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004

SCREW MOTION SURFACES IN $\widetilde{PSL}_2(\mathbb{R},\tau)$*

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Abstract. A screw motion surfaces in $\widetilde{PSL}_2(\mathbb{R},\tau)$ is either a minimal or a constant mean curvature surface which is invariant by helicoidal isometries. In this paper, we study the geometric behavior of such screw motion surfaces.

Key words. Constant mean curvature, invariant surfaces, one-parameter group of isometries, screw motion, helicoids.

AMS subject classifications. 53A35.

1. Introduction. The study of constant man curvature surfaces immersed in the product spaces $M^2 \times \mathbb{R}$, where M^2 is a complete 2-dimensional manifold, has grown in the last years. In particular the study of such surfaces invariant by one parameter group of isometries in the product spaces $\mathbb{D}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{D}^2$ denotes the 2-dimensional hyperbolic disk and $\mathbb{S}^2$ denotes the Euclidean sphere of radius one, was given in [7]. More precisely, in [7], the authors consider screw motion surfaces, that is, surfaces which are invariant by the composition of rotational isometries (originated from the space $\mathbb{D}^2$) together with translations along the real line $\mathbb{R}$. The study of such screw motion surfaces immersed in the 3-dimensional spaces form was given in [3]. On the other hand, the screw motion surfaces immersed in the Heisenberg space Nil_3 was given in [1]

The 3-dimensional spaces form, the product spaces and the Heisenberg space are homogeneous simple connected 3-dimensional manifolds, the natural questions is, what happen with the other homogeneous simple connected 3-dimensional manifolds?. To answer this question notice that the space $\mathbb{D}^2 \times \mathbb{R}$ form part of a family of homogeneous simple connected 3-dimensional manifolds $E(\tau)$ of parameter τ, that is of the family

$$E(\tau) = \begin{cases} \mathbb{D}^2 \times \mathbb{R}, & \tau = 0, \\ \widetilde{PSL}_2(\mathbb{R},\tau), & \tau \neq 0. \end{cases}$$

In this paper we show that the geometric behavior of the screw motion surfaces immersed in the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ are similar to this one immersed in the space $\mathbb{D}^2 \times \mathbb{R}$.

A screw motion surface immersed in $\widetilde{PSL}_2(\mathbb{R},\tau)$ is a constant mean curvature surface which is invariant by one-parameter group of isometries Γ, where Γ is the composition of rotational isometries together with vertical translation in a proportional way, the study of rotational surfaces having constant mean curvature which are immersed in the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ was given in [5], when the space is $\mathbb{D}^2 \times \mathbb{R}$, the study of such rotational surfaces was given in [2].

The paper is organized as follows. In section 2, we introduce the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ and in section 3 we give the geometric behavior of the screw motion surfaces immersed in the space $\widetilde{PSL}_2(\mathbb{R},\tau)$.

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2. The space $\widetilde{PSL}_2(\mathbb{R},\tau)$. We denote by $\mathbb{D}^2$ the hyperbolic space, that is,

$$\mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$$

endowed with metric

$$d\sigma^2 = \lambda^2(dx^2 + dy^2), \qquad \lambda = \frac{2}{1-(x^2+y^2)}.$$

The natural orthonormal frame field on $\mathbb{D}^2$ is given by $\{e_1 = \lambda^{-1}\partial_x, e_2 = \lambda^{-1}\partial_y\}$.

The space $\widetilde{PSL}_2(\mathbb{R},\tau)$ (for each fixed $\tau \neq 0$) is a complete simply connected homogeneous 3-manifold and there exists a Riemannian submersion

$$\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \to \mathbb{D}^2,$$

over the 2-dimensional hyperbolic space. For each $p \in \mathbb{D}^2$, the fiber passing by p is a complete 1-manifold diffeomorphic to the real line, the translations along each fiber are isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$, we denote by E_3 the unitary vector field tangent to this fiber and we call E_3 a vertical vector field. Considering the horizontal lift $\{E_1, E_2\}$ of $\{e_1, e_2\}$ we obtain the orthonormal frame $\{E_1, E_2, E_3\}$, where $\{E_1, E_2\}$ are horizontal vector fields.

More precisely the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ is given by,

$$\widetilde{PSL}_2(\mathbb{R},\tau) = \{(x,y,t) \in \mathbb{R}^3; (x,y) \in \mathbb{D}^2, t \in \mathbb{R}\}$$

endowed with metric,

$$g := ds^2 = \lambda^2(dx^2+dy^2) + \left(2\tau\left(\frac{\lambda_y}{\lambda}dx - \frac{\lambda_x}{\lambda}dy\right) + dt\right)^2, \ \ \lambda = \frac{2}{1-(x^2+y^2)}.$$

By considering the Riemannian submersion $\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow \mathbb{D}^2$, we obtain the next Lemma.

LEMMA 2.1. *The fields $\{E_1, E_2, E_3\}$ in the referential $\{\partial_x, \partial_y, \partial_t\}$ are given by,*

$$E_1 = \frac{1}{\lambda}\partial_x - 2\tau\frac{\lambda_y}{\lambda^2}\partial_t, \qquad E_2 = \frac{1}{\lambda}\partial_y + 2\tau\frac{\lambda_x}{\lambda^2}\partial_t, \qquad E_3 = \partial_t.$$

For more detail about the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ see [4].

2.1. Isometries in $\widetilde{PSL}_2(\mathbb{R},\tau)$. The isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ are strongly related with the isometries of the hyperbolic space $\mathbb{D}^2$.

From now on we identify the Euclidean space $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$, more precisely $z = x + iy \approx (x,y)$. Thus

$$\widetilde{PSL}_2(\mathbb{R}) = \{(z,t) \in \mathbb{R}^3; x^2+y^2 < 1, t \in \mathbb{R}\}$$

is endowed with metric

$$ds^2 = \lambda^2(z)|dz|^2 + (i\tau\lambda(\overline{z}dz - zd\overline{z}) + dt)^2.$$

PROPOSITION 2.2. *[8, Theorem 9] The isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ are given by:*

$$F(z,t) = (f(z), t - 2\tau \arg f' + c)$$

or

$$G(z,t) = (\overline{f}(z), -t + 2\tau \arg f' + c)$$

where f is a positive isometry of $\mathbb{D}^2$ and c is a real number.

2.2. Graphs in $\widetilde{PSL}_2(\mathbb{R},\tau)$. Since, $\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow \mathbb{D}^2$ is a Riemannian submersion, it is possible to consider graphs in $\widetilde{PSL}_2(\mathbb{R},\tau)$.

Let $\Omega \in \mathbb{D}^2$ a open domain in $\mathbb{D}^2$, a section

$$s : \Omega \subset \mathbb{D}^2 \longrightarrow \widetilde{PSL}_2(\mathbb{R},\tau)$$

is a map such that $\pi \circ s = Id_\Omega$, where Id denotes the identity in $\mathbb{D}^2$.

DEFINITION 2.3. A graph in $\widetilde{PSL}_2(\mathbb{R},\tau)$ is the image of a section s defined over a domain Ω.

Given a domain $\Omega \subset \mathbb{D}^2$ we denote by Ω its lift to $\mathbb{D}^2 \times \{0\}$, if we consider the section over Ω given by $t = u(x,y)$, where $u \in C^0(\partial\Omega) \cap C^\infty(\Omega)$ and $(x,y) \in \Omega$, then the graph of u (which we denote by $\Sigma(u)$) is given by

$$\Sigma(u) = \{(x,y,u(x,y)) \in \widetilde{PSL}_2(\mathbb{R},\tau); (x,y) \in \Omega\}.$$

LEMMA 2.4. *[4, Lemma 3.2] Suppose that the surface $\Sigma(u)$ has constant mean curvature H. Then, the function u satisfies the equation*

$$2H = div_{\mathbb{D}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right),$$

where $W = \sqrt{1+\alpha^2+\beta^2}$ and

- $\alpha = \dfrac{u_x}{\lambda} + 2\tau\dfrac{\lambda_y}{\lambda^2}$,
- $\beta = \dfrac{u_y}{\lambda} - 2\tau\dfrac{\lambda_x}{\lambda^2}$.

3. Screw motion surfaces in $\widetilde{PSL}_2(\mathbb{R},\tau)$. The idea to obtain screw motion surfaces is simple, we will take a curve C in the xt-plane and we will apply to C a one-parameter group Γ of helicoidal isometries, that is Γ is the composition of rotational isometries and vertical translations of pitch $\widetilde{l} > 0$. More precisely, let $r_\theta(z) = e^{i\theta}z$ be the rotation of angle θ around the origin $(0,0)$ in the hyperbolic space $\mathbb{D}^2$. From Proposition 2.2, the isometry R_θ in $\widetilde{PSL}_2(\mathbb{R},\tau)$ generated by r_θ is given by

$$R_\theta(z,t) = (z, t - 2\tau\theta).$$

Now, for a fixed $\widetilde{l} \in \mathbb{R}$, the vertical translation of height $\widetilde{l}$ is given by

$$L(z,t) = (z, t + \widetilde{l}).$$

The helicoidal isometry is the composition

$$E(z,\theta) = (L_\theta \circ R_\theta)(z,t) = (z, t - 2\tau\theta + \widetilde{l}\theta),$$

where $L_\theta(z,t) = (z, t+\widetilde{l}\theta)$ for a fixed θ. Denoting by $Isom(\widetilde{PSL}_2(\mathbb{R},\tau))$ the isometries group of $\widetilde{PSL}_2(\mathbb{R},\tau)$, the group Γ is given by

$$\Gamma = \{E(z,\theta) \in Isom(\widetilde{PSL}_2(\mathbb{R},\tau)); \theta \in \mathbb{R}\}.$$

Remark 3.1. When $\widetilde{l} = 2\tau$, we obtain a rotational isometry. The surfaces invariant by rotational ismetries was treated in [5].

First we suppose that the curve C is the graph of a function u (we denote this curve by $C(u)$) defined in an small open interval I, with I in the x-axis. We denote by $\Sigma(u)$, the surface generated by C and the group Γ.

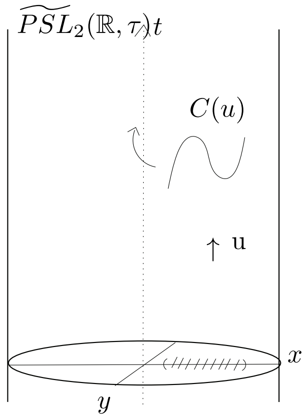


Fig. 1. *The generating curve of a screw motion surface.*

To study the geometric behavior of $\Sigma(u)$, we re-parameterized the hyperbolic disk whit generalized polar coordinates ρ, θ, that is,

$$x = \tanh\left(\frac{\rho}{2}\right)\cos(\theta)$$
$$y = \tanh\left(\frac{\rho}{2}\right)\sin(\theta)$$

where $\rho > 0$ is the hyperbolic distance measure from the origin of $\mathbb{D}^2$ and $\theta \in \mathbb{R}$. Locally the surface $\Sigma(u)$ is a graph over a domain in the hyperbolic disk $\mathbb{D}^2$, we still call $\Sigma(u)$ the graph of the function u in the polar coordinates. Therefore the surface $\Sigma(u)$ can be parameterized by

$$\varphi(\rho,\theta) = \left(\tanh\left(\frac{\rho}{2}\right)\cos(\theta), \tanh\left(\frac{\rho}{2}\right)\sin(\theta), u(\rho) + (\widetilde{l} - 2\tau)\theta\right) \tag{3.1}$$

for convenience we assume $l = \widetilde{l} - 2\tau > 0$ and we call l the pitch of $\Sigma(u)$. The case $\widetilde{l} = 2\tau$ gives a rotational surfaces, see the Remark 3.1. For $l = \widetilde{l} - 2\tau < 0$, the vertical translation is in the downwards direction, since the surfaces are complete, this surface is the same as in the $l > 0$ case.

The proof of the next lemma is a straightforward computation.

LEMMA 3.2. *In polar coordinates we obtain,*
(1) $\partial_x = 2\cosh^2(\frac{\rho}{2})\cos(\theta)\partial_\rho - \coth(\frac{\rho}{2})\sin\theta\partial_\theta$
(2) $\partial_y = 2\cosh^2(\frac{\rho}{2})\sin(\theta)\partial_\rho + \coth(\frac{\rho}{2})\cos\theta\partial_\theta$
(3) $\rho_x = 2\cosh^2(\frac{\rho}{2})\cos(\theta)$
(4) $\rho_y = 2\cosh^2(\frac{\rho}{2})\sin(\theta)$
(5) $\theta_x = -\coth(\frac{\rho}{2})\sin(\theta)$
(6) $\theta_y = \coth(\frac{\rho}{2})\cos(\theta)$
(7) $\lambda = 2\cosh^2(\frac{\rho}{2})$
(8) $d\sigma^2 = d\rho^2 + \sinh^2(\rho)d\theta^2$

The next lemma is crucial for our study of screw motion surfaces.

LEMMA 3.3. *(Main lemma) Supposing that $\Sigma(u)$ has constant mean curvature $H > 0$. Then, the function u (which depends on the parameter $d \in \mathbb{R}$) satisfies*
(3.2)
$$u^d(\rho) = \int_*^\rho \frac{|2H\cosh(r) + d|}{\sqrt{\sinh^2(r) - (2H\cosh(r) + d)^2}}\sqrt{1 + \left(\frac{l}{\sinh(r)} - 2\tau\tanh\left(\frac{r}{2}\right)\right)^2}\,dr$$
where d is a real number.

REMARK 3.4. When $\tau \equiv 0$ we are in the space $\mathbb{D}^2 \times \mathbb{R}$. In this case, Ricardo Sa Earp and Eric Toubiana found explicit formulas for rotational-screw motion surfaces, see [7].

Proof. Notice that, if the surface $\Sigma(v)$ has mean curvature H, then by lemma 2.4 the function v satisfies the equation

$$2H = div_{\mathbb{D}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right), \tag{3.3}$$

where $W = \sqrt{1 + \alpha^2 + \beta^2}$, $\alpha = \dfrac{v_x}{\lambda} + 2\tau y$, and $\beta = \dfrac{v_y}{\lambda} - 2\tau x$.

In our case, we consider $v(x, y) = v(\rho(x, y), \theta(x, y)) = u(\rho) + l\theta$. And we call the surface $\Sigma(v)$ simply by $\Sigma(u)$. Since that $\Sigma(u)$ is a screw motion surface, u does not depend of θ, it is clear from the parametrization (3.1), thus

$$\begin{cases} v_x = u_\rho\rho_x + u_\theta\theta_x + l\theta_x = u_\rho\rho_x + l\theta_x, \\ v_y = u_\rho\rho_y + u_\theta\theta_y + l\theta_y = u_\rho\rho_y + l\theta_y. \end{cases}$$

Notice that

$$\begin{cases} \alpha = \dfrac{v_x}{\lambda} + 2\tau y = u_\rho\cos\theta - l\dfrac{\sin\theta}{\sinh\rho} + 2\tau\tanh(\rho/2)\sin\theta, \\ \beta = \dfrac{v_y}{\lambda} - 2\tau x = u_\rho\sin\theta + l\dfrac{\cos\theta}{\sinh\rho} - 2\tau\tanh(\rho/2)\cos\theta. \end{cases}$$

Where

$$\alpha^2 + \beta^2 = u_\rho^2 + \left(\frac{l}{\sinh\rho} - 2\tau\tanh(\rho/2)\right)^2,$$
$$W^2 = 1 + u_\rho^2 + \left(\frac{l}{\sinh\rho} - 2\tau\tanh(\rho/2)\right)^2.$$

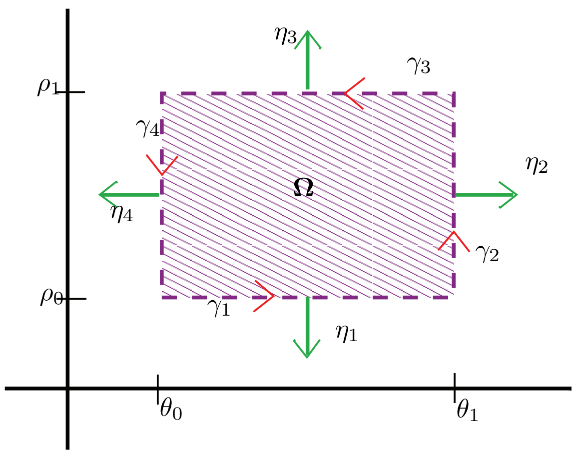
Setting $X_u = \dfrac{\alpha}{W}e_1 + \dfrac{\beta}{W}e_2 = \dfrac{1}{W}\left(\alpha\dfrac{\partial_x}{\lambda} + \beta\dfrac{\partial_y}{\lambda}\right)$.
From Lemma 3.2, we obtain

$$X_u = \frac{1}{W}\left[u_\rho \partial_\rho + \left(\frac{l}{\sinh^2\rho} - 2\tau\frac{\tanh(\rho/2)}{\sinh(\rho)}\right)\partial_\theta\right]$$

and

$$W = \sqrt{1 + 4\tau^2\tanh^2(\rho/2) + u_\rho^2}.$$

Let $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 < \theta_1$ and $\rho_0, \rho_1 \in \mathbb{R}_+$, with $\rho_0 < \rho_1$ and consider the domain $\Omega = [\theta_0, \theta_1] \times [\rho_0, \rho_1]$ in the plane $\rho\theta$.



By integrating the equation (3.3), we obtain,

$$\int_{\partial(\Omega)} \langle X_u, \eta\rangle = 2H Area([\theta_0, \theta_1] \times [\rho_0, \rho_1])$$

where η is the outer co-normal.

Since

$$\int_{\partial\Omega} \langle X_u, \eta\rangle = \int_{\gamma_1} \langle X_u, \eta_1\rangle + \int_{\gamma_2} \langle X_u, \eta_2\rangle + \int_{\gamma_3} \langle X_u, \eta_3\rangle + \int_{\gamma_4} \langle X_u, \eta_4\rangle$$

and

- $\displaystyle\int_{\gamma_1} \langle X_u, \eta_1\rangle = -\int_{\theta_0}^{\theta_0} \frac{u_\rho}{W}(\rho_0)\sinh\rho_0 d\theta$
- $\displaystyle\int_{\gamma_2} \langle X_u, \eta_2\rangle = \int_{\rho_0}^{\rho_1} \left(\frac{l}{\sinh^2\rho} - 2\tau\frac{\tanh(\rho/2)}{\sinh\rho}\right)\frac{\sinh\rho}{W}d\rho$
- $\displaystyle\int_{\gamma_3} \langle X_u, \eta_3\rangle = \int_{\theta_0}^{\theta_1} \frac{u_\rho}{W}(\rho_1)\sinh\rho_1 d\theta$
- $\displaystyle\int_{\gamma_4} \langle X_u, \eta_4\rangle = -\int_{\rho_0}^{\rho_1} \left(\frac{l}{\sinh^2\rho} - 2\tau\frac{\tanh(\rho/2)}{\sinh\rho}\right)\frac{\sinh\rho}{W}d\rho,$

we conclude that

$$\int_{\theta_0}^{\theta_1} \frac{u_\rho}{W}(\rho_1)\sinh\rho_1 d\theta - \int_{\theta_0}^{\theta_1} \frac{u_\rho}{W}(\rho_0)\sinh\rho_0 d\theta = 2H\int_{\theta_0}^{\theta_1}\int_{\rho_0}^{\rho_1}\sinh\rho d\rho d\theta.$$

As ρ and θ are arbitrary, we get

$$\frac{\partial}{\partial_\rho}\left(\frac{u_\rho\sinh\rho}{W}\right) = 2H\sinh\rho,$$

that is

$$\frac{u_\rho^d\sinh\rho}{W} = 2H\cosh\rho + d,$$

for some constant d.

A straightforward computation gives

$$u^d(\rho) = \int_*^\rho \frac{|2H\cosh(r)+d|}{\sqrt{\sinh^2(r)-(2H\cosh(r)+d)^2}}\sqrt{1+\left(\frac{l}{\sinh(r)}-2\tau\tanh\left(\frac{r}{2}\right)\right)^2}dr$$

□

3.1. Minimal screw motion in $\widetilde{PSL}_2(\mathbb{R},\tau)$. In this section, we study the geometric behavior of minimal screw motion surfaces.

Making $H\equiv 0$ in (3.2), we obtain a family (depending on the parameter d) of functions u^d which gives rise to a family of complete minimal surfaces (with fixed pitch $l>0$),

(3.4)

$$u^d(\rho) = \pm d\int_a^\rho \frac{1}{\sqrt{\sinh^2(r)-d^2}}\sqrt{1+\left(\frac{l}{\sinh(r)}-2\tau\tanh\left(\frac{r}{2}\right)\right)^2}dr, \quad d=\sinh(a)$$

We focus our attention on the case $d>0$. For technical computations, we drop up the super-index d in u^d.

Consider the Euclidean radius R in the unitary disk, measure from the origin of $\mathbb{D}^2$, the coordinates ρ and R are related by $R=\tanh\left(\frac{\rho}{2}\right)$. Thus the function u can be see as a function of R, that is $t=(u\circ\rho)(R)$ and its derivatives are

$$\begin{aligned}\frac{du\circ\rho}{dR} &= u_\rho(1+\cosh(\rho)),\\ \frac{d^2u\circ\rho}{dR^2} &= u_{\rho\rho}(1+\cosh(\rho))^2+u_\rho\sinh(\rho)(1+\cosh(\rho)).\end{aligned} \tag{3.5}$$

In the next theorem, for each $d>0$, we consider the number $c_d=\dfrac{\sqrt{1+d^2}-1}{d}$.

Theorem 3.5. *For $l>0$ and $\tau<0$, the function $t^d=u^d\circ\rho(R)$ gives rise to a complete generating curve $\mathfrak{G}(u)$ of a complete minimal immersed screw motion* *surface* Υ_d*, the curve* $\mathfrak{G}(u)$ *is symmetric with respect to the slice* $\{t = 0\}$*. The function* $t^d = u^d \circ \rho(R)$ *is increasing and concave for* $R > c_d$ *cutting orthogonally the* x*-axis at* c_d*. When* $R \longrightarrow 1$ *the graph of* u^d *has a finite value in* $\partial_\infty \mathbb{D}^2 \times \mathbb{R}$*. In that finite value, the graph has a limit tangent line, which makes with the* x*-axis an angle* α *such that* $\tan(\alpha) = d\sqrt{1+4\tau^2}$*. See figure 2.*

Proof. Consider the functions

$$f(\rho) = 1 + \left(\frac{l}{\sinh \rho} - 2\tau \tanh(\rho/2)\right)^2$$
$$g(\rho) = \sinh^2 \rho - d^2$$

with derivatives

$$f_\rho(\rho) = -2\cosh^{-2}(\rho/2)\left(\frac{l}{\sinh \rho} - 2\tau \tanh \rho/2\right)\left(\frac{l}{2}\frac{\cosh \rho}{\cosh \rho - 1} + \tau\right),$$
$$g_\rho(\rho) = 2 \sinh \rho \cosh \rho.$$

The formulae (3.4) is writing as

$$u(\rho) = d \int_*^\rho \sqrt{\frac{f(r)}{g(r)}} dr \tag{3.6}$$

therefore $u_\rho(\rho) > 0$ and since $f(\rho) > 0$, $g(\rho) > 0$, then:

$$u_{\rho\rho}(\rho) = \frac{1}{2}\sqrt{\frac{g(\rho)}{f(\rho)}} \frac{f_\rho g - f g_\rho}{g^2} < 0.$$

It is clear that $\dfrac{du \circ \rho}{dR} > 0$ and

$$\frac{d^2 u \circ \rho}{dR^2} = [u_{\rho\rho}(1 + \cosh \rho) + u_\rho \sinh(\rho)](1 + \cosh \rho)$$

but

$$u_{\rho\rho}(1 + \cosh \rho) + u_\rho = \frac{dg}{2g^2\sqrt{fg}} [f_\rho g(1 + \cosh \rho) - f g_\rho \cosh \rho + (2fg - f g_\rho)] < 0.$$

That is, the function $t^d = (u^d \circ \rho)(R)$ is an increasing concave function for $R > c_d$. Notice that the curvature,

$$k(R) = \frac{\frac{d^2 t^d}{dR^2}}{(1 + (\frac{dt^d}{dR})^2)^{3/2}}$$

has a finite limit at the vertical point $R = c_d$. Therefore, considering the rotation by π around the geodesic $\{y = 0, t = 0\}$, we obtain a complete C^2 curve $\mathfrak{C}(u)$, which generates a complete minimal surfaces Υ_d invariant by screw motion translations. That is, the surface Υ_d is invariant by the one-paremeter isometries group Γ, where Γ is given in the introduction of this section.

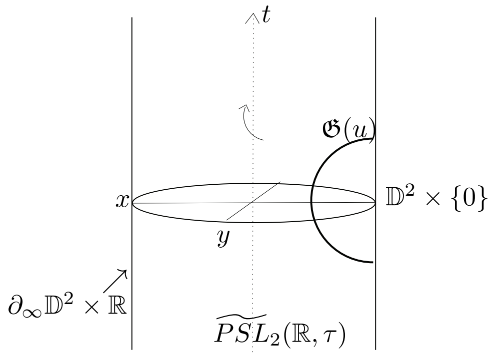


FIG. 2. *$\mathfrak{G}(u)$ is the generating curve of a screw motion surface.*

From $R = \tanh(\rho/2)$ we have $\cosh(\rho) = (1+R^2)/(1-R^2)$ this implies:

$$\frac{d(u^d \circ \rho)}{dR} = \frac{2d}{\sqrt{4R^2 - d^2(1-R^2)^2}} \sqrt{1 + \left(\frac{l(1-R^2)}{2R} - 2\tau R\right)^2}$$

thus

$$\lim_{R \longrightarrow 1} \frac{d(u^d \circ \rho)}{dR} = d\sqrt{1+4\tau^2}$$

where $\tan(\alpha) = d\sqrt{1+4\tau^2}$. $\square$

3.2. Screw motion surfaces in $\widetilde{PSL}_2(\mathbb{R},\tau)$. In this section, we study the geometric behavior of the screw motion surfaces having constant mean curvature $H > 0$. Following the ideas presented as for the minimal screw motion we will study the function

$$u^d(\rho) = \int_*^\rho \frac{(2H\cosh(r) + d)}{\sqrt{\sinh^2(r) - (2H\cosh(r) + d)^2}} \sqrt{1 + \left(\frac{l}{\sinh(r)} - 2\tau \tanh\left(\frac{r}{2}\right)\right)^2}\, dr$$

when d takes the values $d = 0$, $d = -2H$ and $d = 2H$.

3.3. First case $d = 0$. We denote u^0 simply by u, therefore

$$u(\rho) = \int_*^\rho \frac{2H\cosh(r)}{\sqrt{(1-4H^2)\cosh^2(r) - 1}} \sqrt{1 + \left(\frac{l}{\sinh(r)} - 2\tau \tanh\left(\frac{r}{2}\right)\right)^2}\, dr \tag{3.7}$$

here necessarily $0 < H < 1/2$.

Again we consider the Euclidean coordinate $R = \tanh(\rho/2)$, and for each fixed $0 < H < 1/2$, the value $R_0 = \dfrac{2H}{1+\sqrt{1-4H^2}}$.

THEOREM 3.6. *For each $l > 0$, the function u is defined for $R > R_0$ and is strictly increasing, for R near R_0 the function is strictly concave and when R goes to*

1 the function u goes to $+\infty$. The graph of u gives rise to a complete curve $\mathfrak{C}(u)$ lying in the xt-plane, the curve $\mathfrak{C}(u)$ is symmetric with respect to the slice $\{t = 0\}$, and $\mathfrak{C}(u)$ has a vertical tangent line at the point R_0. Furthermore, the curve $\mathfrak{C}(u)$ is the generating curve of a complete immersed screw motion surface Υ of pitch l, having constant mean curvature $0 < H < 1/2$. See figure 3.

In addition if $H \to 0$ then $R_0 \to 0$ and if $H \to 1/2$ then $R_0 \to 1$.

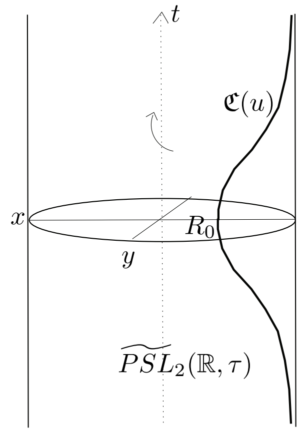


FIG. 3. $\mathfrak{C}(u)$ *is the generating curve of a screw motion surface.*

Proof. Consider the functions

$$f(\rho) = 1 + \left(\frac{l}{\sinh\rho} - 2\tau\tanh\rho/2\right)^2$$
$$h(\rho) = a\cosh^2\rho - 1$$

where $a = 1 - 4H^2 > 0$. The correspondent derivatives are

$$f_\rho(\rho) = -2\cosh^{-2}(\rho/2)\left(\frac{l}{\sinh\rho} - 2\tau\tanh\rho/2\right)\left(\frac{l}{2}\frac{\cosh\rho}{\cosh\rho - 1} + \tau\right)$$
$$h_\rho(\rho) = 2a\sinh\rho\cosh\rho.$$

Since that $R = \tanh(\rho/2)$ then

$$\cosh\rho = \frac{1+R^2}{1-R^2}$$

(3.8)

$$\sinh\rho = \frac{2R}{1-R^2}.$$

Let ρ_0 be such that $h(\rho_0) = 0$. Since $R_0 = \dfrac{2H}{1+\sqrt{1-4H^2}}$, we obtain $R_0 = \tanh\rho_0$. By formula (3.7), the function $u(\rho)$ is defined for $\rho > \rho_0$.

On the other hand,

$$u_\rho(\rho) = 2H\cosh\rho\sqrt{\frac{f(\rho)}{h(\rho)}} > 0$$

therefore $\dfrac{d(u\circ\rho)}{dR} > 0$ and $t = (u\circ\rho)(R)$ is strictly increasing.

To see that $t = (u\circ\rho)(R)$ goes to $+\infty$ when R goes to 1, observe that

$$I(\rho) = \int_{\rho_0}^{\rho} \frac{2H\cosh(r)}{\sqrt{a\cosh^2(r) - 1}} dr \leq u(\rho)$$

and setting $w = \sinh\rho$ and $b^2 = \dfrac{1-a}{a}$ then the integral $I(\rho)$ is easily solved

$$I(\rho) = \frac{2H}{\sqrt{a}}\left[\ln|\sinh\rho + \sqrt{\sinh^2\rho - b^2}| - \ln(b)\right]$$

thus whenever ρ goes to $+\infty$, $I(\rho)$ goes to $+\infty$, this implies $u(\rho)$ goes to $+\infty$, now using equations (3.5) and the fact that R goes to 1 if ρ goes to $+\infty$, we conclude that $t = (u\circ\rho)(R)$ goes to $+\infty$ if R goes to 1.

Notice that the curvature,

$$k(R) = \frac{\frac{d^2(u\circ\rho)}{dR^2}}{(1 + (\frac{d(u\circ\rho)}{dR})^2)^{3/2}}$$

has a finite limit at the vertical point R_0, so considering the rotation by π around the geodesic $\{y = 0, t = 0\}$, the graph of u gives rise to a complete and C^2 curve $\mathfrak{C}(u)$. $\mathfrak{C}(u)$ is the generating curve a complete properly immersed screw motion surface Υ. $\square$

3.4. Second case $d = -2H$. We have the function

$$u^{-2H}(\rho) = \int_0^{\rho} \frac{2H(\cosh(r) - 1)}{\sqrt{\sinh^2(r) - 4H^2(\cosh(r) - 1)^2}} \sqrt{1 + \left(\frac{l}{\sinh(r)} - 2\tau\tanh\left(\frac{r}{2}\right)\right)^2} dr$$

and we denote by $C(-2H) = C(u^{-2H})$ the graph of the function u^{-2H} in the xt-plane. Before enunciate the two theorems we prove the next lemma.

LEMMA 3.7. *The curve $C(-2H)$ has a tangent line at the origin which makes an angle α such that* $\tan(\alpha) = 2Hl$. *Furthermore, for each $\tau < 0$, $C(-2H)$ is strictly convex.*

Proof. From (3.5) and (3.8)

$$\frac{du^{-2H}\circ\rho}{dR} = \frac{2H}{1-R^2}\sqrt{\frac{4R^2 + [l(1-R^2) + 2R^2]^2}{1 - 4H^2R^2}}$$

thus

$$\lim_{R\to 0}\frac{du^{-2H}\circ\rho}{dR} = 2Hl.$$

In order to prove the second affirmation, notice that

$$\frac{du^{-2H}\circ\rho}{dR} = \frac{2H}{1-R^2}\sqrt{\frac{4R^2 + [l(1-R^2) + 2R^2]^2}{1 - 4H^2R^2}} = \frac{2H}{1-R^2}\sqrt{A(R)}$$

where

$$A(R) = \frac{4R^2 + [l(1 - R^2) + 2R^2]^2}{1 - 4H^2R^2}$$

hence

$$\frac{d^2 u^{-2H} \circ \rho}{dR^2} = \frac{H}{\sqrt{A(R)}(1 - R^2)^2}[A_R(R)(1 - R^2) + 4RA(R)].$$

Setting $B(R) = A_R(R)(1 - R^2) + 4RA(R)$, observing that $\tau < 0$ and $0 < R < 1$, we obtain

$$\begin{aligned} B(R) = & \frac{4R}{(1 - 4H^2R^2)}[(2 - 4\tau l + (l + 4\tau)^2 R^2 - 8\tau l R^2)(1 - R^2) + \\ & +(1 + 4\tau^2)4R^2 + [l(1 - R^2) - 1/2]^2 - 1/2] + \\ & + \frac{8H^2R(1 - R^2)[4R^2 + (l(1 - R^2) - 4\tau R^2)^2]}{(1 - 4H^2R^2)^2} > 0 \end{aligned}$$

thus

$$\frac{d^2 u^{-2H} \circ \rho}{dR^2} > 0$$

this completes the prove. ☐

This lemma help us to proof the next two theorems.

THEOREM 3.8. *Assume* $0 < H \leq 1/2$. *For each* $l > 0$, *the curve* $C(-2H)$ *is strictly convex and gives rise to a complete curve* $\mathfrak{C}(-2H)$ *which is the generating curve of a complete properly immersed screw motion surface* $\Phi^{(-2H)}$ *having pitch* $l > 0$, *this surface contains the* t*-axis. The curve* $\mathfrak{C}(-2H)$ *has a tangent line at the origin which makes an angle* α *such that* $\tan(\alpha) = 2Hl$. *Furthermore, when* R *goes to* 1, *the function* $t = (u^{-2H} \circ \rho)$ *goes to* $+\infty$. *See figure 4.*

Proof. Notice that the curvature,

$$k(R) = \frac{\frac{d^2(u^{-2H} \circ \rho)}{dR^2}}{(1 + (\frac{d(u^{-2H} \circ \rho)}{dR})^2)^{3/2}}$$

has a finite limit at the origin $(0, 0)$ in the xt-plane, so considering the rotation by π around the x-axis followed by the rotation by π around the t-axis, the graph of $t = u^{-2H} \circ \rho$ gives rise a complete curve $\mathfrak{C}(-2H)$. The properties of the curve $\mathfrak{C}(-2H)$ are given by the Lemma 3.7, this complete curve is the generating curve of a complete properly immersed screw motion surface $\Phi^{(-2H)}$ of pitch $l > 0$.

On the other hand, notice that

$$\frac{l^2}{\sinh^2(r)} \leq \left(\frac{l}{\sinh(r)} - 2\tau \tanh(\rho/2)\right)^2$$

this implies

$$I(\rho) = \int_0^\rho \frac{2H(\cosh(r) - 1)}{\sqrt{\sinh^2(r) - 4H^2(\cosh(r) - 1)^2}} \sqrt{1 + \frac{l^2}{\sinh^2(r)}} dr \leq u^{-2H}(\rho). \tag{3.9}$$

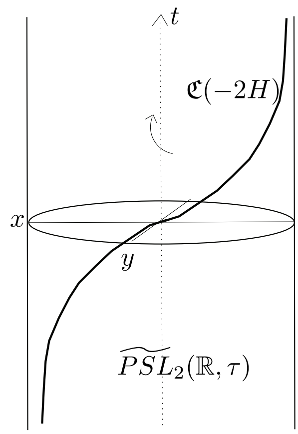


FIG. 4. $\mathfrak{C}(-2H)$ *is the generating curve of a screw motion surface.*

It was showed in [7, Theorem 17.A] that $I(\rho)$ goes to $+\infty$ when ρ goes to $+\infty$, thus $t = u^{-2H} \circ \rho$ goes to $+\infty$ when R goes to 1. $\square$

THEOREM 3.9. *Assume $H > 1/2$. In this case, for each $l > 0$, the curve $C(-2H)$ is defined for $0 < R < 1/(2H)$, has a vertical tangent line at $R_0 = 1/(2H)$ and in this point $C(-2H)$ has a finite height. The curve $C(-2H)$ is strictly convex and can be extended to a complete embedded curve $\zeta(-2H)$ by a rotation by π around the x-axis at this finite height together with the composition of a rotation by π around the t-axis and the x-axis. The curve $\zeta(-2H)$ has a tangent line at the origin which makes an angle α such that $\tan(\alpha) = 2Hl$. The curve $\zeta(-2H)$ is the generating curve of a complete properly immersed screw motion surface $\Psi^{(-2H)}$ having pitch $l > 0$, this surface contains the t-axis. See figure 5.*

Proof. For $H > 1/2$ the function u^{-2h} is defined for $0 < R < R_0$ and in the points 0 and R_0 the curvature

$$k(R) = \frac{\frac{d^2(u^{-2H}\circ\rho)}{dR^2}}{(1+(\frac{d(u^{-2H}\circ\rho)}{dR})^2)^{3/2}}$$

has a finite limit. Notice that (see equation (3.9))

$$I(\rho) \leq u^{-2H}(\rho) \leq I(\rho)\sqrt{3\tau^2 + (\tau-1)^2}. \tag{3.10}$$

It was showed in [7, Theorem 17.B] that $I(\rho)$ has a finite height over all its domain, this implies that the function $t = u^{-2H} \circ \rho$ has the finite value $t_0(\tau)$ at the point R_0. By the lemma 3.7, the curve $C(-2H)$ is strictly convex and applying the rotation by π around the x-axis which is at height $t_0(\tau)$ together with the composition of the rotation by π around the x-axis and the t-axis, we obtain a curve which is symmetric with respect to the curve $t = t_0(\tau)$ in the xt-plane. Continuing this process, we obtain the curve $\zeta(-2H)$ with the desired properties. $\square$

3.5. Second case $d = 2H$. We have the function

$$u^{2H}(\rho) = \int_0^\rho \frac{2H\sqrt{\cosh(r)+1}}{\sqrt{(1-4H^2)\cosh(r)-(1+4H^2)}}\sqrt{1+\left(\frac{l}{\sinh(r)} - 2\tau\tanh\left(\frac{r}{2}\right)\right)^2}\,dr$$

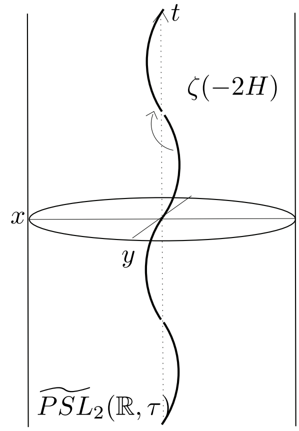


FIG. 5. $\zeta(-2H)$ *is the generating curve of a screw motion surface.*

here, necessarily $0 < H < 1/2$.

Following the same arguments as in the previous cases, we have the next theorem.

THEOREM 3.10. *The function u^{2H} is defined for $2H < R$ and the graph of the function u^{2H} is a curve $C(2H)$ which cut orthogonally the x-axis at $R_0 = 2H$. When R goes to 1 the function u^{2H} goes to $+\infty$. The curve $C(2H)$ gives rise to a complete curve $\mathfrak{R}(2H)$ which is symmetric with respect to the slice $\{t = 0\}$ and it is the generating curve of an immersed screw motion surface $\Pi(2H)$ having pitch $l > 0$. See figure 6.*

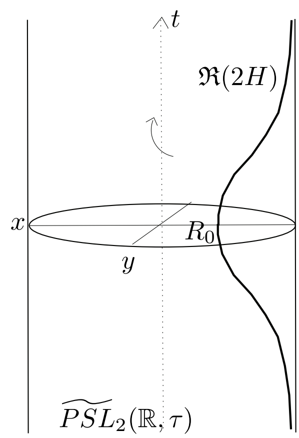


FIG. 6. $\mathfrak{R}(2H)$ *is the generating curve of a screw motion surface.*

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