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TRANSVERSALITY OF COMPLEX LINEAR DISTRIBUTIONS WITH SPHERES, CONTACT FORMS AND MORSE TYPE FOLIATIONS – II*

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Abstract. The study of holomorphic foliations transverse to real submanifolds has its own interest, as for instance its connections with the construction or existence of complex structures. The comprehension of the transverse dynamics of such foliations is also granted by that study. As for the non-integrable case, the study of contact forms in the holomorphic framework is related to the study of (non-integrable) codimension one distributions which are transverse to spheres in the affine space.

The starting point for our work is the following question: *Is there any codimension one holomorphic foliation $\mathcal{F}$ in a neighborhood of the closed unit disk $\overline{D^{2n}(1)} \subset \mathbb{C}^n$ such that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(1)$ for $n \geq 3$?* In this paper we study transversality of (integrable or not) holomorphic perturbations of codimension one linear distributions, with spheres in the complex affine space. So far, the examples of such distributions are related to contact forms and are as a kind of counterpart of the integrable case. Based on an extension of Takagi's factorization theorem for nonsingular matrices in terms of Jordan canonical forms of its generalized coneigenvectors, we prove that given a generic nonsingular $n \times n$ complex matrix A and any holomorphic one-form ω having its linear part at the origin given by A, the corresponding distribution $\mathcal{K}(\omega)\colon \{\omega = 0\}$ is not transverse to the spheres $S^{2n-1}(r)$ for small $r > 0$. Here, by generic we mean that $\bar{A}A$ has a simple positive eigenvalue $\lambda > 0$, and any other eigenvalue has absolute value different from λ. Using this, we are able to conclude that, in $\mathbb{C}^n$, *a distribution which is a perturbation of a linear Morse foliation, is not transverse to small spheres and its variety of contacts has at least n branches in a neighborhood of the origin.*

Key words. Takagi's factorization theorem, holomorphic distribution, holonomy group, linear Morse foliation.

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1. Introduction.

The qualitative aspects of the geometric theory of smooth codimension one foliations has among its main tools the relation between transverse sections and the dynamics of vector fields in the 2-disk or in the 2-sphere as it is clear from the classical proofs of Haefliger's one-sided holonomy theorem and Novikov compact leaf theorem. Indeed, in both cases the use of Poincaré-Bendixson's theorem is the key point in the description of the transverse dynamics of the original foliation. In the case of holomorphic vector fields, A. Douady and the first named author discovered a Poincaré-Bendixson type theorem [9]. Following this line of research we have addressed the question ([10, 11]): *Is there a codimension one holomorphic foliation $\mathcal{F}$ in a neighborhood of the closed unit disk $\overline{D^{2n}(1)} \subset \mathbb{C}^n$ such that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(1)$ for $n \geq 3$?* The general question is related also to (not necessarily integrable) codimension one holomorphic distributions.

Regarding the integrable case, we remark that for $n = 2$ there are linear examples giving a positive answer to the question above, and the situation is well-understood ([4, 9]). In [10] it is shown that *a linear foliation $\mathcal{F}$ on $\mathbb{C}^n$, $n \geq 3$, is not transverse to the sphere $S^{2n-1}(1)$. Moreover, $\mathcal{F}$ is transverse to the sphere $S^{2n-1}(1)$ off the*

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singular set $sing(\mathcal{F}) \cap S^{2n-1}(1)$ *if and only if* $\mathcal{F}$ *is a product* $\mathcal{L}_\lambda \times \mathbb{C}^{n-2}$ *for some linear foliation* $\mathcal{L}_\lambda \colon x\,dy - \lambda y\,dx = 0$*, in the Poincaré domain on* $\mathbb{C}^2$*.*

In [12], we introduced a class of of generic foliations, where "generic" stands for a generic set of tangency points of the leaves of the foliation with the spheres centered at the origin as follows. This follows original ideas of R. Thom ([19]). Such foliations are called *Morse type holomorphic foliations* and we have given a negative answer to the above question also in the case of Morse type foliations. The precise definition is as follows:

Definition 1.1 (Morse type, [12] Definition 1.1 page 437). Let $\mathcal{F}$ be as above and denote by φ the canonical distance function with respect to the origin $0 \in \mathbb{C}^n$. The foliation $\mathcal{F}$ is of *Morse type* if for each leaf $L \in \mathcal{F}$, each critical point $p \in L$ of the restriction of the distance function on L, $\varphi\big|_L$, on L is nondegenerate.

The geometry of Morse type foliations is studied in detail in [4] and [16] for isolated singularities of dimension one foliations, and in [12], [13] and [14] for the case of codimension one. In [12] we prove that there is no Morse type holomorphic foliation $\mathcal{F}(\omega)$ of codimension one in a neighborhood U of the closed unit disk $\overline{D^{2n}(1)} \subset \mathbb{C}^n$, $n \geq 3$, such that $\mathcal{F}(\omega)$ is transverse to the boundary sphere $S^{2n-1}(1)$. A complex linear foliation $\mathcal{F}(\omega_A)$ is of Morse type if and only if $\bar{A}A$ has distinct (positive) eigenvalues. In this case the variety of contacts $\Sigma(\omega_A, \varphi)$, is the union of the n lines given by the eigenvectors of $\bar{A}A$. On the other hand, it is shown in [13] that a linear distribution $\mathcal{K}(\omega_A)$ is transverse to $S^{2n-1}(1) \subset \mathbb{C}^n$ if and only if $\bar{A}A$ has no positive eigenvalues. We also show that the tangential property is robust under a small perturbation: A linear distribution $\mathcal{K}(\omega_A)$ with $\omega_A = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} z_j) dz_i$ is not transverse to the sphere $S^{2n-1}(1)$ provided that it is close enough to a linear foliation of Morse type. If this is the case, let $f_i = f_i(z_1, \ldots, z_n)$, $i = 1, \ldots, n$, be holomorphic functions defined in a neighborhood of the origin, such that $f_i(0) = 0$, $\frac{\partial}{\partial z_j} f_i(0) = 0$, $i, j = 1, \ldots, n$. Then the holomorphic distribution $\mathcal{K}(\tilde{\omega})$, given by $\tilde{\omega} = 0$, for $\tilde{\omega} = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} z_j + f_i) dz_i$, is not transverse to the spheres $S^{2n-1}(r)$ for small $r > 0$.

In [14] we prove that a point of non-degenerate contact of a leaf with a sphere is a hyperbolic fixed point of the corresponding dynamics. Around a point of degenerate contact, the intersection of branches of the variety of contacts is described as a bifurcation diagram of a neutral fixed point of dynamics. The Morse index for the distance function from the origin is computed as the complex dimension of an unstable manifold.

In the present manuscript we extend some results in [13] and [14] as well, and open a new perspective in the context, by bringing some concepts of Linear Algebra to our framework. First we prove (cf. Theorem 2.7) an *extension* of classical Takagi's factorization theorem ([18], [1]) on the diagonalization of complex symmetric matrices by unitary transformations, to the Jordan canonical forms of generalized *coneigenvectors* (see [3]) under linear transformations.

The above mentioned canonical Jordan form is obtained as a consequence of Proposition 2.4 for the real case and of Theorem 2.5 for the complex case.

Always keeping in mind that a linear foliation $\mathcal{F}(\omega_A)$ is of Morse type if (A is symmetric and) the hermitian matrix $\bar{A}A$ has distinct (positive) eigenvalues we shall say that a (not necessarily symmetric) nonsingular complex matrix $A \in \mathrm{GL}(n, \mathbb{C})$ is *generic* if $\bar{A}A$ has a simple positive eigenvalue $\lambda > 0$, and any other eigenvalue has absolute value different from λ. Our main result below is a consequence of the above mentioned extended factorization lemma and reads as follows:

Theorem 1.2. *Let $A = (a_{ij}) \in GL(n, \mathbb{C})$ be a generic nonsingular complex matrix. If ω is any holomorphic one-form having ω_A as linear part at the origin then the distribution $\mathcal{K}(\omega)$ is not transverse to the spheres $S^{2n-1}(r)$ for small $r > 0$.*

We find that the proofs of Theorems 1.2 and 2.7 are interesting on their own, and they are similar to the classical proof of existence of a hyperbolic fixed point in dynamical systems ([17]).

In the same line of reasoning we prove a Parametrization theorem (cf. Theorem 3.2) for the variety of contacts (definition in Section 2) of perturbations of linear distributions as above and extend Theorem 3 in [13] as follows:

Theorem 1.3. *In $\mathbb{C}^n$, a distribution which is a perturbation of a linear Morse foliation, is not transverse to small spheres and its variety of contacts exhibits at least n branches in a neighborhood of the origin.*

A more precise statement is given in Corollary 3.7. We point-out that it is not yet clear whether the variety of contacts Σ "consists only of" the branches which are perturbations of the eigenspaces.

Some of our Linear Algebra results are essentially known or may seem natural to specialists. For instance, in Section 2 we give a canonical form of a square complex matrix up to transformations and thus we give a canonical form of the matrix of a semilinear operator on a complex vector space. Nevertheless, we think that the proofs we provide for them are important in our framework and also for their dynamical systems interpretation, which we think is new.

As it is known by specialists, semilinear operators over fields and skew fields are classified in Section 2.8 of [15], and pseudo-linear operators, which generalize semilinear operators, are classified in Section 8.4 of [2]. Consimilarity transformations are studied in Sections 4.5 and 4.6 of [5] and canonical forms of complex matrices for consimilarity transformations are also constructed in [6] (see also [7] and [8]). Nevertheless, we highlight our belief that our proofs of some of these facts, those in which we -are interested, are quite simple and even geometrical, which is in the basic spirit of our applications to the theory of Foliations.

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2. Jordan canonical form of a linear distribution. In this section we first recall some notations and definitions. Given a non-singular complex one-form ω in a complex manifold M, we define the distribution $\mathcal{K}(\omega)$ on M as follows: to each point $p \in M$ we associate the codimension one linear subspace $\mathcal{K}(\omega)(p) < T_p(M)$ of vectors $v_p \in T_p(M)$ such that $\omega(p) \cdot v_p = 0$.

Definition 2.1 (Transversality). Let $\omega = \sum_{j=1}^{n} f_j(z)\ dz_j$ be a holomorphic one-form in an open subset $U \subset \mathbb{C}^n$. Given a smooth (real) submanifold $M \subset U$, we say that the distribution $\mathcal{K}(\omega)$ *is transverse* to M if $\operatorname{Sing}(\mathcal{K}(\omega)) \cap M = \emptyset$ and for every $p \in M$ we have $T_pM + \mathcal{K}(\omega)(p) = T_p\mathbb{R}^{2n}$ as real linear spaces. Otherwise we say that $\mathcal{K}(\omega)$ *is not transverse* to M.

In this article we consider a codimension one holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)$ defined by an integrable one-form $\omega = \sum_{j=1}^{n} f_j(z)dz_j$ on (an open set of) $\mathbb{C}^n$, and the distance function from the origin $\varphi(z) = \|z\|^2 = \sum_{j=1}^{n} |z_j|^2$. A level surface is a sphere $\varphi^{-1}(r^2) = S^{2n-1}(r)$, for $r > 0$. Let $\mathcal{K}(\omega)(p) = \{v \in T_p\mathbb{C}^n :\ \omega(p) \cdot v = 0\}$, $p \in \mathbb{C}^n$, and $\mathrm{Sing}(\omega) = \{p \in \mathbb{C}^n : \omega(p) = 0\}$. The distribution $\mathcal{K}(\omega)$ is called *transverse* to the sphere at a point p if $p \neq 0$, $p \notin \mathrm{Sing}(\omega)$ and $\mathcal{K}(\omega)(p) + T_p(S^{2n-1}(r)) = T_p\mathbb{R}^{2n}$, $r = \|p\|^{1/2}$, as a real subspace. The set $\Sigma(\omega)$ of the points $p \in \mathbb{C}^n$ where $\mathcal{F} = \mathcal{F}(\omega)$ is not transverse to the spheres is called *variety of contacts* ([19]), or *polar variety* ([16]). We have shown in [10] and [12] that $\mathcal{F}$ is not transverse to the spheres at a point $z \in \mathbb{C}^n \setminus 0$ if and only if there exists $\lambda \in \mathbb{C}$ such that

$$\overline{f(z)} = \lambda z, \tag{1}$$

where $f(z) = (f_1(z), \dots, f_n(z))$, $z = (z_1, \dots, z_n)$. Thus $\Sigma(\omega)$ is a real analytic set given by

$$\Sigma(\omega) = \left\{ z \in \mathbb{C}^n : \ z_j\overline{f_k(z)} = z_k\overline{f_j(z)}, \forall j, k = 1, \dots, n \right\}. \tag{2}$$

The scalar $\lambda \in \mathbb{C}$ in (1) is called *conjugate-multiplier.* A point $p \in \Sigma^0 := \Sigma \setminus (\mathrm{Sing}(\omega) \cup \{0\})$ is called a *contact point.* It is called a *degenerate* contact point if it is a degenerate critical point of the distance function $\varphi|L_p$.

If A is an $n \times n$ complex matrix, $z \in \mathbb{C}^n$ is called a *conjugate-eigenvector* or *coneigenvector* of A if

$$Az = \mu\bar{z}$$

for some $\mu \in \mathbb{C}$ (see also the beginning of Section 3). If ω is a linear one-form $\omega = \sum_{i=1}^n (\sum_{j=1}^n a_{ij}z_j)dz_i$, then from equations (1) and (2) above, $z \in \Sigma(\omega)$ if and only if z is a coneigenvector of $A = (a_{ij})$.

We consider a linear transformation of holomorphic distributions which preserve transversality property to the spheres $S^{2n-1}(r)$.

LEMMA 2.2. *Let V be an open set in $\mathbb{C}^n$ containing a sphere $S^{2n-1}(r)$ of radius $r > 0$ and center at the origin. Let $\omega = \sum_{i=1}^n f_i(z)\ dz_i$ be a holomorphic one-form in $V \supset S^{2n-1}(r)$ in $\mathbb{C}^n$ and let $F\colon V \to \mathbb{C}^n$ be the holomorphic map given by $F = (f_1, \dots, f_n)$. Given a linear transformation $P \in GL(n, \mathbb{C})$, consider the holomorphic map $G = \bar{P}^{-1} \circ F \circ P = (g_1, \dots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ and the one-form $\omega' = \sum_{i=1}^n g_i(w)\ dw_i$. The distribution $\mathcal{K}(\omega')$ is transverse to the foliation $\mathcal{F}(d\varphi)$ at $\zeta' \in \mathbb{C}^n$ if and only if the distribution $\mathcal{K}(\omega)$ is transverse to $\mathcal{F}(d\varphi)$ at $\zeta = P\zeta'$.*

Proof. The distribution $\mathcal{K}(\omega')$ is not transverse to $\mathcal{F}(d\varphi)$ at $w = \zeta'$

$$\begin{aligned}
&\Leftrightarrow \overline{g_i(\zeta')} = c\zeta'_i, \ 1 \leq i \leq n, \ \exists c \in \mathbb{C} \\
&\Leftrightarrow P^{-1}(\overline{F(P\zeta')}) = c\zeta' \\
&\Leftrightarrow \overline{F(P\zeta')} = cP\zeta' \\
&\Leftrightarrow \overline{F(\zeta)} = c\zeta
\end{aligned}$$

$\Leftrightarrow \mathcal{K}(\omega)$ is not transverse to $\mathcal{F}(d\varphi)$ at $\zeta = P\zeta'$. □

REMARK 2.3. A usual linear transformation ${}^tPFP$ preserves the integrability, but not necessarily the transversality to the spheres. Our linear transformation $\bar{P}^{-1}FP$ does not necessarily preserve the integrability, but it does preserve the transversality.

The remaining part of this section is an specialized Linear Algebra theory on Jordan canonical forms of a complex $n \times n$ matrix A with respect to the linear transformation considered above. Denote by $m(k,\lambda,\delta) = \begin{pmatrix} \lambda & \delta & & \\ & \lambda & \delta & \\ & & \dots & \delta \\ & & & \lambda \end{pmatrix} \in M(k,\mathbb{C})$, $\lambda \in \mathbb{C}$, $\delta > 0$, a *Jordan cell* of size $k > 0$. Let $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$, $\iota(x + \sqrt{-1}y) = \begin{pmatrix} x \\ y \end{pmatrix}$, be the canonical isomorphism. For $A = A_1 + \sqrt{-1}A_2 \in M(n,\mathbb{C})$, denote by $A_{\mathbb{R}} = \begin{pmatrix} A_1 & -A_2 \\ -A_2 & -A_1 \end{pmatrix} \in M(2n,\mathbb{R})$. We have $\iota(\overline{Az}) = A_{\mathbb{R}} \begin{pmatrix} x \\ y \end{pmatrix}$. Let $\sigma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then we have $\sigma^2 = -1$, and $A_{\mathbb{R}}\sigma = -\sigma A_{\mathbb{R}}$.

The next proposition shows the existence of a decomposition in Jordan cells for a linear map $A_{\mathbb{R}} \in M(2n,\mathbb{R})$ up to a similarity in $\mathrm{GL}(2n,\mathbb{C})$.

PROPOSITION 2.4. *Let $\delta > 0$. For each $A_{\mathbb{R}} = \begin{pmatrix} A_1 & -A_2 \\ -A_2 & -A_1 \end{pmatrix} \in M(2n,\mathbb{R})$ there exists $Q \in GL(2n,\mathbb{C})$ such that*

$$C = Q^{-1} A_{\mathbb{R}}\, Q = \begin{pmatrix} C_1 & & \\ & C_2 & \\ & & \ddots \end{pmatrix} \in M(2n,\mathbb{C}),$$

where either

$$C_j = \begin{pmatrix} m(k_j,\lambda_j,\delta) & 0 \\ 0 & -m(k_j,\lambda_j,\delta) \end{pmatrix}, \ \lambda_j \geq 0, \tag{3}$$

or

$$C_j = \begin{pmatrix} m(k_j,\lambda_j,\delta) & & & \\ & \overline{m(k_j,\lambda_j,\delta)} & & \\ & & -m(k_j,\lambda_j,\delta) & \\ & & & -\overline{m(k_j,\lambda_j,\delta)} \end{pmatrix}, \ \lambda_j \in \mathbb{C} \setminus \mathbb{R}. \tag{4}$$

Proof. For a matrix $A_{\mathbb{R}} \in M(2n,\mathbb{R})$ fixed, consider the generalized eigenspace $E(\lambda) = \cup_{k>0} E_k(\lambda)$, $E_k(\lambda) = \mathcal{K}((A_{\mathbb{R}} - \lambda)^k) \subset \mathbb{C}^{2n}$, that belongs to the eigenvalue $\lambda \in \mathbb{C}$. Since $(A - \lambda)\sigma = -\sigma(A + \lambda)$, we have an isomorphism $\sigma : E(\lambda) \to E(-\lambda)$. Thus, in the case $\lambda \in \mathbb{R}$, we obtain the Jordan cells (3), where λ may be nonnegative without loss of generality.

Let $\tau : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, $\tau(\zeta) = \bar{\zeta}$, be an isomorphism of complex conjugacy. Since $A_{\mathbb{R}}$ is a real matrix, we have $\tau(A_{\mathbb{R}} - \lambda) = (A_{\mathbb{R}} - \bar{\lambda})\tau$. So $\tau : E(\lambda) \to E(\bar{\lambda})$ is an isomorphism of eigenspaces. Thus, in the case $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \sqrt{-1}\mathbb{R})$, the four eigenspaces $E(\lambda)$, $E(-\lambda)$, $E(\bar{\lambda})$, $E(-\bar{\lambda})$ are isomorphic, and we obtain the Jordan cells (4).

In the case of a purely imaginary $\lambda \in \sqrt{-1}\mathbb{R}$, we have $\bar{\lambda} = -\lambda$, but we are still going to show that the Jordan cells (4) exist. First, there exists a $k > 0$ such that $E(\lambda) = E_k(\lambda) \neq E_{k-1}(\lambda)$. Let V_k be the orthogonal complement of $E_{k-1}(\lambda)$ in $E_k(\lambda)$, that is, $V_k \perp E_{k-1}(\lambda)$ and $V_k \oplus E_{k-1}(\lambda) = E_k(\lambda)$. We shall obtain an orthogonal basis $\zeta_1^k, \sigma\tau(\zeta_1^k), \dots, \zeta_{\ell_k}^k, \sigma\tau(\zeta_{\ell_k}^k)$ of V_k by an induction as follows. For an arbitrary $\zeta_1^k \in V_k$, $\zeta_1^k \neq 0$, $\sigma\tau(\zeta_1^k)$ is orthogonal to ζ_1^k. If

$$\zeta_1^k, \sigma\tau(\zeta_1^k), \dots, \zeta_j^k, \sigma\tau(\zeta_j^k) \tag{5}$$

do not span V_k, take a $\zeta_{j+1}^k \in V_k$ which is orthogonal to (5). Then ζ_1^k, $\sigma\tau(\zeta_1^k)$, $\dots$, ζ_{j+1}^k, $\sigma\tau(\zeta_{j+1}^k)$ is a set of mutually orthogonal vectors in V_k. This process can be repeated until (5) spans V_k.

The set of vectors

$$\left\{(A_{\mathbb{R}}-\lambda)^i\zeta_j^k,\ (A_{\mathbb{R}}-\lambda)^i\sigma\tau(\zeta_j^k);\ 0\le i<k, 1\le j\le \ell_k\right\} \tag{6}$$

is linearly independent, which may not in general be an orthogonal system. Denote by $\tilde{V}_k$ the linear subspace of $E_k(\lambda)$ spanned by (6).

By an induction in $0<\ell<k$, let $V_{k-\ell}\subset E_{k-\ell}(\lambda)$ be the orthogonal complement of $\tilde{V}_k\oplus\cdots\oplus\tilde{V}_{k-\ell+1}\oplus E_{k-\ell-1}(\lambda)$ in $E_k(\lambda)$. Let $\zeta_j^{k-\ell}, \sigma\tau(\zeta_j^{k-\ell})$, $1\le j\le \ell_{k-\ell}$, be an orthogonal basis of $V_{k-\ell}$. The set of vectors

$$\begin{aligned}&\left\{(A_{\mathbb{R}}-\lambda)^i\zeta_j^{k-\ell};\ 0\le i<k-\ell, 1\le j\le \ell_{k-\ell}\right\}\\ &\cup\left\{(A_{\mathbb{R}}-\lambda)^i\sigma\tau(\zeta_j^{k-\ell});\ 0\le i<k-\ell, 1\le j\le \ell_{k-\ell}\right\}\end{aligned} \tag{7}$$

is linearly independent. Let $\tilde{V}_{k-\ell}\subset E_{k-\ell}(\lambda)$ be the subspace spanned by (7). As a consequence we obtain

$$E_k(\lambda)=\tilde{V}_k\oplus\cdots\oplus\tilde{V}_1. \tag{8}$$

By multiplying the vectors in (7) by $\delta^{k-\ell-i}$, it is easy to obtain a basis of (8) that exhibits the Jordan form (4). ∎

Theorem 2.5. *For $\delta>0$ and $A\in M(n,\mathbb{C})$, there exists $P\in GL(n,\mathbb{C})$ such that*

$$\bar{P}^{-1}AP=B=\begin{pmatrix}B_1 & & \\ & B_2 & \\ & & \ddots\end{pmatrix},$$

where either

$$B_j=\overline{B_j}=m(k_j,\lambda_j,\delta),\ \lambda_j\ge 0, \tag{9}$$

or

$$B_j=\begin{pmatrix}0 & \overline{m(k_j,\lambda_j,\delta)}\\ m(k_j,\lambda_j,\delta) & 0\end{pmatrix},\ \lambda_j\in\mathbb{C}\setminus\mathbb{R}. \tag{10}$$

Proof. First we apply Proposition 2.4 to the linear map $A_{\mathbb{R}}$. For each Jordan cell

$$\begin{pmatrix}m(k,\lambda,\delta) & 0\\ 0 & -m(k,\lambda,\delta)\end{pmatrix}$$

of $A_{\mathbb{R}}$ with $\lambda\ge 0$, there correspond $\zeta_1,\dots,\zeta_k\in\mathbb{R}^{2n}\subset\mathbb{C}^{2n}$ such that

$$A_{\mathbb{R}}\zeta_1=\lambda\zeta_1,\quad A_{\mathbb{R}}\zeta_i=\lambda\zeta_i+\delta\zeta_{i-1},\ 1<i\le k.$$

If we denote by $u_i=\iota^{-1}(\zeta_i)\in\mathbb{C}^n$, $1\le i\le k$, we have

$$\overline{Au_1}=\lambda u_1,\quad \overline{Au_i}=\lambda u_i+\delta u_{i-1},\ 1<i\le k,$$

to obtain (9).

For each Jordan cell

$$C = \begin{pmatrix} m(k,\lambda,\delta) & & & \\ & \overline{m(k,\lambda,\delta)} & & \\ & & -m(k,\lambda,\delta) & \\ & & & -\overline{m(k,\lambda,\delta)} \end{pmatrix}$$

of $A_{\mathbb{R}}$ with $\lambda = a + \sqrt{-1}b \in \mathbb{C} \setminus \mathbb{R}$, there correspond $\zeta_i = \xi_i + \sqrt{-1}\xi_{i+k} \in \mathbb{C}^{2n}$, $1 \le i \le k$, such that

$$\begin{aligned} A_{\mathbb{R}}\zeta_1 &= (a + \sqrt{-1}b)\zeta_1, \\ A_{\mathbb{R}}\zeta_i &= (a + \sqrt{-1}b)\zeta_i + \delta\zeta_{i-1} \quad 1 < i \le k. \end{aligned}$$

Note that $\xi_i \in \mathbb{R}^{2n}$. If we denote by $u_i = \iota^{-1}(\xi_i) \in \mathbb{C}^n$, $1 \le i \le 2k$, we have

$$\begin{aligned} \overline{Au_1} &= au_1 - bu_{1+k}, \\ \overline{Au_{1+k}} &= bu_1 + au_{1+k}, \\ \overline{Au_i} &= au_i - bu_{i+k} + \delta u_{i-1}, \\ \overline{Au_{i+k}} &= bu_i + au_{i+k} + \delta u_{i-1+k}, \quad 1 < i \le k. \end{aligned}$$

If we moreover denote by $v_i = u_i + \sqrt{-1}u_{i+k}$, $v_{i+k} = u_i - \sqrt{-1}u_{i+k}$, $1 \le i \le k$, we have

$$\begin{aligned} \overline{Av_1} &= \bar{\lambda}v_{1+k}, \\ \overline{Av_{1+k}} &= \lambda v_1, \\ \overline{Av_i} &= \bar{\lambda}v_{i+k} + \delta v_{i-1+k}, \\ \overline{Av_{i+k}} &= \lambda v_i + \delta v_{i-1}, \quad 1 < i \le k, \end{aligned}$$

to obtain (10). ∎

It was shown in [10, 11] that in an odd dimensional space $\mathbb{C}^{2n+1}$ there is no holomorphic codimension one distribution in an open neighborhood of the unit ball, which is transverse to the sphere $S^{4n+1}(1)$. In the case of linear distributions, the theorem above gives another proof of that statement, since $\bar{P}^{-1}AP$ have a Jordan cell (9) with real $\lambda_j \ge 0$ if n is odd.

As a consequence of Theorem 2.5 we obtain:

LEMMA 2.6. *Let $A = A_1 + \sqrt{-1}A_2 \in M(n,\mathbb{C})$. Let $\varphi_H(t)$, $\varphi_{A_{\mathbb{R}}}(t)$ be the characteristic polynomials of $H = \bar{A}A \in M(n,\mathbb{C})$ and $A_{\mathbb{R}} = \begin{pmatrix} A_1 & -A_2 \\ -A_2 & -A_1 \end{pmatrix} \in M(2n,\mathbb{R})$. Then*

$$\varphi_H(t^2) = \varphi_{A_{\mathbb{R}}}(t) \in \mathbb{R}[t].$$

In particular, $\varphi_H(t)$ is a real polynomial.

Proof. $P^{-1}HP = \bar{B}B = \begin{pmatrix} \bar{B}_1B_1 & & \\ & \bar{B}_2B_2 & \\ & & \dots \end{pmatrix}$, where either

$$\bar{B}_jB_j = (m(k,\lambda,\delta))^2, \ \lambda \ge 0,$$

or

$$\bar{B}_j B_j = \begin{pmatrix} (m(k,\lambda,\delta))^2 & 0 \\ 0 & \overline{(m(k,\lambda,\delta))^2} \end{pmatrix}, \ \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

□

As a consequence of the above results we can state:

THEOREM 2.7 (Extended Takagi's Factorization lemma). *Given a nonsingular complex matrix $A = (a_{ij}) \in GL(n,\mathbb{C})$ the affine space $\mathbb{C}^n$ splits into generalized coneigenspaces of A. Under some linear transformation A is then equivalent to some Jordan canonical form.*

The above theorem holds for a not necessarily symmetric matrix provided it is nonsingular.

3. Small perturbations of linear distributions. In order to better embed our next results into the classical Linear Algebra framework, we recall the definition of coneigenvalues of a complex matrix. Given a $n \times n$ complex matrix A, by definition, the *coneigenvalues* of A are the scalar numbers $\mu_1, \dots, \mu_n$ defined as follows: If an eigenvalue λ of $\bar{A}A$ does not lie on the negative real axis, then the corresponding coneigenvalue $\mu(\lambda)$ is defined as a square root of λ with nonnegative real part, the multiplicity of $\mu(\lambda)$ is set to that of λ with $\mu(\lambda) = \lambda^{\frac{1}{2}}$, $\Re(\mu(\lambda)) \geq 0$. To a real negative eigenvalue λ of $\bar{A}A$ we associate two conjugate purely imaginary coneigenvalues $\mu(\lambda) = \pm\lambda^{\frac{1}{2}}$, the multiplicity of each is by definition half the multiplicity of λ. Most complex matrices have no coneigenvectors. Nevertheless, in our case we have the following result, concerning the existence of coneigenvectors associated to the equation $\overline{Az} = \lambda z$ and to the variety of contacts of a linear distribution ω_A in $\mathbb{C}^n$.

LEMMA 3.1 ([13]). *Let $A = (a_{ij}) \in GL(n,\mathbb{C})$. Let $E = \{z \in \mathbb{C};\ \bar{A}Az = \lambda^2 z\}$ be an eigenspace of $\bar{A}A$ that belongs to the eigenvalue λ^2, $\lambda > 0$. Then $\Sigma(\lambda) := \{z \in \mathbb{C};\ \overline{Az} = \lambda z\}$ is an $\mathbb{R}$-subspace of E, and the $\mathbb{C}$-linear subspace spanned by $\Sigma(\lambda)$ is equal to E. In particular, $E \neq 0 \Leftrightarrow \Sigma(\lambda) \neq 0$.*

The following result, also motivated by Lemma 2.6, describes the variety of contacts of a distribution having a generic linear part as in Theorem 1.2.

THEOREM 3.2 (Parametrization theorem). *Let $\omega = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} z_j) dz_i$ be a complex linear one-form such that $A = (a_{ij}) \in GL(n,\mathbb{C})$. Suppose that the characteristic polynomial $\varphi_H(t)$ of $H = \bar{A}A$ has roots $\lambda_1^2, \dots, \lambda_n^2$ such that $\lambda_1 > 0$ and $|\lambda_j| \neq \lambda_1$, $1 < j \leq n$. Take $u \in \mathbb{C}^n$, $u \neq 0$, such that $\overline{Au} = \lambda_1 u$. Let $f_i = f_i(z_1, \dots, z_n)$, $i = 1, \dots, n$, be holomorphic functions defined in a neighborhood of the origin, such that $f_i(0) = 0$, $\frac{\partial}{\partial z_j} f_i(0) = 0$, $i, j = 1, \dots, n$ and put $\tilde{\omega} = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} z_j + f_i) dz_i$. Then:*

1. *The holomorphic distribution $\mathcal{K}(\tilde{\omega})$ is not transverse to the spheres $S^{2n-1}(r)$ for small $r > 0$.*
2. *There exists a C^∞ map $z = z(t) \in \mathbb{C}^n$, $t \in \mathbb{D}(r) \subset \mathbb{C}$, such that $z(t)$ lands on the variety of contacts $\Sigma \subset \mathbb{C}^n$, and $z(t) = tu + O(|t|^2)$ as $t \to 0$.*

In the proof of Theorem 3.2 we shall use the following lemma.

LEMMA 3.3. *Let $U \subset \mathbb{C}^m$ be an open set. Let $T : U \to T(U) \subset \mathbb{C}^m$ be a homeomorphism and suppose that T^{-1} is Lipschitz. Let $F : U \to \mathbb{C}^m$ be a continuous* *map and suppose that* $\mathrm{Lip}(F-T)\mathrm{Lip}(T^{-1}) < 1$. *Then F is a homeomorphism onto its image, and F^{-1} is Lipschitz:*

$$\mathrm{Lip}(F^{-1}) \le (\frac{1}{\mathrm{Lip}(T^{-1})} - \mathrm{Lip}(F-T))^{-1}.$$

If $L_x(r) = \{x' \in \mathbb{C}^m;\ \|x'-x\| \le r\}$, $r > 0$, *denotes a closed neighborhood of* $x \in \mathbb{C}^m$, *then*

$$L_{F(x)}(r') \subset F(L_x(r))$$

where $r' = r(\frac{1}{\mathrm{Lip}(T^{-1})} - \mathrm{Lip}(F-T))$.

Proof. See the Local Lipschitz Inverse Function Theorem in [17, Appendix I]. □

Proof of Theorem 3.2. Let $\delta > 0$ be small enough such that $|\lambda_j| \notin (\lambda_1 - \delta, \lambda_1 + \delta)$, $1 < j \le n$. By a linear coordinate change if necessary, we may suppose without loss of generality that

$$A = (a_{ij}) = \begin{pmatrix} \lambda_1 & & \\ & B_2 & \\ & & \ddots \end{pmatrix}, \tag{11}$$

where either $B_j = m(k_j, \mu_j, \delta)$ or $B_j = \begin{pmatrix} 0 & \overline{m(k_j,\mu_j,\delta)} \\ m(k_j,\mu_j,\delta) & 0 \end{pmatrix}$, $j > 1$, with μ_j^2 a characteristic root of $H = \bar{A}A$.

Let $T, F : \mathbb{C}^n \to \mathbb{C}^n$ be the mappings defined by

$$\begin{aligned} T(z_1, z_2, \dots, z_n) &= (\overline{z_1}, \frac{\overline{z_1}(\sum_{j=2}^n a_{2j}z_j)}{\lambda_1 z_1}, \dots, \frac{\overline{z_1}(\sum_{j=2}^n a_{nj}z_j)}{\lambda_1 z_1}), \\ F(z_1, z_2, \dots, z_n) &= (\overline{z_1}, \frac{\overline{z_1}(\sum_{j=2}^n a_{2j}z_j + f_2)}{\lambda_1 z_1 + f_1}, \dots, \frac{\overline{z_1}(\sum_{j=2}^n a_{nj}z_j + f_n)}{\lambda_1 z_1 + f_1}). \end{aligned}$$

Although the images of the origin $T(0)$, $F(0)$ are indeterminate, the limits

$$\lim_{t\to 0} T(tz_1, \dots, tz_n) = \lim_{t \to 0} F(tz_1, \dots, tz_n) = 0$$

exist if $z_1 \ne 0$. Let

$$L_{\zeta_1,(\zeta_2,\dots,\zeta_n)}(r) = L_{(\zeta_1,\zeta_2,\dots,\zeta_n)}(r) = \left\{ z = (\zeta_1, z_2, \dots, z_n) \in \mathbb{C}^n;\ \max_{1<j\le n} |z_j - \zeta_j| \le r \right\}$$

be a closed polydisk of codimension 1 with radius $r > 0$ and the center $(\zeta_1, \zeta_2 \dots, \zeta_n)$. Denote by $L_{\zeta_1} = L_{\zeta_1,0}(|\zeta_1|)$ for the simplicity in later use. There exist $c_1, r_1 > 0$ such that

$$|f_i(z)| \le c_1|z_1|^2,\ |\frac{\partial}{\partial z_j} f_i(z)| \le c_1 |z_1|, \quad 1 \le i, j \le n,$$

whenever $z \in L_{\zeta_1}$ and $|\zeta_1| \le r_1$. Then we have

$$|\lambda_1 z_1 + f_1(z)| \ge \lambda_1 |z_1| - c_1|z_1|^2 \ge \frac{1}{2}\lambda_1|z_1|$$

whenever $z \in L_{z_1}$ and $|z_1| \le r_2 = \min\left\{r_1, \frac{\lambda_1}{2c_1}\right\}$.

The partial derivatives of the mapping $(F-T)(z) = F(z)-T(z)$ by z_j, $1 < j \le n$, can be expressed as:

$$\begin{aligned}
&\frac{\partial}{\partial z_j}((F-T)(z))_i \\
&= \overline{z_1}\left(\frac{\partial f_i/\partial z_j}{\lambda_1 z_1 + f_1(z)} - \frac{(\sum_{k=1}^n a_{ik}z_k + f_i(z))(\partial f_1/\partial z_j)}{(\lambda_1 z_1 + f_1(z))^2} - \frac{a_{ij}f_1(z)}{(\lambda_1 z_1 + f_1(z))\lambda_1 z_1}\right), \\
&\qquad\qquad 1 < i, j \le n.
\end{aligned}$$

So, for $z \in L_{z_1}$ with $|z_1| \le r_2$, we have

$$\left|\frac{\partial}{\partial z_j}((F-T)(z))_i\right| \le |z_1|\left(\frac{2c_1}{\lambda_1} + \frac{4c_1}{\lambda_1^2}(|\lambda_i| + \delta + c_1|z_1|) + \frac{2c_1}{\lambda_1^2}|a_{ij}|\right),$$

and

$$\begin{aligned}
&\sum_{j=2}^n \left|\frac{\partial}{\partial z_j}((F-T)(z))_i\right| \\
&\le |z_1|\left(\frac{2(n-1)c_1}{\lambda_1} + \frac{4(n-1)c_1}{\lambda_1^2}(|\lambda_i| + \delta + \frac{1}{2}\lambda_1) + \frac{2c_1}{\lambda_1^2}(|\lambda_i| + \delta)\right) \\
&\le c_2|z_1|, \qquad c_2 = \frac{(4n-2)c_1}{\lambda_1^2}(\lambda_1 + \max_{1<i\le n}|\lambda_i| + \delta).
\end{aligned}$$

If we use the max norm $\|(z_2, \dots, z_n)\| = \max_{1<i\le n}|z_i|$ on L_{z_1}, the Lipschitz constant of the mapping $F - T|L_{\zeta_1} : L_{\zeta_1} \to L_{\bar{\zeta}_1,0}(\infty)$, $|\zeta_1| \le r_2$, denoted by $\mathrm{Lip}_{\zeta_1}(F-T)$, has an upper bound

$$\mathrm{Lip}_{\zeta_1}(F-T) \le c_2|\zeta_1|.$$

Without loss of generality we may assume that

$$\begin{aligned}
\lambda_1 + \delta &< |\lambda_j|, \quad 1 < j \le k, \\
\lambda_1 - \delta &> |\lambda_j|, \quad k < j \le n,
\end{aligned}$$

for some $1 \le k \le n$. Let

$$\rho = \max\left\{\frac{\lambda_1}{\min_{1<j\le k}|\lambda_j| - \delta}, \frac{\max_{k<j\le n}|\lambda_j| + \delta}{\lambda_1}\right\} < 1.$$

We have a splitting $\mathbb{C}^n = E^1 \oplus E^u \oplus E^s$, $E^1 \simeq \mathbb{C}$, $E^u \simeq \mathbb{C}^{k-1}$, $E^s \simeq \mathbb{C}^{n-k}$, and $T = T_1 \oplus T_u \oplus T_s$, such that $T_1 = T|E^1 : z_1 \mapsto \bar{z}_1$ is a complex conjugacy, $T_u = T|E^u$ is an expansion $\|(T_u)^{-1}\| = \frac{\lambda_1}{\min_{1<j\le k}|\lambda_j| - \delta} \le \rho < 1$, and $T_s = T|E^s$ is a contraction $\|T_s\| = \frac{\max_{k<j\le n}|\lambda_j| + \delta}{\lambda_1} \le \rho < 1$.

Let

$$D = \left\{(z_1, z_u) \in E^1 \oplus E^u;\ \|z_u\| \le c_3|z_1|^2,\ \ |z_1| \le r_3\right\}$$

be a 'quadratic' cone in $E^1 \oplus E^u$, where $c_3 = \frac{6c_1}{\lambda_1(1-\rho)}$, $r_3 = \min\left\{r_2, \frac{1-\rho}{3c_2}, \frac{1}{c_3}\right\}$. Note that $\|z_u\| \le |z_1|$ when $(z_1, z_u) \in D$. Consider the space

$$\Psi = \left\{\psi_s \in C^\infty(D, E^s);\ \|\psi_s(z_1, z_u)\| \le c_3|z_1|^2,\ \ \mathrm{Lip}_{z_1}(\psi_s) \le 1\right\}$$

of $\mathbb{C}^{n-k}$-valued C^∞-functions of $(z_1, z_u) \in D$. We shall define the graph transform $\Gamma^s : \Psi \to \Psi$ by

$$\Gamma^s(\psi_s) = \pi^s \circ \overline{F} \circ (\mathrm{id}, \psi_s) \circ [\pi^{1u} \circ \overline{F} \circ (\mathrm{id}, \psi_s)]^{-1}|D.$$

Claim 3.4. *The map $\Gamma^s(\psi_s) : D \to E^s$ is well-defined.*

Proof of Claim 3.4. If $\psi_s \in \Psi$, then

$$\begin{aligned}\mathrm{Lip}_{z_1}(\pi^u F(\mathrm{id}, \psi_s) - T^u) &\le \mathrm{Lip}(\pi^u)\mathrm{Lip}_{z_1}(F - T)\mathrm{Lip}_{z_1}(\mathrm{id}, \psi_s)\\ &\le \mathrm{Lip}_{z_1}(F - T) \le c_2|z_1| < \frac{1}{\rho}.\end{aligned}$$

So Lemma 3.3 applies to see that

$$\begin{aligned}\mathrm{Lip}_{z_1}(\pi^u \circ F \circ (\mathrm{id}, \psi_s)) &\le (\frac{1}{\|(T^u)^{-1}\|} - \mathrm{Lip}_{z_1}(\pi^u F(\mathrm{id}, \psi_s) - T^u))^{-1}\\ &\le (\frac{1}{\rho} - c_2|z_1|)^{-1} < \frac{3}{5 - 2\rho} < 1\end{aligned}$$

and

$$L_{(\bar{\zeta}_1,0)}(c_3|\zeta_1|^2) \subset L_{F(\zeta_1,0)}(\frac{(5-2\rho)c_3|\zeta_1|^2}{3}) \subset F(L_{(\zeta_1,0)}(c_3|\zeta_1|^2)),$$

because

$$\frac{(5-2\rho)c_3|\zeta_1|^2}{3} - |F(\zeta_1, 0)| - c_3|\zeta_1|^2 \ge (\frac{(5-2\rho)c_3}{3} - \frac{2c_1}{\lambda_1} - c_3)|\zeta_1|^2 = \frac{2c_1}{\lambda_1}|\zeta_1|^2 > 0.$$

Thus $\pi^{1u}\overline{F}(\mathrm{id}, \psi_s)$ overflows D, that is, $\pi^{1u}\overline{F}(\mathrm{id}, \psi_s)(D) \supset D$. Hence the mapping

$$\Gamma^s(\psi_s) = \pi^s\overline{F}(\mathrm{id}, \psi_s)[\pi^{1u}\overline{F}(\mathrm{id}, \psi_s)]^{-1}|D : D \to E^s$$

is well-defined. □

Claim 3.5. *$\Gamma^s(\psi_s) \in \Psi$.*

Proof of Claim 3.5. We see that

$$\begin{aligned}\mathrm{Lip}_{z_1}(\Gamma^s(\psi_s)) &\le \mathrm{Lip}_{z_1}(\pi^s F(\mathrm{id}, \psi_s))\mathrm{Lip}_{z_1}([\pi^{1u} \circ \overline{F} \circ (\mathrm{id}, \psi_s)]^{-1})\\ &\le \mathrm{Lip}_{z_1}(\pi^s F(\mathrm{id}, \psi_s))\\ &\le \|T^s\| + \mathrm{Lip}_{z_1}(\pi^s(F - T)(\mathrm{id}, \psi_s))\\ &\le \rho + c_2|z_1| \le 1,\end{aligned}$$

and

$$\begin{aligned}&\|\pi^s F(z_1, z_u, \psi_s(z_1, z_u))\|\\ &\le \|\pi^s F(z_1, z_u, \psi_s) - \pi^s T(z_1, z_u, \psi_s)\| + \|\pi^s T(z_1, z_u, \psi_s)\|\\ &\le \|(F - T)(z_1, z_u, \psi_s)\| + \|T^s\|\|\psi_s\|\\ &\le \|(F - T)(z_1, z_u, \psi_s) - (F - T)(z_1, 0, 0)\| + \|(F - T)(z_1, 0, 0)\| + \rho c_3|z_1|^2\\ &\le \mathrm{Lip}_{z_1}(F - T)\|(z_u, \psi_s)\| + \|F(z_1, 0, 0)\| + \rho c_3|z_1|^2\\ &\le \frac{1-\rho}{3}c_3|z_1|^2 + \frac{2c_1}{\lambda_1}|z_1|^2 + \rho c_3|z_1|^2 \le c_3|z_1|^2.\end{aligned}$$

Thus we obtain $\Gamma^s(\psi_s) \in \Psi$. $\square$

CLAIM 3.6. *For all ψ_s, $\tilde{\psi}_s \in \Psi$ we have:*

$$\|\Gamma^s(\tilde{\psi}_s) - \Gamma^s(\psi_s)\| \le \|\tilde{\psi}_s - \psi_s\|.$$

Hence Γ^s is a contraction of Ψ.

Proof of Claim 3.6. Since

$$\mathrm{Lip}_{z_1}(\pi^s F) \le \mathrm{Lip}_{z_1}(\pi^s F - \pi^s T) + \|T^s\| \le c_2|z_1| + \rho \le \frac{1+2\rho}{3} < 1,$$

we have

$$\begin{aligned}
&\|\pi^s F(z_1, z_u, z_s) - \Gamma^s(\psi_s)(\pi^{1s} F(z_1, z_u, z_s))\| \\
&\le \|\pi^s F(z_1, z_u, z_s) - \pi^s F(z_1, z_u, \psi_s))\| \\
&\qquad + \|\pi^s F(z_1, z_u, \psi_s)) - \Gamma^s(\psi_s)(\pi^{1u} F(z_1, z_u, z_s))\| \\
&\le \|\pi^s F(z_1, z_u, z_s) - \pi^s F(z_1, z_u, \psi_s))\| \\
&\qquad + \|\Gamma^s(\psi_s)(\pi^{1u} F(z_1, z_u, \psi_s)) - \Gamma^s(\psi_s)(\pi^{1u} F(z_1, z_u, z_s))\| \\
&\le \mathrm{Lip}_{z_1}(\pi^s F)\|z_s - \psi_s\| + \mathrm{Lip}_{z_1}(\Gamma^s(\psi_s))\|\pi^{1u} F(z_1, z_u, \psi_s) - \pi^{1u} F(z_1, z_u, z_s)\| \\
&\le \frac{1+2\rho}{3}\|z_s - \psi_s\| + \|\pi^{1u}(F-T)(z_1, z_u, \psi_s) - \pi^{1u}(F-T)(z_1, z_u, z_s)\| \\
&\le \frac{1+2\rho}{3}\|z_s - \psi_s\| + \frac{1-\rho}{3}\|\psi_s - z_s\| \\
&= \frac{2+\rho}{3}\|z_s - \psi_s\|.
\end{aligned}$$

By letting $z_s = \tilde{\psi}_s(z_1, z_u)$, we obtain

$$\|\Gamma^s(\tilde{\psi}_s) - \Gamma^s(\psi_s)\| \le \frac{2+\rho}{3}\|\tilde{\psi}_s - \psi_s\|, \quad \psi_s, \tilde{\psi}_s \in \Psi.$$

Since $\rho < 1$ by definition, then Γ^s is a contraction of Ψ. $\square$

As a consequence, there exists a unique $\psi_s \in \Psi$ such that $\Gamma^s(\psi_s) = \psi_s$.

Next we consider the space

$$\Phi = \{\varphi \in C^\infty(\mathbb{D}(r_3), E^u);\ \mathrm{graph}(\varphi) \subset D\}$$

and define the graph transform $\Gamma^u : \Phi \to \Phi$ by

$$\pi^{1u}\overline{F}(\mathrm{id}, \psi_s)(\mathrm{graph}\Gamma^u(\varphi)) = \mathrm{graph}\varphi,$$

that is,

$$(z_1, \Gamma^u(\varphi)(z_1)) = [\pi^{1u}\overline{F}(\mathrm{id}, \psi_s)]^{-1}(z_1, \varphi(z_1)).$$

Then Γ^u is a contraction of Φ with the contraction constant $\frac{3}{5-2\rho} < 1$. There exists a unique $\varphi \in \Phi$ such that $\Gamma^u(\varphi) = \varphi$. If we denote by $z(t) = (t, \varphi(t), \psi_s(t, \varphi(t)))$ for $t \in \mathbb{D}(r_3)$, we obtain

$$\overline{F(z(t))} = z(t),$$

so that $z(t)$ lies on the variety of contacts Σ. It is clear that $\varphi(t) = O(|t|^2)$, $\psi_s(t, \varphi(t)) = O(|t|^2)$, as $t \to 0$. This ends the proof of Theorem 3.2. $\square$

Applying then Theorem 3.2 to each eigenvalue λ_i we obtain:

COROLLARY 3.7. *Let $\omega = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} z_j) dz_i$ be a complex linear one-form such that $A = (a_{ij}) \in GL(n, \mathbb{C})$. Suppose that the characteristic polynomial $\varphi_H(t)$ of $H = \bar{A}A$ has distinct roots $\lambda_1^2 > \cdots > \lambda_n^2 > 0$. Then in a neighborhood of the origin the variety of contacts Σ contains a union of finitely many 'branches' which are parametrized by C^∞ maps $z = z_i(t) \subset \Sigma \subset \mathbb{C}^n$, $t \in \mathbb{D}(r) \subset \mathbb{C}$, such that $z_i(t) = tu_i + O(|t|^2)$ as $t \to 0$, where $\overline{Au_i} = \lambda_i u_i$, $1 \leq i \leq n$.*

Theorem 1.2 is now an immediate consequence of Theorem 3.2 above while Theorem 1.3 follows immediately from Corollary 3.7 above.

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