Asian J. Math.
Vol. 19, No. 3, pp. 377–390, July 2015

001

TOPOLOGY OF GENERIC LINE ARRANGEMENTS*

ARNAUD BODIN†

Abstract. Our aim is to generalize the result that two generic complex line arrangements are equivalent. In fact for a line arrangement $\mathcal{A}$ we associate a defining polynomial $f = \prod_i (a_i x + b_i y + c_i)$, so that $\mathcal{A} = (f = 0)$. We prove that the defining polynomials of two generic line arrangements are, up to a small deformation, topologically equivalent. In higher dimension the related result is that within a family of equivalent hyperplane arrangements the defining polynomials are topologically equivalent.

Key words. Line arrangement, hyperplane arrangement, polynomial in several variables.

AMS subject classifications. 32S22 (14N20, 32S15, 57M25).

Introduction. We start with a basic result that says that generic complex line arrangements are topologically unique. *Generic* means that there is no triple point. For a generic line arrangements its *combinatorics* is $\{p_1, \ldots, p_\ell\}$ where p_j denotes the number of parallel lines in a given direction indexed by j.

We start by recalling a special case of result of Jiang and Yau [7].

Theorem A. *Any two generic complex line arrangements having the same combinatorics are topologically equivalent.*

This result is known to be false for real line arrangements, even in the case where there is no parallel lines.

One of the key-point is to construct a family of generic line arrangements that links the two arrangements, see Jiang and Yau [7]. If we already start from a family $\mathcal{A}_t$ of hyperplane arrangements (or even of subspace arrangements) then the constancy of the combinatorics structure implies the topological equivalence, see Randell [12], Jiang and Yau [7].

Theorem B. *Let $\mathcal{A}_t$ be a smooth family of hyperplane arrangements. If the lattice of $\mathcal{A}_t$ remains the same for all $t \in [0,1]$ then the arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ are topologically equivalent.*

We recall that two arrangements $\{H_1, \ldots, H_d\}$ and $\{H'_1, \ldots, H'_d\}$ have the *same lattice* if for any $I \subset \{1, \ldots, d\}$, $\dim \bigcap_{i \in I} H_i = \dim \bigcap_{i \in I} H'_i$.

Our goal is to study not only the arrangement but also the global behavior of the defining polynomial of the arrangement. To an arrangement composed with lines of equation $a_i x + b_i y + c_i = 0$ we associated a *defining polynomial* $f(x,y) = \prod_{i=1}^d (a_i x + b_i y + c_i)$. We are interested in the arrangement $\mathcal{A} = f^{-1}(0)$, i.e. the union of lines given by the equation $(f = 0)$ but also in the other curve levels $f^{-1}(c)$, i.e. the algebraic curves of equation $(f = c)$. Two polynomials having all their curve levels homeomorphic are called *topologically equivalent* (see definition 8). In particular some algebraic curves $f^{-1}(c)$ may have singularities. We say that f is *Morse outside the arrangement* if the other singularities are ordinary double points, with distinct critical

*Received September 4, 2013; accepted for publication March 12, 2014.

†Laboratoire Paul Painlevé, Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France (Arnaud.Bodin@math.univ-lille1.fr).

values (see definition 3). A family $(f_t)_{t\in[0,1]}$ of polynomials, whose coefficients depend analytically on t, is called a *deformation* of the polynomial f_0.

THEOREM A'. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two generic line arrangements with the same combinatorics and f_0 and f_1 their defining polynomials.*

- *Up to a small deformation of $\mathcal{A}_0$ and $\mathcal{A}_1$ the polynomials f_0 and f_1 are topologically equivalent.*
- *If f_0 and f_1 are Morse outside the arrangement, then the polynomials f_0 and f_1 are topologically equivalent.*

One step of the proof is to connect f_0 and f_1 by a family (f_t) of polynomials defining generic arrangements. Then one can apply a global version of Lê-Ramanujam theorem to prove the topological equivalence.

What happens to hyperplane arrangements in higher dimension or to non-generic arrangements? We have a partial answer to this questions:

THEOREM B'. *If $(f_t)_{t\in[0,1]}$ is a smooth family of topologically equivalent hyperplane arrangements in $\mathbb{C}^n$ (with $n \neq 3$) that are Morse outside the arrangement, then the polynomials f_0 and f_1 are topologically equivalent.*

This result is more in the spirit of Randell's theorem B, since we start from a family. Note also that in the hypothesis we can substitute "topologically equivalent arrangements" by "the same lattice".

Acknowledgments. I thank Michael Falk for discussions and the reference to Randell's work and the referee for valuable comments. This work is supported by the French grant ANR-12-JS01-002-01 "SUSI".

1. Definitions.

DEFINITION 1 (Topologically equivalent arrangements). An *arrangement* in $\mathbb{C}^n$ is a finite collection of hyperplanes $\mathcal{A} = \{H_i\}$. Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are *topologically equivalent* if the pair $(\mathbb{C}^n, \mathcal{A})$ is homeomorphic to the pair $(\mathbb{C}^n, \mathcal{A}')$, in other words there exists a homeomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\phi(\mathcal{A}) = \mathcal{A}'$.

DEFINITION 2 (Defining polynomial). To a hyperplane H_i we associate an affine equation $\ell_i(x) = 0$ for some $\ell_i(x) = a_{i,1}x_1 + \cdots + a_{i,n}x_n + c_i$. The *defining polynomial* of $\mathcal{A} = \{H_i\}_{i=1,\ldots,d}$ is $f = \prod_{i=1}^d \ell_i$.

Our point of view is not only to consider the arrangement $(f = 0)$ but also all other hypersurfaces defined by equations of type $(f = c)$.

DEFINITION 3 (Morse outside the arrangement). We will say that the arrangement (or f) is *Morse outside the arrangement* if:

- its defining polynomial f has only a finite number of singular points in $\mathbb{C}^n \setminus \mathcal{A}$, denote them by $\{q_1, \ldots, q_k\}$;
- this singular points $\{q_1, \ldots, q_k\}$ are ordinary double points (i.e. the local Milnor number at each q_i is 1; equivalently the Hessian matrix at q_i is invertible);
- the critical values are distinct (i.e. $f(q_i) \neq f(q_j)$ for $i \neq j$).

This hypothesis does not concern the points of the arrangement $\mathcal{A}$ identified with $(f = 0)$.

In a first part we will focus on complex line arrangements, that is to say configurations of lines in $\mathbb{C}^2$.

Definition 4 (Generic line arrangement). A line arrangement is *generic* if the following conditions hold:

- there are no triple points,
- not all the lines are parallel.

The first condition means that three lines cannot have a common point of intersection. To a generic line arrangement $\prod_{i=1}^{d}(a_i x + b_i y + c_i)$ we associate a set of integers. Let the set of *directions* be $\{(a_i : b_i)\}_{i=1,\dots,d} \subset \mathbb{P}^1$. To each direction δ_j, let p_j be the number of lines parallel to δ_j. The *combinatorics* of a generic line arrangement is $(p_1, \dots, p_\ell)$. This list is unique up to permutation and we have $\sum_{i=1}^{\ell} p_i = d$.

For example if no lines are parallel then $\ell = d$ and the combinatorics is $(1, 1, \dots, 1)$. A combinatorics equals to $(3, 2, 1, 1)$ denotes a generic line arrangement of 7 lines (see figure 1 below), three of them being parallel (in one direction) another two being parallel (in another direction) and the two last ones not being parallel to any other lines.

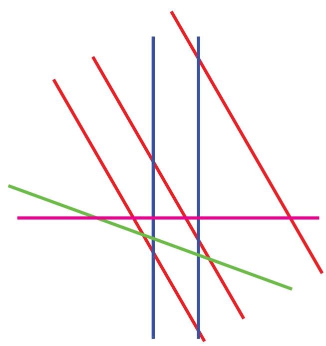

Fig. 1. *A generic line arrangement with combinatorics* $(3, 2, 1, 1)$

2. Short review on topology of polynomials. We will recall the definition of topological equivalence and other facts about the topology of polynomials in this paragraph. Let f be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$.

Definition 5 (Bifurcation set and generic fibre). There exists a finite *bifurcation set* $\mathcal{B} \subset \mathbb{C}$ such that

$$f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$$

is a locally trivial fibration. In particular for $c_{\text{gen}} \notin \mathcal{B}$, $f^{-1}(c_{\text{gen}})$ is a *generic fibre*.

Definition 6 (Affine critical value). The point $(x_1, \dots, x_n)$ is a *critical point* of f, if $\left(\frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n)\right) = (0, \dots, 0)$. If $(x_1, \dots, x_n)$ is a critical point then $f(x_1, \dots, x_n)$ is an *affine critical value*. We denote by $\mathcal{B}_{\mathit{aff}}$ the set of all *affine critical values*.

If c is an affine critical value, the fibre $f^{-1}(c)$ is singular and then it is not a generic fibre.

We denote $D^2_\delta(c) = \big\{ z \in \mathbb{C} \mid |z-c| \leqslant \delta \big\}$ the disk of radius δ centered at c and $B^{2n}_R(0) = \big\{ (x_1,\ldots,x_n) \in \mathbb{C} \mid |x_1|^2 + \cdots + |x_n|^2 \leqslant R^2 \big\}$ the ball of radius R centered at the origin.

Definition 7 (Irregular value at infinity). For a value $c \in \mathbb{C}$, if there exists $0 < \delta \ll 1$ and $R \gg 1$ such that $f : f^{-1}\big(D^2_\delta(c)\big) \setminus B^{2n}_R(0) \longrightarrow D^2_\delta(c)$ is a locally trivial fibration we say that c is a *regular value at infinity*. The set of *irregular values at infinity* is denoted by $\mathcal{B}_{\mathit{inf}}$. We have that

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_{\mathit{inf}}.$$

Definition 8 (Topologically equivalent polynomial). Two polynomials $f, g \in \mathbb{C}[x_1,\ldots,x_n]$ are *topologically equivalent* if there exist homeomorphisms Φ and Ψ with a commutative diagram:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\Phi"] \arrow[d, "f"'] & \mathbb{C}^n \arrow[d, "g"] \\ \mathbb{C} \arrow[r, "\Psi"'] & \mathbb{C} \end{tikzcd}$$

In particular, if $\Psi(0)=0$, it implies that the pair $(\mathbb{C}^n, f^{-1}(0))$ is homeomorphic to $(\mathbb{C}^n, g^{-1}(0))$.

The distinction between the topological equivalence of arrangements and of polynomials is important. If two polynomials are topologically equivalent then the arrangements are topologically equivalent, but also any fibre $f_0^{-1}(c)$ is equivalent to some fibre $f_1^{-1}(c')$.

The reciprocal is false: two arrangements can be equivalent but not their defining polynomials. Let $f_t = xy(x+y-4)(x-ty)$. For non-zero t, t' the arrangements of f_t and $f_{t'}$ are equivalent. Let $j, \bar{j}$ the roots of z^2+z+1. Then for $t, t' \notin \{-1, 0, j, \bar{j}\}$, f_t is Morse outside the arrangement: f has two critical values (outside 0), that correspond to double points. For $t = j$ or $t = \bar{j}$, f_t has only one critical value outside 0 which is $3(t-1)$, the corresponding fibre $f_t^{-1}(3(t-1))$ has two double points. Then for $t = j$ (or $t = \bar{j}$) and for $t' \notin \{-1, 0, j, \bar{j}\}$ the polynomials f_t and $f_{t'}$ are not topologically equivalent but they have equivalent arrangements.

A simple criterion to construct topologically equivalent polynomials is to consider a family (f_t) with some numerical invariants not depending on the parameter t. We then apply the following global Lê-Ramanujam theorem ([1], [3]):

Theorem 9. *Let $n \neq 3$. If $\big(f_t(x_1,\ldots,x_n)\big)_{t\in[0,1]}$ is a continuous family of polynomials with isolated singularities such that the degree* $\deg f_t$*, the number of critical values (affine and at infinity) $\#\mathcal{B}(t)$ and the Euler characteristic of a generic fibre $\chi(f^{-1}(c_{\mathrm{gen}}))$ remain constant for all* $t \in [0,1]$, *then f_0 and f_1 are topologically equivalent.*

This result will be our main tool to prove theorem 10 and theorem 15.

3. Generic complex line arrangements are unique.

THEOREM 10. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two generic complex line arrangements with the same combinatorics. Then the arrangements are equivalent, that is to say the pairs $(\mathbb{C}^2, \mathcal{A}_0)$ and $(\mathbb{C}^2, \mathcal{A}_1)$ are homeomorphic.*

Suppose moreover that their defining polynomials f_0 and f_1 are Morse outside the arrangement. Then the polynomials f_0 and f_1 are topologically equivalent.

REMARK. Theorem 10 is false in the realm of real line arrangements. For example in the case of 6 lines in the real plane there exist four different generic configurations that are not equivalent. See [6, appendix].

The following lemma connects two generic line arrangements in a family. This is a well-known result, see Jiang and Yau [7], and [11] for related recent results.

LEMMA 11. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two generic complex line arrangements having the same combinatorics. Then $\mathcal{A}_0$ and $\mathcal{A}_1$ can be linked by a continuous family of generic complex line arrangements $\{\mathcal{A}_t\}_{t\in[0,1]}$ having all the same combinatorics.*

"Continuous family" means that the equation of each line of $\mathcal{A}_t$, $a_i(t)x + b_i(t)y + c_i(t)$ has coefficients depending continuously on $t \in [0, 1]$.

LEMMA 12. *The defining polynomial of a complex line arrangement (not all lines being parallel) has no irregular value at infinity.*

It holds in any dimension as we will prove latter in corollary 18. But in the two dimensional case we can provide simpler arguments.

Proof. Let $f(x, y) = \prod_{i=1}^{d}(a_i x + b_i y + c_i)$ be the defining polynomial of a line arrangement, each direction corresponds to a point at infinity of $(f = 0)$. We homogenize $f(x, y) - c$ in $F(x, y, z) - cz^d = \prod_{i=1}^{d}(a_i x + b_i y + c_i z) - cz^d$ and localize at a point at infinity. For instance if $\delta < d$ lines are parallel to the y-axis, they have intersection at infinity at $P = (0 : 1 : 0)$. We may suppose that these lines have equations $x + c_i = 0$, $i = 1, \dots, \delta$, the c_i being pairwise distinct. Then $F(x, y, z) - cz^d = \prod_{i=1}^{\delta}(x + c_i z) \times \prod_{i=\delta+1}^{d}(a_i x + b_i y + c_i z) - cz^d$, with $b_i \neq 0$ for $i = \delta + 1, \dots, d$. The localization of F at P is $f_P(x, z) = F(x, 1, z) = \prod_{i=1}^{\delta}(x+c_i z) \times \prod_{i=\delta+1}^{d}(b_i + a_i x + c_i z)$. As $b_i \neq 0$, $u = \prod_{i=\delta+1}^{d}(b_i + a_i x + c_i z)$ is a unit (in the ring of convergent power series $\mathbb{C}\{x, z\}$). So that $f_P(x, z) - cz^d = u \prod_{i=1}^{\delta}(x + c_i z) - cz^d$.

It remains to prove that the topology of the germ $f_P - cz^d$ is independent of c. We will use the following characterization of irregular value at infinity: "c is an irregular value at infinity for f (at P) if the local Milnor number $\mu_P(f_P - cz^d)$ is greater than the local Milnor number $\mu_P(f_P - c_{\text{gen}} z^d)$, for some generic value c_{gen}". A good reference for different characterizations of irregular values at infinity is [5].

A first method to prove that there is no jump of Milnor number is to see that its Newton polygon is independent of c (because $\delta < d$) and is Newton non-degenerate (because all the c_i are distinct), then by Kouchnirenko's theorem [8] the local Milnor number $\mu_P(f_P - cz^d)$ of the germ is constant (with respect to c).

Another method is to compute the resolution of f, by blow-ups of $f_P - cz^d$ at P, and see that no critical value occurs (see [5]). ☐

LEMMA 13. *Let f be the defining polynomial of a generic line arrangement having combinatorics $(p_1, \dots, p_\ell)$. Then the Euler characteristic of the generic fibre*

$\chi(f^{-1}(c_{\mathrm{gen}}))$ is

$$\chi(f^{-1}(c_{\mathrm{gen}})) = -d(d-2) + \sum_{j=1}^{\ell} p_j(p_j-1)$$

and

$$\chi(f^{-1}(0)) = 1 - \frac{(d-1)(d-2)}{2} + \sum_{j=1}^{\ell} \frac{p_j(p_j-1)}{2}.$$

Notice that in proposition 21 below we will be able to recover $\chi(f^{-1}(c_{\mathrm{gen}}))$ by proving that the generic fibre can be obtained by d disks, and each pair of disks is connected by two bands.

Proof. As $(f=0)$ is a generic line arrangement, $(f=0)$ has the homotopy type of a bouquet of $\frac{(d-1)(d-2)}{2} - \sum_{j=1}^{\ell} \frac{p_j(p_j-1)}{2}$ circles: (i) $\frac{(d-1)(d-2)}{2}$ is for the homotopy type of generic line arrangement such that no pair of lines are parallel; (ii) for configurations with parallel lines the terms $\frac{p_j(p_j-1)}{2}$ correct the formulas to take into account that some cycles go to infinity. It yields the formula for $\chi(f^{-1}(0))$.

Let $\mu(0)$ be the sum of Milnor numbers at singular points of $f^{-1}(0)$. As each singularity in the arrangement is an ordinary double points, $\mu(0)$ equals the number of those double points. Hence $\mu(0) = \frac{d(d-1)}{2} - \sum_{j=1}^{\ell} \frac{p_j(p_j-1)}{2}$. Now, as there is no irregular value at infinity we have the formula:

$$\chi(f^{-1}(0)) - \chi(f^{-1}(c_{\mathrm{gen}})) = \mu(0)$$

that expresses that the number of vanishing cycles equals the number of singular points (counted with multiplicity); it yields the result for $\chi(f^{-1}(c_{\mathrm{gen}}))$. ◻

Lemma 14. *For the defining polynomial f of a hyperplane arrangement that is Morse outside the arrangement we have:*

$$\#\mathcal{B} = 2 - \chi(f^{-1}(0)).$$

So that by lemma 13 we have an explicit formula for $\#\mathcal{B}$.

Proof. We start by recalling the formula:

$$1 - \chi(f^{-1}(c_{\mathrm{gen}})) = \sum_{c\in\mathcal{B}} \big(\chi(f^{-1}(c)) - \chi(f^{-1}(c_{\mathrm{gen}}))\big) \tag{1}$$

that reformulates that the global Milnor number is the sum of the local Milnor numbers. A proof can be formulate as follows: by additivity of the Euler characteristic we have

$$1 = \chi(\mathbb{C}^n) = \chi(f^{-1}(\mathbb{C}\setminus\mathcal{B})) + \sum_{c\in\mathcal{B}} \chi(f^{-1}(c)) \tag{2}$$

but because $f : f^{-1}(\mathbb{C}\setminus\mathcal{B}) \to \mathbb{C}\setminus\mathcal{B}$ is a locally trivial fibration, with fibre $f^{-1}(c_{\mathrm{gen}})$, we have

$$\chi(f^{-1}(\mathbb{C}\setminus\mathcal{B})) = (1-\#\mathcal{B})\chi(f^{-1}(c_{\mathrm{gen}})). \tag{3}$$

Formulas (2) and (3) prove (1).

Formula (1) can now be decomposed in

$$1-\chi(f^{-1}(c_{\text{gen}}))=\chi(f^{-1}(0))-\chi(f^{-1}(c_{\text{gen}}))+\sum_{c\in\mathcal{B}\setminus\{0\}}\left(\chi(f^{-1}(c))-\chi(f^{-1}(c_{\text{gen}}))\right).$$

By lemma 12 there is no irregular value at infinity (in fact we only need $\mathcal{B}_{inf}\setminus\{0\}=\varnothing$). As $\mathcal{B}\setminus\{0\}$ is composed only of affine critical values, that moreover are Morse critical values, we get $\chi(f^{-1}(c))-\chi(f^{-1}(c_{\text{gen}}))=1$ for all $c\in\mathcal{B}\setminus\{0\}$; that implies

$$\#\mathcal{B}=2-\chi(f^{-1}(0))$$

□

Proof of theorem 10. The first step is to link the defining polynomials f_0 and f_1 by a family $(f_t)_{t\in[0,1]}$ of polynomial such that the corresponding arrangements $\mathcal{A}_t$ are generic and have the same combinatorics, this is possible by lemma 11.

To apply the global μ-constant theorem (theorem 9) to our family (f_t) we need also to choose (f_t) such that $\deg f_t$, $\chi(f_t^{-1}(c_{\text{gen}}))$ and $\#\mathcal{B}(t)$ are independent of t. This is clear for the degree, that equals the number of lines and in lemma 13 we already proved that $\chi(f_t^{-1}(c_{\text{gen}}))$ depends only on the combinatorics.

For $\#\mathcal{B}(t)$ the situation is more complicated. It is not always possible to find (f_t) such that $\#\mathcal{B}(t)$ remains constant (see the example of section 2). However we will prove that in the set of generic arrangements of a given combinatorics there exists a dense subset such that for these arrangements $\#\mathcal{B}$ is constant.

We fix a combinatorics $(p_1,\ldots,p_\ell)$ and consider the set $\mathcal{C}$ of all polynomials defining a generic d-line arrangement having combinatorics equal to $(p_1,\ldots,p_\ell)$. First of all $\mathcal{C}$ is a connected set (see lemma 11) and is a constructible set of the set of polynomials defining a d-line arrangement. Moreover $\mathcal{C}$ is a smooth manifold because each $\mathcal{A}\in\mathcal{C}$ has a neighborhood diffeomorphic to an open ball of $\mathbb{C}^{3\ell+(p_1-1)+\cdots+(p_\ell-1)}$ (the configurations of ℓ non-parallel lines form an open subset of $\mathbb{C}^{3\ell}$, adding a parallel line to this configuration add one dimension).

Now $\mathcal{C}$ is stratified by a finite number of constructible subsets $\mathcal{C}_i$ such that for each $f\in\mathcal{C}_i$, $\#\mathcal{B}(f)=i$: there are only a finite number of critical values, by lemma 12 there is no irregular value at infinity and affine critical values are given by the vanishing of a resultant. Hence one of this constructible subset, say $\mathcal{C}_{i_0}$, is dense (and contains a non-empty Zariski open set) in $\mathcal{C}$. Moreover by the smoothness of $\mathcal{C}$ and its connectedness (lemma 11), $\mathcal{C}_{i_0}$ is also a connected set.

We now can prove theorems A, A' and theorem 10.

By density of $\mathcal{C}_{i_0}$ in $\mathcal{C}$ any polynomial $f\in\mathcal{C}$ can be approximated (by a small perturbation of some coefficients of the lines) by a polynomial $\tilde{f}\in\mathcal{C}_{i_0}$. The arrangements defined by $(f=0)$ and $(\tilde{f}=0)$ are topologically equivalent.

Hence for $f_0,f_1\in\mathcal{C}$ there exist approximations $\tilde{f}_0,\tilde{f}_1$ belonging to $\mathcal{C}_{i_0}$ such that the arrangements defined by f_0 and $\tilde{f}_0$ (resp. f_1 and $\tilde{f}_1$) are topologically equivalent. But $\mathcal{C}_{i_0}$ is a connected set, so we can link $\tilde{f}_0$ to $\tilde{f}_1$ by a continuous family $(\tilde{f}_t)_{t\in[0,1]}$ of generic arrangements having the same combinatorics and belonging to $\mathcal{C}_{i_0}$.

For $(\tilde{f}_t)$ we have the constancy of $\deg\tilde{f}_t$, $\chi(\tilde{f}_t^{-1}(c_{\text{gen}}))$ and $\#\mathcal{B}(t)$ so that we can now apply theorem 9, to conclude that $\tilde{f}_0$ and $\tilde{f}_1$ are topologically equivalent. In particular the arrangement $\tilde{f}_0^{-1}(0)$ is topologically equivalent to $\tilde{f}_1^{-1}(0)$ so that the arrangement $f_0^{-1}(0)$ is topologically equivalent to $f_1^{-1}(0)$.

On the other hand if we start with f_0 (and f_1) satisfying the additional hypothesis of being Morse outside the arrangement, then we know that the corresponding constructible set $\mathcal{C}_{i_0}$ is exactly the set of functions Morse outside the arrangement. To prove that, we first notice that by hypothesis the set of functions Morse outside the arrangement is non-empty (because it contains f_0 and f_1). Being Morse is an open condition in the set of polynomials of degree d endowed with the natural topology, because a singular point is an ordinary double point if and only the determinant of the Hessian matrix at this point is non-zero. So to be dense in $\mathcal{C}$, $\mathcal{C}_{i_0}$ must contain one function, say g, that is Morse outside the arrangement. Secondly, $\mathcal{C}_{i_0}$ is defined by the condition $\#\mathcal{B} = i_0$. But lemma 14 applied to g implies $\#\mathcal{B} = 2 - \chi(g^{-1}(0))$ as $g \in \mathcal{C}_{i_0}$. So that $\mathcal{C}_{i_0}$ is defined by the condition $\#\mathcal{B} = 2 - \chi(g^{-1}(0))$. Now, again by lemma 14, any function in $\mathcal{C}$ that is Morse outside the arrangement belongs to $\mathcal{C}_{i_0}$.

Then we can directly build a family (f_t) of functions Morse outside the arrangement linking f_0 to f_1. It is then clear that $\deg f_t$, $\chi(f_t^{-1}(c_{\text{gen}}))$ and $\#\mathcal{B}(t)$ are constant and theorem 9 implies that f_0 and f_1 are topologically equivalent. $\square$

4. Hyperplane arrangements. We state a generalization in higher dimension of the equivalence of generic line arrangements. We have to strengthen the hypothesis: we start with a continuous family of *equivalent* hyperplane arrangements (instead of constructing this family). On the other hand the conclusion is valid for all hyperplane arrangements (that are Morse outside the arrangement); we do not assume here that hyperplanes are in generic position.

THEOREM 15. *Let the dimension be $n \neq 3$. Let $(f_t)_{t\in[0,1]}$ be a smooth family of polynomials of hyperplane arrangements that are functions Morse outside the arrangement. If all the arrangements $(f_t = 0)$ are equivalent then the polynomial f_0 is topologically equivalent to f_1.*

As seen in section 2, the Morse condition is necessary. For the proof we can not directly apply the μ-constant theorem because singularities above 0 are non-isolated. However the following lemmas prove that nothing happens at infinity.

Let us introduce some notations. Let B_R be the closed $2n$-ball of radius R centred at $(0,\ldots,0)$ in $\mathbb{C}^n$ and let $S_R = \partial B_R$. Let D_r be the closed disk of radius r centred at $0 \in \mathbb{C}$. For the polynomial map $f : \mathbb{C}^n \to \mathbb{C}$, let $T_r = f^{-1}(D_r)$ (resp. $T_r^* = f^{-1}(D_r \setminus \{0\})$) be the tube (resp. punctured tube) around the fibre $f^{-1}(0)$.

LEMMA 16. *Let f be the polynomial of a hyperplane arrangement. For all $r > 0$ there exists a sufficiently large R, such that for $x \in T_r^* \setminus B_R$, $f^{-1}(f(x))$ is transversal at x to the sphere $S_{x/\|x\|}$ passing through x.*

Then classical arguments of transversality enable the construction of vector fields, whose integration gives the following trivialization diffeomorphism Φ:

LEMMA 17. *There exist a diffeomorphism Φ and $R \gg 1$ such that the diagram is commutative:*

$$\begin{tikzcd} \mathring{B}_R \arrow[r, "\Phi"] \arrow[d, "f"'] & T_r \arrow[d, "f"] \\ D_r \arrow[r, "id"'] & D_r. \end{tikzcd}$$

COROLLARY 18. *$\mathcal{B}_{inf} = \{0\}$ or $\mathcal{B}_{inf} = \varnothing$.*

Proof of lemma 16. Let $f = \prod_{j=1}^{d} \ell_j$. We suppose that the vector space generated by $\{\operatorname{grad} \ell_j, j = 1, \dots, d\}$ is n-dimensional (if not then we first diminish the dimension of the ambient space). Notice that we use the classical convention for the gradient: $\operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$.

Suppose that there exists a sequence (z_k) of points in $\mathbb{C}^n$ where the fibres are non transversal to the spheres; that is to say such that: $\|z_k\| \to +\infty$, $z_k \in T_r^*$, $\operatorname{grad} f(z_k) = \lambda_k \bar{z_k}$, with $\lambda_k \in \mathbb{C}$.

The hypothesis of the beginning of the proof can be reformulated as follows: there exists ℓ_j such that $|\ell_j(z_k)| \to +\infty$. We denote by $\ell_1, \dots, \ell_q$ the linear forms such that $\ell_j(z_k) \to 0$. We set $g = \prod_{j=1}^{q} \ell_j$ and $h = \prod_{j>q} \ell_j$. We have $\deg h > 0$. And as $|f(z_k)|$ is bounded by r, we have $q = \deg g > 0$. We can assume that $\bigcap_{j=1,\dots,q} (\ell_j = 0)$ is given by $(x_1 = 0, \dots, x_p = 0)$. Then ℓ_j, $j = 1, \dots, q$ are linear forms in $(x_1, \dots, x_p)$ with a constant term equal to 0; that is to say g is a homogeneous polynomial in $\mathbb{C}[x_1, \dots, x_p]$. We write $z_k = (z_1^k, \dots, z_n^k)$. Hence the p first terms $z_1^k, \dots, z_p^k$ tend towards 0 as $k \to +\infty$. And there exists $i_1 > p$ such that $|z_{i_1}^k| \to +\infty$.

We now compute $\operatorname{grad} f$. We have $f = g \cdot h$ so that $\frac{\operatorname{grad} f}{f} = \frac{\operatorname{grad} g}{g} + \frac{\operatorname{grad} h}{h}$.

But $\frac{\partial h}{\partial x_i}/h = \sum_{j>q} \frac{\partial \ell_j}{\partial x_i}/\ell_j$ where $\frac{\partial \ell_j}{\partial x_i}$ is a constant; $\ell_j(z_k)$ is bounded away from 0 because $j > q$. Whence $(\frac{\partial h}{\partial x_i}/h)(z_k)$ and $(\operatorname{grad} h/h)(z_k)$ are bounded as $k \to +\infty$.

Moreover if $i > p$ then $\frac{\partial g}{\partial x_i} = 0$, it implies that for $i > p$, $(\frac{\partial f}{\partial x_i}/f)(z_k)$ is bounded as $k \to +\infty$.

Partial conclusion: there exists $i_1 > p$ such that $|z_{i_1}^k| \to +\infty$ and $(\frac{\partial f}{\partial x_{i_1}}/f)(z_k)$ is bounded (as $k \to +\infty$).

We now consider the projection $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^p$ defined by $\pi(x_1, \dots, x_n) = (x_1, \dots, x_p)$. We denote $\langle (x_1, \dots, x_p) | (y_1, \dots, y_p) \rangle = \sum_{i=1}^{p} x_i \bar{y_i}$ the inner Hermitian product. Then

$$\langle \pi(\operatorname{grad} f/f) | \pi(z_k) \rangle = \langle \pi(\operatorname{grad} g/g) | \pi(z_k) \rangle + \langle \pi(\operatorname{grad} h/h) | \pi(z_k) \rangle.$$

As $k \to +\infty$ we have $\pi(z_k) \to (0, \dots, 0)$. Then $\langle \pi(\operatorname{grad} h/h) | \pi(z_k) \rangle \to 0$. By Euler's relation for the homogeneous polynomial g of degree q we have $\langle \pi(\operatorname{grad} g/g) | \pi(z_k) \rangle = q > 0$. It implies that at least the module of one component of $\pi(\operatorname{grad} f/f)$ tends towards $+\infty$. We call $i_2 \leqslant p$ the index of this component.

Partial conclusion: there exists $i_2 \leqslant p$ such that $z_{i_2}^k \to 0$ and $(\frac{\partial f}{\partial x_{i_2}}/f)(z_k) \to +\infty$ (as $k \to +\infty$).

We have $\operatorname{grad} f(z_k) = \lambda_k \bar{z_k}$, hence we have the equality, for all k:

$$\overline{z_{i_2}^k} \frac{\frac{\partial f}{\partial x_{i_1}}}{f}(z_k) = \overline{z_{i_1}^k} \frac{\frac{\partial f}{\partial x_{i_2}}}{f}(z_k).$$

And we know that, as $k \to +\infty$, we have: $\overline{z_{i_2}^k} \to 0$, $(\frac{\partial f}{\partial x_{i_1}}/f)(z_k)$ is bounded, $\overline{z_{i_1}^k} \to \infty$, $(\frac{\partial f}{\partial x_{i_2}}/f)(z_k) \to \infty$.

Then, as $k \to +\infty$, the left-hand side tends towards 0 while the right-hand side tends towards ∞. It gives the contradiction. ∎

Proof of the theorem. We decompose the proof in different steps.

1. We only have to prove that f_0 and f_1 are topologically equivalent when restricted to $T_r \cap B_R$. Lemma 17 will then extend the equivalence to T_r.
2. Lemma 14 is valid in any dimension under the hypothesis $\mathcal{B}_{inf} \setminus \{0\} = \varnothing$ (see the remark in the proof) so that $\#\mathcal{B} = \#\mathcal{B}_{aff} = 2 - \chi(f^{-1}(0))$, while the sum $\sum_{c\neq 0} \mu_c = \#\mathcal{B}_{aff} - 1$.
3. Around the fibres above 0. We fix $R \gg 1$ and $0 < \epsilon \ll 1$. We define $T_\epsilon = f^{-1}(D_\epsilon)$. We denote $\mathcal{T} = \bigcup_{t\in[0,1]}(T_\epsilon \cap B_R) \times \{t\}$. The space $\mathcal{T}$ has a natural Whitney stratification given by the intersections of the hyperplanes. The function $F : \mathcal{T} \longrightarrow D_\epsilon \times [0,1]$ defined by $F(x,t) = (f_t(x), t)$ is a function transversal to $\bigcup_{t\in[0,1]} B_R \times \{t\}$ (see the proof of lemma 16) that satisfies Thom a_F condition. From Thom-Mather's second isotopy lemma we have that F is a trivial fibration in t.
4. Outside a neighborhood of the fibres $f_t^{-1}(0)$. We can apply the methods of the proof of the global μ-constant theorem (theorem 9). It provides a trivialization, which can be glued with the one obtained around the fibre above 0. It proves that the polynomials f_0 and f_1 are topologically equivalent on $T_r \cap B_R$. As nothing happens at infinity the polynomials f_0 and f_1 are topologically equivalent on $\mathbb{C}^n$.

□

5. Nearby fibre and intermediate links. We end with a topological description of the generic fibre of a line arrangement. In fact we will firstly prove a result for all polynomials and next will apply this to compute the topology of generic fibres of a line arrangement intersected with any a ball.

5.1. Nearby fibre. It is useful to consider a smooth deformation $(f = \delta)$ of $(f = 0)$. But we shall also consider a deformation f_s of the polynomial f. We will prove that the generic fibres $(f = \delta)$ and $(f_s = \delta)$, restricted to a ball, are diffeomorphic.

LEMMA 19. *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Suppose that $(f = 0)$ has isolated (affine) singularities. For any $r > 0$ such that the intersection $(f = 0)$ with the sphere $S_r^{2n-1}(0)$ is transversal, there exists $0 < \delta \ll 1$ such that:*

1. *The intersection of $(f = c)$ with $S_r^{2n-1}(0)$ is transversal, for any $c \in D_\delta^2(0)$.*
2. *The restriction $f : f^{-1}(D_\delta^2(0) \setminus \{0\}) \cap B_r^{2n}(0) \to D_\delta^2(0) \setminus \{0\}$ is a locally trivial fibration.*

This lemma is global version of classical results by Milnor to prove the Milnor fibration theorem, see [9]. More precisely the first conclusion follows from the continuity of the transversality, while the second one follows from Ehresmann fibration theorem.

It is crucial that we first choose the radius of the sphere and then the radius of the disk. This is the opposite of the situation for singularity at infinity.

5.2. Deformation of f. It is useful not to consider $(f = 0)$ but to study a deformation $(f_s = 0)$ of f, that is to say a family $(f_s)_{s\in[0,1]}$ of polynomials, whose coefficients depend analytically on s, such that $f_0 = f$. The following proposition proves that the nearby fibre $(f = \delta)$ and $(f_s = \delta)$ are diffeomorphic.

PROPOSITION 20. *Let f, $r > 0$, $0 < \delta \ll 1$ as in Lemma 19. Consider a deformation f_s, $s \in [0,1]$ with $f_0 = f$. For all sufficiently small $0 < s \ll \delta$ we have :*

1. *The intersection of $(f_s = c)$ with $S_r^{2n-1}(0)$ is transversal, for any $c \in D_\delta^2(0)$.*

2. *The sum of Milnor numbers of the critical points of* f_s *inside* $f^{-1}(D^2_\delta(0)) \cap B^{2n}_r(0)$ *equals the sum of Milnor numbers of the critical points of* $(f = 0)$ *inside* $B^{2n}_r(0)$.
3. *The restriction* $f_s : f_s^{-1}(D^2_\delta(0) \setminus \mathcal{B}_s) \cap B^{2n}_r(0) \to D^2_\delta(0) \setminus \mathcal{B}_s$ *is a locally trivial fibration, where* $\mathcal{B}_s$ *is a finite number of points (the critical values of* f_s *corresponding to the critical points of* f_s *inside* $f^{-1}(D^2_\delta(0)) \cap B^{2n}_r(0)$*).*
4. *The fibrations* $f : f^{-1}(S^1_\delta(0)) \cap B^{2n}_r(0) \to S^1_\delta(0)$ *and* $f_s : f_s^{-1}(S^1_\delta(0)) \cap B^{2n}_r(0) \to S^1_\delta(0)$ *are diffeomorphic. In particular the fibres* $(f = \delta)$ *and* $(f_s = \delta)$ *are diffeomorphic.*

The proofs of theses items are very standard in singularity theory, the main ingredients are: continuity of the critical points, continuity of transversality, Ehresmann fibration theorem and integration of vector fields.

Be careful! The order for choosing the constants is crucial. We fix $f = f_0$, then r, then δ, then s. The δ is chosen small for f_0 but is not small for f_s. In other words the fibration $f_s : f_s^{-1}(S^1_\delta(0)) \cap B^{2n-1}_r(0) \to S^1_\delta(0)$ is **not** diffeomorphic to the fibration $f_s : f_s^{-1}(S^1_{\delta'}(0)) \cap B^{2n-1}_r(0) \to S^1_{\delta'}(0)$ for all $0 < \delta' \ll 1$. Even if the fibres are diffeomorphic!

Remark. It should also be compared to the work of Neumann and Rudolph, [10]. Another interesting idea is to include the smooth part of $(f = 0)$ into the nearby fibre $(f = \delta)$, see [2].

6. Nearby fibres of line arrangements. The case where $(f = 0)$ is an affine line arrangement is of particular interest. Let $f(x, y) = \prod_i (a_i x + b_i y + c_i)$, $a_i, b_i, c_i \in \mathbb{C}$. Growing the radius of the sphere yields only two phenomena: birth or joint (see [4]). There is no other point of non-transversality for $(f = 0)$. We can compute the topology of a nearby fibre $(f = \delta) \cap B_r$, where $B_r = B^4_r(0)$.

Algorithm 21. *The topology of* $(f = \delta) \cap B_r$ *can be computed as follows: for each line* ℓ_i *of* $(f = 0)$ *that intersects* B_r*, associate a 2-disk* D_i*. For any two lines* ℓ_i *and* ℓ_j *that intersect in* B_r *glue the disks* D_i *to* D_j *by two twisted-bands* $[0, 1] \times [0, 1]$.

A fundamental example is an arrangement of only two lines that intersect inside B_r. Then $(f = \delta)$ is two disks joined by two twisted bands. See figure 1 from left to right. (i) a real picture of an arrangement with two lines that intersect in the ball, and a real picture of one generic fibre. (ii) and (iii) a topological picture of the generic fibre as the gluing of two disks by two bands.

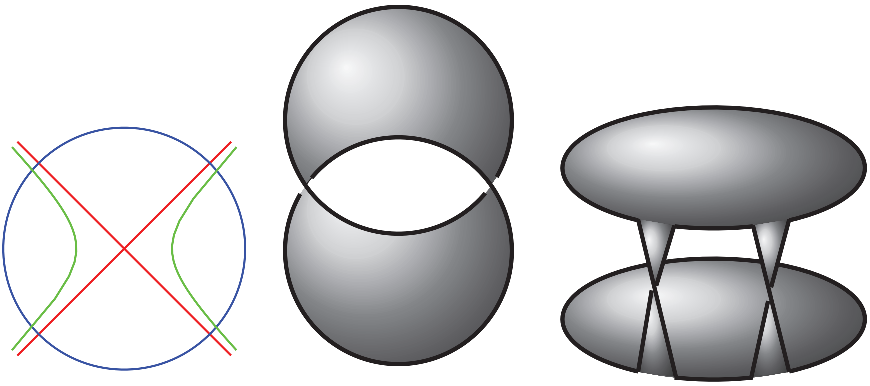

FIG. 2. *Topology of a generic fibre*

Proof. First of all we make a deformation of the arrangement f to another arrangement f_s, such that $(f_s = 0)$ is a union of lines with only double points. Of course, some other singular fibres appear, but by proposition 20, the nearby fibres $(f = \delta)$ and $(f_s = \delta)$ are diffeomorphic.

Then we have to consider only gluing of nearby fibres of intersection of two lines as in the example above. □

Illustration. To compute the topology of the generic fibre of this arrangement with 4 lines inside a ball (figure 3, left), we first shift one line in order to remove the triple point (figure 3, right). We also have drawn the real trace of a generic fibre. The topology is built as follows: starts from 4 disks labeled $D_1, \ldots, D_4$ (corresponding to the lines $\ell_1, \ldots, \ell_4$). Between each pair (D_1, D_2), (D_1, D_3), (D_2, D_3) (D_1, D_4) attach two bands (each pair of bands corresponds to one intersection inside the ball). The Euler characteristic of a generic fibre is then -4 and as a surface it is a torus with 4 punctures.

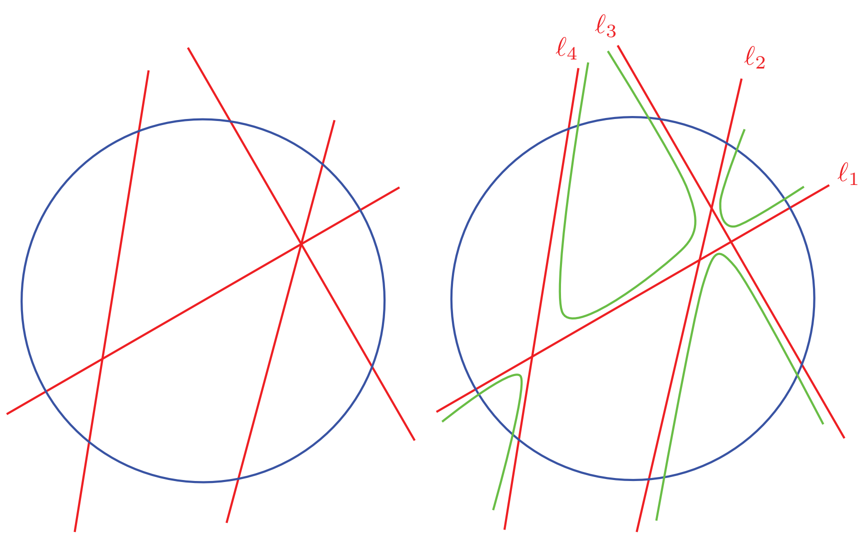

FIG. 3. *A 4-lines arrangement*

REFERENCES

[1] A. Bodin, *Invariance of Milnor numbers and topology of complex polynomials*, Comment. Math. Helv., 78 (2003), pp. 134–152.

[2] A. Bodin, *Irregular fiber of complex polynomials in two variables*, Rev. Mat. Complut., 17 (2004), pp. 67–81.

[3] A. Bodin and M. Tibar, *Topological equivalence of complex polynomials,* Adv. Math., 199 (2006), pp. 136–150.

[4] M. Borodzik, *Morse theory for plane algebraic curves*, J. Topology, 5 (2012), pp. 341–365.

[5] A. Durfee, *Five definitions of critical point at infinity,* Singularities (Oberwolfach, 1996), pp. 345–360, Progr. Math. 162, Birkhäuser, 1998.

[6] S. M. Finashin, *Configuration of seven points in* $\mathbb{R}P^3$*,* Topology and geometry–Rohlin Seminar, pp. 501–526, Lecture Notes in Math. 1346, Springer, 1988.

[7] T. Jiang and S. S.-T. Yau, *Diffeomorphic types of the complements of arrangements of hyperplanes,* Compositio Math., 92 (1994), pp. 133–155.

[8] A. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor,* Invent. Math., 32 (1976), pp. 1–31.

[9] J. Milnor, *Singular points of complex hypersurfaces,* Ann. of Maths. Studies 61, 1968, Princeton Univ. Press.

[10] W. Neumann and L. Rudolph, *Unfolding in knot theory,* Math. Ann., 278 (1987), pp. 409–39.

[11] S. Nazir and M. Yoshinaga, *On the connectivity of the realization space of line arrangements,* Ann. Sc. Norm. Super. Pisa Cl. Sci., 11 (2012), pp. 921–937.

[12] R. Randell, *Lattice-isotopic arrangements are topologically isomorphic,* Proc. Amer. Math. Soc., 107 (1989), pp. 555–559.

[13] A. Varchenko, *Critical points of the product of powers of linear functions and families of bases of singular vectors,* Compositio Math., 97 (1995), pp. 385–401.