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006

AFFINE E_8 BASIC REPRESENTATION BUNDLES OVER RATIONAL SURFACES WITH $c_1^2 = 0$*

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Abstract. Rational surfaces with $c_1^2 = 0$ are $\mathbb{P}^2$ blown up at $n = 9$ points (or $\mathbb{F}_m$ blown up at 8 points). When $n \leq 8$, lines on such surfaces ($\mathbb{P}^2$ blown up at n points) give rise to E_n representation bundles over them. When $n = 9$, there exists infinitely many lines. We use them to construct an affine E_8 basic representation bundle. Its VOA structure is determined by line configuration of such surfaces.

Key words. Rational surface, basic representation, bundle.

AMS subject classifications. 14J26, 14H60.

1. Introduction. In this paper, we consider rational surfaces with $c_1^2 = 0$, i.e. $\mathbb{P}^2$ blown up at 9 points or Hirzebruch surfaces $\mathbb{F}_m$ blown up at 8 points. Since all the results of $\mathbb{F}_m$ blown up 8 points are similar to $\mathbb{P}^2$ blown up 9 points, we will mainly consider $\mathbb{P}^2$ blown up at 9 points, for those results of $\mathbb{F}_m$ blown up at 8 points, please refer to Remark 1 and Remark 10.

There is a deep connection between the exceptional Lie algebra E_n and the surface X_n which is a blowup of $\mathbb{P}^2$ at n points when $n \leq 9$ (Here $E_9 = \widehat{E}_8$ is the affine E_8 Lie algebra). Let K_{X_n} be the canonical divisor class of X_n, then it is well-known that the sub-lattice $\langle K_{X_n} \rangle^{\perp}$ of $Pic\,(X_n)$ is isomorphic to Λ_{E_n}, the root lattice of type E_n (Here Λ_{E_9} means the lattice generated by the affine real root system of type $\widehat{E}_8$). Thus there is a natural Lie algebra bundle $\mathcal{E}_n$ of type E_n over X_n [3][6][11][12][13][17].

It is also well-known that, when $n \leq 7$, the exceptional divisors (also known as (-1)-curves or lines) on X_n will form a representation of E_n; when $n = 8$, we can use the exceptional divisors together with $-K_{X_8}$ to construct the adjoint representation of E_8. Hence, we can construct the corresponding representation bundle $\mathcal{L}_n$ over X_n when $n \leq 8$ [12]. What about the $n = 9$ case?

For the affine Lie algebra $\widehat{E}_8$, its representations are much more complicated. But its simplest non-trivial highest weight representation (the so called basic representation) is well-known [4][5]. In this paper, we will construct the basic representation bundle over X_9. To do this, we first give a uniform description of $\mathcal{L}_n$ ($n \leq 8$) when the blow up points are in general position. It turns out that these $\mathcal{L}_n$'s are constructed from those $D \in H^2(X_n, \mathbb{Z})$ which are effective and $D \cdot K_{X_n} = -1$ (Lemma 2 and Lemma 3). So we consider all such divisors when $n = 9$ and the 9 blow up points are in general position, they can be written as $D = l - mK_{X_9}$, where l is an exceptional divisor and $m \in \mathbb{Z}$, $m \geq 0$ (Lemma 4). Inspired by these, we construct a vector bundle $\mathcal{L}_9$ over X_9 as a sum of $\mathcal{O}(D)$'s with multiplicities:

$$\mathcal{L}_9 := Symm(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}) \otimes (\bigoplus \mathcal{O}(l)).$$

Then we prove this $\mathcal{L}_9$ is the basic representation bundle over X_9 (Theorem 8).

Note that the basic representations of affine Lie algebras can be described in terms of the vertex operators and these basic representations admit vertex operator algebra

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structures [16][18]. We will show that in our $\widehat{E}_8$ case, fix any exceptional divisor l_0, the bundle $\mathcal{L}_9(-l_0)$ has a fiberwise vertex operator algebra structure. Hence we have a vertex operator algebra bundle $\mathcal{L}_9(-l_0)$ over X_9 (Theorem 14).

The organization of this paper is as follows. In section 2, we consider X_n with $n \le 8$. We review the construction of the E_n-bundle $\mathcal{E}_n$ and the representation bundle $\mathcal{L}_n$ over X_n. Section 3 deals with $n = 9$ case, we first construct an $\widehat{E}_8$-bundle $\mathcal{E}_9$ and a vector bundle $\mathcal{L}_9$ over X_9, then show this $\mathcal{L}_9$ is indeed the basic representation bundle. The review of vertex operator algebra and the construction of the vertex operator algebra bundle $\mathcal{U}$ over X_9 is in section 4.

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2. E_n-bundle and E_n representation bundle over X_n with $n \le 8$, a review. When $X = X_n$ is a blowup of $\mathbb{P}^2$ at n points $x_1, ..., x_n$ with $n \le 8$, there is a canonical Lie algebra bundle $\mathcal{E}_n$ of type E_n and a representation bundle $\mathcal{L}_n$ over X_n. In this section, we review the construction of such bundles.

2.1. E_n-bundle over X_n. The Picard group $Pic(X_n) \cong H^2(X_n, \mathbb{Z})$ is a rank $n+1$ lattice with generators $h, l_1, \cdots, l_n$, where h is the class of lines in $\mathbb{P}^2$ and l_i is the exceptional class of the blow-up at x_i. So $h^2 = 1 = -l_i^2$ and $h \cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_n, \mathbb{Z}) \cong \mathbb{Z}^{1,n}$. The canonical class is $K_{X_n} = -3h + l_1 + \cdots + l_n$.

Denote

$$P_n = \{\alpha \in H^2(X_n, \mathbb{Z}) | \alpha \cdot K_{X_n} = 0\}.$$

$$R_n = \{\alpha \in H^2(X_n, \mathbb{Z}) | \alpha \cdot K_{X_n} = 0,\ \alpha^2 = -2\}.$$

It is well-known that P_n is a root lattice of type E_n and R_n is a root system of type E_n with a system of simple roots $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3$,...., $\alpha_{n-3} = l_{n-3} - l_{n-2}$, $\alpha_{n-2} = h - l_n - l_{n-1} - l_{n-2}$, $\alpha_{n-1} = l_{n-2} - l_{n-1}$, $\alpha_n = l_{n-1} - l_n$ (see Mannin's book [14]).

The corresponding Dynkin diagram is the following:

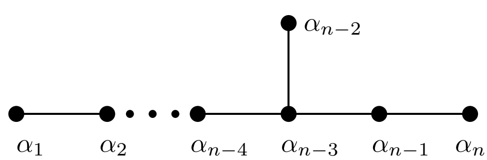

FIG. 1. *The root system of E_n*

Since we have a root system of type E_n attached to X_n, inspired by the Cartan decomposition of a complex simple Lie algebra, we can construct a Lie algebra bundle over X_n as follows:

$$\mathcal{E}_n := \mathcal{O}^{\oplus n} \oplus \bigoplus_{\alpha \in R_n} \mathcal{O}(\alpha).$$

The fiberwise Lie algebra structure on $\mathcal{E}_n$ is defined as the following. Fix the system of simple roots of R_n as $\Delta(E_n) = \{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, ..., \alpha_{n-3} =$

$l_{n-3} - l_{n-2}, \alpha_{n-2} = h - l_n - l_{n-1} - l_{n-2}, \alpha_{n-1} = l_{n-2} - l_{n-1}, \alpha_n = l_{n-1} - l_n\}$, and take a trivialization of $\mathcal{E}_n$. For every trivialization open subset U of X, $\mathcal{E}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \oplus_{\alpha \in R_n} \mathbb{C}_\alpha)$, we take x_α^U to be a nonvanishing section of $\mathcal{O}_U(\alpha)$ and h_i^U ($1 \leq i \leq n$) nonvanishing sections of $\mathcal{O}_U^{\oplus n}$. Define a Lie algebra structure $[\ ,\ ]$ on $\mathcal{E}_n$ such that $\{x_\alpha^U$'s, h_i^U's$\}$ is the Chevalley basis [8], i.e.

(a) $[h_i^U,\ h_j^U] = 0,\ 1 \leq i,\ j \leq n$.

(b) $[h_i^U,\ x_\alpha^U] = \langle \alpha,\ C_i \rangle\, x_\alpha^U,\ 1 \leq i \leq n,\ \alpha \in R_n$.

(c) $[x_\alpha^U,\ x_{-\alpha}^U] = h_\alpha^U$ is a $\mathbb{Z}$-linear combination of h_i^U.

(d) If α, β are independent roots, and $\beta - p\alpha, \cdots, \beta + q\alpha$ is the α-string through β, then $[x_\alpha^U,\ x_\beta^U] = 0$ if $q = 0$, otherwise $[x_\alpha^U,\ x_\beta^U] = \pm(p+1)x_{\alpha+\beta}^U$.

In (b), the inner product $\langle , \rangle$ on R_n is defined by $\langle \alpha, \beta \rangle := -\alpha \cdot \beta$, negative of the intersection form.

Note that h_i^U, $1 \leq i \leq n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathcal{E}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \oplus_{\alpha \in R_n} \mathbb{C}_\alpha)$ is another trivialization, and f_α^{UV} is the transition function for the line bundle $\mathcal{O}(\alpha)$ ($\alpha \in R_n$), that is, $x_\alpha^U = f_\alpha^{UV}\ x_\alpha^V$, then the relation (b) is

$$[h_i^U, f_\alpha^{UV} x_\alpha^V] = \langle \alpha,\ C_i \rangle\, f_\alpha^{UV} x_\alpha^V,$$

that is

$$[h_i^V, x_\alpha^V] = \langle \alpha,\ C_i \rangle\, x_\alpha^V.$$

So (b) is invariant. (c) is also invariant since $(f_\alpha^{UV})^{-1}$ is the transition function for $\mathcal{O}(-\alpha)$ ($\alpha \in R_n$). Finally, (d) is invariant since $f_\alpha^{UV} f_\beta^{UV}$ is the transition function for $\mathcal{O}(\alpha + \beta)$ ($\alpha, \beta \in R_n$).

Therefore, the Lie algebra structure is compatible with the trivialization. In other words, we can construct globally a Lie algebra bundle over a surface once we are given a root system consisting of divisors on this surface.

2.2. E_n representation bundle over X_n. Denote

$$I_n = \{l \in H^2(X_n, \mathbb{Z}) | l \cdot l = -1 = l \cdot K_{X_n}\}.$$

An element of I_n is called an *exceptional divisor* or a (-1)-curve. The configuration of these divisors is well-known [10].

Let V_n be the fundamental representation of E_n corresponding to α_1 (Figure 1), then we have

n	1	2	3	4	5	6	7	8
$\dim V_n$	1	3	6	10	16	27	56	248
$\|I_n\|$	1	3	6	10	16	27	56	240

Inspired by these, we can construct the representation bundle $\mathcal{L}_n$ using the exceptional divisors on X_n as follows:

$$\mathcal{L}_n = \bigoplus_{l \in I_n} \mathcal{O}(l) \text{ when } n \leq 7,$$

$$\mathcal{L}_8 = \bigoplus_{l \in I_8} \mathcal{O}(l) \oplus \mathcal{O}(-K_{X_8})^{\oplus 8}.$$

For $\mathcal{L}_n$ with $n \leq 7$, the fiberwise action is defined naturally. Over a trivialization open subset U, $\mathcal{L}_n|_U \cong U \times (\oplus_{l\in I_n}\mathbb{C}_l)$, take e_l^U to be a nonvanishing holomorphic section of $\mathcal{O}_U(l)$. Define $x_\alpha^U \cdot e_l^U$ to be equal to $e_{l'}^U$ if $l' = l+\alpha \in I_n$ and be equal to 0 otherwise. And define $h_\alpha^U \cdot e_l^U = -(\alpha\cdot l)e_l^U$. Under this action, each fiber of $\mathcal{L}_n$ is the fundamental representation of E_n corresponding to α_1.

To show that the fiberwise action is compatible with any trivialization, we take U, V to be two trivialization open subsets of both $\mathcal{L}_n$ and $\mathcal{E}_n$. Suppose g_l^{UV} is the transition function for the line bundle $\mathcal{O}(l)$, that is, $e_l^U = g_l^{UV}e_l^V$. Then the action

$$x_\alpha^U \cdot e_l^U = \begin{cases} e_{l'}^U & \text{if } l' = l+\alpha \in I_n \\ 0 & \text{otherwise} \end{cases}$$

is

$$(f_\alpha^{UV}x_\alpha^V)\cdot(g_l^{UV}e_l^V) = \begin{cases} g_{l'}^{UV}e_{l'}^V & \text{if } l' = l+\alpha \in I_n \\ 0 & \text{otherwise} \end{cases}.$$

Since $f_\alpha^{UV}g_l^{UV}$ is the transition function for the line bundle $\mathcal{O}(l') = \mathcal{O}(l+\alpha)$, this action is invariant. Note that h_α^U, $\alpha \in R_n$, is independent of any trivialization, so $h_\alpha^U \cdot e_l^U = -(\alpha\cdot l)e_l^U$ is always invariant under different trivializations.

Hence $\mathcal{L}_n$ with $n \leq 7$ is a representation bundle of E_n over X_n.

For $\mathcal{L}_8$, the bijection $I_8 \to R_8$ given by $l \mapsto l + K_{X_8}$ induces an isomorphism

$$\mathcal{E}_8 \cong \mathcal{L}_8 \otimes \mathcal{O}(K_{X_8}).$$

This implies $\mathcal{L}_8$ is just the adjoint representation bundle.

3. $\widehat{E}_8$-bundle and basic representation bundle over X_9. In this section, we consider X_9, a blowup of $\mathbb{P}^2$ at 9 points $x_1, ..., x_9$. There is a canonical Lie algebra bundle $\mathcal{E}_9$ of type $\widehat{E}_8$ over X_9. For the representation bundle, similar to $\mathcal{L}_n$ with $n \leq 8$, we will construct a vector bundle $\mathcal{L}_9$ using exceptional divisors and $-K_{X_9}$ on X_9, and then show it is the basic representation bundle of $\widehat{E}_8$ over X_9.

3.1. $\widehat{E}_8$-bundle over X_9. The Picard group $Pic(X_9) \cong H^2(X_9,\mathbb{Z})$ is a rank 10 lattice with generators $h, l_1, \cdots, l_9$, where h is the class of lines in $\mathbb{P}^2$ and l_i is the exceptional class of the blowup at x_i. So $h^2 = 1 = -l_i^2$ and $h\cdot l_i = 0 = l_i \cdot l_j$, $i \neq j$. Thus $H^2(X_n,\mathbb{Z}) \cong \mathbb{Z}^{1,9}$. The canonical class is $K_{X_9} = -3h + l_1 + \cdots + l_9$.

Denote R_9 as before, i.e.

$$R_9 = \{\alpha \in H^2(X_9,\mathbb{Z}) | \alpha \cdot K_{X_9} = 0,\ \alpha^2 = -2\}.$$

It is well-known that the set $R_9 \cup \{m(-K_{X_9}) | m \neq 0, m \in \mathbb{Z}\}$ forms a root system of (untwisted) affine E_8-type (that is, $\widehat{E}_8$ type), with real roots $\Delta^{re} = R_9$ and imaginary roots $\Delta^{im} = \{m(-K_{X_9}) | m \neq 0, m \in \mathbb{Z}\}$ [7][9][11]. Here the system of simple roots of R_9 is $\{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, ..., \alpha_6 = l_6 - l_7, \alpha_7 = h - l_9 - l_8 - l_7, \alpha_8 = l_7 - l_8, \alpha_9 = l_8 - l_9\}$.

Inspired by this, we can construct the $\widehat{E}_8$-bundle $\mathcal{E}_9$ over X_9 as follows:

$$\mathcal{E}_9 := (\mathcal{O}^{\oplus 8} \oplus \mathcal{O}) \oplus \bigoplus_{\alpha\in\Delta^{re}} \mathcal{O}(\alpha) \oplus \bigoplus_{\beta\in\Delta^{im}} \mathcal{O}(\beta)^{\oplus 8}.$$

Note that $\widehat{E}_8 = E_8 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}_c$. Let $\mathcal{E}_8$ be the pull-back of the E_8-bundle over X_8 (the blowup at $x_1, ..., x_8$) via the blowup map $\pi_9 : X_9 \to X_8$, then

$$\mathcal{E}_9 \cong \mathcal{E}_8 \otimes (\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(nK_{X_9})) \oplus \mathcal{O}.$$

The fiberwise Lie algebra structure of $\mathcal{E}_9$ is defined as following. Over a trivialization open subset U, $\mathcal{E}_9|_U \cong \mathcal{E}_8|_U \otimes (U \times \oplus_{n\in\mathbb{Z}}\mathbb{C}_{nK}) \oplus (U \times \mathbb{C})$. Take a local basis e_i^U of $\mathcal{E}_8|_U$ (e_i^U is just h_j^U or x_α^U in section 2.1), e_{nK}^U of $\mathcal{O}_U(nK)$, e_c^U of $\mathcal{O}_U$, compatible with the tensor product, for example, $e_{nK}^U \otimes e_{mK}^U = e_{(n+m)K}^U$. Then define

$$[e_i^U e_{nK}^U + \lambda e_c^U, e_j^U e_{mK}^U + \mu e_c^U] := [e_i^U, e_j^U]_0 e_{(n+m)K}^U + n\delta_{n+m,0}\langle e_i^U, e_j^U\rangle e_c^U.$$

Here $[\ ,\ ]_0$ is the Lie bracket on $\mathcal{E}_8$ and $\langle , \rangle$ is the Killing form on E_8.

The above formula defines a fiberwise affine Lie algebra structure which is compatible with any trivialization (see Proposition 23 in [11]). Hence, $\mathcal{E}_9$ is an $\widehat{E}_8$-bundle over X_9.

REMARK 1. We can construct an $\widehat{E}_8$-bundle over a blowup of $\mathbb{F}_m$ (Hirzebruch surface) at 8 points similarly. In more detail, for the Hirzebruch surface $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$ with $m \geq 0$, it is geometrically ruled over $\mathbb{P}^1$, we denote by s (resp. f) the class in $Pic(\mathbb{F}_m)$ of the tautological bundle $\mathcal{O}_{\mathbb{F}_m}(-1)$ (resp. of a fiber). After blowing up 8 points in $\mathbb{F}_m$ (we denote the exceptional classes by $l_1, \cdots, l_8$), the Picard group of the new surface is a rank 10 lattice with generators $s, f, l_1, \cdots, l_8$, where $s \cdot s = -m$, $f \cdot f = 0$, $l_i \cdot l_i = -1$, $s \cdot f = 1$, $s \cdot l_i = 0$, $f \cdot l_i = 0$ and $l_i \cdot l_j = 0$ for $i \neq j$. The canonical class of the new surface is $K_m = -(m+2)f - 2s + l_1 + \cdots + l_8$. We consider the sub-lattice $\langle K_m\rangle^\perp$ inside the picard lattice, when m is even, we can take $\{\alpha_1 = s + \frac{n-2}{2}f, \alpha_2 = f - l_1 - l_2, \alpha_k = l_{k-2} - l_{k-1}, k = 3, ...9\}$ as generators of $\langle K_m\rangle^\perp$; when m is odd, we can take $\{\alpha_1 = s + \frac{n-2}{2}f - l_1, \alpha_2 = f - l_1 - l_2, \alpha_k = l_{k-2} - l_{k-1}, k = 3, ...9\}$ as generators of $\langle K_m\rangle^\perp$. From the intersection patterns of these divisors, we know $\langle K_m\rangle^\perp$ is a lattice generated by the affine real root system of type $\widehat{E}_8$ no matter m is even or odd. Hence, we can construct an $\widehat{E}_8$-bundle over a blowup of $\mathbb{F}_m$ at 8 points.

3.2. Construction of $\mathcal{L}_9$. Denote I_9 as before, i.e.

$$I_9 = \{l \in H^2(X_9, \mathbb{Z}) | l \cdot l = -1 = l \cdot K_{X_9}\},$$

then I_9 is an infinite set (Lemma 5 in [11]).

Recall in the construction of $\mathcal{L}_n$ with $n \leq 8$, we only need to use I_n when $n \leq 7$, but for $n = 8$, we need to add some $\mathcal{O}(-K_{X_8})$'s. To give a uniform description of these $\mathcal{L}_n$ with $n \leq 8$, we consider the following lemmas. Note that every element in I_n ($n \leq 9$) is effective (Lemma 4 in [11]). We say that 8 points in $\mathbb{P}^2$ are in general position if no lines pass through three of them, no conics pass through six of them and no cubic curves pass through eight of them with one of the eight points being a double point.

LEMMA 2. *For X_n with $n \leq 7$ and the n blowup points in general position, if $D \in H^2(X_n, \mathbb{Z})$ is an effective divisor and $D \cdot K_{X_n} = -1$, then $D \in I_n$.*

Proof. Let $D = ah - \sum_{i=1}^n a_i l_i$, from $D \cdot K_{X_n} = -3a + \sum_{i=1}^n a_i = -1$, we have $\sum_{i=1}^n a_i^2 \geq \frac{(\sum_{i=1}^n a_i)^2}{n} = \frac{(3a-1)^2}{n}$. Hence $D \cdot D = a^2 - \sum_{i=1}^n a_i^2 \leq a^2 - \frac{(3a-1)^2}{n} \leq \frac{1}{9-n}$. Since $n \leq 7$, we know $D \cdot D \leq 0$. From the genus formula, $D \cdot D$ must be an odd number, so $D \cdot D \leq -1$. Because the n blowup points are in general position, $-K$ is ample, we know D is irreducible and $D \cdot D \geq -1$ [11]. Hence $D \cdot D = -1$, $D \in I_n$. ☐

For X_n with $n = 8$, we have the following lemma.

Lemma 3. *For X_8 with the 8 blowup points in general position, if $D \in H^2(X_8, \mathbb{Z})$ is an effective divisor and $D \cdot K_{X_8} = -1$, then $D \in I_8$ or $D = -K_{X_8}$.*

Proof. Similar to the proof of above lemma, we will get $D \cdot D = -1$ or $D \cdot D = 1$. If $D \cdot D = -1$, then $D \in I_8$. If $D \cdot D = 1$, then $D = -K_{X_8}$. □

From the above two lemmas, we know that to construct $\mathcal{L}_n$ $(n \leq 8)$ on X_n whose n blowup points are in general position, we just need to consider all those effective divisors D with $D \cdot K_{X_n} = -1$. So we consider all such divisors on X_9. For the definition and properties of 9 points in $\mathbb{P}^2$ are in general position, please refer to [3][15].

Lemma 4. *For X_9 with the 9 blowup points in general position, if $D \in H^2(X_9, \mathbb{Z})$ is an effective divisor and $D \cdot K_{X_9} = -1$, then $D = l - mK_{X_9}$, where $l \in I_9$ and $m \in \mathbb{Z}$, $m \geq 0$.*

Proof. Since the 9 blowup points are in general position, we have $-K_{X_9}$ is an effective divisor. From $D \cdot K_{X_9} = -1$, we know $D \cdot D$ is an odd number. Suppose $D \cdot D = 2m - 1$, then $(D + mK) \cdot (D + mK) = -1$. Also $(D + mK) \cdot K = -1$, hence $D = l - mK$ with $l \in I_9$. Because D is effective, we have $m \geq 0$. □

Inspired by the above lemma, we will use the line bundles $\mathcal{O}(l - mK)$'s ($l \in I_9$ and $m \in \mathbb{Z}$, $m \geq 0$) to construct a bundle $\mathcal{L}_9$ over X_9:

$$\mathcal{L}_9 := Symm(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}) \otimes (\bigoplus_{l \in I_9} \mathcal{O}(l)).$$

We will show that $\mathcal{L}_9$ is in fact the basic representation bundle of $\widehat{E}_8$ over X_9 in section 3.4.

3.3. Review of basic representations of affine Lie algebras. In this subsection, we give a brief review of basic representations of affine Lie algebras. For more details, please refer to Frenkel and Kac's paper [4][5].

Let's first recall the definition of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a complex finite dimensional simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, R be the root system and Q be the lattice in $\mathfrak{h}^*$ generated by R. Let $\langle , \rangle$ denote the killing form on $\mathfrak{g}$, normalized in such a way that the square length of a long root is 2. We identify $\mathfrak{h}^*$ and $\mathfrak{h}$ by the form $\langle , \rangle$. The affine Lie algebra $\widehat{\mathfrak{g}}$ is the complex vector space:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\langle c \rangle$$

provided with the bracket

$$[xt^n + \lambda c, yt^m + \mu c] := [x, y]_0 t^{n+m} + n\delta_{n+m,0}\langle x, y\rangle c,$$

where $x, y \in \mathfrak{g}$, $[,]_0$ denotes the Lie bracket induced from $\mathfrak{g}$, $\lambda, \mu \in \mathbb{C}$ [9].

The basic representation π in a vector space V of an affine Lie algebra $\widehat{\mathfrak{g}}$ is defined by the properties that it is irreducible and there exists a non-zero vector $v \in V$ such that

$$\pi(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v = 0 \text{ and } \pi(c) \cdot v = v.$$

This is the simplest among the irreducible highest weight representations. We will give the explicit construction of the basic representation without proof.

We set $V = S(\mathfrak{s}_-)\otimes\mathbb{C}(Q)$, where $\mathfrak{s}_- = t^{-1}\mathbb{C}[t^{-1}]\otimes\mathfrak{h}$, $S(\mathfrak{s}_-)$ is the symmetric algebra of the space $\mathfrak{s}_-$, $\mathbb{C}(Q)$ is the group algebra of the lattice Q.

We define the representation π as follows. $\pi(c)$ acts as identity on V. For $\pi(t^k\otimes h)$ with $h\in\mathfrak{h}$ and $k\in\mathbb{Z}$, if $k<0$, $\pi(t^k\otimes h)$ acts trivially on $\mathbb{C}(Q)$ and acts as the operator of multiplication by $t^k\otimes h$ on $S(\mathfrak{s}_-)$ (the creation operator); if $k>0$, $\pi(t^k\otimes h)$ acts trivially on $\mathbb{C}(Q)$ and acts as the derivation on $S(\mathfrak{s}_-)$ defined by:

$$\pi(t^k\otimes h)\cdot(t^{-k_1}\otimes h_1) := k\delta_{k,k_1}\langle h,h_1\rangle,$$

(the annihilation operator); if $k=0$, $\pi(1\otimes h)$ acts trivially on $S(\mathfrak{s}_-)$ and acts as the derivation ∂_h on the algebra $\mathbb{C}(Q)$ defined by

$$\partial_h e^\gamma := \gamma(h)e^\gamma,\ \gamma\in Q.$$

We choose Chevalley basis $\{x_\alpha\text{'s}, h_i\text{'s}\}$ in $\mathfrak{g}$ as before. To define the action $\pi(x_\alpha\otimes t^k)$ with $\alpha\in\Delta$, we need to define the vertex operator $X(\alpha,z)$ first.

For any $\gamma\in Q$ and any non-zero complex number z, we define a *vertex operator* $X(\gamma,z)$ by:

$$X(\gamma,z) := \exp\Big(\sum_{k\geq 1}\frac{z^k}{k}\gamma(-k)\Big)\exp(\gamma+(\ln z)\partial_\gamma)\exp\Big(-\sum_{k\geq 1}\frac{z^{-k}}{k}\gamma(k)\Big)$$

where $\gamma(k)$ denotes $\pi(t^k\otimes h_\gamma)$ with $k\in\mathbb{Z}$, $k\neq 0$. Exactly in this form the vertex operator appears in physics. The operator $X(\gamma,z)$ is a linear map from the space V into its completion $\overline{V}$. However, developing this operator by power of z we obtain:

$$X(\gamma,z) = \sum_{k\in\mathbb{Z}} X_k(\gamma)z^{-k},$$

where each $X_k(\gamma)$ maps V into itself.

Now, for $\alpha\in\Delta$, $\pi(x_\alpha\otimes t^k) := X_k(\alpha)c_\alpha$, where $c_\alpha(e^\beta) := \varepsilon(\beta,\alpha)e^\beta$, ε is a 2-cocycle of the group Q with values in $\{\pm 1\}$ satisfying some properties [5].

Under the above action π of $\widehat{\mathfrak{g}}$ on V, V is equivalent to the basic representation of $\widehat{\mathfrak{g}}$.

3.4. $\mathcal{L}_9$ as the basic representation bundle. From the above subsection, to construct the basic representation V of $\widehat{E}_8$, we need to find the E_8 root lattice Q.

LEMMA 5. *Fix any $l_0\in I_9$, $Q\cong\langle K_{X_9}, l_0\rangle^\perp\subset H^2(X_9,\mathbb{Z})$.*

Proof. Fix any $l_0\in I_9$, if we contract this l_0, then we will get X_8. Over this X_8, we know $\langle K_{X_8}\rangle^\perp$ is a root lattice of E_8 type. But now $\langle K_{X_8}\rangle^\perp$ is the same with $\langle K_{X_9}, l_0\rangle^\perp$, hence $Q\cong\langle K_{X_9}, l_0\rangle^\perp$. □

The relationship between I_9 and Q is given by the following lemma.

LEMMA 6. *Fix any $l_0\in I_9$, $I_9\cong Q$.*

Proof. Define $f: I_9\to Q$ by $l\mapsto l-l_0+(1+l\cdot l_0)K$. It is obvious that f is injective. For any $\alpha\in Q$, we have $(\alpha+l_0+\frac{\alpha^2}{2}K)\in I_9$ and $f(\alpha+l_0+\frac{\alpha^2}{2}K)=\alpha$. Hence f is also surjective. □

Linearly extend this f in the above lemma to $\mathbb{C}(I_9)\to\mathbb{C}(Q)$, then we have a bijection between $\mathbb{C}(I_9)$ and $\mathbb{C}(Q)$. Recall that

$$\mathcal{L}_9 := Symm\Big(\bigoplus_{m<0}\mathcal{O}(mK)^{\oplus 8}\Big)\otimes\Big(\bigoplus_{l\in I_9}\mathcal{O}(l)\Big),\text{ and}$$

$$\mathcal{E}_9 \cong \mathcal{E}_8 \otimes (\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(nK_{X_9})) \oplus \mathcal{O},$$

where $\mathcal{E}_8$ is the E_8-bundle over X_8 (here $\pi_9 : X_9 \to X_8$ is given by contracting the exceptional divisor $l_0 \in I_9$).

Compare the definition of the basic representation V and the vector bundle $\mathcal{L}_9$, we know each fiber L_9 of $\mathcal{L}_9$ is a basic representation of $\widehat{E}_8$ under the following action:

$$\rho : \widehat{E}_8 \times L_9 \to L_9,$$

$$\rho(x, v) := (id \otimes f^{-1}) \circ \pi(x, (id \otimes f) \cdot v).$$

Note that we will sometimes write $\rho(x, v)$ as $\rho(x) \cdot v$, and similarly for π.

Take a trivialization open subset U for both $\mathcal{E}_9$ and $\mathcal{L}_9$, i.e. $\mathcal{E}_9|_U \cong \mathcal{E}_8|_U \otimes (U \times \oplus_{n\in\mathbb{Z}}\mathbb{C}_{nK}) \oplus (U \times \mathbb{C})$ and $\mathcal{L}_9|_U \cong U \times (Symm(\oplus_{m<0}\mathbb{C}_{mK}^{\oplus 8}) \otimes (\oplus_{l\in I_9}\mathbb{C}_l))$. Then we have the action

$$\rho_U : \mathcal{E}_9|_U \times \mathcal{L}_9|_U \to \mathcal{L}_9|_U$$

induced from $\rho : \widehat{E}_8 \times L_9 \to L_9$.

Lemma 7. *$\rho_U : \mathcal{E}_9|_U \times \mathcal{L}_9|_U \to \mathcal{L}_9|_U$ satisfies $\mathcal{O}_U(x) \times \mathcal{O}_U(v) \to \bigoplus \mathcal{O}_U(x + v)$ for any direct summand $\mathcal{O}(x)$ of $\mathcal{E}_9$ and $\mathcal{O}(v)$ of $\mathcal{L}_9$.*

Proof. If $x = nK$ with $n \in \mathbb{Z}$, then the corresponding action is just $\rho(c)$ or $\rho(t^k \otimes h)$. In this case, the conclusion is obviously.

If $x = \alpha + nK$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then the corresponding action is $\rho(x_\alpha \otimes t^n)$. We need to show that $\rho(x_\alpha \otimes t^n)$ maps from

$$\Gamma(\mathcal{O}_U(l + mK))$$

to

$$\Gamma(\oplus\mathcal{O}_U(l + \alpha + (m + n)K)).$$

Recall that $\rho(x, v) := (id \otimes f^{-1}) \circ \pi(x, (id \otimes f) \cdot v)$.

First, under the action of $id \otimes f$, $\Gamma(\mathcal{O}_U(l + mK))$ is mapped to $\Gamma(\mathcal{O}_U(mK + l - l_0 + (1 + l \cdot l_0)K))$. We denote $\beta := l - l_0 + (1 + l \cdot l_0)K \in Q$.

Second, we consider the action of $\pi(x_\alpha \otimes t^n) = X_n(\alpha)c_\alpha$, where $X(\alpha, z) = \sum_{k\in\mathbb{Z}} X_k(\alpha)z^{-k}$. Recall the definition of the vertex operator:

$$X(\alpha, z) := \exp(\sum_{k\geq 1} \frac{z^k}{k}\alpha(-k)) \exp(\alpha + (\ln z)\partial_\alpha) \exp(-\sum_{k\geq 1} \frac{z^{-k}}{k}\alpha(k)),$$

where the middle factor is just $z^{\frac{1}{2}\langle\alpha,\alpha\rangle}e^\alpha \exp(\ln z)\partial_\alpha$ and $\exp(\ln z)\partial_\alpha = \sum_{k\in\mathbb{Z}} z^k P_k$, where $P_k(e^\beta) = \delta_{k,\langle\alpha,\beta\rangle}e^\beta$. So for this middle factor, we can only have z with degree $\frac{1}{2}\langle\alpha, \alpha\rangle + \langle\alpha, \beta\rangle$, and the corresponding operator maps from any $\Gamma(\mathcal{O}_U(D))$ to $\Gamma(\mathcal{O}_U(D + \alpha))$. For the first and third factor, to get a term z^k with degree k, the corresponding operators will map from any $\Gamma(\mathcal{O}_U(D))$ to $\Gamma(\oplus\mathcal{O}_U(D - kK))$. Since we need to get z with degree $-n$, we need to get $z^{-n-\frac{1}{2}\langle\alpha,\alpha\rangle-\langle\alpha,\beta\rangle}$ from the first and third factors. Hence $\pi(x_\alpha \otimes t^n)$ maps from

$$\Gamma(\mathcal{O}_U(mK + \beta))$$

to

$$\Gamma(\oplus\mathcal{O}_U(mK+\beta+\alpha+(n+\frac{1}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\beta\rangle)K)).$$

We denote $\beta'=\beta+\alpha\in Q$.

Third, under the action of $id\otimes f^{-1}$,

$$\Gamma(\oplus\mathcal{O}_U(mK+\beta'+(n+\frac{1}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\beta\rangle)K))$$

is mapped to

$$\Gamma(\oplus\mathcal{O}_U(mK+\beta'+l_0+\frac{(\beta')^2}{2}K+(n+\frac{1}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\beta\rangle)K)).$$

By direct computations, we have

$$\mathcal{O}(mK+\beta'+l_0+\frac{(\beta')^2}{2}K+(n+\frac{1}{2}\langle\alpha,\alpha\rangle+\langle\alpha,\beta\rangle)K)=\mathcal{O}(l+\alpha+(m+n)K).$$

Hence $\rho(x_\alpha\otimes t^n)$ maps from $\Gamma(\mathcal{O}_U(l+mK))$ to $\Gamma(\oplus\mathcal{O}_U(l+\alpha+(m+n)K))$. We are done. □

From the above lemma and the relationship between the transition functions of these direct summand line bundles, we know that the fiberwise action ρ is compatible with any trivialization of $\mathcal{E}_9$ and $\mathcal{L}_9$, i.e.

THEOREM 8. *$\mathcal{L}_9$ is the basic representation bundle of $\widehat{E}_8$ over X_9.*

REMARK 9. Note that if we use the root lattice Q instead of I to construct the bundle, *i.e.*

$$\mathcal{V}:=Symm(\bigoplus_{m<0}\mathcal{O}(mK)^{\oplus 8})\otimes(\bigoplus_{\alpha\in Q}\mathcal{O}(\alpha)),$$

though each fiber of $\mathcal{V}$ is a basic representation of $\widehat{E}_8$, the fiberwise action is not compatible with different trivializations of $\mathcal{E}_9$ and $\mathcal{V}$.

REMARK 10. For the Hirzebruch surfaces case, we only need to show that over a blowup of $\mathbb{F}_m$ at 7 points, the sub-lattice $\langle K_m\rangle^\perp$ is isomorphic to the root lattice of E_8 type. The proof is similar to Remark 1. From this, we can construct the basic representation bundle of $\widehat{E}_8$ over a blowup of $\mathbb{F}_m$ at 8 points.

4. Vertex operator algebra bundle over X_9. It is well-known that the basic representations of affine Lie algebras admit vertex operator algebra structures [16][18]. We will review the vertex operator structure on the basic representation of $\widehat{E}_8$ and construct a vertex operator algebra bundle $\mathcal{U}$ (means this bundle has a globally defined vertex operator algebra structure) over X_9.

4.1. Review of Vertex operator algebras. For more details about this subsection, please refer to [16] and [18].

DEFINITION 11. ([18]) A vertex operator algebra (VOA) is a graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ equipped with a linear map:

$$V \otimes V \to V((z))$$

$$(a, b) \mapsto Y(a, z)b = \sum_{n\in\mathbb{Z}} (a(n)b)z^{-n-1},\ (a(n) \in End(V))$$

(we call $Y(a, z)$ the vertex operator of a) and with two distinguished vectors $1 \in V_0$ (called the identity element), $\omega \in V_2$ (called the Virasoro element) satisfying the following conditions for $a, b \in V$:

(1) $a(n)b = 0$ for n sufficiently large;

(2) $Y(1, z) = 1$;

(3) $Y(a, z)1 \in V[[z]]$ and $\lim_{z\to 0} Y(a, z)1 = a$;

the vertex operator $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ generates a copy of Virasoro algebra:

(4) $[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c$, where c is a constant called the rank of V; and

(5) $L_0 a = na = (\deg a)a$ for $a \in V_n$;

(6) $Y(L_{-1}a, z) = \frac{d}{dz}Y(a, z)$;

and the following Jacobi identity holds for every $m, n, l \in \mathbb{Z}$:

(7) $\sum_{i=0}^{\infty} \binom{m}{i} Y(a(l + i)b, w)w^{m+n-i} = \sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m + l - i)Y(b, w)w^{n+i} - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}Y(b, w)a(m + i)w^{n+l-i}$.

This completes the definition.

Back to the basic representation $V = S(\mathfrak{s}_-)\otimes\mathbb{C}(Q)$ of $\widehat{E}_8$. We define a grading on V with the degree defined as:

$$\deg(h_1^{-n_1}h_2^{-n_2}\cdots h_N^{-n_N}e^{\alpha}) := n_1 + n_2 + \cdots + n_N + \frac{\langle\alpha, \alpha\rangle}{2},$$

(here we write $t^k \otimes h$ as h^k for simplicity). Then V is graded by $\mathbb{Z}$ and we have $V = \bigoplus_{n=0}^{\infty} V_n$ with $V_0 = \{1 \otimes e^0\}$. For the vertex operator $Y(v, z)$ for $v \in V$, if $v = 1 \otimes e^{\alpha}$ with $\alpha \in Q$, then $Y(1 \otimes e^{\alpha}, z) = X(\alpha, z)z^{-\frac{\langle\alpha,\alpha\rangle}{2}}$, where $X(\alpha, z)$ as defined in section 3.3. To define $Y(v, z)$ for general $v \in V$, we need to define an ordering on V, the so called boson Wick ordering (also called the normal ordering) $\circ\atop\circ$ $\circ\atop\circ$. For the explicitly definition of this ordering, please refer to [16].

DEFINITION 12. ([16]) For every element $v \in V$ and $z \in \mathbb{C}-\{0\}$, we define an operator

$$Y(v, z) : V \to V,$$

by the following rules:

(1) $Y(1 \otimes e^{\alpha}, z) = X(\alpha, z)z^{-\frac{\langle\alpha,\alpha\rangle}{2}}$ for $\alpha \in Q$,

(2) $Y(v + u, z) = Y(v, z) + Y(u, z)$, $Y(cv, z) = cY(v, z)$ for $c \in \mathbb{C}$,

(3) $Y(v \cdot u, z) = {\circ\atop\circ} Y(v, z)Y(u, z) {\circ\atop\circ}$,

(4) $Y(L_{-1}v, z) = \frac{d}{dz}Y(v, z)$.

The above operators $Y(v, z)$ are also called vertex operators, and they satisfy all the conditions in the definition of VOA. Hence the basic representation V of $\widehat{E}_8$ together with these operators $Y(v, z)$ is a VOA.

4.2. Vertex operator algebra bundle over X_9. Fix $l_0 \in I_9$, we define a vector bundle $\mathcal{L}_9(-l_0)$ over X_9 as follows:

$$\mathcal{L}_9(-l_0) := Symm(\bigoplus_{m<0} \mathcal{O}(mK)^{\oplus 8}) \otimes (\bigoplus_{l\in I_9} \mathcal{O}(l-l_0)),$$

i.e. $\mathcal{L}_9(-l_0)= \mathcal{L}_9 \otimes \mathcal{O}(-l_0)$. From the above subsection, we know that each fiber of $\mathcal{L}_9(-l_0)$ admits a VOA structure (through the map $f : I_9 \to Q$).

For any trivialization open subset U of $\mathcal{L}_9(-l_0)$, we have a linear map

$$\mathcal{L}_9(-l_0)|_U \times \mathcal{L}_9(-l_0)|_U \to \bigoplus_n \mathcal{L}_9(-l_0)|_U \otimes \mathcal{O}_U(nK)$$

(here we view z^n as a section of $\mathcal{O}_U(nK)$) induced from the vertex operator $Y : V \otimes V \to V((z))$.

Lemma 13. *$\mathcal{L}_9(-l_0)|_U \times \mathcal{L}_9(-l_0)|_U \to \bigoplus_n \mathcal{L}_9(-l_0)|_U \otimes \mathcal{O}_U(nK)$ satisfies $\mathcal{O}_U(x) \times \mathcal{O}_U(y) \to \bigoplus \mathcal{O}_U(x+y)$ for any two direct summands $\mathcal{O}(x)$ and $\mathcal{O}(y)$ of $\mathcal{U}$.*

Proof. Because $Y(v \cdot u, z) =^{\circ}_{\circ} Y(v,z)Y(u,z)^{\circ}_{\circ}$, we only need to prove:

$$\mathcal{O}_U(l_1 - l_0) \times \mathcal{O}_U(l_2 - l_0 + mK) \to \bigoplus \mathcal{O}_U(l_1 + l_2 - 2l_0 + mK),\ \forall l_1, l_2 \in I_9 \tag{1}$$

$$\mathcal{O}_U(-K) \times \mathcal{O}_U(l_2 - l_0 + mK) \to \bigoplus \mathcal{O}_U(l_2 - l_0 + (m-1)K),\ \forall l_2 \in I_9. \tag{2}$$

Prove formula (1) : first, under the map $f : I_9 \to Q$,

$$\mathcal{O}_U(l_1 - l_0) \times \mathcal{O}_U(l_2 - l_0 + mK)$$

becomes

$$\mathcal{O}_U(l_1 - l_0 + (1 + l_1 \cdot l_0)K) \times \mathcal{O}_U(l_2 - l_0 + (1 + l_2 \cdot l_0)K + mK),$$

we denote $\alpha_1 = l_1 - l_0 + (1 + l_1 \cdot l_0)K$ and $\alpha_2 = l_2 - l_0 + (1 + l_2 \cdot l_0)K$.

Second, under $Y(e^{\alpha_1}, z) = X(\alpha_1, z) z^{-\frac{\langle \alpha_1, \alpha_1 \rangle}{2}}$,

$$\mathcal{O}_U(\alpha_2 + mK)$$

becomes

$$\bigoplus_n \mathcal{O}_U(\alpha_1 + \alpha_2 + (m + n + \langle \alpha_1, \alpha_2 \rangle)K) z^{-n},$$

(see the proof of Theorem 8).

Third, under $f^{-1} : Q \to I_9$, we will finally get

$$\bigoplus_n \mathcal{O}_U(\alpha_1 + \alpha_2 + \frac{(\alpha_1 + \alpha_2)^2}{2} K + (m + n + \langle \alpha_1, \alpha_2 \rangle)K) z^{-n}$$

which is the same with

$$\bigoplus_n \mathcal{O}_U(l_1 + l_2 - 2l_0 + (m+n)K) z^{-n}.$$

Since we view z^n as a section of $\mathcal{O}_U(nK)$, we get formula (1).

Prove formula (2) : first, under the map $f : I_9 \to Q$,

$$\mathcal{O}_U(-K) \times \mathcal{O}_U(l_2 - l_0 + mK)$$

becomes

$$\mathcal{O}_U(-K) \times \mathcal{O}_U(l_2 - l_0 + (1 + l_2 \cdot l_0)K + mK),$$

we denote $\alpha_2 = l_2 - l_0 + (1 + l_2 \cdot l_0)K$.

Second, we need to use the operator $Y(h_\alpha^{-1}, z)$, where h_α^{-1} is a section of $\mathcal{O}_U(-K)$ with $\alpha \in R$ is a root. Note that

$$Y(h_\alpha^{-1}, z) = Y(L_{-1}e^\alpha \cdot e^{-\alpha}, z) =^\circ_\circ Y(L_{-1}e^\alpha, z)Y(e^{-\alpha}, z)^\circ_\circ.$$

Under $Y(e^{-\alpha}, z)$,

$$\mathcal{O}_U(\alpha_2 + mK)$$

becomes

$$\bigoplus_n \mathcal{O}_U(-\alpha + \alpha_2 + (m + n - \langle \alpha, \alpha_2 \rangle)K)z^{-n},$$

then becomes

$$\bigoplus_n \mathcal{O}_U(\alpha_2 + (m + n - \langle \alpha, \alpha_2 \rangle)K + (k + \langle \alpha, \alpha_2 - \alpha \rangle)K)z^{-n-k-1}$$

under $Y(L_{-1}e^\alpha, z)$ since $Y(L_{-1}e^\alpha, z) = \frac{d}{dz}Y(e^\alpha, z)$ (Note that the ordering of these operators will not affect where the coefficients of each z^n lie in).

Third, under $f^{-1} : Q \to I_9$, we will finally get

$$\bigoplus_n \mathcal{O}_U(\alpha_2 + \frac{\alpha_2^2}{2}K + (m + n + k - \langle \alpha, \alpha \rangle)K)z^{-n-k-1}$$

which is the same with

$$\bigoplus_n \mathcal{O}_U(l_2 - l_0 + (m + n + k - 2)K)z^{-n-k-1}$$

since $\langle \alpha, \alpha \rangle = 2$. We view z^n as a section of $\mathcal{O}_U(nK)$, then get formula (2). □

From the above lemma and the relationship between the transition functions of these direct summand line bundles, we know that the fiberwise VOA structure on $\mathcal{L}_9(-l_0)$ is compatible with any trivialization of $\mathcal{L}_9(-l_0)$, i.e.

THEOREM 14. *$\mathcal{L}_9(-l_0)$ is a vertex operator algebra bundle over X_9.*

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