Q_V^{(n)})^{-1}(\beta)$. Take any element $x \in (Q_V^{(n)})^{-1}(\beta)$. Let

$$\mathrm{St}_{\mathrm{O}(V)}(x) := \{h \in \mathrm{O}(V) : hx = x\}.$$

Then $\mathrm{St}_{\mathrm{O}(V)}(x)\backslash \mathrm{O}(V)$ is isomorphic (as a variety over k) to $(Q_V^{(n)})^{-1}(\beta)$ by

$$h \longmapsto h^{-1}x.$$

We take the measure $\tilde{d}h$ on $\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)\backslash \mathrm{O}(V)(\mathbb{A}_k)$ to be the measure on $(Q_V^{(n)})^{-1}(\beta)(\mathbb{A}_k)$ determined by the gauge form δ_β, with a set $\{\lambda_v(\beta)\}_v$ of convergence factors. The measures dh on $\mathrm{O}(V)(\mathbb{A}_k)$ and $\tilde{d}h$ on $\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)\backslash \mathrm{O}(V)(\mathbb{A}_k)$ induce a unique measure dh' on $\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)$ such that

$$dh = dh'\tilde{d}h.$$

Then we can write

$$I_\beta(g,\varphi) = \mathrm{vol}\big(\mathrm{St}_{\mathrm{O}(V)}(x)(k)\backslash \mathrm{St}_{\mathrm{O}(V)}(\mathbb{A}_k), dh'\big) \cdot \int_{\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)\backslash \mathrm{O}(V)(\mathbb{A}_k)} \omega(g)\varphi(h^{-1}x)\tilde{d}h.$$

When φ is a pure tensor, say $\varphi = \otimes_v \varphi_v$, write $\tilde{d}h$ as $\prod_v \lambda_v(\beta)^{-1}\tilde{d}h_v$ we get

$$\begin{aligned} I_\beta(g,\varphi) = {} & \mathrm{vol}\big(\mathrm{St}_{\mathrm{O}(V)}(x)(k)\backslash \mathrm{St}_{\mathrm{O}(V)}(\mathbb{A}_k), dh'\big) \\ & \cdot \prod_v \left(\lambda_v(\beta)^{-1} \int_{\mathrm{St}_{O(V)}(x)(k_v)\backslash O(V)(k_v)} \omega_v(g_v)\varphi_v(h_v^{-1}x)\tilde{d}h_v \right). \end{aligned}$$

LEMMA A.1. *Suppose* $\dim(V) \geq n$. *For each* $\beta_v \in \mathrm{Sym}_n(k_v)$ *with* $\det \beta_v \neq 0$, *let*

$$\mathbb{T}_{\beta_v} := \left\{ T \in \mathrm{Hom}_{\mathbb{C}}(S(V(k_v)^n), \mathbb{C}) \;\middle|\; \begin{array}{c} T(\omega_v(\mathfrak{n}(b),h)\varphi_v) = \psi_v(Tr(b\beta_v))T(\varphi_v), \\ \forall b \in \mathrm{Sym}_n(k_v), h \in \mathrm{O}(V)(k_v), \varphi_v \in S(V(k_v)^n) \end{array} \right\}.$$

Then $\mathbb{T}_{\beta_v} = 0$ *if* $(Q_V^{(n)})^{-1}(\beta_v)(k_v)$ *is empty. When* $(Q_V^{(n)})^{-1}(\beta_v)(k_v)$ *is not empty,* $\mathbb{T}_{\beta_v}$ *is a one dimensional* $\mathbb{C}$*-vector space spanned by the following functional*

$$T_{\beta_v} : \varphi_v \longmapsto \int_{(Q_V^{(n)})^{-1}(\beta_v)(k_v)} \varphi_v \delta_{\beta_v} = \int_{\mathrm{St}_{\mathrm{O}(V)}(x_v^0)(k_v)\backslash \mathrm{O}(V)(k_v)} \varphi_v(h_v^{-1}x_v^0)\tilde{d}h_v$$

where x_v^0 *is in* $(Q_V^{(n)})^{-1}(\beta_v)(k_v)$.

Proof. It is observed that the restriction map $S(V(k_v)^n) \to S\big((Q_V^{(n)})^{-1}(\beta_v)\big)$ induces an embedding from $\mathbb{T}_{\beta_v}$ into $\mathrm{Hom}_{\mathbb{C}}\Big(S\big((Q_V^{(n)})^{-1}(\beta_v)\big), \mathbb{C}\Big)$. Since every $\mathrm{O}(V)(k_v)$-invariant functional on $S\big((Q_V^{(n)})^{-1}(\beta_v)\big)$ must be a scalar multiple of T_{β_v}, the result holds. ∎

REMARK A.2. Given $x \in V^n$, $Q_V^{(n)}$ is *submersive at* x if $dQ_V^{(n)}(x) : V^n \to \mathrm{Sym}_n$ is surjective. Take $\varphi_v \in S(V(k_v)^n)$ which is supported on the vectors in $V(k_v)^n$ where $dQ_V^{(n)}$ is submersive. Define the function $T_{\varphi_v} \in S(\mathrm{Sym}_n(k_v))$ by

$$T_{\varphi_v}(b_v) = \begin{cases} 0, & \text{if } (Q_V^{(n)})^{-1}(b_v)(k_v) \text{ is empty,} \\ \int_{(Q_V^{(n)})^{-1}_{\mathrm{reg}}(b_v)(k_v)} \varphi_v \delta_{b_v}, & \text{otherwise.} \end{cases}$$

Then for every $\beta \in \mathrm{Sym}_n(k)$ with $\det \beta \neq 0$ and $a_v \in \mathrm{GL}_n(k_v)$,

$$\begin{aligned} &\int_{\mathrm{Sym}_n(k_v)} \Phi_{\varphi_v}(w_n \mathfrak{n}(b_v)\mathfrak{m}(a_v), s_{(n)})\psi_v(-\,Tr(b_v\beta))db_v \\ &= \int_{\mathrm{Sym}_n(k_v)} T^{\wedge}_{\varphi_v}(b_v)\psi_v(-\,Tr(b_v \cdot {}^t a_v \beta a_v))db_v \\ &= T_{\varphi_v}({}^t a_v \beta a_v). \end{aligned}$$

Here $T^{\wedge}_{\varphi_v}(b_v)$ is the Fourier transform of T_{φ_v}, i.e.

$$T^{\wedge}_{\varphi_v}(b_v) = \int_{\mathrm{Sym}_n(k_v)} T_{\varphi_v}(b'_v)\psi_v\big(\,Tr(b_v b'_v)\big)db'_v.$$

On the other hand, we have

LEMMA A.3. *For* $g \in \mathrm{Sp}_n(\mathbb{A}_k)$, $\Phi \in I_{\mathbb{A}_k}(s)$, *and* $\beta = \begin{pmatrix} 0 & 0 \\ 0 & \beta_0 \end{pmatrix}$ *with* $\beta_0 \in \mathrm{Sym}_{n-r}(k_v)$, $0 \leq r < n$, *and* $\det \beta_0 \neq 0$,

$$\begin{aligned} &E_\beta(g, s, \Phi) \\ &= \sum_{i \leq r} \sum_{a' \in Q^{(r)}_{r-i}(k) \backslash \mathrm{GL}_r(k)} \int_{\mathrm{Sym}_{n-i}(\mathbb{A}_k)} \Phi(w_{n-i}\mathfrak{n}(b')\gamma(a')g, s)\psi(-\,Tr(b'\beta_0))db'. \end{aligned}$$

Here $\gamma(a') = \mathfrak{m}\begin{pmatrix} a' & 0 \\ 0 & I_{n-r} \end{pmatrix}$ *and we embed* Sym_{n-i} *into* Sym_n *by sending* b' *to* $\begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}$.

Proof. The argument is similar to Lemma 3.1. Therefore we omit the proof. ∎

Now, we arrive at the main result of this section.

THEOREM A.4. *There exists a non-zero constant* c *such that the function*

$$(I^n)'(g, \varphi) := I(g, \varphi) - cE(g, s_{(n)}, \Phi_\varphi)$$

satisfies $(I^n)'_\beta(g, \varphi) = 0$ *for every* $g \in \mathrm{Sp}_n(\mathbb{A}_k)$, $\varphi \in S(V(\mathbb{A}_k)^n)$, *and* $\beta \in \mathrm{Sym}_n(k)$ *with* $\det \beta \neq 0$.

Proof. Without loss of generality, assume $g = \mathfrak{m}(a)$ for $a \in \mathrm{GL}_n(\mathbb{A}_k)$ and $\varphi = \otimes_v \varphi_v$ is a pure tensor. It is clear that the functional

$$T'_a := \Big(\varphi \longmapsto E_\beta(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi), \quad \forall \varphi \in S(V(\mathbb{A}_k)^n)\Big)$$

satisfies that

$$T'_a\big(\omega(\mathfrak{n}(b))\varphi\big) = \psi(\mathit{Tr}(b \cdot {}^t a \beta a)) T'_a(\varphi), \quad \forall b \in \mathrm{Sym}_n(\mathbb{A}_k).$$

Therefore Lemma A.1 implies that

$$E_\beta(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = I^n_\beta(\mathfrak{m}(a), \varphi) = 0$$

if $(Q_V^{(n)})^{-1}(\beta)(k)$ is empty.

When $(Q_V^{(n)})^{-1}(\beta)(k)$ is not empty, $\dim(V) \geq n$ and by Lemma A.1 again we can find a constant $c = c(\beta, a) \in \mathbb{C}$ such that

$$E_\beta(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = c(\beta, a) \cdot I^n_\beta(\mathfrak{m}(a), \varphi).$$

It remains to show that c is non-zero and is independent of the choices of β and a.

Lemma A.3 says that

$$E_\beta(\mathfrak{m}(a), s_{(n)}, \Phi_\varphi) = \prod_v \left(\int_{\mathrm{Sym}_n(k_v)} \Phi_\varphi(w_n \mathfrak{n}(b_v)\mathfrak{m}(a_v), s_{(n)}) \psi_v(-\mathit{Tr}(b_v \beta)) db_v \right).$$

Let $\Sigma(a, Q_V, db)$ be the finite set of places of k such that when $v \notin \Sigma(a, Q_V, db)$, we have $a_v \in \mathrm{GL}_n(O_v)$, $\beta_v \in \mathrm{Sym}_n(O_v) \cap \mathrm{GL}_n(O_v)$, Q_V is unramified at v and $\mathrm{vol}(\mathrm{Sym}_n(O_v), db_v) = 1$. For $v \notin \Sigma(a, Q_V, db)$, take $\varphi_v^0 \in S(V(k_v)^n)$ such that $\Phi_{\varphi_v^0} = \Phi_v^0$ (recall that $\Phi_v^0(\kappa_v, s) = 1$ for all $\kappa_v \in \mathrm{Sp}_n(O_v)$). Then we get

$$\begin{aligned}
&\int_{\mathrm{Sym}_n(k_v)} \Phi_{\varphi_v^0}(w_n \mathfrak{n}(b_v)\mathfrak{m}(a_v), s) \psi_v(-\mathit{Tr}(b_v \beta)) db_v \\
&= L_v(s + \frac{n+1}{2}, \chi_{V,v})^{-1} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \zeta_v(2s + n + 1 - 2i)^{-1} \\
&\quad \cdot \begin{cases} L_v(s + 1/2, \chi_{V,v}\chi_{\beta,n,v}), & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Here $\chi_{\beta,n,v} : k_v^\times \to \{\pm 1\}$ is defined by

$$\chi_{\beta,n,v}(\alpha_v) := ((-1)^{n/2} \det \beta, \alpha_v)_v, \quad \forall \alpha_v \in k_v^\times.$$

Given $a_1,\ a_2 \in \mathrm{GL}_n(\mathbb{A}_k)$, we choose $\varphi_v^1,\ \varphi_v^2 \in S(V(k_v)^n)$ for every place v as follows. Let $\Sigma := \Sigma(a_1, Q_V, db) \cup \Sigma(a_2, Q_V, db)$. For $v \notin \Sigma$ we let $\varphi_v^1 = \varphi_v^2 = \varphi_v^0$; and for $v \in \Sigma$ we choose φ_v^i which are supported on the vectors in $V(k_v)^n$ where $dQ_V^{(n)}$ is submersive and $T_{\varphi_v^i}({}^t a_{v,i} \beta a_{v,i}) \neq 0$. Define

$$\begin{aligned}
\Lambda_\Sigma(s) := L_\Sigma(s + \frac{n+1}{2}, \chi_V)^{-1} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \zeta_\Sigma(2s + n + 1 - 2i)^{-1} \\
\cdot \begin{cases} L_\Sigma(s + 1/2, \chi_V \chi_{\beta,n}), & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Then the discussion in Remark A.2 implies that

$$\Lambda_\Sigma(s_{(n)}) = c(\beta, a_i)\left(\prod_{v\in\Sigma}\frac{1}{\lambda_v(\beta)}\right)\mathrm{vol}\big(\mathrm{St}_{\mathrm{O}(V)}(x)(k)\backslash \mathrm{St}_{\mathrm{O}(V)}(\mathbb{A}_k), dh'\big)\prod_{v\notin\Sigma}\frac{T_{\beta_v}(\varphi_v^0)}{\lambda_v(\beta)},$$

where $\{\lambda_v(\beta)\}$ is a convenient set of convergent factors for the gauge on $(Q_V^{(n)})^{-1}(\beta)$ and x is a chosen element in $(Q_V^{(n)})^{-1}(\beta)$. Hence c is independent of the choice of a. Finally, let $\{\lambda'_v\}$ be a set of convergence factors for the measure dh on $\mathrm{O}(V)(\mathbb{A}_k)$, then $\{\lambda''_v := \lambda'_v/\lambda_v(\beta)\}$ is a set of convergence factors for the measure dh' on $\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)$. Choosing suitable convergence factors $\{\lambda''_v\}$ for the measure dh' on $\mathrm{St}_{\mathrm{O}(V)}(x)(\mathbb{A}_k)$, it can be shown that c is also independent of the choice of β. Therefore the proof is complete. □

One consequence of Theorem A.4 is:

Theorem A.5. *Given a Schwartz function $\varphi \in S(V(\mathbb{A}_k)^n)$, we have that for every cusp form f on $\mathrm{Sp}_n(\mathbb{A}_k)$,*

$$\int_{\mathrm{Sp}_n(k)\backslash \mathrm{Sp}_n(\mathbb{A}_k)} I^n(g,\varphi)f(g)dg = 0.$$

Proof. The argument is similar to Theorem 2.7 in [9]. We recall the key steps for the sake of completeness.

Consider the auxiliary Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_k)$:

$$\mathcal{E}^0(g,s) := \sum_{\gamma\in P_n(k)\backslash \mathrm{Sp}_n(k)} \Phi^0(\gamma g, s)$$

where for $g = \begin{pmatrix} a & * \\ 0 & {}^t a^{-1}\end{pmatrix}\kappa$ with $a \in \mathrm{GL}_n(\mathbb{A}_k)$ and $\kappa \in \mathrm{Sp}_n(O_{\mathbb{A}_k})$,

$$\Phi^0(g,s) := |\det a|_{\mathbb{A}_k}^{s+(n+1)/2}.$$

It is known that $\mathcal{E}^0(g,s)$ has meromorphic continuation to the whole s-plane, and has a simple pole at $s = (n+1)/2$ with residue c_1 independent of g. Then for any cusp form f on $\mathrm{Sp}_n(\mathbb{A}_k)$,

$$\mathrm{Res}_{s=(n+1)/2}\ \langle (I^n)'(\varphi), \mathcal{E}^0(s)f\rangle = c_1\langle (I^n)'(\varphi), f\rangle = c_1\langle I^n(\varphi), f\rangle. \tag{A.1}$$

Here

$$\langle f_1, f_2\rangle := \int_{\mathrm{Sp}_n(k)\backslash \mathrm{Sp}_n(\mathbb{A}_k)} f_1(g)f_2(g)dg.$$

The second equality in (A.1) holds by Proposition 3.2. It suffices to show that

$$\langle (I^n)'(\varphi), \mathcal{E}^0(s)f\rangle = 0.$$

It is observed that

$$\begin{aligned}&\langle (I^n)'(\varphi), \mathcal{E}^0(s)f\rangle \\ &= \sum_{\beta \bmod \mathrm{GL}_n(k)} \int_{M_\beta(\mathbb{A}_k)N_n(\mathbb{A}_k)\backslash \mathrm{Sp}_n(\mathbb{A}_k)} \int_{M_\beta(k)\backslash M_\beta(\mathbb{A}_k)} (I^n)'_\beta(mg)f_{-\beta}(mg)\Phi^0(mg,s)dmdg.\end{aligned}$$

The sum runs over representatives β mod $\mathrm{GL}_n(k)$ in $\mathrm{Sym}_n(k)$. The action of $\mathrm{GL}_n(k)$ on $\mathrm{Sym}_n(k)$ is defined by $a * \beta := {}^t a\beta a$, and

$$M_\beta := \{\mathfrak{m}(a) \in M_n : a \in \mathrm{GL}_n \text{ such that } a * \beta = \beta\}.$$

Theorem A.5 tells us that the sum runs over singular β. For convention, we choose β to be of the form $\begin{pmatrix} 0 & 0 \\ 0 & \beta_0 \end{pmatrix}$ where $\beta_0 \in \mathrm{Sym}_{n-r}(k)$ with $0 < r \leq n$. Note that $M_\beta = L_\beta \cdot U_\beta$ where

$$U_\beta \cong \left\{ u(x) = \begin{pmatrix} I_r & x \\ 0 & I_{n-r} \end{pmatrix} \;\middle|\; x \in \mathrm{Mat}_{r\times(n-r)} \right\}$$

and

$$L_\beta \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r,\ d \in \mathrm{GL}_{n-r} \text{ with } {}^t d\beta_0 d = \beta_0 \right\}.$$

Thus

$$\int_{M_\beta(k)\backslash M_\beta(\mathbb{A}_k)} (I^n)'_\beta(mg) f_{-\beta}(mg)\Phi^0(mg,s)dm$$
$$= \int_{L_\beta(k)\backslash L_\beta(\mathbb{A}_k)} \left(\int_{U_\beta(k)\backslash U_\beta(\mathbb{A}_k)} (I^n)'_\beta(ulg) f_{-\beta}(ulg)du \right) \Phi^0(lg,s)dl.$$

For $\alpha \in \mathrm{Mat}_{r\times(n-r)}(k)$, denote

$$(I^n)'_{\alpha,\beta}(g) := \int_{\mathrm{Mat}_{r\times(n-r)}(k)\backslash \mathrm{Mat}_{r\times(n-r)}(\mathbb{A}_k)} (I^n)'_\beta(u(x)g)\psi(-\,Tr({}^t\alpha x))dx.$$

Then

$$\int_{U_\beta(k)\backslash U_\beta(\mathbb{A}_k)} (I^n)'_\beta(ulg) f_{-\beta}(ulg)du = \sum_{\alpha\in \mathrm{Mat}_{r\times(n-r)}(k)} (I^n)'_{\alpha,\beta}(lg) f_{-\alpha,-\beta}(lg).$$

By straightforward calculation, we obtain:
(i) If $\beta = 0$, then $f_\beta = 0$.
(ii) If singular $\beta \neq 0$, then $f_{0,\beta} = 0$.
(iii) If β is singular and $\alpha \neq 0$, then $E_{\alpha,\beta}(\cdot, s, \Phi_\varphi) = 0$.
(iv) If β is singular and $\alpha \neq 0$, then $I^n_{\alpha,\beta}(\cdot, \varphi) = 0$.
We point out that (iii) is deduced from the expression of $E_\beta(\cdot, s, \Phi_\varphi)$ in Lemma A.3. These observation completes the proof. □

Appendix B. On the Jacquet module of $S(V(k_v)^n)$ with respect to P_n.

In this section, we describe the Jordan structure of the Jacquet module of $S(V(k_v)^n)$ (which is studied in [8] for the number field case). Recall that (V, Q_V) is an anisotropic quadratic space over k, and ω_v is the Weil representation of $\mathrm{Sp}_n(k_v) \times \mathrm{O}(V)(k_v)$ on the Schwartz space $S(V(k_v)^n)$ for each place v of k. The Jacquet module $\mathcal{J}_n$ of $S(V(k_v)^n)$ with respect to P_n is the quotient space

$$\frac{S(V(k_v)^n)}{\mathrm{Span}\Big\{\omega_v(\mathfrak{n}(b))\varphi - \varphi \;\Big|\; b \in \mathrm{Sym}_n(k_v),\ \varphi \in S(V(k_v)^n)\Big\}}.$$

We modify the action of $\mathrm{GL}_n(k_v)$ on $\mathcal{J}_n$ by:

$$\tilde{\omega}_v(a)\bar{\varphi} := |\det a|_v^{-\frac{n+1}{2}} \cdot \overline{\omega_v(\mathfrak{m}(a))\varphi}, \quad \forall a \in \mathrm{GL}_n(k_v) \text{ and } \bar{\varphi} \in \mathcal{J}_n.$$

Define $V(k_v)_0^n := \{x \in V(k_v)^n \mid Q_V^{(n)}(x) = 0\}$, where $Q_V^{(n)} : V^n \to \mathrm{Sym}_n$ is the moment map introduced in Section 1.3. The Schwartz space $S(V(k_v)_0^n)$ is invariant under the action of $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$ defined by

$$(a, h) \cdot \varphi(x) = \chi_{V,v}(\det a)|\det a|_v^{\frac{\dim(V)-(n+1)}{2}} \varphi(h^{-1}xa).$$

Let $l_0 := \min(l, n)$ where l is the dimension of a maximal isotropic subspace of $V(k_v)$ (which is 0 or $\dim(V)/2$). It is clear that every $x = (x_1, ..., x_n) \in V(k_v)_0^n$ satisfies $\dim(\mathrm{Span}\ x) \leq l_0$, where $\mathrm{Span}\ x := \mathrm{Span}\{x_1, ..., x_n\}$. Thus

$$V(k_v)_0^n = \coprod_{i=0}^{l_0} V(k_v)_{0,i}^n$$

where

$$V(k_v)_{0,i}^n := \Big\{x \in V(k_v)_0^n \ \Big|\ \dim(\mathrm{Span}\ x) = i\Big\}.$$

Proposition B.1. (1) *As a* $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$*-module,* $\mathcal{J}_n$ *is isomorphic to* $S(V(k_v)_0^n)$*, where the isomorphism is induced by the restriction from* $S(V(k_v)^n)$ *to* $S(V(k_v)_0^n)$*.*
(2) *We have a* $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$*-invariant filtration*

$$\mathcal{J}_n = \mathcal{J}_n^{(0)} \supset \mathcal{J}_n^{(1)} \supset \cdots \supset \mathcal{J}_n^{(l_0)} \supset \mathcal{J}_n^{(l_0+1)} = \{0\}$$

such that as $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$*-modules,*

$$\widetilde{\mathcal{J}}_n^{(i)} := \mathcal{J}_n^{(i)}/\mathcal{J}_n^{(i+1)} \cong S(V(k_v)_{0,i}^n).$$

Proof. Consider the following exact sequence:

$$0 \to S(V(k_v)^n - V(k_v)_0^n) \to S(V(k_v)^n) \to S(V(k_v)_0^n) \to 0.$$

It is clear that $\omega_v(\mathfrak{n}(b))\varphi - \varphi \in S(V(k_v)^n - V(k_v)_0^n)$ for every $b \in \mathrm{Sym}_n(k_v)$ and $\varphi \in S(V(k_v)^n)$. On the other hand, for $x \in V(k_v)^n - V(k_v)_0^n$, we can find $b \in \mathrm{Sym}_n(k_v)$ such that $\psi_v\big(\mathrm{Trace}(bQ_V^{(n)}(x))\big) \neq 1$. Choose a sufficiently small neighborhood U_x of x in $V(k_v)^n - V(k_v)_0^n$ such that

$$\psi_v\big(\mathrm{Trace}(bQ_V^{(n)}(x'))\big) = \psi_v\big(\mathrm{Trace}(bQ_V^{(n)}(x))\big), \quad \forall x' \in U_x.$$

Let φ_x be the characteristic function of U_x. Then

$$\omega_v(\mathfrak{n}(b))\varphi_x - \varphi_x = \Big(\psi_v\big(\mathrm{Trace}(bQ_V^{(n)}(x))\big) - 1\Big)\varphi_x.$$

This implies that the Schwartz space $S(V(k_v)^n - V(k_v)_0^n)$ coincides with

$$\mathrm{Span}\Big\{\omega_v(\mathfrak{n}(b))\varphi - \varphi \ \Big|\ b \in \mathrm{Sym}_n(k_v),\ \varphi \in S(V(k_v)^n)\Big\}.$$

Therefore the proof of (1) is complete.

Identifying $\mathcal{J}_n = \mathcal{J}_n^{(0)}$ with $S(V(k_v)_0^n)$, for $0 < i \leq l_0$ we let

$$\mathcal{J}_n^{(i)} := \left\{ \varphi \in S(V(k_v)_0^n) \;\middle|\; \varphi(x) = 0,\ \forall x \in \coprod_{j=0}^{i-1} V(k_v)_{0,j}^n, \right\}.$$

Then for $0 \leq i \leq l_0$,

$$\mathcal{J}_n / \mathcal{J}_n^{(i+1)} \cong S\Big(\coprod_{j=0}^{i} V(k_v)_{0,j}^n \Big).$$

This assures that for $0 \leq i \leq l_0$,

$$\mathcal{J}_n^{(i)} / \mathcal{J}_n^{(i+1)} \cong S(V(k_v)_{0,i}^n)$$

and completes the proof of (2). $\square$

Choose $x_1, ..., x_l \in V(k_v)$ such that $\mathrm{Span}\{x_1, ..., x_l\}$ is a maximal isotropic subspace. Then there exist $x_1', ..., x_l' \in V(k_v)$ such that

$$< x_i', x_j' >_V = 0, \quad 1 \leq i, j \leq l,$$

$$< x_i, x_j' >_V = 0, \quad i \neq j, \qquad \text{and } < x_i, x_i' >_V = 1.$$

For $0 \leq i \leq l$, let $V(k_v)^{(i)}$ be the orthogonal complement of $\mathrm{Span}\{x_1, ..., x_i, x_1', ..., x_i'\}$ in $V(k_v)$. Define

$$P_i' := \Big\{ h \in \mathrm{O}(V)(k_v) \;\Big|\; h \cdot \mathrm{Span}\{x_1, ..., x_i\} = \mathrm{Span}\{x_1, ..., x_i\} \Big\},$$

which is a parabolic subgroup of $\mathrm{O}(V)(k_v)$ whose Levi subgroup M_i' is isomorphic to $\mathrm{GL}_i(k_v) \times \mathrm{O}(V(k_v)^{(i)})$. More precisely, let $\{x_1', ..., x_l', x_1'', ..., x''_{\dim(V)-2l}, x_1, ..., x_l\}$ be a basis of V. Then with respect to this basis, elements in M_i' are of the form

$$(a, b) = \begin{pmatrix} {}^t a^{-1} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix},$$

where $a \in \mathrm{GL}_i(k_v)$ and $b \in \mathrm{O}(V(k_v)^{(i)})$.

Take $0 \leq i \leq l_0$. Recall

$$Q_i = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in \mathrm{GL}_n \;\middle|\; a_1 \in \mathrm{GL}_{n-i},\ a_2 \in \mathrm{GL}_i \right\}.$$

Define an action ρ_i of $Q_i(k_v) \times P_i'$ on $S(\mathrm{GL}_i(k_v))$ by

$$\rho_i \left(\left(\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix}, (a_2', h') \cdot n' \right) \varphi \right)(g) = \varphi(a_2'^{-1} g a_2),$$

where $(a_2', h') \in M_i'$ and n' is in the unipotent radical of P_i'. Let $\mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big)$ be the space of smooth functions f from $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$ to $S(\mathrm{GL}_i(k_v))$ satisfying that for every element $(g, h) \in \mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$,

$$f\big((b, b')(g, h)\big) = \mu_i(b) \cdot \rho_i(b, b')\big(f(g, h)\big), \quad \forall (b, b') \in Q_i \times P_i',$$

where for $b = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in Q_i(k_v)$,

$$\mu_i(b) = \chi_{V,v}(\det a_1 \det a_2)|\det a_1 \det a_2|_v^{\frac{\dim(V)-(n+1)}{2}}.$$

The action of $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$ on $\mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big)$ is defined by right translation.

Proposition B.2. *For $0 \leq i \leq l_0$, we have*

$$\widetilde{\mathcal{J}}_n^{(i)} \cong S(V(k_v)^n_{0,i}) \cong \mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big)$$

as $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$ modules.

Proof. For $0 \leq i \leq l_0$, Set $x^{(i)} := (0, ..., 0, x_1, x_2, ..., x_i) \in V(k_v)^n_{0,i}$. Let ι_1 and ι_2 be the embeddings from $\mathrm{GL}_i(k_v)$ into $P_i' \subset \mathrm{O}(V)(k_v)$ and $Q_i(k_v) \subset \mathrm{GL}_n(k_v)$, respectively. Then we have

$$\iota_1(g')x^{(i)} = x^{(i)}\iota_2(g'), \quad \forall g' \in \mathrm{GL}_i(k_v).$$

Define a map F from $S(V(k_v)^n_{0,i})$ to $\mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big)$ by

$$\varphi \longmapsto F_\varphi,$$

where for $(g,h) \in \mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$,

$$F_\varphi(g,h) = \Big(g' \mapsto \chi_{V,v}(\det g)|\det g|_v^{\frac{\dim(V)-(n+1)}{2}} \cdot \varphi(h^{-1}x^{(i)}\iota_2(g')g)\Big) \in S(\mathrm{GL}_i(k_v)).$$

Then it is clear that F is $\mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$-equivariant. Since for every element $x \in V(k_v)^n_{0,i}$ we can find $(g,h) \in \mathrm{GL}_n(k_v) \times \mathrm{O}(V)(k_v)$ such that $x = h^{-1}x^{(i)}g$, the inverse map $F^{-1} : \mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big) \to S(V(k_v)^n_{0,i})$ can be defined by the following: for $f \in \mathrm{Ind}\Big(S(\mathrm{GL}_i(k_v))\Big)$,

$$F_f^{-1}(x) := \chi_{V,v}(\det g)^{-1}|\det g|_v^{\frac{-\dim(V)+(n+1)}{2}} \cdot f(g,h)(1), \quad \forall x = h^{-1}x^{(i)}g \in V(k_v)^n_{0,i}.$$

□

We remark that the modulus character δ_i of the parabolic subgroup $Q_i(k_v) \times P_i'$ is:

$$\delta_i\left(\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} {}^t a_2'^{-1} & * & * \\ 0 & h' & * \\ 0 & 0 & a_2' \end{pmatrix}\right) = \frac{|\det a_1|_v^i}{|\det a_2|_v^{n-i}} \cdot |\det a_2'|_v^{-\dim(V)+i+1}.$$

Recall that $s_{(n)} = \dim(V)/2 - (n+1)/2$, and $I_v(-s_{(n)})$ is the space of smooth functions f on $\mathrm{Sp}_n(k_v)$ satisfying that

$$f\left(\begin{pmatrix} a & * \\ 0 & {}^t a^{-1} \end{pmatrix} g\right) = \chi_{V,v}(\det a)|\det a|_v^{-\frac{\dim(V)}{2}} f(g), \quad \forall a \in \mathrm{GL}_n(k_v),\ g \in \mathrm{Sp}_n(k_v).$$

In other words, $I_v(-s_{(n)})$ is the Siegel-parabolic induction from the character $|\cdot|_v^{-s_{(n)}}$ on $P_n(k_v)$. Therefore the Frobenius reciprocity (cf. [1] Proposition 4.5.1) gives us

$$\mathrm{Hom}_{\mathrm{Sp}_n(k_v)\times \mathrm{O}(V)(k_v)}\big(S(V(k_v)^n), I_v(-s_{(n)}) \otimes \mathbf{1}\big) = \mathrm{Hom}_{\mathrm{GL}_n(k_v)\times \mathrm{O}(V)(k_v)}\big(\mathcal{J}_n, |\cdot|_v^{-s_{(n)}} \otimes \mathbf{1}\big).$$

Note that $\ell(n,v) = \dim\left(\mathrm{Hom}_{\mathrm{GL}_n(k_v)\times \mathrm{O}(V)(k_v)}\left(\mathcal{J}_n, |\cdot|_v^{-s_{(n)}} \otimes \mathbf{1}\right)\right)$ is bounded by

$$\sum_{i=0}^{l_0} \dim\left(\mathrm{Hom}_{\mathrm{GL}_n(k_v)\times \mathrm{O}(V)(k_v)}\left(\widetilde{\mathcal{J}}_n^{(i)}, |\cdot|_v^{-s_{(n)}} \otimes \mathbf{1}\right)\right).$$

For each i, Proposition B.2 tells us that $\widetilde{\mathcal{J}}_n^{(i)}$ is also a parabolic induction. Hence by Frobenius reciprocity again, we get

$$\mathrm{Hom}_{\mathrm{GL}_n(k_v)\times \mathrm{O}(V)(k_v)}\left(\widetilde{\mathcal{J}}_n^{(i)}, |\cdot|_v^{-s_{(n)}} \otimes \mathbf{1}\right) = \mathrm{Hom}_{R_i}\left((\mu_i\delta_i^{-1})\otimes \rho_i, |\cdot|_v^{-s_{(n)}} \otimes \mathbf{1}\right),$$

where $R_i = \Big(\mathrm{GL}_{n-i}(k_v)\times \mathrm{GL}_i(k_v)\Big) \times \Big(\mathrm{GL}_i(k_v)\times O(V(k_v)^{(i)})\Big)$ is the Levi subgroup of $Q_i(k_v)\times P_i'$.

For $m = (a_1, a_2, a_2', h) \in R_i$,

$$\mu_i\delta_i^{-1}(m) = \chi_{V,v}(\det a_1 \det a_2)|\det a_1|_v^{s_{(n)}-i}|\det a_2|_v^{s_{(n)}+n-i}\cdot |\det a_2'|_v^{\dim(V)-i-1}.$$

Hence $\mathrm{Hom}_{R_i}\left((\mu_i\delta_i^{-1})\otimes\rho_i, |\cdot|_v^{-s_{(n)}}\otimes\mathbf{1}\right) = 0$ unless (i) $i = 2s_{(n)} = \dim(V) - (n+1)$ or (ii) $i = n$. In both cases we can get

$$\dim\left(\mathrm{Hom}_{R_i}\left((\mu_i\delta_i^{-1})\otimes\rho_i, |\cdot|_v^{-s_{(n)}}\otimes\mathbf{1}\right)\right) = 1.$$

Note that i is bounded by $l_0 = \min(l, n)$. We then arrive at

LEMMA B.3. *Let*

$$\ell(n,v) := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}_n(k_v)\times \mathrm{O}(V)(k_v)}\Big(S(V(k_v)^n), I_v(-s_{(n)})\otimes \mathbf{1}\Big),$$

where $\mathbf{1}$ *is the trivial representation of* $\mathrm{O}(V)(k_v)$*. Then*

(i) When $\dim(V) < n+1$*, we have* $\ell(n,v) = 0$.

(ii) When $\dim(V) = 2$ *and* $n = 1$,

$$\ell(n,v) \le \begin{cases} 1, & \textit{if } V(k_v) \textit{ is anisotropic,} \\ 2, & \textit{if } V(k_v) \textit{ is isotropic.} \end{cases}$$

(iii) When $\dim(V) = 4$ *and* $n = 1$ *or* 2,

$$\ell(n,v) \le \begin{cases} n, & \textit{if } V(k_v) \textit{ is isotropic,} \\ 0, & \textit{if } V(k_v) \textit{ is anisotropic.} \end{cases}$$

(iv) When $\dim(V) = 4$ *and* $n = 3$*, we have* $\ell(n,v) \le 1$.

REMARK B.4. When $\dim(V) = 2$, $n = 1$, and $V(k_v)$ is isotropic, the above discussion says that

$$\ell(n,v) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_1(k_v)\times \mathrm{GL}_1(k_v)}\left(\mathcal{J}_1(k_v), \mathbf{1}\right).$$

In this case, $\mathcal{J}_1(k_v) \cong S(k_v)$, and every homomorphism in $\mathrm{Hom}_{k_v^\times \times k_v^\times}\left(S(k_v), \mathbf{1}\right)$ must be of the form

$$c\cdot(\varphi \longmapsto \varphi(0)), \quad c \in \mathbb{C}.$$

Therefore $\ell(n,v)=1$ in this case, and the proof of Lemma 4.4 is complete.

Appendix C. Maass-Jacquet-Shalika Eisenstein series on $\mathrm{GL}_n(\mathbb{A}_k)$. Fix an integer r with $0<r<n$. Set $X_r := \mathrm{Mat}_{r\times n}$ as an affine algebraic variety over k. Let $\mu_{1,v}$, $\mu_{2,v}$ be two characters on $k_v^\times$. For any $g\in\mathrm{GL}_n(k_v)$ and Schwartz function $f_v\in S(X_r(k_v))$, define $F_v(g)=F_v(g,\mu_{1,v},\mu_{2,v},f_v)$ to be

$$\mu_{1,v}(\det g)|\det g|_v^{r/2}\int_{\mathrm{GL}_r(k_v)} f_v(h_v^{-1}(0,I_r)g)\mu_v^{-1}(\det h_v)d^\times h_v.$$

Here $\mu_v=\mu_{1,v}\mu_{2,v}^{-1}|\cdot|_v^{n/2}$ and the Haar measure $d^\times h_v$ is normalized so that the volume of $\mathrm{GL}_r(O_v)$ is 1 for all v.

Lemma C.1. *The integral $F_v(g)$ is absolutely convergent for every $g\in\mathrm{GL}_n(k_v)$ if $|\mu_{1,v}\mu_{2,v}^{-1}|=|\cdot|_v^\sigma$ where $\sigma>r-n/2-1$.*

Proof. Without loss of generality, assume f_v is the characteristic function of $X_r(O_v)$ and $g=1$. Then by straightforward computation we get

$$\begin{aligned}&\int_{\mathrm{GL}_r(k_v)} \left|f_v(h_v^{-1}(0,I_r))\mu_v^{-1}(\det h_v)\right|d^\times h_v\\ &=\int_{\mathrm{GL}_r(k_v)\cap\mathrm{Mat}_r(O_v)} |\mu_v|^{-1}d^\times h_v\\ &=\prod_{i=0}^{r-1}\zeta_v(\sigma-\frac{n}{2}-i).\end{aligned}$$

This assures the result. □

Recall that

$$Q_r=\left\{\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\in\mathrm{GL}_n \,\middle|\, a_1\in\mathrm{GL}_{n-r},\ a_2\in\mathrm{GL}_r\right\}.$$

For $g\in\mathrm{GL}_n(k_v)$ and $b=\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\in Q_r(k_v)$, it is clear that

$$F_v(bg)=\mu_{1,v}(\det a_1)\mu_{2,v}(\det a_2)|\delta_{Q_r}(b)|_v^{1/2}F_v(g)$$

where $\delta_{Q_r}(b)=(\det a_1)^r\cdot(\det a_2)^{r-n}$. Let $\tilde{I}_v(\mu_{1,v},\mu_{2,v})$ be the space of smooth functions Ψ_v on $\mathrm{GL}_n(k_v)$ satisfying that for $g\in\mathrm{GL}_n(k_v)$ and $b=\begin{pmatrix}a_1 & *\\ 0 & a_2\end{pmatrix}\in Q_r(k_v)$,

$$\Psi_v(bg)=\mu_{1,v}(\det a_1)\mu_{2,v}(\det a_2)|\delta_{Q_r}(b)|_v^{1/2}\Psi_v(g).$$

Then the map $(f_v\mapsto F_v)$ gives us a $\mathrm{GL}_n(k_v)$-equivariant homomorphism from $S(X_r(k_v))$ to $\tilde{I}_v(\mu_{1,v},\mu_{2,v})$.

Lemma C.2. (1) *Given $\Psi_v\in\tilde{I}_v(\mu_{1,v},\mu_{2,v})$, there exists a Schwartz function f on $X_r(k_v)$ supported on elements of rank r such that*

$$F_v(g,\mu_{1,v},\mu_{2,v},f_v)=\Psi_v(g),\quad \forall g\in\mathrm{GL}_n(k_v).$$

In other words, the map $(f_v \mapsto F_v)$ *from* $S(X_r(k_v))$ *to* $\tilde{I}_v(\mu_{1,v},\mu_{2,v})$ *is surjective.*
(2) *Suppose* $\mu_{1,v}$ *and* $\mu_{2,v}$ *are both unramified, i.e.* $\mu_{1,v}(O_v)=\mu_{2,v}(O_v)=1$. *Let* f_v^0 *be the charateristic function of* $X_r(O_v)$. *Then for* $g \in \mathrm{GL}_n(k_v)$

$$F_v^0(g) := F_v(g,\mu_{1,v},\mu_{2,v},f_v^0) = \prod_{i=0}^{r-1} \zeta_v(\sigma - \frac{n}{2} - r - i)\Psi_v^0(g)$$

where $\mu_{1,v}\mu_{2,v}^{-1} = |\cdot|_v^{\sigma}$ *and* $\Psi_v^0 \in \widetilde{I}_v(\mu_{1,v},\mu_{2,v})$ *satisfies*

$$\Psi_v^0(\kappa) = 1, \quad \forall \kappa \in \mathrm{GL}_n(O_v).$$

Proof. The Iwasawa decomposition allows us to write g as $b\kappa$, where $b \in Q_r(k_v)$ and $\kappa \in \mathrm{GL}_n(O_v)$. Hence we can assume $g = \kappa \in \mathrm{GL}_n(O_V)$. Then (2) follows from the proof of Lemma C.1. To prove (1), we take $f_v \in S(X_r(k_v))$ such that the support of f_v is contained in $(0,I_r)\cdot \mathrm{GL}_n(O_v)$ and

$$f_v((0,I_r)\kappa) := \mu_{1,v}(\det \kappa)^{-1}\psi_v(\kappa).$$

Then for each $\kappa \in \mathrm{GL}_n(O_v)$,

$$\begin{aligned} F_v(\kappa) &= \mu_{1,v}(\kappa)\int_{\mathrm{GL}_r(k_v)} f_v(h_v^{-1}(0,I_r)\kappa)\mu_v^{-1}(\det h_v)d^\times h_v \\ &= \mu_{1,v}(\kappa)\int_{\mathrm{GL}_r(O_v)} \mu_{1,v}(\det h_v^{-1}\det\kappa)^{-1}\Psi_v\left(\begin{pmatrix} I_{n-r} & 0 \\ 0 & h_v^{-1}\end{pmatrix}\kappa\right)\mu_v^{-1}(\det h_v)d^\times h_v \\ &= \Psi_v(\kappa). \end{aligned}$$

Therefore the proof is complete. ∎

Let μ_1,μ_2 be two Hecke characters on $k^\times\backslash\mathbb{A}_k^\times$. For any $g \in \mathrm{GL}_n(\mathbb{A}_k)$ and Schwartz function $f \in S(X_r(\mathbb{A}_k))$, we set

$$F(g) = F(g,\mu_1,\mu_2,f) := \mu_1(\det g)|\det g|_{\mathbb{A}_k}^{r/2}\int_{\mathrm{GL}_r(\mathbb{A}_k)} f(h^{-1}(0,I_r)g)\mu^{-1}(\det h)d^\times h.$$

Here $\mu = \mu_1\mu_2^{-1}|\cdot|_{\mathbb{A}_k}^{n/2}$ and the Haar measure $d^\times h = \prod_v d^\times h_v$. In particular, the volume of $\mathrm{GL}_r(O_{\mathbb{A}_k})$ is 1. By Lemma C.1, this integral is absolutely convergent if $|\mu_1\mu_2^{-1}| = |\cdot|_{\mathbb{A}_k}^{\sigma}$ where $\sigma > r - n/2$. The *Maass-Jacquet-Shalika Eisenstein series associated to* f, μ_1, μ_2 is defined by (cf. [7] and [13])

$$E(g,\mu_1,\mu_2,f) = \sum_{\gamma \in Q_r(k)\backslash \mathrm{GL}_n(k)} F(\gamma g).$$

This series converges absolutely when $|\mu_1\mu_2^{-1}|_{\mathbb{A}_k} = |\cdot|_{\mathbb{A}_k}^{\sigma}$ with $\sigma > n/2$ (cf. [14] II.1.5).

Theorem C.3. *Suppose* $\mu_1 \cdot \mu_2^{-1} = |\cdot|_{\mathbb{A}_k}^{\sigma}$. *Then*

(1) (*Continuation*) $E(g,\mu_1,\mu_2,f)$ *can be extended to a meromorphic function in* σ (*in fact, a rational function in* $q^{-\sigma}$), *and every possible pole can only be a simple pole. Let* $P(\sigma) := P^+(\sigma)\cdot P^-(\sigma)$, *where*

$$P^{\pm}(\sigma) = \prod_{i=0}^{r-1}(1 - q^{-\sigma \pm (\frac{n}{2}-i)}).$$

Then $P(\sigma)\cdot E(g,\mu_1,\mu_2,f)$ *is entire.*

(2) (*Functional equation*) *For each* $f \in S(X_r(\mathbb{A}_k))$*, we have*

$$E(g, \mu_1, \mu_2, f) = E({}^t g^{-1}, \mu_1^{-1}, \mu_2^{-1}, f^\wedge)$$

where $f^\wedge$ *is the Fourier transform of* f*:*

$$f^\wedge(x) := \int_{X_r(\mathbb{A}_k)} f(y)\psi(-\operatorname{Tr}(x^t y))dy.$$

The Haar measure dy *is chosen to be self-dual, i.e.* $f^{\wedge\wedge}(x) = f(-x)$*.*

(3) *Suppose that there exists a place* v *of* k *such that the support of the restriction of* f *on* $X_r(k_v)$ *is contained in the set of elements with rank* r *in* $X_r(k_v)$*. Then*

$$P^+(\sigma) \cdot E(g, \mu_1, \mu_2, f) \text{ is entire.}$$

Proof. Replacing f to the Schwartz function $f(\cdot g)$, we can assume $g = 1$ and set $E_r(\sigma, f) := E(1, \mu_1, \mu_2, f)$. Let

$$X_r^{(i)}(k) := \{x \in X_r(k) : \operatorname{rank}(x) = i\}, \quad 0 \leq i \leq r.$$

For $h \in \mathrm{GL}_r(\mathbb{A}_k)$, $f \in S(X_r(\mathbb{A}_k))$ we define

$$\theta_r^{(i)}(h; f) := \sum_{x \in X_r^{(i)}(k)} f(h^{-1}x)$$

and

$$\theta_r(h; f) := \sum_{i=0}^{r} \theta_r^{(i)}(h; f) = \sum_{x \in X_r(k)} f(h^{-1}x).$$

Then we have

$$E_r(\sigma, f) = \int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)} \theta_r^{(r)}(h; f)\mu^{-1}(h)d^\times h.$$

For $* \in \{>,\ <,\ \geq,\ \leq,\ =\}$, let

$$\mathrm{GL}_r(\mathbb{A}_k)^{*1} := \{h \in \mathrm{GL}_r(\mathbb{A}_k) : |\det h|_{\mathbb{A}_k} * 1\}$$

Note that

$$\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{\leq 1}} \theta_r^{(r)}(h; f)\mu^{-1}(h)d^\times h$$

always converges absolutely for every σ. On the other hand, by Poisson summation formula we get

$$\theta_r(h; f) = |\det h|_{\mathbb{A}_k}^n \theta_r({}^t h^{-1}; f^\wedge).$$

Hence

$$\theta_r^{(r)}(h; f) = \sum_{i=0}^{r-1} \Big(|\det h|_{\mathbb{A}_k}^n \theta_r^{(i)}({}^t h^{-1}; f^\wedge) - \theta_r^{(i)}(h; f) \Big).$$

Let $Q_r^{(i)} = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \in \mathrm{GL}_r \;\middle|\; a_1 \in \mathrm{GL}_i,\ a_2 \in \mathrm{GL}_{r-i} \right\}$. It is clear that

$$\theta_r^{(i)}(h;f) = \sum_{h_{(i)} \in Q_r^{(i)}(k) \backslash \mathrm{GL}_r(k)} \left(\sum_{x_i \in X_i^{(i)}(k)} f\left(h^{-1} h_{(i)}^{-1} \begin{pmatrix} x_i \\ 0 \end{pmatrix} \right) \right)$$

and

$$\theta_r^{(i)}({}^t h^{-1}; f^\wedge) = \sum_{h_{(i)} \in Q_r^{(i)}(k) \backslash \mathrm{GL}_r(k)} \left(\sum_{x_i \in X_i^{(i)}(k)} f\left({}^t h {}^t h_{(i)} \begin{pmatrix} 0 \\ x_i \end{pmatrix} \right) \right).$$

For each $\kappa \in \mathrm{GL}_r(O_{\mathbb{A}_k})$, let $f_\kappa(x) := f(\kappa^{-1}x)$. We then observe that

$$\begin{aligned} &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{>1}} \theta_r^{(r)}(h;f)\mu^{-1}(h) d^\times h \\ = &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{>1}} \theta_r^{(r)}({}^t h^{-1}; f^\wedge)\mu^{-1}({}^t h^{-1}) d^\times h \\ &+ \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} \sum_{i=0}^{r-1} \mathrm{Vol}\big(\mathrm{Mat}_{i\times(r-i)}(k)\backslash \mathrm{Mat}_{i\times(r-i)}(\mathbb{A}_k)\big) \cdot \mathrm{Vol}\big(\mathrm{GL}_i(k)\backslash \mathrm{GL}_i(\mathbb{A}_k)^{=1}\big) \\ &\quad \cdot \Bigg(E_i(i-\frac{n}{2}, (f_\kappa)^\wedge_{(i,2)}) \cdot \sum_{\ell=1}^{\infty} q^{(-\sigma+\frac{n}{2}-i)\ell} \\ &\qquad - E_i(i-\frac{n}{2}, (f_\kappa)_{(i,1)}) \cdot \sum_{\ell=1}^{\infty} q^{(-\sigma-\frac{n}{2}+i)\ell} \Bigg) d\kappa. \end{aligned}$$

Here for every $f \in S(X_r(\mathbb{A}_k))$, $f_{(i,1)}$ and $f_{(i,2)} \in S(X_i(\mathbb{A}_k))$ are defined by

$$f_{(i,1)}(x_i) := f\begin{pmatrix} x_i \\ 0 \end{pmatrix}, \quad f_{(i,2)}(x_i) := f\begin{pmatrix} 0 \\ x_i \end{pmatrix}.$$

Therefore when $\sigma > n/2$,

$$\begin{aligned} &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{>1}} \theta_r^{(r)}(h;f)\mu^{-1}(h) d^\times h \\ = &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{<1}} \theta_r^{(r)}(h; f^\wedge)\mu^{-1}(h) d^\times h \\ &+ \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} \sum_{i=0}^{r-1} \mathrm{Vol}\big(\mathrm{Mat}_{i\times(r-i)}(k)\backslash \mathrm{Mat}_{i\times(r-i)}(\mathbb{A}_k)\big) \cdot \mathrm{Vol}\big(\mathrm{GL}_i(k)\backslash \mathrm{GL}_i(\mathbb{A}_k)^{=1}\big) \\ &\cdot \left(E_i(i-\frac{n}{2}, (f_\kappa)^\wedge_{(i,2)}) \cdot \frac{q^{-\sigma+\frac{n}{2}-i}}{1-q^{-\sigma+\frac{n}{2}-i}} + E_i(i-\frac{n}{2}, (f_\kappa)_{(i,1)}) \cdot \frac{1}{1-q^{\sigma+\frac{n}{2}-i}} \right) d\kappa. \end{aligned}$$

This gives the meromorphic continuation of $E(\sigma, f)$ (by induction on r) and (1) holds. In particular, suppose there is a place v of k such that the support of the restriction of f on $X_r(k_v)$ is contained in the set of elements with rank r in $X_r(k_v)$. Then

$$\theta_r(1,h;f) = \theta_r^{(r)}(1,h;f)$$

and

$$E_i(i - \frac{n}{2}, (f_\kappa)_{i,1}) = 0, \quad \forall 0 \le i \le r-1.$$

This completes the proof of (3).

Note that

$$\begin{aligned}
&\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{=1}} \theta_r^{(r)}(h; f) d^\times h \\
= &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{=1}} \theta_r^{(r)}(h; f^\wedge) d^\times h \\
&+ \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} \sum_{i=0}^{r-1} \mathrm{Vol}\big(\mathrm{Mat}_{i\times(r-i)}(k)\backslash \mathrm{Mat}_{i\times(r-i)}(\mathbb{A}_k)\big) \cdot \mathrm{Vol}\big(\mathrm{GL}_i(k)\backslash \mathrm{GL}_i(\mathbb{A}_k)^{=1}\big) \\
&\cdot \left(E_i(i - \frac{n}{2}, (f_\kappa)^\wedge_{(i,2)}) - E_i(i - \frac{n}{2}, (f_\kappa)_{(i,1)}) \right) d\kappa.
\end{aligned}$$

Moreover, from

$$\begin{pmatrix} x_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & I_i \\ I_{r-i} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_i \end{pmatrix}$$

we get

$$\int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} E_i(i - \frac{n}{2}, (f_\kappa)_{(i,1)}) d\kappa = \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} E_i(i - \frac{n}{2}, (f_\kappa)_{(i,2)}) d\kappa.$$

Thus by induction on r we have

$$\begin{aligned}
&E(\sigma, f) \\
= &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{<1}} \theta_r^{(r)}(h; f) + \theta_r^{(r)}(h; f^\wedge)\mu^{-1}(h) d^\times h \\
&+ \int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{=1}} \theta_r^{(r)}(h; f) d^\times h \\
&+ \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} \sum_{i=0}^{r-1} \mathrm{Vol}\big(\mathrm{Mat}_{i\times(r-i)}(k)\backslash \mathrm{Mat}_{i\times(r-i)}(\mathbb{A}_k)\big) \cdot \mathrm{Vol}\big(\mathrm{GL}_i(k)\backslash \mathrm{GL}_i(\mathbb{A}_k)^{=1}\big) \\
&\cdot \left(E_i(i - \frac{n}{2}, (f_\kappa)^\wedge_{(i,2)}) \cdot \frac{q^{-\sigma+\frac{n}{2}-i}}{1-q^{-\sigma+\frac{n}{2}-i}} + E_i(i - \frac{n}{2}, (f_\kappa)_{(i,1)}) \cdot \frac{1}{1-q^{\sigma+\frac{n}{2}-i}} \right) d\kappa. \\
= &\int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{<1}} \theta_r^{(r)}(h; f) + \theta_r^{(r)}(h; f^\wedge)\mu^{-1}(h) d^\times h \\
&+ \int_{\mathrm{GL}_r(k)\backslash \mathrm{GL}_r(\mathbb{A}_k)^{=1}} \theta_r^{(r)}(h; f^\wedge) d^\times h \\
&+ \int_{\mathrm{GL}_r(O_{\mathbb{A}_k})} \sum_{i=0}^{r-1} \mathrm{Vol}\big(\mathrm{Mat}_{i\times(r-i)}(k)\backslash \mathrm{Mat}_{i\times(r-i)}(\mathbb{A}_k)\big) \cdot \mathrm{Vol}\big(\mathrm{GL}_i(k)\backslash \mathrm{GL}_i(\mathbb{A}_k)^{=1}\big) \\
&\cdot \left(E_i(i - \frac{n}{2}, (f_\kappa)^\wedge_{(i,1)}) \cdot \frac{1}{1-q^{-\sigma+\frac{n}{2}-i}} + E_i(i - \frac{n}{2}, (f_\kappa)_{(i,2)}) \cdot \frac{q^{\sigma+\frac{n}{2}-i}}{1-q^{\sigma+\frac{n}{2}-i}} \right) d\kappa. \\
= &E(-\sigma, f^\wedge).
\end{aligned}$$

Therefore the proof of (2) is complete. □

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