ASIAN J. MATH.
Vol. 19, No. 2, pp. 321–342, April 2015

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G_2-STRUCTURES ON EINSTEIN SOLVMANIFOLDS*

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Abstract. We study the G_2 analogue of the Goldberg conjecture on non-compact solvmanifolds. In contrast to the almost-Kähler case we prove that a 7-dimensional solvmanifold cannot admit any left-invariant calibrated G_2-structure φ such that the induced metric g_φ is Einstein, unless g_φ is flat. We give an example of 7-dimensional solvmanifold admitting a left-invariant calibrated G_2-structure φ such that g_φ is Ricci-soliton. Moreover, we show that a 7-dimensional (non-flat) Einstein solvmanifold (S, g) cannot admit any left-invariant cocalibrated G_2-structure φ such that the induced metric $g_\varphi = g$.

Key words. Calibrated G_2-structures, Cocalibrated G_2-structures, Einstein metrics, Ricci-solitons, Kähler-Einstein metrics, solvable Lie groups.

AMS subject classifications. Primary 53C25, 53C38; Secondary 22E25, 53C55.

1. Introduction. A 7-dimensional smooth manifold M^7 is said to admit a G_2–structure if there is a reduction of the structure group of its frame bundle from $GL(7, \mathbb{R})$ to the exceptional Lie group G_2 which can actually be viewed naturally as a subgroup of $SO(7)$. Therefore a G_2-structure determines a Riemannian metric and an orientation. In fact, one can prove that the presence of a G_2-structure is equivalent to the existence of a certain type of a non-degenerate 3-form φ on the manifold. By [11] a manifold M^7 with a G_2-structure comes equipped with a Riemannian metric g, a cross product P, a 3-form φ, and orientation, which satisfy the relation

$$\varphi(X, Y, Z) = g(P(X, Y), Z),$$

for every vector field X, Y, Z.

This is exactly analogue to the data of an almost Hermitian manifold, which comes with a Riemannian metric, an almost complex structure J, a 2-form F, and an orientation, which satisfy the relation $F(X, Y) = g(JX, Y)$.

Whenever this 3-form φ is covariantly constant with respect to the Levi-Civita connection then the holonomy group is contained in G_2 and the 3-form φ is closed and co-closed.

A G_2-structure is called *calibrated* if the 3-form φ is closed and it can be viewed as the G_2 analogous of an almost-Käher structure in almost Hermitian geometry. By the results in [6, 8] no compact 7-dimensional manifold M^7 can support a calibrated G_2-structure φ whose underlying metric g_φ is Einstein unless g_φ has holomomy contained in G_2. This could be considered to be a G_2 analogue of the Goldberg conjecture in almost-Kähler geometry. The result was generalized by R.L. Bryant to calibrated G_2-structures with too tightly pinched Ricci tensor and by R. Cleyton and S. Ivanov to calibrated G_2-structures with divergence-free Weyl tensor.

A non-compact complete Einstein (non-Kähler) almost-Kähler manifold with negative scalar curvature was constructed in [3] and in [14] it was shown that it is an

*Received December 10, 2012; accepted for publication December 30, 2013.
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almost-Kähler solvmanifold, that is, a simply connected solvable Lie group S endowed with a left-invariant almost-Kähler structure [14]. In Section 3 we show that in dimension six this is the unique example of Einstein almost-Kähler (non-Kähler) solvmanifold and we classify the 6-dimensional solvmanifolds admitting a left-invariant (non-flat) Kähler-Einstein structure.

A natural problem is then to study the existence of calibrated G_2-structures inducing Einstein metrics on non-compact homogeneous Einstein manifolds. All the known examples of non-compact homogeneous Einstein manifolds belong to the class of solvmanifolds, that is, they are simply connected solvable Lie groups S endowed with a left invariant metric (see for instance the survey [19]). A left-invariant metric on a Lie group S will be always identified with the inner product $\langle \cdot, \cdot \rangle$ determined on the Lie algebra $\mathfrak{s}$ of S. According to a long standing conjecture attributed to D. Alekseevskii (see [4, 7.57]), these might exhaust the class of non-compact homogeneous Einstein manifolds.

On the other hand, Lauret in [20] showed that the Einstein solvmanifolds are *standard*, i.e. satisfy the following additional condition: if $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition of the Lie algebra $\mathfrak{s}$ of S with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, then $\mathfrak{a}$ is abelian.

A left-invariant Ricci-flat metric on a solvmanifold is necessarily flat [2], but solvmanifolds can admit incomplete metrics with holonomy contained in G_2 as shown in [12, 7].

In Section 4 by using the classification of 7-dimensional Einstein solvmanifolds and some obstructions to the existence of calibrated G_2-structures, in contrast to the almost-Kähler case, we prove that a 7-dimensional solvmanifold cannot admit any left-invariant calibrated G_2-structure φ such that the induced metric g_φ is Einstein, unless g_φ is flat.

If φ is co-closed, then the G_2-structure is called *cocalibrated*. In Section 5 we show that a 7-dimensional (non-flat) Einstein solvmanifold (S, g) cannot admit any left-invariant cocalibrated G_2-structure φ such that the induced metric $g_\varphi = g$.

2. Preliminaries on Einstein solvmanifolds. By [20] all the Einstein solvmanifolds are standard. Standard Einstein solvmanifolds constitute a distinguished class that has been deeply studied by J. Heber, who has obtained many remarkable structural and uniqueness results, by assuming only the standard condition (see [13]). In contrast to the compact case, a standard Einstein metric is unique up to isometry and scaling among left-invariant metrics [13, Theorem E]. The study of standard Einstein solvmanifolds can be reduced to the rank-one case, that is, to the ones with $\dim \mathfrak{a} = 1$ (see [13, Sections 4.5,4.6]) and everything is determined by the nilpotent Lie algebra $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. Indeed, a nilpotent Lie algebra $\mathfrak{n}$ is the nilradical of a rank-one Einstein solvmanifold if and only if $\mathfrak{n}$ admits a nilsoliton metric (also called a minimal metric), meaning that its Ricci operator is a multiple of the identity modulo a derivation of $\mathfrak{n}$.

Any standard Einstein solvmanifold is isometric to a solvmanifold whose underlying metric Lie algebra resembles an Iwasawa subalgebra of a semisimple Lie algebra in the sense that ad_A is symmetric and nonzero for any $A \in \mathfrak{a}$, $A \neq 0$. Moreover, if H denotes the mean curvature vector of S (i.e., the only element $H \in \mathfrak{a}$ such that $\mathrm{tr}(ad_A) = \langle A, H \rangle$, for every $A \in \mathfrak{a}$), then the eigenvalues of $ad_H|_{\mathfrak{n}}$ are all positive integers without a common divisor, say $k_1 < \ldots < k_r$. If $d_1, \ldots, d_r$ denote the corresponding multiplicities, then the tuple

$$(k; d) = (k_1 < \ldots < k_r; d_1, \ldots, d_r)$$

is called the *eigenvalue type* of S. It turns out that $\mathbb{R}H \oplus \mathfrak{n}$ is also an Einstein solvmanifold (with inner product the restriction of $\langle\cdot,\cdot\rangle$ on it). It is thus enough to consider rank-one (i.e. $\dim \mathfrak{a} = 1$) metric solvable Lie algebras since every higher rank Einstein solvmanifold will correspond to a unique rank-one Einstein solvmanifold and to a certain abelian subalgebra of derivations of $\mathfrak{n}$ containing ad_H. In every dimension, only finitely many eigenvalue types occur.

By [22, Lemma 11], [1]) and [13, Proposition 6.12] it follows that if $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$ is an Einstein (non-flat) solvable Lie algebra, such that $\dim \mathfrak{a} = m$ and $[\mathfrak{s}, \mathfrak{s}]$ is abelian, then the eigenvalue type is $(1; k)$, with $k = \dim[\mathfrak{s}, \mathfrak{s}] \geq m$.

In the case that $\mathfrak{n}$ is non abelian, it is proved in [21] that any nilpotent Lie algebra of dimension ≤ 5 admits an Einstein solvable extension. In [24] it is shown that the same is true for any of the 34 nilpotent Lie algebras of dimension 6, obtaining then a classification of all 7-dimensional rank-one Einstein solvmanifolds (see Table 2). A classification of 6 and 7-dimensional Einstein solvmanifolds of higher rank can be obtained by [25], where more in general there is a study of Ricci solitons up to dimension 7 on solvmanifolds. We recall that a Riemannian manifold (M, g) is called Ricci soliton if the metric g is such that $Ric(g) = \lambda g + L_X g$ for some $\lambda \in \mathbb{R}$, and $X \in \mathfrak{X}(M)$. Ricci solitons are called expanding, steady, or shrinking depending on whether $\lambda < 0, \lambda = 0$, or $\lambda > 0$. Any nontrivial homogeneous Ricci soliton must be non-compact, expanding and non-gradient (see for instance [21]). Up to now, all known examples are isometric to a left-invariant metric g on a simply connected Lie group G such that

$$Ric(g) = \lambda I + D, \tag{1}$$

for some $\lambda \in \mathbb{R}$ and some derivation D of the Lie algebra $\mathfrak{g}$ of G. Conversely, any left-invariant metric g which satisfies (1) is automatically a Ricci soliton. If G is solvable, these metrics are also called solvsolitons.

3. Almost-Kähler structures. An almost Hermitian manifold (M, J, g) is called an almost-Kähler manifold if the corresponding Kähler form $F(\cdot,\cdot) = g(\cdot, J\cdot)$ is a closed 2-form. In this section we study the existence of Einstein almost-Kähler structures (J, g, F) on 6-dimensional solvmanifolds.

Along all this work, the coefficient appearing in the rank-one Einstein extension of a Lie algebra will be denoted by a while the coefficients of the extension up to dimension 6 for almost-Kähler, and up to dimension 7 for G_2 manifolds, will be denoted by b_i.

THEOREM 3.1. *A 6-dimensional solvmanifold (S, g) admits a left-invariant Einstein (non-Kähler) almost-Kähler metric if and only if its Lie algebra $(\mathfrak{s}, g)$ is isometric to the rank-two Einstein solvable Lie algebra* (3) *defined below.*

A 6-dimensional solvmanifold (S, g) admits a left-invariant Kähler-Einstein structure if and only if the Lie algebra $(\mathfrak{s}, g)$ is isometric either to the rank-one Einstein solvable Lie algebra $\mathfrak{k}_4$ or to the rank-two Einstein solvable Lie algebra (2) *or to the rank-three Einstein solvable Lie algebra* (4)*; both Lie algebras* (2) *and* (4) *are given below.*

Proof. A 6-dimensional Einstein solvable Lie algebra $(\mathfrak{s}, g)$ is necessarily standard, so one has the orthogonal decomposition (with respect to g)

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},$$

with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ nilpotent and $\mathfrak{a}$ abelian. We will consider separately the different cases according to the rank of $\mathfrak{s}$, i.e., to the dimension of $\mathfrak{a}$.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is abelian, then we know by [13, Proposition 6.12] that $\mathfrak{s}$ has structure equations

$$(ae^{16}, ae^{26}, ae^{36}, ae^{46}, ae^{56}, 0),$$

where a is a non-zero real number. For this Lie algebra we get that any closed 2-form F is degenerate, i.e. satisfies $F^3 = 0$ and so it does not admit symplectic forms.

If $\dim \mathfrak{a} = 1$ but $\mathfrak{n}$ is nilpotent (non-abelian), then $(\mathfrak{s}, g)$ is isometric to one of the solvable Lie algebras $\mathfrak{k}_i$ $(i = 1, \ldots, 8)$ defined below in Table 1, endowed with the inner product g such that the basis $\{e_1, \ldots, e_6\}$ is orthonormal.

For $\mathfrak{k}_1$, $\mathfrak{k}_j$, $5 \leq j \leq 8$, we get again that any closed 2-form F is degenerate.

The Lie algebras $\mathfrak{k}_2$ and $\mathfrak{k}_3$ admit symplectic forms. However, one can check that any almost complex structure J on $\mathfrak{k}_i$ $(i = 2, 3)$ is such that $g(\cdot, \cdot) \neq F(\cdot, J\cdot)$.

For $\mathfrak{k}_4$ we get that a symplectic form is

$$F = \mu_1 e^{12} + \mu_2 e^{16} + \mu_3 e^{26} + \mu_1 e^{34} + \mu_4 e^{36} + \mu_5 e^{46} + \mu_1 e^{56},$$

where μ_i are real numbers satisfying $\mu_1 \neq 0$. The almost complex structures J such that $g(\cdot, \cdot) = F(\cdot, J\cdot)$ are given, with respect to the basis $\{e_1, \ldots, e_6\}$, by

$$Je_1 = \pm e_2, \quad Je_3 = \pm e_4, \quad Je_5 = \pm e_6,$$

with e_i the dual of e^i via the inner product, and they are integrable. Therefore, (J, g, F) are Kähler-Einsten structures on $\mathfrak{k}_4$.

In order to determine all the 6-dimensional rank-two Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions $\mathfrak{n}_4 \oplus \mathbb{R}\langle e_5 \rangle$ of the 4-dimensional nilpotent Lie algebras $\mathfrak{n}_4$.

Then we consider the standard solvable Lie algebra $\mathfrak{s}_6 = \mathfrak{n}_4 \oplus \mathfrak{a}$, with $\mathfrak{a} = \mathbb{R}\langle e_5, e_6 \rangle$ abelian and such that the basis $\{e_1, \ldots, e_6\}$ is orthonormal.

If $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is abelian and dimension of $\mathfrak{a} = 2$, we have to consider the structure equations

$$\begin{cases} de^1 = ae^{15} + b_1 e^{16}, \\ de^2 = ae^{25} + b_2 e^{26}, \\ de^3 = ae^{35} + b_3 e^{36}, \\ de^4 = ae^{45} + b_4 e^{46}, \\ de^5 = de^6 = 0 \end{cases}$$

and then to impose that the inner product for which $\{e_1, \ldots, e_6\}$ is orthonormal has to be Einstein and $d^2 e^j = 0$, $j = 1, \ldots, 6$. Solving these conditions we find that the structure equations are:

$$\begin{cases} de^1 = ae^{15} + b_1 e^{16}, \\ de^2 = ae^{25} + (-b_1 - b_3 - b_4)\, e^{26}, \\ de^3 = ae^{35} + b_3 e^{36}, \\ de^4 = ae^{45} + b_4 e^{46}, \\ de^5 = de^6 = 0. \end{cases}$$

where $a = \frac{\sqrt{2(b_1^2+b_3^2+b_4^2+b_1b_3+b_1b_4+b_3b_4)}}{2}$. This Lie algebra does not admit any symplectic form.

If $\mathfrak{n}$ is nilpotent (non-abelian) and $\dim \mathfrak{a} = 2$, two cases should be considered for $\mathfrak{n}$: $(0,0,e^{12},0)$ and $(0,0,e^{12},e^{13})$. We find that they have the following rank-one Einstein solvable extensions

$$(\tfrac{1}{2}ae^{15}, \tfrac{1}{2}ae^{25}, \tfrac{1}{4}\sqrt{22}ae^{12} + ae^{35}, \tfrac{3}{4}ae^{45}, 0),$$

if $\mathfrak{n} = (0,0,e^{12},0)$; and

$$(\tfrac{1}{4}ae^{15}, \tfrac{1}{2}ae^{25}, \tfrac{1}{2}\sqrt{5}ae^{12} + \tfrac{3}{4}ae^{35}, \tfrac{1}{2}\sqrt{5}ae^{13} + ae^{45}, 0),$$

if $\mathfrak{n} = (0,0,e^{12},e^{13})$. Now, to compute the rank-two Einstein extension of $\mathfrak{n} = (0,0,e^{12},0)$ we should consider the Lie algebra

$$\begin{cases} de^1 = \frac{1}{2}ae^{15} + b_1e^{16} + b_2e^{26} + b_3e^{36} + b_4e^{46}, \\ de^2 = \frac{1}{2}ae^{25} + b_5e^{16} + b_6e^{26} + b_7e^{36} + b_8e^{46}, \\ de^3 = \frac{1}{4}\sqrt{22}ae^{12} + ae^{35} + b_9e^{16} + b_{10}e^{26} + b_{11}e^{36} + b_{12}e^{46}, \\ de^4 = \frac{3}{4}ae^{45} + b_{13}e^{16} + b_{14}e^{26} + b_{15}e^{36} + b_{16}e^{46}, \\ de^5 = de^6 = 0. \end{cases}$$

Then we have to impose the Jacobi identity and that the inner product, such that the basis $\{e_1,\dots,e_6\}$ is orthonormal, has to be Einstein. We obtain the Einstein extension:

$$(2) \qquad \begin{cases} de^1 = \frac{1}{2}ae^{15} + b_1e^{16} + b_2e^{26}, \\ de^2 = \frac{1}{2}ae^{25} + b_2e^{16} + b_{10}e^{26}, \\ de^3 = \frac{1}{4}\sqrt{22}ae^{12} + ae^{35} + (b_1+b_{10})e^{36}, \\ de^4 = \frac{3}{4}ae^{45} - 2(b_1+b_{10})e^{46}, \\ de^5 = de^6 = 0, \end{cases}$$

where $a = \frac{4\sqrt{66}}{33}\sqrt{3b_1^2 + 5b_{10}b_1 + b_2^2 + 3b_{10}^2}$, which admits the Kähler-Einstein structures given, in terms of the orthonormal basis $\{e_1,\dots,e_6\}$, by

$$F = \mu_1\big(ae^{12} + 2\sqrt{\tfrac{2}{11}}ae^{35} + 2\sqrt{\tfrac{2}{11}}(b_1+b_{10})e^{36}\big) + \mu_2(ae^{15} + 2b_1e^{16} + 2b_2e^{26}) \\ +\mu_3(2b_2e^{16} + ae^{25} + 2b_{10}e^{26}) + \mu_4\big(3ae^{45} - 8(b_1 + b_{10}e^{46})\big) + \mu_5e^{56},$$

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = 2\sqrt{\tfrac{2}{11}}e_5 + \sqrt{\tfrac{3}{11}}e_6, \quad Je_4 = \sqrt{\tfrac{3}{11}}e_5 - 2\sqrt{\tfrac{2}{11}}e_6,$$

$$Je_5 = -2\sqrt{\tfrac{2}{11}}e_3 - \sqrt{\tfrac{3}{11}}e_4, \qquad Je_6 = -\sqrt{\tfrac{3}{11}}e_3 + 2\sqrt{\tfrac{2}{11}}e_4,$$

where μ_i are real parameters satisfying $(b_1+b_{10})\mu_1^2\mu_4 \neq 0$. The almost complex structure J is indeed complex i.e., the Nijenhuis tensor of J vanishes.

From the rank-one Einstein solvable extension of $\mathfrak{n} = (0,0,e^{12},e^{13})$ we get the

6-dimensional Einstein solvable Lie algebra of rank two:

$$(3)\qquad \begin{cases} de^1 = \frac{a}{4}e^{15} + \frac{3}{4}ae^{16}, \\ de^2 = \frac{a}{2}e^{25} - ae^{26}, \\ de^3 = \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35} - \frac{a}{4}e^{36}, \\ de^4 = \frac{1}{2}\sqrt{5}ae^{13} + ae^{45} + \frac{a}{2}e^{46}, \\ de^5 = de^6 = 0, \end{cases}$$

which admit the Einstein (non-Kähler) almost-Kähler structure given by

$$\begin{aligned} F &= \mu_1(-2\sqrt{5}e^{12} - 3e^{35} + e^{36}) + \mu_2(\sqrt{5}e^{13} + 2e^{45} + e^{46}) + \mu_3(e^{15} + 3e^{16}) \\ &\quad + \mu_4(-e^{25} + 2e^{26}) + \mu_5 e^{56}, \end{aligned}$$

$$Je_1 = e_3, \quad Je_3 = -e_1, \quad Je_2 = -\tfrac{1}{\sqrt{5}}e_5 + \tfrac{2}{\sqrt{5}}e_6, \quad Je_4 = \tfrac{2}{\sqrt{5}}e_5 + \tfrac{1}{\sqrt{5}}e_6,$$

$$Je_5 = \tfrac{1}{\sqrt{5}}e_2 - \tfrac{2}{\sqrt{5}}e_4, \qquad Je_6 = -\tfrac{1}{\sqrt{5}}e_2 - \tfrac{2}{\sqrt{5}}e_4,$$

where $\mu_2(4\mu_1^2 + \mu_2\mu_4) \neq 0$. The almost-Kähler structure is not integrable since

$$N_J(e_1, e_2) = -\sqrt{5}ae_3, \; N_J(e_1, e_5) = ae_1, \; N_J(e_1, e_6) = -2ae_1.$$

Now for the rank-three extensions we proceed as for the previous ones.

If $\dim \mathfrak{a} = 3$ and $\mathfrak{n}$ is abelian, we have the Einstein solvable Lie algebra

$$(4)\qquad \begin{cases} de^1 = ae^{14} - \frac{\sqrt{6}}{2}ae^{15} + \frac{\sqrt{2}}{2}e^{16}, \\ de^2 = ae^{24} + \frac{\sqrt{6}}{2}ae^{25} + \frac{\sqrt{2}}{2}ae^{26}, \\ de^3 = ae^{34} - \sqrt{2}ae^{36}, \\ de^4 = de^5 = de^6 = 0, \end{cases}$$

which admits the almost-Kähler structure given by

$$\begin{aligned} F &= \mu_1(\sqrt{2}e^{14} - \sqrt{3}e^{15} + e^{16}) + \mu_2(\sqrt{2}e^{24} + \sqrt{3}e^{25} + e^{26}) + \mu_3(-e^{34} + \sqrt{2}e^{36}) \\ &\quad + \mu_4 e^{45} + \mu_5 e^{46} + \mu_6 e^{56}, \end{aligned}$$

$$Je_1 = \tfrac{1}{\sqrt{3}}e_4 - \tfrac{1}{\sqrt{2}}e_5 + \tfrac{1}{\sqrt{6}}e_6, \quad Je_2 = \tfrac{1}{\sqrt{3}}e_4 + \tfrac{1}{\sqrt{2}}e_5 + \tfrac{1}{\sqrt{6}}e_6, \quad Je_3 = -\tfrac{1}{\sqrt{3}}e_4 + \sqrt{\tfrac{2}{3}}e_6,$$

$$Je_4 = -\tfrac{1}{\sqrt{3}}e_1 - \tfrac{1}{\sqrt{3}}e_2 + \tfrac{1}{\sqrt{3}}e_3, \quad Je_5 = \tfrac{1}{\sqrt{2}}e_1 - \tfrac{1}{\sqrt{2}}e_2, \quad Je_6 = -\tfrac{1}{\sqrt{6}}e_1 - \tfrac{1}{\sqrt{6}}e_2 - \sqrt{\tfrac{2}{3}}e_3,$$

where $\mu_1\mu_2\mu_3 \neq 0$ and actually the almost complex structure is complex i.e., $N_J = 0$.

If $\dim \mathfrak{a} = 3$ and $\mathfrak{n}$ is nilpotent (non-abelian) $\mathfrak{n}$ is exactly $\mathfrak{h}_3$ (the 3-dimensional Heisenberg Lie algebra), having structure equations:

$$(0, 0, e^{12}).$$

We find the following rank-one Einstein solvable extension

$$(\tfrac{a}{2}e^{14}, \tfrac{a}{2}e^{24}, ae^{12} + ae^{34}, 0),$$

proceeding in the same way as in the previous examples we find that $\mathfrak{h}_3$ does not admit a rank-three Einstein solvable extension unless it is flat. □

$\mathfrak{s}_6$	6-dimensional Einstein solvable Lie algebras of rank one
$\mathfrak{k}_1$	$(\frac{2}{13}ae^{16}, \frac{9}{13}ae^{26}, \frac{10}{13}\sqrt{3}ae^{12}+\frac{11}{13}ae^{36}, \frac{20}{13}ae^{13}+ae^{46}, \frac{10}{13}\sqrt{3}ae^{14}+\frac{15}{13}ae^{56}, 0)$
$\mathfrak{k}_2$	$(\frac{1}{4}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{4}\sqrt{30}ae^{12}+\frac{3}{4}ae^{36}, \frac{1}{4}\sqrt{30}ae^{13}+ae^{46}, -\frac{1}{2}\sqrt{5}ae^{14}-\frac{1}{2}\sqrt{5}ae^{23}+\frac{5}{4}ae^{56}, 0)$
$\mathfrak{k}_3$	$(\frac{3}{10}ae^{16}, \frac{2}{5}ae^{26}, \frac{3}{5}ae^{36}, \frac{1}{5}\sqrt{30}ae^{12}+\frac{7}{10}ae^{46}, \frac{1}{5}\sqrt{15}ae^{23}+\frac{1}{5}\sqrt{30}ae^{14}+ae^{56}, 0)$
$\mathfrak{k}_4$	$(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}ae^{36}, \frac{1}{2}ae^{46}, ae^{12}+ae^{34}+ae^{56}, 0)$
$\mathfrak{k}_5$	$(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, 2ae^{12}+ae^{36}, \sqrt{3}ae^{13}+\frac{3}{2}ae^{46}, \sqrt{3}ae^{23}+\frac{3}{2}ae^{56}, 0)$
$\mathfrak{k}_6$	$(\frac{1}{3}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}ae^{36}, ae^{12}+\frac{5}{6}ae^{46}, ae^{13}+\frac{5}{6}ae^{56}, 0)$
$\mathfrak{k}_7$	$(\frac{1}{2}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{2}\sqrt{7}ae^{12}+ae^{36}, \frac{3}{4}ae^{46}, \frac{3}{4}ae^{56}, 0)$
$\mathfrak{k}_8$	$(\frac{1}{4}ae^{16}, \frac{1}{2}ae^{26}, \frac{1}{4}\sqrt{26}ae^{12}+\frac{3}{4}ae^{36}, \frac{1}{4}\sqrt{26}ae^{13}+ae^{46}, \frac{3}{4}ae^{56}, 0)$

TABLE 1. *Rank-one Einstein 6-dimensional solvable Lie algebras.*

4. Calibrated G_2-structures. In this section we study the existence of calibrated G_2-structures φ on 7-dimensional solvable Lie algebras whose underlying Riemannian metric g_φ is Einstein. We will use the classification of the 7-dimensional Einstein solvable Lie algebras and the following obstructions.

LEMMA 4.1. *[9] If there is a non zero vector X in a 7-dimensional Lie algebra $\mathfrak{g}$ such that $(i_X\varphi)^3 = 0$ for all closed 3-form $\varphi \in Z^3(\mathfrak{g}^*)$, then $\mathfrak{g}$ does not admit any calibrated G_2-structure.*

LEMMA 4.2.
Let $\mathfrak{g}$ be a 7-dimensional Lie algebra and φ a G_2-structure on $\mathfrak{g}$. Then the bilinear form $g_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$g_\varphi(X,Y)vol = \frac{1}{6}(i_X\varphi \wedge i_Y\varphi \wedge \varphi)$$

has to be a Remannian metric.

Proof. It follows by the fact that in general there is a $1-1$ correspondence between G_2-structures on a 7-manifold and 3-forms φ for which the 7-form-valued bilinear form B_φ defined by

$$B_\varphi(X,Y) = (i_X\varphi \wedge i_Y\varphi \wedge \varphi)$$

is positive definite (see [5], [15]). ☐

LEMMA 4.3. *Let $(\mathfrak{s}, g)$ be a 7-dimensional Enstein solvable Lie algebra endowed with a G_2-structure φ, then, for any $A \in \mathfrak{a} = [\mathfrak{s},\mathfrak{s}]^\perp$ such that $g_\varphi(A,A) = 1$, the forms*

$$\alpha = i_A\varphi, \quad \beta = \varphi - \alpha \wedge A^*,$$

define an $SU(3)$-structure on $(\mathbb{R}\langle A\rangle)^\perp$, where by $A^* \in \mathfrak{s}^*$ we denote the dual of A. So in particular $\alpha \wedge \beta = 0$ and $\alpha^3 \neq 0$.

Proof. It follows by Proposition 4.5 in [23]. ☐

In contrast with the almost-Kähler case, we can prove the following theorem

THEOREM 4.4. *A 7-dimensional solvmanifold cannot admit any left-invariant calibrated G_2-structure φ such that g_φ is Einstein, unless g_φ is flat.*

In particular, if the 7-dimensional Einstein (non-flat) solvmanifold (S,g) has rank one, then (S,g) hasa calibrated G_2-structure if and only if the Lie algebra $\mathfrak{s}$ of S is isometric to the Einstein solvable Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_4$, $\mathfrak{g}_9$, $\mathfrak{g}_{18}$, $\mathfrak{g}_{28}$ in Table 2.

Proof. A 7-dimensional Einstein solvable Lie algebra $(\mathfrak{s}, g)$ is necessarily standard, so one has the orthogonal decomposition (with respect to g)

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},$$

with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ nilpotent and $\mathfrak{a}$ abelian. We will consider separately the different cases according to the rank of $\mathfrak{s}$, i.e., to the dimension of $\mathfrak{a}$.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is abelian, then we know by [13, Proposition 6.12] that $\mathfrak{s}$ has structure equations

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where a is a non-zero real number. Computing the generic closed 3-form on $\mathfrak{s}$ it is easy to check that $\mathfrak{s}$ cannot admit any calibrated G_2-structure.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{n}$ is nilpotent (non-abelian), then $(\mathfrak{s}, g)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i, i = 1, \dots, 33$, in Table 2, endowed with the inner product such that the basis $\{e_1, \dots, e_7\}$ is orthonormal. We may apply Lemma 4.1 with $X = e_6$ to all the Lie algebras $\mathfrak{g}_i, i = 1, \dots, 33$, except to the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_4$, $\mathfrak{g}_9$, $\mathfrak{g}_{18}$, $\mathfrak{g}_{28}$, showing in this way that they do not admit any calibrated G_2-structure. For the remaining Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$ we first determine the generic closed 3-form φ and then, by applying Lemma 4.3, we impose, that $\alpha \wedge \alpha \wedge \alpha \neq 0$ and $\alpha \wedge \beta = 0$, where

$$\alpha = i_{e_7}\varphi, \quad \beta = \varphi - e^7 \wedge \beta. \tag{5}$$

Moreover, we have that the closed 3-form φ defines a G_2-structure if and only the matrix associated to the symmetric bilinear form g_φ, with respect to the orthonormal basis $\{e_1, \dots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a calibrated G_2-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the basis $\{e_1, \dots, e_7\}$, is a multiple of the identity matrix. By a direct computation we have that then the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$ admit a calibrated G_2-structure (see Table 3) but they do not admit any calibrated G_2-structure inducing a Einstein (non-flat) metric.

Next, we show that result for the Lie algebra $\mathfrak{g}_{28}$. To this end, we see that any closed 3-form φ on $\mathfrak{g}_{28}$ has the following expression:

$$\begin{aligned}\varphi = {}& \rho_{1,2,7}e^{127} - \frac{1}{2}\rho_{5,6,7}e^{136} + \rho_{2,4,7}e^{137} + \frac{1}{2}\rho_{5,6,7}e^{145} - \rho_{2,3,7}e^{147} - \rho_{2,6,7}e^{157}\\ &+ \rho_{2,5,7}e^{167} + \frac{1}{2}\rho_{5,6,7}e^{235} + \rho_{2,3,7}e^{237} + \frac{1}{2}\rho_{5,6,7}e^{246} + \rho_{2,4,7}e^{247} + \rho_{2,5,7}e^{257}\\ &+ \rho_{2,6,7}e^{267} + \rho_{3,4,7}e^{347} + \rho_{3,5,7}e^{357} + \rho_{3,6,7}e^{367} + \rho_{3,6,7}e^{457} - \rho_{3,5,7}e^{467}\\ &+ \rho_{5,6,7}e^{567},\end{aligned}$$

where $\rho_{i,j,k}$ are arbitrary constants denoting the coefficients of e^{ijk}.

In this case, one can check that the induced metric is given by the matrix G with elements

$$\begin{aligned}&g(e_1,e_1) = -\tfrac{1}{4}\rho_{1,2,7}\rho_{5,6,7}^2, \quad g(e_1,e_2) = 0, \quad g(e_1,e_3) = \tfrac{1}{4}\rho_{2,3,7}\rho_{5,6,7}^2,\\ &g(e_1,e_4) = \tfrac{1}{4}\rho_{2,4,7}\rho_{5,6,7}^2, \quad g(e_1,e_5) = \tfrac{1}{4}\rho_{2,5,6}\rho_{5,6,7}^2, \quad g(e_1,e_6) = \tfrac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2,\\ &g(e_1,e_7) = 0, \quad g(e_2,e_2) = -\tfrac{1}{4}\rho_{1,2,7}\rho_{5,6,7}^2, \quad g(e_2,e_3) = -\tfrac{1}{4}\rho_{2,4,7}\rho_{5,6,7}^2,\\ &g(e_2,e_4) = \tfrac{1}{4}\rho_{2,3,7}\rho_{5,6,7}^2, \quad g(e_2,e_5) = \tfrac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2, \quad g(e_2,e_6) = -\tfrac{1}{4}\rho_{2,5,7}\rho_{5,6,7}^2,\end{aligned}$$

$g(e_2,e_7)=0, \quad g(e_3,e_3)=-\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2, \quad g(e_3,e_4)=0, \quad g(e_3,e_5)=\frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2,$
$g(e_3,e_6)=-\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, \quad g(e_3,e_7)=0, \quad g(e_4,e_4)=-\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2,$
$g(e_4,e_5)=-\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, \quad g(e_4,e_6)=-\frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2, \quad g(e_4,e_7)=0,$
$g(e_5,e_5)=\frac{\rho_{5,6,7}^3}{4}, \quad g(e_5,e_6)=0, \quad g(e_5,e_7)=0, \quad g(e_6,e_6)=\frac{\rho_{5,6,7}^3}{4}, \quad g(e_6,e_7)=0,$

$$\begin{aligned} g(e_7,e_7)= & -\rho_{5,6,7}\rho_{2,3,7}^2+\rho_{1,2,7}\rho_{3,5,7}^2+\rho_{1,2,7}\rho_{3,6,7}^2+\rho_{2,5,7}^2\rho_{3,4,7}+\rho_{2,6,7}^2\rho_{3,4,7} \\ & +\rho_{2,5,7}\left(2\rho_{2,3,7}\rho_{3,6,7}-2\rho_{2,4,7}\rho_{3,5,7}\right)-2\rho_{2,6,7}\left(\rho_{2,3,7}\rho_{3,5,7}+\rho_{2,4,7}\rho_{3,6,7}\right) \\ & -\rho_{2,4,7}^2\rho_{5,6,7}+\rho_{1,2,7}\rho_{3,4,7}\rho_{5,6,7}. \end{aligned}$$

Now we have that the system $G=k\cdot I_7$ does not have solution, for any real number k, where I_7 is the identity matrix. This means that the Lie algebra $\mathfrak{g}_{28}$ does not admit any calibrated G_2-structure defining an Einstein metric. However, we can solve 48 from the 49 equations of the system $G=k\cdot I_7$, and we obtain the metric defined by the matrix

$$G=2\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&4 \end{pmatrix}.$$

Since this matrix is positive definite, the Lie algebra $\mathfrak{g}_{28}$ has a calibrated G_2 form

$$\varphi=-2e^{127}-2e^{347}-e^{136}+e^{145}+e^{235}+e^{246}+2e^{567},$$

which induces the metric defined by G.

In order to determine all the 7-dimensional rank-two Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions $\mathfrak{s}_6=\mathfrak{n}_5\oplus\mathbb{R}\langle e_6\rangle$ of any of the eight 5-dimensional nilpotent Lie algebras $\mathfrak{n}_5$ (see Table 1) and then consider the standard solvable Lie algebra $\mathfrak{s}_7=\mathfrak{n}_5\oplus\mathfrak{a}$, with $\mathfrak{a}=\mathbb{R}\langle e_6,e_7\rangle$ abelian and such that the basis $\{e_1,\dots,e_7\}$ is orthonormal. From $\mathfrak{k}_1$ we get the 7-dimensional Einstein Lie algebra of rank two with structure equations

$$\begin{cases} de^1=\frac{1}{3}\sqrt{6}ae^{16}+4ae^{17},\\ de^2=\frac{3}{2}\sqrt{6}ae^{26}-7ae^{27},\\ de^3=\frac{5}{3}\sqrt{18}ae^{12}+\frac{11}{6}\sqrt{6}ae^{36}-3ae^{37},\\ de^4=\frac{10}{3}\sqrt{6}ae^{13}+\frac{13}{6}\sqrt{6}ae^{46}+ae^{47},\\ de^5=\frac{5}{3}\sqrt{18}ae^{14}+\frac{5}{2}\sqrt{6}ae^{56}+5ae^{57},\\ de^6=de^7=0. \end{cases}$$

By computing the generic closed 3-form φ and by using Lemma 4.2 and Lemma 4.3, we get that the matrix associated to g_φ, with respect to the basis $\{e_1,\dots,e_7\}$, cannot be a multiple of the identity matrix.

From $\mathfrak{k}_2$ we do not get any 7-dimensional Einstein Lie algebra of rank two. From

$\mathfrak{k}_3$ we get the 7-dimensional Einstein Lie algebra of rank two with structure equations

$$\begin{cases} de^1 = \frac{1}{7}\sqrt{21}ae^{16} - ae^{17}, \\ de^2 = \frac{4}{21}\sqrt{21}ae^{26} + 2ae^{27}, \\ de^3 = \frac{2}{7}\sqrt{21}ae^{36} - 2ae^{37}, \\ de^4 = \frac{2}{21}\sqrt{30}\sqrt{21}ae^{12} + \frac{1}{3}\sqrt{21}ae^{46} + ae^{47}, \\ de^5 = \frac{2}{21}\sqrt{30}\sqrt{21}ae^{14} + \frac{2}{21}\sqrt{15}\sqrt{21}ae^{23} + \frac{10}{21}\sqrt{21}ae^{56}, \\ de^6 = de^7 = 0. \end{cases}$$

By computing the generic closed 3-form φ and by using Lemma 4.2 and Lemma 4.3, we get that the matrix associated to g_φ, with respect to the basis $\{e_1, \dots, e_7\}$, cannot be a multiple of the identity matrix. From $\mathfrak{k}_4$ we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

$$\begin{cases} de^1 = ae^{16} - b_7e^{17} + b_2e^{27} + b_3e^{37} + b_4e^{47}, \\ de^2 = ae^{26} + b_2e^{17} + b_7e^{27} + b_4e^{37} - b_3e^{47}, \\ de^3 = ae^{36} + b_3e^{17} + b_4e^{27} - b_{19}e^{37} + b_{14}e^{47}, \\ de^4 = ae^{46} + b_4e^{17} - b_3e^{27} + b_{14}e^{37} + b_{19}e^{47}, \\ de^5 = 2a(e^{12} + e^{34} + e^{56}), \\ de^6 = de^7 = 0 \end{cases}$$

where $a = \frac{1}{2}\sqrt{b_7^2 + b_2^2 + 2b_3^2 + 2b_4^2 + b_{14}^2 + b_{19}^2}$. We may then apply Lemma 4.1 with $X = e_5$.

From $\mathfrak{k}_5$ we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

$$\begin{cases} de^1 = ae^{16} + b_{19}e^{17} + b_{20}e^{27}, \\ de^2 = ae^{26} + b_{20}e^{17} - b_{19}e^{27}, \\ de^3 = 4ae^{12} + 2ae^{36}, \\ de^4 = 2\sqrt{3}ae^{13} + 3ae^{46} + b_{19}e^{47} + b_{20}e^{57}, \\ de^5 = 2\sqrt{3}ae^{23} + 3ae^{56} + b_{20}e^{47} - b_{19}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

For these Lie algebras in order to study completely all possibilities we will study separately the two cases $b_{20} \neq 0$ and $b_{20} = 0$. By computing the generic closed 3-form φ and using Lemma 4.3 we have that the system $g_\varphi(e_i, e_j) - k\delta_i^j = 0$ (where k is a non zero positive real number) with variables the coefficients c_{ijk} of e^{ijk} in φ has no solutions.

From $\mathfrak{k}_6$ we get the two families of 7-dimensional Einstein Lie algebras of rank two, namely $\mathfrak{k}_{6,1}$ and $\mathfrak{k}_{6,2}$ with respectively structure equations

$$1) \begin{cases} de^1 = 2ae^{16} + 2(b_{19} + b_{25})e^{17}, \\ de^2 = 3ae^{26} - (b_{19} + 2b_{25})e^{27} + b_{12}e^{37}, \\ de^3 = 3ae^{36} + b_{12}e^{27} - (b_{19} + 2b_{25})e^{37}, \\ de^4 = 6ae^{12} + 5ae^{46} + b_{19}e^{47} + b_{12}e^{57}, \\ de^5 = 6ae^{13} + 5ae^{56} + b_{12}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0 \end{cases}$$

and

$$2)\begin{cases} de^1 = \sqrt{2}b_{25}e^{16} + 4b_{25}e^{17}, \\ de^2 = \frac{3}{2}\sqrt{2}b_{25}e^{26} - 3b_{25}e^{27} - b_{12}e^{37}, \\ de^3 = \frac{3}{2}\sqrt{2}b_{25}e^{36} + b_{12}e^{27} - 3b_{25}e^{37}, \\ de^4 = 3\sqrt{2}b_{25}e^{12} + \frac{5}{2}\sqrt{2}b_{25}e^{46} + b_{25}e^{47} - b_{12}e^{57}, \\ de^5 = 3\sqrt{2}b_{25}e^{13} + \frac{5}{2}\sqrt{2}b_{25}e^{56} + b_{12}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

For 1) we compute first the generic closed 3-forms φ and then, using Lemma 4.3 for $A = e_7$, we impose the condition $\alpha \wedge \beta = 0$. By this condition we get in particular that

$$\rho_{1,2,3}\rho_{1,3,4}(b_{19} + b_{25}) = 0,$$

where by $\rho_{i,j,k}$ we denote the coefficient of e^{ijk} in φ. One can immediately exclude the case $\rho_{1,3,4} = 0$, since otherwise the element of the matrix associated to the metric g_φ has to be zero. Then we study separately the cases $\rho_{1,2,3} = 0$ and $b_{19} = -b_{25}$. In both cases we do not find any solution for the system $g_\varphi(e_i, e_j) - k\delta_i^j = 0$. For 2) we study separately the cases $b_{12}b_{25} \neq 0$, $b_{12} = 0$ and $b_{25} = 0$.

In the case $b_{12}b_{25} \neq 0$ we compute first the generic closed 3-forms φ and then, using Lemma 4.3 for $A = e_7$, we impose the condition $\alpha \wedge \alpha \wedge \alpha \neq 0$, getting the condition $\rho_{1,2,5} \neq 0$. Thus, we take the system $S_{ij} = g_\varphi(e_i, e_j) - k\delta_i^j = 0$ and get the values of $\rho_{2,5,6}, \rho_{3,4,6}, \rho_{3,5,6}$ and $\rho_{2,3,6}$ from $S_{5,5}, S_{3,5}, S_{4,4}$ and $S_{3,4}$. Now $S_{3,3} = -k$, and the system does not admit any solution.

In the case $b_{12} = 0$ we first compute the generic closed 3-forms φ and then we use Lemma 4.3 for $A = e_7$, obtaining that $\rho_{2,3,6}, \rho_{2,4,6}, \rho_{3,5,6}$ and $\rho_{4,5,6}$ are all different from zero.

In the case $b_{25} = 0$ we first compute the generic closed 3-forms φ and then we may apply Lemma 4.1 with $X = e_1, \dots, e_5$.

From $\mathfrak{k}_7$ we get the four families of 7-dimensional Einstein Lie algebras of rank two, namely $\mathfrak{k}_{7,1}, \mathfrak{k}_{7,2}, \mathfrak{k}_{7,3}$ and $\mathfrak{k}_{7,4}$ with respectively structure equations

$$i)\begin{cases} de^1 = ae^{16} + (-b_7 + b_{13})e^{17} + b_6e^{27}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27}, \\ de^3 = a(\sqrt{7}e^{12} + 2e^{36}) + b_{13}e^{37}, \\ de^4 = \frac{3}{2}ae^{46} - (2b_{13} + b_{25})e^{47} + b_{24}e^{57}, \\ de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} + b_{25}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \frac{2}{21}\sqrt{21b_7^2 - 21b_7f_{13} + 63b_{13}^2 + 21b_6^2 + 42b_{25}b_{13} + 21b_{25}^2 + 21b_{24}^2}$,

$$ii)\begin{cases} de^1 = ae^{16} + (-b_7 + b_{13})e^{17} + b_6e^{27}, \\ de^2 = ae^{26} + b_6e^{17} + b_7e^{27}, \\ de^3 = a(\sqrt{7}e^{12} + 2e^{36}) + b_{13}e^{37}, \\ de^4 = \frac{3}{2}ae^{46} - b_{13}e^{47} - b_{24}e^{57}, \\ de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} - b_{13}e^{57}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a=\frac{2}{21}\sqrt{21b_7^2-21b_7b_{13}+42b_{13}^2+21b_6^2}$

$$iii)\begin{cases} de^1 = ae^{16}+\frac{1}{2}b_{13}e^{17}-b_6e^{27},\\ de^2 = ae^{26}+b_6e^{17}+\frac{1}{2}b_{13}e^{27},\\ de^3 = a(\sqrt{7}e^{12}+2e^{36})+b_{13}e^{37},\\ de^4 = \frac{3}{2}ae^{46}-(2b_{13}+b_{25})e^{47}+b_{24}e^{57},\\ de^5 = \frac{3}{2}ae^{56}+b_{24}e^{47}+b_{25}e^{57},\\ de^6 = de^7 = 0, \end{cases}$$

where $a=\frac{1}{21}\sqrt{231b_{13}^2+168b_{13}b_{25}+84b_{25}^2+84b_{24}^2}$

$$iv)\begin{cases} de^1 = \frac{1}{3}\sqrt{3}b_{25}e^{16}-\frac{1}{2}b_{25}e^{17}-b_6e^{27},\\ de^2 = \frac{1}{3}\sqrt{3}b_{25}e^{26}+b_6e^{17}-\frac{1}{2}b_{25}e^{27},\\ de^3 = \frac{1}{3}\sqrt{3}b_{25}(\sqrt{7}e^{12}+2e^{36})-b_{25}e^{37},\\ de^4 = \frac{1}{2}\sqrt{3}b_{25}e^{46}+b_{25}e^{47}-b_{24}e^{57},\\ de^5 = \frac{1}{2}\sqrt{3}b_{25}e^{46}+b_{24}e^{47}+b_{25}e^{57},\\ de^6 = de^7 = 0. \end{cases}$$

For all of them after computing the generic closed 3-forms we may apply Lemma 4.1 with $X=e_3$.

From $\mathfrak{k}_8$ we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

$$\begin{cases} de^1 = ae^{16}+(-b_{13}+b_{19})e^{17},\\ de^2 = ae^{26}+(2b_{13}-b_{19})e^{27},\\ de^3 = a(\sqrt{26}e^{12}+3e^{36})+b_{13}e^{37},\\ de^4 = a(\sqrt{26}e^{13}+4e^{46})+b_{19}e^{47},\\ de^5 = 3ae^{56}-(2b_{13}+b_{19})e^{57},\\ de^6 = de^7 = 0. \end{cases}$$

where $a=\frac{1}{39}\sqrt{390b_{13}^2-78b_{13}b_{19}+156b_{19}^2}$.

We study separately the cases $b_{13}b_{19}\neq 0$, $b_{13}=0$ and $b_{19}=0$.

In the case $b_{13}b_{19}\neq 0$ using Lemma 4.3 (i.e. $\alpha\wedge\alpha\wedge\alpha\neq 0$, with $A=e_7$) we may suppose $\rho_{1,2,4}\rho_{1,3,5}\neq 0$ for the generic closed 3-form. Now, we consider the system $S_{ij}=g_\varphi(e_i,e_j)-k\delta_i^j=0$. We take $\rho_{1,2,5},\rho_{1,2,3}$ and a from $S_{4,5}=0=S_{3,4}$ and $S_{2,2}=0$. In the new system we can conclude that from equations $S_{2,4}=0$ and $S_{3,5}=0$ that there is no solution. Indeed from the two equations it follows that $b_{13}=-\frac{19}{5}b_{19}$ and $b_{13}=\frac{7}{31}b_{19}$, which is a contradiction since $b_{13}b_{19}\neq 0$.

In the case $b_{13}=0$ using Lemma 4.3 we may suppose $\rho_{1,3,5}\rho_{3,4,7}\neq 0$ for the generic closed 3-form. Then we get $\rho_{2,5,7},k$ and $\rho_{2,3,7}$ from $S_{5,5}=0=S_{5,3}=S_{2,3}$. The new system satisfies:

$$S_{4,4}=\frac{7\rho_{1,3,5}\left(49\rho_{1,2,5}^2+152\rho_{3,4,7}^2\right)}{76\sqrt{78}}.$$

In the case $f_{19}=0$ using Lemma 4.3 we may suppose $c_{1,3,5}c_{3,4,7}\neq 0$ for the generic closed 3-form. Then we get $c_{1,2,3},c_{2,5,6},c_{2,3,7}$ and $c_{1,3,5}$ from $S_{3,4}=0=S_{2,3}=S_{2,4}$ and $S_{3,3}=0$. The new system satisfies:

$$S_{4,4}=-\frac{959322c_{1,2,5}^2c_{3,4,7}^4+59711k^2}{59711k}.$$

what implies again $S_{4,4} \neq 0$.

If $[\mathfrak{s},\mathfrak{s}] = \mathfrak{n}$ is abelian and $\dim \mathfrak{n} = 5$, we have to consider the structure equations

$$\begin{cases} de^1 = ae^{16} + b_1 e^{17}, \\ de^2 = ae^{26} + b_2 e^{27}, \\ de^3 = ae^{36} + b_3 e^{37}, \\ de^4 = ae^{46} + b_4 e^{47} \\ de^5 = ae^{56} + b_5 e^{57}, \\ de^6 = de^7 = 0. \end{cases}$$

By imposing that $\mathfrak{s}$ is a Einstein Lie algebra (the inner product is the one for which $\{e_1,\dots,e_7\}$ is orthonormal), we get the Lie algebras with structure equations

$$\begin{cases} de^1 = ae^{16} + (-b_2 - b_3 - b_4)e^{17}, \\ de^2 = ae^{26} + b_2 e^{27}, \\ de^3 = ae^{36} + b_3 e^{37}, \\ de^4 = ae^{46} + b_4 e^{47} \\ de^5 = ae^{56}, \\ de^6 = de^7 = 0, \end{cases}$$

where $a = \sqrt{10b_7^2 + 10b_7b_{13} + 10b_7b_{19} + 10b_{13}^2 + 10b_{13}b_{19} + 10b_{19}^2}$. For these Lie algebras we first compute the generic closed 3-forms φ and then we may apply Lemma 4.1 with $X = e_1,\dots,e_5$.

In order to determine all the 7-dimensional rank-three Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions $\mathfrak{n}_4 \oplus \mathbb{R}\langle e_5\rangle$ of the two 4-dimensional nilpotent Lie algebras $\mathfrak{n}_4$ and then consider the standard solvable Lie algebra $\mathfrak{s}_7 = \mathfrak{n}_4 \oplus \mathfrak{a}$, with $\mathfrak{a} = \mathbb{R}\langle e_5, e_6, e_7\rangle$ abelian and such that the basis $\{e_1,\dots,e_7\}$ is orthonormal. For any of the nilpotent Lie algebras $\mathfrak{n}_4$ we find the following rank-one Einstein solvable extensions

1) $(\frac{1}{4}ae^{15}, \frac{1}{2}ae^{25}, \frac{1}{2}\sqrt{5}ae^{12} + \frac{3}{4}ae^{35}, \frac{1}{2}\sqrt{5}ae^{13} + ae^{45}, 0)$
2) $(\frac{1}{2}ae^{15}, \frac{1}{2}ae^{25}, \frac{1}{4}\sqrt{22}ae^{12} + ae^{35}, \frac{3}{4}ae^{45}, 0)$

From 1) we do not get any 7-dimensional Einstein Lie algebra of rank three. From 2) we get the 7-dimensional Einstein Lie algebra of rank three with structure equations

$$(6) \qquad \begin{cases} de^1 = \frac{1}{2}ae^{15} - (b_{10} + \frac{1}{2}b_{28})e^{16} + b_2 e^{26} + (-b_{14} + b_{23})e^{17} + b_6 e^{27}, \\ de^2 = \frac{1}{2}ae^{25} + b_9 e^{16} + b_{10}e^{26} + b_{13}e^{17} + b_{14}e^{27}, \\ de^3 = \frac{1}{4}\sqrt{22}ae^{12} + ae^{35} - \frac{1}{2}b_{28}e^{36} + b_{23}e^{37}, \\ de^4 = \frac{3}{4}ae^{45} + b_{28}e^{46} - 2b_{23}e^{47}, \\ de^5 = de^6 = de^7 = 0. \end{cases}$$

satisfying the conditions $d^2e^i = 0, i = 1,\dots,4$, and one of the following:

(i) $a = \sqrt{\frac{32b_{14}^2 - 32b_{14}b_{23} + 96b_{23}^2 + 32b_{13}^2}{33}}$, $b_2 = b_9 = \pm\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}\,b_{13}b_{23}$, $b_6 = b_{13}, b_{10} = \mp\frac{1}{11}\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}(-13b_{14}b_{23} + 6b_{23}^2 + 2b_{14}^2 + 2b_{13}^2)$, $b_{28} = \pm\frac{2}{11}\sqrt{\frac{11}{4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2}}(4b_{14}^2 - 4b_{14}b_{23} + b_{23}^2 + 4b_{13}^2)$

(ii) $a = 2\sqrt{\frac{2}{3}}b_{23}, b_2 = \pm\frac{1}{2}\sqrt{11b_{23}^2 - 4b_{10}^2}, b_6 = b_{13} = b_{28} = 0, b_{14} = \frac{1}{2}b_{23}$.

For the case (i) we consider the generic closed 3-form φ and use all the time that $a \neq 0$ and the condition $\alpha \wedge \beta = 0$ (with $\alpha = i_{e_7}\varphi$). By imposing the vanishing of $g_\varphi(e_3, e_4)$ and the condition $g_\varphi(e_3, e_3) \neq 0$ we have always that either $g_\varphi(e_2, e_2) = 0$ or $g_\varphi(e_3, e_3) = 0$ so we cannot have calibrated G_2-structures associated to the Einstein metric.

For the case (ii) we start only to impose the conditions

$$a = \frac{2}{3}\sqrt{6}b_{23}, \qquad b_6 = b_{13} = b_{28} = 0, \qquad b_{14} = \frac{1}{2}b_{23},$$

one needs for the Einstein condition still to impose that $b_2 = \pm\frac{1}{2}\sqrt{11b_{23}^2 - 4b_{10}^2}$. We consider the generic closed 3-form φ, using all the time that $b_{23} \neq 0$ (since $a \neq 0$) and we impose $\alpha \wedge \beta = 0$, where $\alpha = i_{e_7}\varphi$. We use that $g_\varphi(e_3, e_3) \neq 0$ and the equations

$$g_\varphi(e_3, e_4) = g_\varphi(e_2, e_3) = g_\varphi(e_1, e_3) = 0.$$

Studying separately the solutions of the above system we show that no calibrated G_2-structure can induce the Einstein metric.

If $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ is abelian and $\dim \mathfrak{n} = 4$, we get the Einstein solvable Lie algebras of rank three with the structure equations

$$\begin{cases} de^1 = ae^{15} + b_1e^{16} + b_2e^{17}, \\ de^2 = ae^{25} + b_3e^{26} + b_4e^{27}, \\ de^3 = ae^{35} + b_5e^{36} + b_6e^{37}, \\ de^4 = ae^{45} + b_7e^{46} + b_8e^{47}, \\ de^5 = de^6 = de^7 = 0, \end{cases} \tag{7}$$

satisfying the conditions

$$b_7 = -b_1 - b_3 - b_5, \qquad b_8 = -b_2 - b_4 - b_6,$$

$$2b_1b_2 + 2b_3b_4 + 2b_5b_6 + b_2b_3 + b_2b_5 + b_1b_4 + b_4b_5 + b_1b_6 + b_3b_6 = 0,$$

$$2b_1^2 + 2b_3^2 + 2b_5^2 + 2b_1b_3 + 2b_1b_5 + 2b_3b_5 = 4a^2.$$

We impose for the generic 3-form φ the conditions $d\varphi = 0$ and $\alpha \wedge \beta = 0$ (with $\alpha = i_{e_7}\varphi$), using all the time that $a \neq 0$. Then we consider the equations

$$g_\varphi(e_1, e_2) = g_\varphi(e_1, e_3) = g_\varphi(e_1, e_4) = g_\varphi(e_2, e_3) = 0. \tag{8}$$

By the conditions

$$g_\varphi(e_1, e_1) \neq 0, \quad g_\varphi(e_2, e_2) \neq 0, \quad g_\varphi(e_3, e_3) \neq 0, \quad g_\varphi(e_4, e_4) \neq 0$$

we get respectively

$$\rho_{1,2,5}\rho_{1,3,5}\rho_{1,4,5} \neq 0,\ \rho_{1,2,5}\rho_{2,3,5}\rho_{2,4,5} \neq 0,\ \rho_{1,3,5}\rho_{2,3,5}\rho_{3,4,5} \neq 0,\ \rho_{1,4,5}\rho_{2,4,5}\rho_{3,4,5} \neq 0.$$

In all the solutions of (8) we have always that either $\rho_{1,2,5} = 0$ or $\rho_{1,3,5} = 0$ or $\rho_{1,4,5} = 0$, which is not possible. □

So, now we can conclude that there is not a counterexample for the G_2 analogue of the Goldberg conjecture in the class of solvmanifolds, but we can show the existence of a calibrated G_2-structure whose underlying metric is a (non trivial) Ricci soliton.

EXAMPLE. *If we take the 6-dimensional nilpotent Lie algebra* $\mathfrak{n}_6 = (0,0,0,0,e^{13}-e^{24}, e^{14}+e^{23})$ *and we compute the rank-one Ricci soliton solvable extension, we obtain the Lie algebra with structure equations:*

$$\mathfrak{s} = (-\frac{1}{2}e^{17}, -\frac{1}{2}e^{27}, e^{37}, e^{47}, e^{13} - e^{24} - \frac{1}{2}e^{57}, e^{14} + e^{23} - \frac{1}{2}e^{67}, 0)$$

where the Ricci soliton metric g *is the one making the basis* $\{e_1, \dots, e_7\}$ *orthonormal. The Lie algebra* $\mathfrak{s}$ *admits the calibrated* G_2 *form*

$$\varphi = -e^{136} + e^{145} + e^{235} + e^{246} + e^{567} - e^{127} - e^{347}$$

such that $g_\varphi = g$ *and* $d * \varphi \neq 0$. *Therefore* g_φ *is a Ricci-soliton on* $\mathfrak{s}$ *but* φ *is not parallel.*

5. Cocalibrated G_2-structures. In this section we study the existence of cocalibrated G_2-structures φ on 7-dimensional Einstein solvable Lie algebras $(\mathfrak{s}, g)$ whose underlying Riemannian metric $g_\varphi = g$.

We will use the classification of the 7-dimensional Einstein solvable Lie algebras of the previous section, Lemma 4.2 and 4.3, together with the following obstructions.

LEMMA 5.1. *Let* $(\mathfrak{g}, g)$ *be a* 7*-dimensional metric Lie algebra. If for every closed* 4*-form* $\Psi \in Z^4(\mathfrak{g}^*)$ *there exists* $X \in \mathfrak{g}$ *such that* $(i_X(*\Psi))^3 = 0$, *then* $\mathfrak{g}$ *does not admit a cocalibrated* G_2*-structure inducing the metric* g.

Proof. It is sufficient to prove that if a 3-form φ defines a G_2-structure on $\mathfrak{g}$ then for any $X \in \mathfrak{g}$ we have

$$\iota_X(*\varphi) \wedge \varphi \neq 0,$$

where * is the Hodge star operator with respect to the metric g_φ associated to φ. Since the 3-form φ defines a G_2-structure on $\mathfrak{g}$, then there exists a basis of $\mathfrak{g}$ $\{f_1, \cdots, f_7\}$ (which is orthonormal with respect to g_φ) such that

$$\varphi = f^{124} + f^{235} + f^{346} + f^{457} + f^{156} + f^{267} + f^{137},$$

where $\{f^1, \cdots, f^7\}$ is the basis of $\mathfrak{g}^*$ dual to $\{f_1, \cdots, f_7\}$. Taking the Hodge star operator with respect to $\{f_1, \cdots, f_7\}$ we have

$$*\varphi = -f^{3567} + f^{1467} - f^{1257} + f^{1236} + f^{2347} - f^{1345} - f^{2456}.$$

Thus, writing ι_X the contraction by X,

$$\iota_{f_1}(*\varphi) = f^{467} - f^{257} + f^{236} - f^{345}$$

and

$$\iota_{f_1}(*\varphi) \wedge \varphi = 4f^{234567}.$$

Similarly, we have

$$\begin{aligned}
\iota_{f_2}(*\varphi) &= f^{157} - f^{136} + f^{347} - f^{456}, \quad \iota_{f_2}(*\varphi) \wedge \varphi = -4f^{134567};\\
\iota_{f_3}(*\varphi) &= -f^{567} + f^{126} - f^{247} + f^{145}, \quad \iota_{f_3}(*\varphi) \wedge \varphi = 4f^{124567};\\
\iota_{f_4}(*\varphi) &= -f^{167} + f^{237} - f^{135} + f^{256}, \quad \iota_{f_4}(*\varphi) \wedge \varphi = -4f^{123567};\\
\iota_{f_5}(*\varphi) &= f^{367} - f^{127} + f^{134} - f^{246}, \quad \iota_{f_5}(*\varphi) \wedge \varphi = 4f^{123467};\\
\iota_{f_6}(*\varphi) &= -f^{357} + f^{147} - f^{123} + f^{245}, \quad \iota_{f_6}(*\varphi) \wedge \varphi = -4f^{123457};\\
\iota_{f_7}(*\varphi) &= f^{356} - f^{146} + f^{125} - f^{234}, \quad \iota_{f_7}(*\varphi) \wedge \varphi = 4f^{123456}.
\end{aligned}$$

In general, for $i = 1, \cdots 7$, we see that

$$\iota_{f_i}(*\varphi) \wedge \varphi = (-1)^{i+1} 4 f^{12\cdots(i-1)(i+1)\cdots 7},$$

which is a non-zero 6-form for any $i = 1, \dots, 7$. □

LEMMA 5.2. *Let $(\mathfrak{g}, g)$ be a 7-dimensional metric Lie algebra. If for any coclosed 3-form φ on $\mathfrak{g}$, the differential form $\tau_3 = *d\varphi|_{\Lambda^3_{27}\mathfrak{g}^*}$ satisfies the conditions*

$$\varphi \wedge \tau_3 \neq 0 \text{ or } (*\varphi) \wedge \tau_3 \neq 0$$

then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure inducing the metric g.

Proof. The expression of the differential and the codifferential of a G_2 form φ are given in terms of the intrinsic torsion forms by

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3$$
$$d * \varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi.$$

with $\tau_0 \in \Lambda^0\mathfrak{g}^*$, $\tau_1 \in \Lambda^1\mathfrak{g}^*$, $\tau_2 \in \Lambda^2_{14}\mathfrak{g}^*$ and $\tau_3 \in \Lambda^3_{27}\mathfrak{g}^*$. The cocalibrated condition $d * \varphi = 0$ implies

$$d\varphi = \tau_0 * \varphi + *\tau_3.$$

Since

$$\Lambda^3_{27}\mathfrak{g}^* = \{\rho \in \Lambda^3\mathfrak{g}^* | \rho \wedge \varphi = 0 = \rho \wedge *\varphi\},$$
$$\Lambda^4_{27}\mathfrak{g}^* = \{\gamma \in \Lambda^4\mathfrak{g}^* | \gamma \wedge \varphi = 0 = \gamma \wedge *\varphi\}$$

it follows that

$$d\varphi \wedge \varphi = \tau_0 |\varphi|^2 e^{1234567}.$$

Therefore

$$\tau_3 = - * (d\varphi - \tau_0 * \varphi).$$

Now as $\tau_3 \in \Lambda^3_{27}\mathfrak{g}^*$ the conditions

$$\tau_3 \wedge \varphi = 0, \quad \tau_3 \wedge *\varphi = 0$$

must be fulfilled. □

We recall that a 5-dimensional manifold N has an $SU(2)$-structure if there exists a quadruplet $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form and ω_i are 2-forms on N satisfying $\omega_i \wedge \omega_j = \delta_{ij} v, \; v \wedge \eta \neq 0$ for some nowhere vanishing 4-form v, and

$$\iota_X \omega_3 = \iota_Y \omega_1 \implies \omega_2(X, Y) \geq 0.$$

PROPOSITION 5.3. *Let $(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, g)$ be a 7-dimensional Einstein Lie algebra of rank two and let $\{e_1, \dots, e_7\}$ be an orthonormal basis of $(\mathfrak{g}, g)$ such that $\mathfrak{a} = \mathbb{R}\langle e_6, e_7\rangle$, then a G_2-structure φ on $\mathfrak{g}$ induces an $SU(2)$-structure on $\mathfrak{n}$ such that the associated metric h is the restriction of g_φ to $\mathfrak{n}$.*

Proof. By Lemma 4.3 we know that the forms $F = \iota_{e_7}\varphi$, $\psi_+ = \varphi - F \wedge e^7$ determine an $SU(3)$-structure on $\mathbb{R}\langle e_1, \dots, e_6\rangle$ and that the associated metric is the restriction of g to $\mathbb{R}\langle e_1, \dots, e_6\rangle$. Now we can write $F = f^{12} + f^{34} + f^5 \wedge e^6$ and $\psi_+ + \imath\psi_- = (f^1 + \imath f^2) \wedge (f^3 + \imath f^4) \wedge (f^5 + \imath e^6)$ where $f_i \in \mathbb{R}\langle e_1, \dots, e_5\rangle$ and $\{f_1, \dots, f_5, e_6\}$ is orthonormal. Then by [10, Proposition 1.4] the forms

$$\eta = f^5, \quad \omega_1 = f^{12} + f^{34}, \quad \omega_2 = f^{13} + f^{42}, \quad \omega_3 = f^{14} + f^{23}$$

define an $SU(2)$-structure on $\mathfrak{n}$. The basis $\{f_1, \dots, f_5\}$ is orthonormal with respect to the metric h induced by the $SU(2)$-structure. So, h coincides with the restriction of g_φ to $\mathfrak{n}$. □

COROLLARY 5.4. *Let $(\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, g)$ be a 7-dimensional Einstein Lie algebra of rank two and let $\{e_1, \dots, e_7\}$ be an orthonormal basis such that $\mathfrak{a} = \mathbb{R}\langle e_6, e_7\rangle$. If for any coclosed 3-form φ one of the following conditions*

- $(\omega_i^2 - \omega_j^2) \wedge \eta \neq 0$ *for some* $i \neq j$*;*
- $\omega_i \wedge \eta \neq *_h\omega_i$ *for some* i*,*

holds, where $(\omega_1, \omega_2, \omega_3, \eta, h)$ is the $SU(2)$-structure as in Proposition 5.3, then $(\mathfrak{g}, g)$ does not admit any cocalibrated G_2-structure φ such that $g_\varphi = g$.

Proof. By Proposition 5.3 the G_2-structure induces an $SU(2)$-structure $(\omega_1, \omega_2, \omega_3, \eta)$ on $\mathfrak{n}$. By definition of $SU(2)$-structure the forms $(\omega_1, \omega_2, \omega_3, \eta)$ have to satisfy the conditions $(\omega_i^2 - \omega_j^2) \wedge \eta \neq 0$ for all i, j and $\omega_i \wedge \eta = *_h\omega_i$ for all $i = 1, 2, 3$. □

We know already that a 7-dimensional Enstein solvable Lie algebra cannot admit nearly-parallel G_2-structures since the scalar curvature has to be positive. For the cocalibrated G_2-structures we can prove the following

THEOREM 5.5. *A 7-dimensional (nonflat) Einstein solvmanifold (S, g) cannot admit any left-invariant cocalibrated G_2-structure φ such that $g_\varphi = g$.*

Proof. For a 7-dimensional rank-one Einstein solvable Lie algebra $(\mathfrak{s}, g)$ we have the orthogonal decomposition (with respect to the Einstein metric g)

$$\mathfrak{s} = \mathfrak{n}_6 \oplus \mathfrak{a},$$

with $\mathfrak{n}_6 = [\mathfrak{s}, \mathfrak{s}]$ a 6-dimensional nilpotent Lie algebra and $\mathfrak{a} = \mathbb{R}\langle e_7\rangle$ abelian.

If $\mathfrak{n}$ is abelian, then we know by [13, Proposition 6.12] that $\mathfrak{s}$ has structure equations

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where a is a non-zero real number. Computing the generic co-closed 4-form on $\mathfrak{s}$ it easy to check that $\mathfrak{s}$ cannot admit any cocalibrated G_2-structure g_φ such that $g_\varphi = g$.

If $\mathfrak{n}_6$ is nilpotent (non-abelian), then $(\mathfrak{s}, g)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i, i = 1, \dots, 33$, in Table 2, endowed with the Riemannian metric such that the basis $\{e_1, \dots, e_7\}$ is orthonormal. We may apply Lemma 5.1 with $X = e_7$ to the Lie algebras $\mathfrak{g}_3$, $\mathfrak{g}_{13}$, $\mathfrak{g}_{23}$ and $\mathfrak{g}_j$, $25 \leq j \leq 33$, showing in this way that they do not admit any cocalibrated G_2-structure φ such that $g_\varphi = g$. For the Lie algebras:

$$\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_{20}$$

we first determine the generic co-closed 3-form φ and then, we compute the values of τ_0, and of the 3-form τ_3. We have that $\tau_3 \wedge \varphi \neq 0$ unless $\varphi = 0$ and so by applying

Lemma 5.2 we have that the Lie algebras do not admit a cocalibrated G_2-structure inducing an Einstein metric. For the Lie algebras

$$\mathfrak{g}_7, \mathfrak{g}_8, \mathfrak{g}_9, \mathfrak{g}_{10}, \mathfrak{g}_{12}, \mathfrak{g}_{14}, \mathfrak{g}_{15}, \mathfrak{g}_{16}, \mathfrak{g}_{17}, \mathfrak{g}_{18}, \mathfrak{g}_{19}, \mathfrak{g}_{21}, \mathfrak{g}_{22}, \mathfrak{g}_{24} \tag{9}$$

we first determine the generic co-closed 3-form φ and then, by applying Lemma 4.3, we impose, that $\alpha \wedge \alpha \wedge \alpha \neq 0$ and $\alpha \wedge \beta = 0$, where α and β are given in (5). Moreover, we have that the closed 3-form φ defines a G_2-structure if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the orthonormal basis $\{e_1, \ldots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a calibrated G_2-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form g_φ, with respect to the basis $\{e_1, \ldots, e_7\}$, is a multiple of the identity matrix. By a direct computation we have thus that the Lie algebras (9) cannot admit any cocalibrated G_2-structures inducing an Einstein metric.

For the 7-dimensional rank-two Einstein solvable Lie algebras, using the same notation as for the calibrated case we obtain the result for $\mathfrak{k}_1, \mathfrak{k}_3, \mathfrak{k}_5, \mathfrak{k}_{6,1}$ and $\mathfrak{k}_{6,2}$ by the first condition of Corollary 5.4. For the remaining Lie algebras, i.e., for $\mathfrak{k}_4, \mathfrak{k}_{7,1}, \mathfrak{k}_{7,2}, \mathfrak{k}_{7,3}, \mathfrak{k}_{7,4}, \mathfrak{k}_8$ and the extension of the abelian one, the result follows by using the second condition of Corollary 5.4.

In the rank-three case we have to study the extensions of the Lie algebras $\mathfrak{n}_4 = (0, 0, e^{12}, 0)$ and the four-dimensional abelian Lie algebra. For the first one we consider the structure quations (6), then we take a generic 3-form φ such that all the coefficients of e^{ijklm} in $d * \varphi$ vanish except those of e^{13467} and e^{23467}. Now if we compute the inner product g_φ induced by φ and we impose the conditions $g_\varphi(e_i, e_i) = g_\varphi(e_j, e_j)$ and $g_\varphi(e_i, e_j) = 0$ for all $i \neq j$ we obtain that $g_\varphi(e_6, e_6) = 0$. For the rank-three Einstein extension of the abelian Lie algebra we consider the structure equations (7) and we take a generic coclosed 3-form φ. By imposing $g_\varphi(e_i, e_i) = g_\varphi(e_j, e_j)$ and $g_\varphi(e_i, e_j) = 0$ for all $i \neq j$ we obtain $g_\varphi(e_6, e_6) = 0$. □

$\mathfrak{g}_1$	$\left(\frac{a}{2}e^{17}, ae^{27}, \sqrt{13}ae^{12} + \frac{3}{2}ae^{37}, 4ae^{13} + 2ae^{47}, 2\sqrt{3}ae^{14} + 2ae^{23} + \frac{5}{2}ae^{57}, -\sqrt{13}ae^{25} + 2\sqrt{3}ae^{34} + \frac{7}{2}ae^{67}, 0\right)$
$\mathfrak{g}_2$	$\left(-\frac{\sqrt{21}}{42}ae^{17}, -\frac{\sqrt{21}}{14}ae^{27}, ae^{12} - 2\frac{\sqrt{21}}{21}ae^{37}, -2\frac{\sqrt{3}}{3}ae^{13} - 5\frac{\sqrt{21}}{42}ae^{47}, ae^{14} - \frac{\sqrt{21}}{7}ae^{57}, ae^{34} - ae^{25} - 3\frac{\sqrt{21}}{14}ae^{67}, 0\right)$
$\mathfrak{g}_3$	$\left(-\frac{\sqrt{14}}{56}ae^{17}, -9\frac{\sqrt{14}}{56}ae^{27}, ae^{12} - 5\frac{\sqrt{14}}{28}ae^{37}, \frac{\sqrt{6}}{2}ae^{13} - 11\frac{\sqrt{14}}{56}ae^{47}, \frac{\sqrt{6}}{2}ae^{14} - 3\frac{\sqrt{14}}{14}ae^{57}, ae^{15} - 13\frac{\sqrt{14}}{56}ae^{67}, 0\right)$
$\mathfrak{g}_4$	$\left(ae^{17}, 2ae^{27}, 2\sqrt{7}ae^{12} + 3ae^{37}, \frac{6\sqrt{154}}{11}ae^{13} + 4ae^{47}, 2\sqrt{7}ae^{14} + 2\frac{\sqrt{1155}}{11}ae^{23} + 5ae^{57}, 2\frac{\sqrt{1155}}{11}ae^{15} + 5\frac{\sqrt{154}}{11}ae^{24} + 6ae^{67}, 0\right)$
$\mathfrak{g}_5$	$\left(ae^{17}, 3ae^{27}, 2\sqrt{14}ae^{12} + 4ae^{37}, 2\sqrt{15}ae^{13} + 5ae^{47}, 6\sqrt{2}ae^{14} + 6ae^{57}, 4\sqrt{2}ae^{15} + 2\sqrt{15}ae^{23} + 7ae^{67}, 0\right)$
$\mathfrak{g}_6$	$\left(\frac{a}{2}e^{17}, ae^{27}, \sqrt{10}ae^{12} + \frac{3}{2}ae^{37}, \sqrt{10}ae^{13} + 2ae^{47}, \sqrt{10}ae^{23} + \frac{5}{2}ae^{57}, \sqrt{10}ae^{14} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_7$	$\left(ae^{17}, ae^{27}, 4ae^{12} + 2ae^{37}, 2\sqrt{5}ae^{13} + 3ae^{47}, 2\sqrt{5}ae^{23} + 3ae^{57}, 4ae^{14} - 4ae^{24} + 4ae^{67}, 0\right)$
$\mathfrak{g}_8$	$\left(ae^{17}, ae^{27}, 4ae^{12} + 2ae^{37}, 2\sqrt{5}ae^{13} + 3ae^{47}, 2\sqrt{5}ae^{23} + 3ae^{57}, 4ae^{14} + 4ae^{24} + 4ae^{67}, 0\right)$
$\mathfrak{g}_9$	$\left(\frac{-3}{14}ae^{17}, \frac{-11}{28}ae^{27}, \frac{-3}{7}ae^{37}, \frac{\sqrt{5}}{2}ae^{12} - \frac{17}{28}ae^{47}, ae^{14} - ae^{23} - \frac{23}{28}ae^{57}, ae^{34} + \frac{\sqrt{5}}{2}ae^{15} - \frac{29}{28}ae^{67}, 0\right)$
$\mathfrak{g}_{10}$	$\left(\frac{4}{9}ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{2\sqrt{114}}{9}ae^{12} + \frac{13}{9}ae^{47}, \frac{2}{9}\sqrt{190}ae^{14} + \frac{17}{9}ae^{57}, \frac{2\sqrt{114}}{9}ae^{15} + \frac{2\sqrt{114}}{9}ae^{23} + \frac{7}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{11}$	$\left(\frac{a}{3}e^{17}, \frac{2}{3}ae^{27}, \frac{10\sqrt{7}}{21}ae^{12} + ae^{37}, \frac{4\sqrt{42}}{21}ae^{12} + ae^{47}, \frac{4\sqrt{105}}{21}ae^{13} + \frac{2\sqrt{70}}{21}ae^{14} + \frac{4}{3}ae^{57}, \frac{2\sqrt{6}}{3}ae^{15} + \frac{2\sqrt{7}}{3}ae^{24} + \frac{5}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{12}$	$\left(\frac{a}{2}e^{17}, ae^{27}, \frac{11}{6}ae^{37}, \frac{2\sqrt{21}}{3}ae^{12} + \frac{3}{2}ae^{47}, \frac{2\sqrt{21}}{3}ae^{14} + 2ae^{57}, \frac{2\sqrt{14}}{3}ae^{15} + \frac{2\sqrt{14}}{3}ae^{24} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{13}$	$\left(\frac{2}{9}ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{2\sqrt{93}}{9}ae^{12} + \frac{33}{27}ae^{47}, \frac{4\sqrt{31}}{9}ae^{14} + \frac{39}{27}ae^{57}, \frac{2\sqrt{93}}{9}ae^{15} + \frac{5}{3}e^{67}, 0\right)$
$\mathfrak{g}_{14}$	$\left(\frac{a}{2}e^{17}, ae^{27}, \frac{3}{4}ae^{37}, \frac{\sqrt{21}}{2}ae^{12} + \frac{3}{2}ae^{47}, \frac{\sqrt{14}}{2}ae^{13} + \frac{5}{4}ae^{57}, \frac{\sqrt{14}}{2}ae^{14} + \frac{\sqrt{21}}{2}ae^{35} + 2ae^{67}, 0\right)$
$\mathfrak{g}_{15}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{10}ae^{12} + 2ae^{47}, \sqrt{10}ae^{23} + 2ae^{57}, \sqrt{10}ae^{14} + \sqrt{10}ae^{35} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{16}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{10}ae^{12} + 2ae^{47}, \sqrt{10}ae^{23} + 2ae^{57}, \sqrt{10}ae^{14} - \sqrt{10}ae^{35} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{17}$	$\left(ae^{17}, ae^{27}, \frac{12}{5}ae^{37}, \frac{4}{5}\sqrt{31}ae^{12} + 2ae^{47}, \frac{2}{5}\sqrt{93}ae^{14} + 3ae^{57}, \frac{2}{5}\sqrt{93}ae^{24} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{18}$	$\left(ae^{17}, ae^{27}, 2ae^{37}, 4ae^{12} + 2ae^{47}, 2ae^{13} - 2\sqrt{3}ae^{24} + 3ae^{57}, 2\sqrt{3}ae^{14} + 2ae^{23} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{19}$	$\left(5ae^{17}, 6ae^{27}, 12ae^{37}, 2\sqrt{134}ae^{12} + 11ae^{47}, \sqrt{402}ae^{14} + 16ae^{57}, \sqrt{134}ae^{13} - \sqrt{402}ae^{24} + 17ae^{67}, 0\right)$
$\mathfrak{g}_{20}$	$\left(ae^{17}, ae^{27}, 2ae^{12} + 2ae^{37}, 2\sqrt{3}ae^{12} + 2ae^{47}, 4ae^{14} + 3ae^{57}, 2ae^{24} + 2\sqrt{3}ae^{23} + 3ae^{67}, 0\right)$
$\mathfrak{g}_{21}$	$\left(3ae^{17}, 5ae^{27}, 6ae^{37}, 2\sqrt{42}ae^{12} + 8ae^{47}, 2\sqrt{21}ae^{13} + 9ae^{57}, 2\sqrt{42}ae^{14} + 2\sqrt{21}ae^{23} + 11ae^{67}, 0\right)$
$\mathfrak{g}_{22}$	$\left(6ae^{17}, 5ae^{27}, 9ae^{37}, 2\sqrt{93}ae^{12} + 11ae^{47}, 2\sqrt{93}ae^{13} + 15ae^{57}, 4\sqrt{31}ae^{24} + 16ae^{67}, 0\right)$
$\mathfrak{g}_{23}$	$\left(ae^{17}, \frac{5}{2}ae^{27}, 3ae^{37}, \sqrt{37}ae^{12} + \frac{7}{2}ae^{47}, \frac{\sqrt{74}}{2}ae^{13} + 4ae^{57}, \sqrt{37}ae^{14} + \frac{9}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{24}$	$\left(ae^{17}, ae^{27}, ae^{37}, \sqrt{6}ae^{12} + 2ae^{47}, \sqrt{6}ae^{13} + 2ae^{57}, \sqrt{6}ae^{23} + 2ae^{67}, 0\right)$
$\mathfrak{g}_{25}$	$\left(\frac{5\sqrt{31}}{124}ae^{17}, \frac{2\sqrt{31}}{31}ae^{27}, \frac{9\sqrt{31}}{124}ae^{37}, \frac{9\sqrt{31}}{124}ae^{47}, ae^{12} + \frac{13\sqrt{31}}{124}ae^{57}, -\frac{\sqrt{3}}{2}ae^{34} - \frac{\sqrt{3}}{2}ae^{15} + \frac{9\sqrt{31}}{62}ae^{67}, 0\right)$
$\mathfrak{g}_{26}$	$\left(ae^{17}, 2ae^{27}, 3ae^{37}, 3ae^{47}, 4\sqrt{2}ae^{12} + 3ae^{57}, 4\sqrt{2}ae^{15} + 4ae^{67}, 0\right)$
$\mathfrak{g}_{27}$	$\left(ae^{17}, \frac{3}{4}ae^{27}, \frac{7}{4}ae^{37}, \frac{3}{2}ae^{47}, \frac{\sqrt{148}}{4}ae^{12} + \frac{7}{4}ae^{57}, \frac{\sqrt{74}}{4}ae^{14} + \frac{\sqrt{37}}{2}ae^{25} + \frac{5}{2}ae^{67}, 0\right)$
$\mathfrak{g}_{28}$	$\left(ae^{17}, ae^{27}, ae^{37}, ae^{47}, 2ae^{13} - 2ae^{24} + 2ae^{57}, 2ae^{14} + 2ae^{23} + 2ae^{67}, 0\right)$
$\mathfrak{g}_{29}$	$\left(ae^{17}, ae^{27}, \frac{4}{3}ae^{37}, \frac{4}{3}ae^{47}, \sqrt{6}ae^{12} + 2ae^{57}, \sqrt{6}ae^{14} + \sqrt{6}ae^{23} + \frac{7}{3}ae^{67}, 0\right)$
$\mathfrak{g}_{30}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{a}{2}e^{37}, \frac{a}{2}e^{47}, \sqrt{2}ae^{12} + ae^{57}, \sqrt{2}ae^{34} + ae^{67}, 0\right)$
$\mathfrak{g}_{31}$	$\left(\frac{\sqrt{11}}{11}ae^{17}, \frac{3\sqrt{11}}{22}ae^{27}, \frac{3\sqrt{11}}{22}ae^{37}, \frac{2\sqrt{11}}{11}ae^{47}, ae^{12} + \frac{5\sqrt{11}}{22}ae^{57}, ae^{13} + \frac{5\sqrt{11}}{22}ae^{67}, 0\right)$
$\mathfrak{g}_{32}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{a}{2}e^{37}, \frac{a}{2}e^{47}, \frac{2}{3}ae^{57}, \frac{\sqrt{11}}{3}ae^{12} + \frac{\sqrt{11}}{3}ae^{34} + ae^{67}, 0\right)$
$\mathfrak{g}_{33}$	$\left(\frac{a}{2}e^{17}, \frac{a}{2}e^{27}, \frac{3}{4}ae^{37}, \frac{3}{4}ae^{47}, \frac{3}{4}ae^{57}, \frac{\sqrt{34}}{4}ae^{12} + ae^{67}, 0\right)$

TABLE 2. *Rank-one Einstein 7-dimensional solvable Lie algebras.*

$\mathfrak{s}_7$	Calibrated G_2-structure
$\mathfrak{g}_1$	$\varphi = \frac{1}{432}\left(1440 - 128\sqrt{3}\right)e^{123} + \frac{\sqrt{13}}{2}e^{125} - \frac{\left(13312\sqrt{3}-748800\right)}{44928\sqrt{3}}e^{127}$ $+\frac{8}{9}e^{135} - 2e^{137} - \frac{1}{\sqrt{3}}e^{146} - \sqrt{3}e^{147} + 10e^{157} - e^{167} - \frac{1}{3}e^{236} + e^{237}$ $+\frac{1}{576}\left(1440 + 128\sqrt{3}\right)e^{247} + \frac{1}{\sqrt{3}}e^{267} + \frac{1}{\sqrt{3}}e^{345} - e^{357} - e^{457} + e^{567}$
$\mathfrak{g}_4$	$\varphi = -\frac{7}{2\sqrt{5}}e^{125} + e^{137} - \frac{7}{13}e^{146} - e^{147} + \frac{1}{2}e^{167} + \frac{7}{13}e^{236} - e^{237}$ $+2e^{247} - e^{267} + \frac{7}{13}e^{345} + \frac{1}{2}e^{357} - e^{457} - e^{567}$
$\mathfrak{g}_9$	$\varphi = -\frac{7}{2\sqrt{5}}e^{125} + e^{137} - \frac{7}{13}e^{146} - e^{147} + \frac{1}{2}e^{167} + \frac{7}{13}e^{236}$ $-e^{237} + 2e^{247} - e^{267} + \frac{7}{13}e^{345} + \frac{1}{2}e^{357} - e^{457} - e^{567}$
$\mathfrak{g}_{18}$	$\varphi = e^{123} - e^{127} - e^{136} + \sqrt{3}e^{145}$ $+3e^{167} + e^{235} + \sqrt{3}e^{246} - \frac{1}{2}e^{347} + 3e^{567}$
$\mathfrak{g}_{28}$	$\varphi = -2e^{127} - 2e^{347} - e^{136} + e^{145} + e^{235} + e^{246} + 2e^{567}$

TABLE 3. *Calibrated G_2-structures on rank-one Einstein solvable Lie algebras.*

Acknowledgments. We are very grateful to the referee for useful comments that helped to improve the paper. This work has been partially supported through Project MICINN (Spain) MTM2011-28326-C02-02, Project UPV/EHU ref. UFI11/52 and by GNSAGA of INdAM.

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