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NORMAL CROSSING SINGULARITIES AND HODGE THEORY OVER ARTIN RINGS*

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Abstract. We introduce the notion of a mixed Hodge structure over an Artin ring, thereby establishing a framework for applying Hodge theoretic arguments to deformation problems. Examples arise from relative simple normal crossing varieties over Artinian base schemes. As an application we prove that the maps between graded pieces of Hodge bundles have constant rank.

Key words. Deformations, local triviality, Hodge theory, normal crossings.

AMS subject classifications. 13D10, 14C30, 32S35.

Introduction. This article aims at constructing a framework for applying Hodge theoretic methods to deformation problems. In the study of Calabi-Yau manifolds, deformation theory – which is controlled by the tangent bundle – is linked to Hodge theory by the CY condition. This has been exploited in the ground–breaking work of Ran and Kawamata [Ra92, Ka92] to prove the unobstructedness of deformations.

As deformation theory is carried out over Artin rings, one is naturally led to the notion of a *mixed Hodge structure over a local Artin* $\mathbb{C}$*-algebra* R, which we introduce in this work. The tools developed here lay the foundation to prove further results about deformations of CY-manifolds in the style of Ran-Kawamata.

Though the CY condition ties deformation theory to Hodge theory, for Hodge theory itself it is not needed and thus none of the results in this note depend on it. We consider locally trivial families of simple normal crossing varieties over an Artinian base and prove Hodge theoretic results in the style of Friedman [Fri83] in this setting. We have

Theorem 4.15. *Let* Y *be a proper, simple normal crossing* $\mathbb{C}$*-variety and let* $f : \mathcal{Y} \longrightarrow S$ *be a locally trivial deformation of* Y *over* $S = \operatorname{Spec} R$ *for an Artinian local* $\mathbb{C}$*-algebra* R *of finite type. Then there is a mixed Hodge structure over* R *on* $H^k(Y^{\mathrm{an}}, \mathbb{R})$*.*

In order to manipulate these objects in a linear algebra fashion as in classical Hodge theory (case $R = \mathbb{C}$) we study the Weil restriction of mixed Hodge structures over R and obtain the following result.

Theorem 4.17. *Let* $f : \mathcal{Y} \longrightarrow S = \operatorname{Spec} R$ *be proper morphism which is a locally trivial deformation of simple normal crossing* $\mathbb{C}$*-variety over an Artinian local* $\mathbb{C}$*-algebra* R *of finite type and let* $g : \mathcal{X} \longrightarrow S$ *be smooth and proper. Let* $i : \mathcal{Y} \longrightarrow \mathcal{X}$ *be an* S*-morphism. We denote by* $\widetilde{\Omega}^p_{\mathcal{Y}/S}$ *the quotient of* $\Omega^p_{\mathcal{Y}/S}$ *by the subsheaf of sections supported on the singular locus of* f*. Then for all* p, q *the morphism* $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ *has a free cokernel.*

My interest in the subject arose from applications to deformations of Lagrangian subvarieties of symplectic manifolds, see [Le11], where these techniques are applied in the study deformations of singular Lagrangian subvarieties of symplectic manifolds.

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In order to motivate the relation with deformation theory, let us recall that for a Calabi-Yau manifold X, freeness of the $R^q g_* \Omega^p_{\mathcal{X}/S}$ for infinitesimal deformations $\mathcal{X} \longrightarrow S$ of X as proven by Deligne [Del68, Thm 5.5] directly implies the unobstructedness of deformations by the result of Kawamata-Ran. In [Le11], their method is used to prove unobstructedness of deformations in the setting mentioned above, and Theorem 4.17 above takes the place of Deligne's theorem as the key ingredient.

A possible field of further applications is the problem of smoothing normal crossing varieties. Though [Fri83] contains interesting results about Hodge theory, the article's main focus were smoothings. This method has been refined in [KN] using logarithmic geometry. It is highly plausible that one can use the tools developed here to obtain interesting results about smoothing of normal crossing varieties in the spirit of [Fri83, KN], see 4.18 for an outlook.

A remark about the connection with the notion of an *infinitesimal variation of Hodge structure* in the sense of [CGGH, Def p. 130] is in order. The latter corresponds to the special case of a Hodge structure over a local Artin ring R whose maximal ideal satisfies $\mathfrak{m}^2 = 0$, but unlike our definition takes into account the $\mathbb{Z}$-structure and the polarization. They focus on Torelli type applications, whereas our goal is to lay foundations for applications to deformation problems. For Torelli one needs a $\mathbb{Z}$-structure, for deformations one is interested in Hodge numbers. Furthermore, to study deformation theory it is crucial to invoke higher order information and so one is naturally led to our notion.

Let us spend some words about the structure of this article. In section 1 we recall the definition of *locally trivial* deformations. The theory of Weil restriction as presented in section 2 relates Hodge- and Hodge-Weil structures. Its exploitation in the infinitesimal setup is the main new feature of this work and its motivation is purely geometric. Mixed Hodge structures over local Artin $\mathbb{C}$-algebras are introduced in section 3. Section 4 provides a construction of a mixed Hodge structure over a local Artin $\mathbb{C}$-algebra R on the cohomology of simple normal crossing varieties over $S = \operatorname{Spec} R$.

Notations and conventions. The term *algebraic variety* will stand for a separated reduced k-scheme of finite type. A k-variety Y of equidimension n is called a *normal crossing variety* if for every closed point $y \in Y$ there is an $r \in \mathbb{N}_0$ such that $\widehat{\mathcal{O}}_{Y,y} \cong k[[y_1, \ldots, y_{n+1}]]/(y_1 \cdot \ldots \cdot y_r)$. It is called a *simple normal crossing variety* if in addition every irreducible component is nonsingular.

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1. Locally trivial deformations. The reader may consult [Ser06] as a general reference on locally trivial deformations. By Art_k we denote the category of local Artinian k-algebras with residue field k. The maximal ideal of an element $R \in \mathrm{Art}_k$ will be denoted by $\mathfrak{m}$. We recall that a deformation $\mathcal{X} \longrightarrow S$ of X over $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_k$, is called *(Zariski resp. analytically) locally trivial*, if for every $x \in X$ there is an open neighbourhood $x \in U \subset X$ in the Zariski- resp. Euclidean topology and an S-isomorphism $\mathcal{X}|_U \xrightarrow{\cong} X|_U \times S$ restricting to the identity on the central fiber. Recall [Har77, Exc II.8.6] that for a regular k-scheme every deformation is locally trivial. Next we will show that irreducible components of a variety extend to flat subschemes on locally trivial deformations. This will take some commutative algebra.

LEMMA 1.1. *Let A be a reduced noetherian ring and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the pairwise distinct minimal primes of A. Then $\operatorname{Ann} \mathfrak{p}_j = \cap_{i \neq j} \mathfrak{p}_i$ for each j.*

Proof. For $A_i = A/\mathfrak{p}_i$ the canonical map $\phi : A \longrightarrow A_1 \times \ldots \times A_n$ is injective, because $\cap_i \mathfrak{p}_i = \operatorname{nil}(A) = 0$. Suppose $a \in \cap_{i \neq j} \mathfrak{p}_i$, $b \in \mathfrak{p}_j$ and write $\phi(a) = (a_1, \ldots, a_n)$ and $\phi(b) = (b_1, \ldots, b_n)$. Then $\phi(ab) = (a_1 b_1, \ldots, a_n b_n) = 0$ because $a_i = 0$ for $i \neq j$ and $b_j = 0$. But ϕ is injective, hence $ab = 0$, in other words, $a \in \operatorname{Ann} \mathfrak{p}_j$, so $\operatorname{Ann} \mathfrak{p}_j \supset \cap_{i \neq j} \mathfrak{p}_i$.

Let $a \in \operatorname{Ann} \mathfrak{p}_j$. Then for every $b \in \mathfrak{p}_j$ we have $0 = \phi(ab) = (a_1 b_1, \ldots, a_n b_n)$ in the above notation, where $b_j = 0$. As the $\mathfrak{p}_i$ are minimal and pairwise distinct, $\mathfrak{p}_j \backslash \mathfrak{p}_k \neq \emptyset$ for every $k \neq j$. If we fix k and choose $b \in \mathfrak{p}_j \backslash \mathfrak{p}_k$, then $b_k \neq 0$. So $a_k b_k = 0$ implies that $a_k = 0$ as A_k is an integral domain, so $a \in \mathfrak{p}_k$. Choosing different b we see that $a \in \cap_{i \neq j} \mathfrak{p}_i$ completing the proof. □

LEMMA 1.2. *Let A be a reduced noetherian ring, $\mathfrak{p} \subset A$ be a minimal prime ideal and $\psi : \mathfrak{p} \longrightarrow A/\mathfrak{p}$ be an A-module homomorphism. Then $\psi = 0$.*

Proof. Let $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the pairwise distinct minimal prime ideals of A and $N := \operatorname{im} \psi \subset A/\mathfrak{p}$. We will show that $N = 0$. By Lemma 1.1 we have $\operatorname{Ann} \mathfrak{p} = \cap_i \mathfrak{p}_i$. So $\mathfrak{p} \notin \operatorname{supp}(\mathfrak{p}) = V(\operatorname{Ann} \mathfrak{p})$, for otherwise $\cap_i \mathfrak{p}_i \subset \mathfrak{p}$ and thus $\mathfrak{p}_i \subset \mathfrak{p}$ for some i as $\mathfrak{p}$ is prime, contradicting the fact that $\mathfrak{p} \neq \mathfrak{p}_i$ and $\mathfrak{p}$ is minimal. Thus, $\mathfrak{p} \otimes_A A_\mathfrak{p} = 0$ and the surjection $0 = \mathfrak{p} \otimes_A A_\mathfrak{p} \twoheadrightarrow N \otimes_A A_\mathfrak{p}$ yields that $N_\mathfrak{p} = N \otimes_A A_\mathfrak{p} = 0$. Therefore, N is torsion and $\subset A/\mathfrak{p}$, hence $N = 0$. □

LEMMA 1.3. *Let A be a reduced noetherian ring, $\mathfrak{p} \subset A$ a minimal prime ideal, $R \in \operatorname{Art}_k$ and $\mathfrak{P} \subset A \otimes_k R$ an ideal such that $A \otimes_k R/\mathfrak{P}$ is a flat deformation of $A/\mathfrak{p}$ over R. Then $\mathfrak{P} = \mathfrak{p} \otimes R$.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal. As R is Artinian, there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$. So we may argue inductively and assume that $\mathfrak{P}/\mathfrak{m}^k = \mathfrak{p} \otimes R/\mathfrak{m}^k \subset A \otimes R/\mathfrak{m}^k$. By flatness, we obtain the commutative diagram

(1.1)
$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A/\mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1} \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathfrak{P}/\mathfrak{m}^{k+1} \arrow[r] \arrow[d] & A \otimes R/\mathfrak{m}^{k+1} \arrow[r, "\varphi"] \arrow[d] & A \otimes R/(\mathfrak{P} + \mathfrak{m}^{k+1}) \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & \mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & A \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & A/\mathfrak{p} \otimes R/\mathfrak{m}^k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

with exact rows and columns.

If we denote the inclusion $\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \hookrightarrow A \otimes R/\mathfrak{m}^{k+1}$ by ψ, then one easily sees that $\varphi \circ \psi$ factors through π. Now observe that this factorization

$\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \longrightarrow A/\mathfrak{p} \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is zero by Lemma 1.2, hence so is $\varphi \circ \psi$. Therefore ψ factors through $\ker \varphi = \mathfrak{P}/\mathfrak{m}^{k+1}$ as $\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{P}/\mathfrak{m}^{k+1} \longrightarrow A \otimes R/\mathfrak{m}^{k+1}$. But $\mathfrak{p} \otimes R/\mathfrak{m}^{k+1} \longrightarrow \mathfrak{P}/\mathfrak{m}^{k+1}$ becomes an isomorphism after tensoring with $R/\mathfrak{m}^k$, thus it is itself an isomorphism by flatness of $\mathfrak{P}/\mathfrak{m}^{k+1}$, see [Ser06, Lem A.4]. □

LEMMA 1.4. *Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of a reduced noetherian scheme Y over an Artinian base $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_k$. Then the irreducible components Y_α of Y lift uniquely to subschemes $\mathcal{Y}_\alpha \hookrightarrow \mathcal{Y}$ flat over S. Moreover, each $\mathcal{Y}_\alpha$ is a locally trivial deformation of Y_α.*

Proof. Let $Y = \cup_i U_i$ be an open affine covering of Y such that there are R-algebra isomorphisms $\theta_i : A_i \otimes_k R \longrightarrow \Gamma(U_i, \mathcal{O}_\mathcal{Y})$ where $A_i := \Gamma(U_i, \mathcal{O}_Y)$. An irreducible component Y_α of Y gives a minimal prime ideal $\mathfrak{p}^i_\alpha$ in each A_i. We define $\mathcal{Y}^i_\alpha$ to be the closed subscheme of $\mathcal{Y}|_U$ whose ideal is $\theta_i(\mathfrak{p}_\alpha)$. Then $\mathcal{Y}^i_\alpha$ is a flat lifting of $Y_\alpha|_{U_i}$ for all i. Therefore, on $U_{ij} := U_i \cap U_j$ also $\mathcal{Y}^j_\alpha|_{U_{ij}}$ is a flat lifting of $Y_\alpha|_{U_{ij}}$ for all j. Then by Lemma 1.3 we conclude that $\mathcal{Y}^i_\alpha|_{U_{ij}} = \mathcal{Y}^j_\alpha|_{U_{ij}}$ and so the $\mathcal{Y}^i_\alpha$ are the restrictions of a closed subscheme $\mathcal{Y}_\alpha$ of $\mathcal{Y}$. The argument also shows that $\mathcal{Y}_\alpha$ is unique. □

2. Weil restriction. For our Hodge theoretical considerations we use the theory of Weil restriction as an essential tool. The foundations of this theory were laid by Grothendieck in [Gro59, Gro60]. The Weil restriction of a $\mathbb{C}$-scheme S will be an $\mathbb{R}$-scheme S_{wl} such that the $\mathbb{R}$-valued points of S_{wl} are exactly the $\mathbb{C}$-valued points of S. In this particular case we interpret Weil restriction as the algebro-geometric analogue of the process of regarding a complex manifold as a differentiable manifold.

We extend the concept of Weil restriction to modules. We are not aware that this has been done systematically before. Nevertheless, it is an elementary byproduct of the functorial treatment. We prove some comparison results between R-modules and their Weil restrictions.

2.1. Weil restriction. Let $S \xrightarrow{f} Z \xrightarrow{p} W$ be morphisms of schemes and consider the functor on the category of W-schemes

$$\underline{S_{\mathrm{wl}}} : (\mathrm{Sch/W})^{\mathrm{op}} \longrightarrow \mathrm{Set}, \quad S' \mapsto \mathrm{Mor}_{\mathrm{Sch/Z}}(S' \times_W Z, S). \tag{2.1}$$

In many cases, it is representable by a W-scheme S_{wl} called the *Weil restriction* of S, see e.g. [Gro60, 4.c., p.20] and [BLR90, 7.6, Thm 4]. In fact, $p_* : S \mapsto S_{\mathrm{wl}}$ is functorial, adjoint to the pullback p^* and sends affine schemes to affine schemes.

We will now specialize to $Z = \operatorname{Spec} \mathbb{C}$ and $W = \operatorname{Spec} \mathbb{R}$, where every quasi-projective $\mathbb{C}$-scheme S has a Weil restriction. If $S = \operatorname{Spec} R$ we will write R_{wl} for the coordinate ring of S_{wl}. Equation (2.1) in particular gives $S(\mathbb{C}) = S_{\mathrm{wl}}(\mathbb{R})$. If $S = \operatorname{Spec} R$, there is a canonical ring homomorphism $\eta : R \longrightarrow R_{\mathrm{wl}} \otimes_\mathbb{R} \mathbb{C}$ by adjunction. For $R = \mathbb{C}[z_1, \ldots, z_n]/(f_1, \ldots, f_k)$ we have

$$R_{\mathrm{wl}} = \mathbb{R}[x_1, y_1, \ldots, x_n, y_n]/(g_1, h_1, \ldots, g_k, h_k) \tag{2.2}$$

where $f_j = g_j + ih_j$ if we evaluate at $z_k = x_k + iy_k$.

If we define $\overline{S} := S \times_\sigma \mathbb{C}$ where $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ is the complex conjugation then (2.1) tells us that there is a canonical isomorphism $\overline{S}_{\mathrm{wl}} \cong S_{\mathrm{wl}}$ and by [Sch94, Ch 1, 4.11.3] there is a canonical isomorphism $S_{\mathrm{wl}} \times_\mathbb{R} \mathbb{C} \longrightarrow S \times_\mathbb{C} \overline{S}$ such that η is identified with projection on the first factor. In particular, η is faithfully flat as the projection $S \times_\mathbb{C} \overline{S} \longrightarrow S$ is faithfully flat.

LEMMA 2.2. *If R is a local Artin $\mathbb{C}$-algebra with residue field $\mathbb{C}$, then R_{wl} is a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$.*

Proof. By (2.2) we see that R_{wl} is an $\mathbb{R}$-algebra of finite type. A maximal ideal $\mathfrak{m} \subset R_{\mathrm{wl}}$ will define a homomorphism $R_{\mathrm{wl}} \longrightarrow R_{\mathrm{wl}}/\mathfrak{m} = k$, where k is a finite field extension of $\mathbb{R}$ by Hilbert's Nullstellensatz. So $k = \mathbb{R}$ or $\mathbb{C}$. By the defining property of Weil restriction we have $\mathrm{Hom}_{\mathbb{R}}(R_{\mathrm{wl}}, \mathbb{R}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{R}}(R_{\mathrm{wl}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C} \times \mathbb{C})$ both of which consist of one element. But the composition of the morphism $R \longrightarrow \mathbb{R}$ with the inclusion $\mathbb{R} \subset \mathbb{C}$ is the unique morphism $R \longrightarrow \mathbb{C}$. Thus, $R_{\mathbb{R}}$ is a local ring with unique maximal ideal $\mathfrak{m}$ and residue field $\mathbb{R}$. As R_{wl} is of finite type, $R_{\mathrm{wl}} = P/I$ where P is a polynomial ring and $I \subset P$ an ideal. The preimage $\mathfrak{n}$ of $\mathfrak{m}$ under the natural map $P \longrightarrow R_{\mathrm{wl}}$ is the unique maximal ideal of P containing I. Let $I \subset \mathfrak{p} \subset \mathfrak{n}$ be a minimal prime ideal containing I. As P is a Jacobson ring by the general form of the Nullstellensatz, see [Eis95, Thm 4.19], the ideal $\mathfrak{p}$ is the intersection of maximal ideals, so that $\mathfrak{p} = \mathfrak{n}$. Taking a primary decomposition of I we see that $\mathfrak{n}^k \subset I$ for some k, so $R_{\mathrm{wl}} = P/I$ is Artinian. ∎

Definition 2.3. Let S be a $\mathbb{C}$-scheme, F be a quasi-coherent sheaf of $\mathcal{O}_S$-modules, denote by $q : S_{\mathrm{wl}} \times_{\mathbb{R}} \mathbb{C} \longrightarrow S_{\mathrm{wl}}$ the canonical projection and let $\eta : S_{\mathrm{wl}} \times_{\mathbb{R}} \mathbb{C} \longrightarrow S$ be the adjunction morphism. We define the S_{wl}-module $F_{\mathrm{wl}} := q_* \eta^* F$ and call it the *Weil restriction* of F.

If $S = \operatorname{Spec} R$ and M is an R-module, then $M_{\mathrm{wl}} = M \otimes_R (R_{\mathrm{wl}} \otimes_{\mathbb{R}} \mathbb{C})$ considered as an R_{wl} module. In the special case $M = H \otimes_{\mathbb{C}} R$ for some $\mathbb{C}$-vector space H, we find $M_{\mathrm{wl}} = H \otimes_{\mathbb{R}} R_{\mathrm{wl}}$. Weil restriction for modules has the following useful property.

Lemma 2.4. *The functor $F \mapsto F_{\mathrm{wl}}$ is faithfully exact, i.e. the sequence $K' \longrightarrow K \longrightarrow K''$ is exact if and only if $K'_{\mathrm{wl}} \longrightarrow K_{\mathrm{wl}} \longrightarrow K''_{\mathrm{wl}}$ is exact.*

Proof. The morphism η is faithfully flat as noted at the end of section 2.1. Therefore, η^* is faithfully exact. Also q_* is faithfully exact, as q is affine. ∎

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local Artin $\mathbb{C}$-algebra and F be a finitely generated R-module. Then F is a free R-module if and only if F_{wl} is a free R_{wl}-module.*

Proof. We will argue separately for η^* and q_*. For brevity we write $(R', \mathfrak{m}')$ instead of $(R_{\mathrm{wl}} \otimes_{\mathbb{R}} \mathbb{C}, \mathfrak{m}_{\mathrm{wl}} \otimes_R \mathbb{C})$. Clearly, $\eta^* F = F \otimes_R R'$ is free if F is. Suppose $\eta^* F$ is free. We take a minimal set of generators for F and obtain a surjection $\varphi : R^n \longrightarrow F$ for some n. By Nakayama's Lemma $n = \dim_{\mathbb{C}} F \otimes_R R/\mathfrak{m}$ and as $F \otimes_R R' \otimes_{R'} R'/\mathfrak{m}' = F \otimes_R R/\mathfrak{m} \otimes_{R/\mathfrak{m}} R'/\mathfrak{m}'$ this is the rank of $\eta^* F$. But as η^* is faithfully exact, $\eta^* \ker \varphi = \ker \eta^* \varphi = 0$. So $\ker \varphi = 0$ and F is free.

Let F' be an R'-module. If F' is free as an R'-module, then it is free as an R_{wl}-module, for R' is free over R_{wl}. Suppose F' is free as an R_{wl}-module. Since F' is an $R' = R_{\mathrm{wl}} \otimes_{\mathbb{R}} \mathbb{C}$-module, the submodule $\mathfrak{m}_{\mathrm{wl}} F'$ is a $\mathbb{C}$-vector space. Thus $\mathfrak{m}_{\mathrm{wl}} F' = \mathfrak{m}' F'$. If we take $x_1, \ldots, x_k \in F'$ whose residue classes modulo $\mathfrak{m}_{\mathrm{wl}}$ form a $\mathbb{C}$-basis of $F'/\mathfrak{m}_{\mathrm{wl}} F'$, then F is freely generated over R_{wl} by $x_1, ix_1, \ldots, x_k, ix_k$. In other words, F is freely generated over R' by $x_1, \ldots, x_k$. So F' is a free R'-module. ∎

3. Hodge-Weil theory. We introduce the notion of a *mixed Hodge structure over R*, where R is a local Artin $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Let $\mathcal{Y} \longrightarrow \operatorname{Spec} R$ be a locally trivial deformation of a simple normal crossing variety and denote by $\underline{R}_{\mathcal{Y}}$ be the constant sheaf R on $\mathcal{Y}$. Then $H^k(\mathcal{Y}, \underline{R}_{\mathcal{Y}})$ carries such a structure, see Theorem 4.15. A mixed Hodge structure over $R = \mathbb{C}$ is just an ordinary mixed Hodge structure. The purpose of this concept is to carry out Hodge theoretic arguments infinitesimally, as e.g. in the proof of Theorem 4.17.

A problem for $R \neq \mathbb{C}$ is that there is no analogue of the complex conjugation on the underlying R-module H. To carry over linear algebra arguments familiar from classical Hodge theory we introduce the notion of a *mixed Hodge-Weil structure over* R', where R' is now a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$. This notion is a formalization of the Weil restriction of a mixed Hodge structure over R and there is canonically a complex conjugation.

DEFINITION 3.1. Let R be a local Artin $\mathbb{C}$-algebra with residue field $\mathbb{C}$. A *mixed Hodge structure over* R is a triple $\mathcal{H} = (H_\mathbb{R}, F^\bullet, W_\bullet)$, which consists of a finite dimensional $\mathbb{R}$-vector space $H_\mathbb{R}$, a finite decreasing filtration $F^\bullet$ and a finite increasing filtration $W_\bullet$ on $H := (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{C} R$ satisfying the following properties.

1. All graded objects $\mathrm{Gr}_F^p \mathrm{Gr}_m^W H$ are free R-modules.
2. The fiber $\mathcal{H} \otimes_R \mathbb{C} = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}, F^\bullet \otimes_R \mathbb{C}, W_\bullet \otimes_R \mathbb{C})$ over the unique point of $S = \operatorname{Spec} R$ is a mixed Hodge structure.

Note that condition (1) implies that the W_m and the F^p are free R-modules. We will also call $\mathcal{H} \otimes_R \mathbb{C}$ the *central fiber* of $\mathcal{H}$. In case $\mathcal{H} \otimes_R \mathbb{C}$ is a pure Hodge structure of weight k, we call $\mathcal{H}$ a *pure Hodge structure* over R of weight k. Morphisms are defined in the obvious way, that is, they are defined over $\mathbb{R}$ and preserve both filtrations.

REMARK 3.2. Everything works out fine if we replace the $\mathbb{R}$-vector space $H_\mathbb{R}$ by a $\mathbb{Z}$-module of finite type, which is useful in the context of moduli theory.

REMARK 3.3. There is a complex conjugation $H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \longrightarrow H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ defined by $\overline{h \otimes \lambda} := h \otimes \overline{\lambda}$. However this does not canonically extend to an $\mathbb{R}$-linear map $H \longrightarrow H$, as H is a tensor product over $\mathbb{C}$ and complex conjugation is only $\mathbb{R}$-linear.

The notion of a Hodge structure over R is an infinitesimal version of a variation of Hodge structures (VHS). The (pointwise) complex conjugates $\overline{F^p}$ of the Hodge filtration of VHS do not in general form holomorphic vector bundles over the base, so there is no algebraic incarnation of $\overline{F^p}$, explaining the above remark. As a substitute we introduce the following notion.

DEFINITION 3.4. Let R be a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$. A *mixed Hodge-Weil structure* over R is a triple $\mathcal{H} = (H_\mathbb{R}, F^\bullet, W_\bullet)$, which consists of a finite dimensional $\mathbb{R}$-vector space $H_\mathbb{R}$, a finite decreasing filtration $F^\bullet$ and a finite increasing filtration $W_\bullet$ on $H := (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{R} R$ satisfying the following properties.

1. All graded objects $\mathrm{Gr}_F^p \mathrm{Gr}_m^W H$ are free R-modules.
2. The fiber $\mathcal{H} \otimes_R \mathbb{R} = (H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}, F^\bullet \otimes_R \mathbb{R}, W_\bullet \otimes_R \mathbb{R})$ over the unique point of $S = \operatorname{Spec} R$ is a mixed Hodge structure.

Note that as in Definition 3.1, condition (1) implies that the W_m and the F^p are free R-modules. We will also call $\mathcal{H} \otimes_R \mathbb{R}$ the *central fiber* of $\mathcal{H}$. In case $\mathcal{H} \otimes_R \mathbb{C}$ is a pure Hodge structure of weight k, we call $\mathcal{H}$ a *pure Hodge-Weil structure* over R of weight k. Again, morphisms are defined in the obvious way.

REMARK 3.5. The complex conjugation $H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \longrightarrow H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ extends canonically to an $\mathbb{R}$-linear map $H \longrightarrow H$. Since morphisms of mixed Hodge-Weil structures are defined over $\mathbb{R}$, they are compatible with complex conjugation.

By Lemma 2.2 the following lemma makes sense.

LEMMA 3.6. *Let* $\mathcal{H} = (H_\mathbb{R}, F^\bullet, W_\bullet)$ *be a mixed Hodge structure over a local Artin* $\mathbb{C}$-*algebra* R. *Then* $\mathcal{H}_{\mathrm{wl}} = (H_\mathbb{R}, F^\bullet_{\mathrm{wl}}, (W_{\mathrm{wl}})_\bullet)$ *is a mixed Hodge-Weil structure over* R_{wl} *and the central fibers of* $\mathcal{H}$ *and* $\mathcal{H}_{\mathrm{wl}}$ *are isomorphic as mixed Hodge structures.*

Proof. The remark after Definition 2.3 tells us that

$$H_{\mathrm{wl}} = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{C}} R)_{\mathrm{wl}} = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} R_{\mathrm{wl}}. \tag{3.1}$$

By Lemma 2.4 we see that the F^p_{wl} and $(W_m)_{\mathrm{wl}}$ are submodules of $H_{\mathrm{wl}} = (H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{R}} R_{\mathrm{wl}}$. By Lemma 2.5 the modules $\left(\mathrm{Gr}^p_F \mathrm{Gr}^W_m H\right)_{\mathrm{wl}}$ are free and by Lemma 2.4 they are the graded objects of the filtrations F^p_{wl} and $(W_m)_{\mathrm{wl}}$. Let $\mathfrak{m}'$ be the maximal ideal of R_{wl}. As $R_{\mathrm{wl}}/\mathfrak{m}' = \mathbb{R}$ we see from 3.1 that $H_{\mathrm{wl}} \otimes_{\mathbb{R}} R_{\mathrm{wl}}/\mathfrak{m}' = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. For the same reason $F^p_{\mathrm{wl}} \otimes \mathbb{R} = F^p \otimes \mathbb{C}$ and $(W_m)_{\mathrm{wl}} \otimes \mathbb{R} = W_m \otimes \mathbb{C}$ so that $\mathcal{H}_{\mathrm{wl}} \otimes \mathbb{R}$ is a mixed Hodge structure. □

LEMMA 3.7. *Let R be a local Artin $\mathbb{R}$-algebra with residue field $\mathbb{R}$ and $\mathcal{H} = (H_{\mathbb{R}}, F^{\bullet})$ a pure Hodge-Weil structure of weight k. Then*

$$H = F^p \oplus \overline{F^{q+1}}, \qquad \forall p, q, p+q = k, \tag{3.2}$$

$$H = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} = F^p \cap \overline{F^q} \qquad \textit{and} \tag{3.3}$$

$$F^p = \bigoplus_{r \geq p} H^{r,k-r}. \tag{3.4}$$

In particular, the last statement implies that the $H^{p,q}$ are free and lift the subquotients $\mathrm{Gr}^p_F H$ to subobjects of H.

Proof. This follows immediately from Nakayama's Lemma and the reasoning for ordinary Hodge structures. □

For the next result we need some terminology. Let R be a noetherian local ring and $\varphi : F \longrightarrow G$ be a morphism between finitely generated free R-modules. We define $I_j(\varphi) = \mathrm{im}(\varphi' : \Lambda^j F \otimes (\Lambda^j G)^{\vee} \longrightarrow R)$, where φ' is induced by $\Lambda^j \varphi : \Lambda^j F \longrightarrow \Lambda^j G$. If we interpret φ as a matrix, then $I_j(\varphi)$ is the ideal generated by all $j \times j$-minors of φ. One defines the *rank of* φ as $\mathrm{rk}\varphi := \max\{i : I_i(\varphi) \neq 0\}$.

DEFINITION 3.8. In the above situation we say that φ has *constant rank* k if $I_k(\varphi) = R$ and $I_{k-1}(\varphi) = 0$. We say that φ has *constant rank* if there is some k such that φ has constant rank k.

An important characterization of this property is given by the [Eis95, Prop 20.8], saying that a morphism is of constant rank if and only if its cokernel is projective, hence free as we supposed R to be local.

LEMMA 3.9. *Let $f : H \longrightarrow H'$ be a morphism of mixed Hodge structures over R. Then $f^{p,q} := f|_{H^{p,q}}$ satisfies $f^{p,q}(H^{p,q}) \subset (H')^{p,q}$ and $f = \sum_{p,q} f^{p,q}$. Moreover, all $f^{p,q}$ have constant rank in the sense of Definition 3.8.*

Proof. By (3.3) the image of $f^{p,q}$ is contained in $(H')^{p,q}$, because f is defined over $\mathbb{R}$ and preserves the Hodge filtration. Moreover, $\mathrm{coker}\, f$ is free as $\mathrm{coker}\, f = (\mathrm{coker}\, f_{\mathbb{R}}) \otimes_{\mathbb{R}} R$. Then $\mathrm{coker}\, f = \bigoplus_{p,q} \mathrm{coker}\, f^{p,q}$ implies that $\mathrm{coker}\, f^{p,q}$ is free. So the claim follows from [Eis95, Prop 20.8]. □

4. Mixed Hodge structures for normal crossing varieties. Let $S = \mathrm{Spec}\, R$ where $R \in \mathrm{Art}_{\mathbb{C}}$ and let $f : \mathcal{Y} \longrightarrow S$ be a proper, locally trivial deformation of a simple normal crossing $\mathbb{C}$-variety. We will construct a complex $\widetilde{\Omega}^{\bullet}_{\mathcal{Y}/S}$, which calculates the cohomology with coefficients in the constant sheaf $\underline{R}_{\mathcal{Y}^{\mathrm{an}}}$ on $\mathcal{Y}^{\mathrm{an}}$ and use it to endow the latter with a mixed Hodge structure over R.

Definition 4.1. Let $R \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S = \operatorname{Spec} R$ be a locally trivial deformation of a variety Y. We define $\tau^k_{\mathcal{Y}/S} \subset \Omega^k_{\mathcal{Y}/S}$ to be the subcomplex of sections whose support is contained in the singular locus of f and $\widetilde{\Omega}^k_{\mathcal{Y}/S} := \Omega^k_{\mathcal{Y}/S}/\tau^k_{\mathcal{Y}/S}$.

4.2. Semi-simplicial resolutions. Recall that a semi-simplicial scheme $Y^\bullet$ is given by schemes Y^n and morphisms $d^j : Y^n \longrightarrow Y^{n-1}$ for $j = 0, \dots, n$ satisfying some compatibility condition. We refer to [PS08, 5.1] for details.

An ordinary scheme Y may be considered as a trivial semi-simplicial scheme with $Y^n = Y$ and all $d^j = \mathrm{id}_Y$. A morphism of semi-simplicial schemes $a : Y^\bullet \longrightarrow Y$ from $Y^\bullet$ to an ordinary scheme is also called an *augmentation* of $Y^\bullet$ to Y or that $Y^\bullet$ is *augmented* towards Y. We will also write an augmented semi-simplicial scheme $Y^\bullet \longrightarrow Y$ in the form

$$\dots \Rrightarrow Y^1 \rightrightarrows Y^0 \rightarrow Y\ .$$

The dual notion is the one of a semi-cosimplicial object.

Definition 4.3. Let S be a $\mathbb{C}$-scheme and $\mathcal{Y} \longrightarrow S$ be a proper scheme over S. A *semi-simplicial resolution* of $\mathcal{Y}$ over S is a semi-simplicial S-scheme $\mathcal{Y}^\bullet$ together with a morphism $a : \mathcal{Y}^\bullet \longrightarrow \mathcal{Y}$ of semi-simplicial S-schemes such that all $a_k : \mathcal{Y}^k \longrightarrow \mathcal{Y}$ are proper and $\mathcal{Y}^k \longrightarrow S$ is smooth for all k.

4.4. Canonical resolution for locally trivial deformations of simple normal crossing varieties. Let Y be a proper simple normal crossing k-variety and let $Y = \cup_i Y_i$ be a decomposition into irreducible components. Let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_k$. Lemma 1.4 allows us to write $\mathcal{Y} = \bigcup_{i=1}^n \mathcal{Y}_i$ with flat S-schemes $\mathcal{Y}_i$. This union is a decomposition into irreducible components and $\mathcal{Y}_i$ is a locally trivial deformation of Y_i. As the $\mathcal{Y}_i \longrightarrow S$ are flat deformations of smooth schemes, $\mathcal{Y}^0 := \coprod_i \mathcal{Y}_i \longrightarrow S$ is smooth as well. For a subset $I \subset [n] := \{1, \dots, n\}$ we put

$$\mathcal{Y}^I := \bigcap_{i \in I} \mathcal{Y}_i, \quad \mathcal{Y}^k := \coprod_{|I|=k+1} \mathcal{Y}^I. \tag{4.1}$$

Here, by $\mathcal{Y}_i \cap \mathcal{Y}_j$ we denote the scheme $\mathcal{Y}_i \times_{\mathcal{Y}} \mathcal{Y}_j$. There exists one map $a_k : \mathcal{Y}^k \longrightarrow \mathcal{Y}$ over S and $k+1$ canonical maps $d_j : \mathcal{Y}^k \longrightarrow \mathcal{Y}^{k-1}$ for $j = 0, \dots, k$ over S coming from the $k+1$ inclusions $[k] \hookrightarrow [k+1]$. In other words, the collection of the $\mathcal{Y}^k$ together with the d_j is a semi-simplicial S-scheme and the a_k form an augmentation of $\mathcal{Y}^\bullet$ to $\mathcal{Y}$.

It follows directly from the definition that $\mathcal{Y}^I \longrightarrow S$ is a locally trivial deformation, thus it is smooth over S. In other words, we have the following

Lemma 4.5. *The semi-simplicial S-scheme $\mathcal{Y}^\bullet$ together with the augmentation $a : \mathcal{Y}^\bullet \longrightarrow \mathcal{Y}$ is a semi-simplicial resolution of $\mathcal{Y}$. We call it the* canonical resolution *of $\mathcal{Y}$ over S.* $\square$

4.6. Semi-cosimplicial resolution for $\widetilde{\Omega}^p_{\mathcal{Y}/S}$. For $\mathcal{Y}$ as in section 4.4 the semi-simplicial S-scheme $\mathcal{Y}^\bullet$ induces semi-cosimplicial $\mathcal{O}_{\mathcal{Y}}$-modules $a_*\Omega^p_{\mathcal{Y}^\bullet/S}$. The formula $\delta_n := \sum_{j=0}^{n+1} (-1)^j d^j$ where $d^j = d_j^*$ makes

$$a_*\Omega^p_{\mathcal{Y}^\bullet/S} : \qquad a_{0*}\Omega^p_{\mathcal{Y}^0/S} \xrightarrow{\delta_0} a_{1*}\Omega^p_{\mathcal{Y}^1/S} \xrightarrow{\delta_1} \dots$$

into a complex. The augmentation $a : \mathcal{Y}^\bullet \longrightarrow \mathcal{Y}$ induces a coagumentation

$$\Omega_{\mathcal{Y}/S} \xrightarrow{a_0^*} a_{0*}\Omega^p_{\mathcal{Y}^0/S} \xrightarrow{\delta_0} a_{1*}\Omega^p_{\mathcal{Y}^1/S} \xrightarrow{\delta_1} \dots .$$

As $\mathcal{Y}^0 \longrightarrow S$ is smooth, the morphism a_0^* factors through $\widetilde{\Omega}^p_{\mathcal{Y}/S}$ from Definition 4.1. Clearly, the composition $\delta_0 \circ a_0^*$ is zero and we obtain a complex

$$0 \longrightarrow \tau^k_{\mathcal{Y}/S} \longrightarrow \Omega^k_{\mathcal{Y}/S} \longrightarrow a_{0*}\Omega^k_{\mathcal{Y}^0/S} \longrightarrow a_{1*}\Omega^k_{\mathcal{Y}^1/S} \longrightarrow \dots \tag{4.2}$$

Denote by $(\cdot)^{\mathrm{an}}$ the analytification functor. All following theory is based on the important

Lemma 4.7. *Let Y be a simple normal crossing $\mathbb{C}$-variety and $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$. Then*

1. *The sequence (4.2) is exact and so is the sequence with $\mathcal{Y}$ replaced by $\mathcal{Y}^{\mathrm{an}}$.*
2. *$\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ is a resolution of the constant sheaf $\underline{R}_{Y^{\mathrm{an}}}$.*
3. *The canonical map $\left(\widetilde{\Omega}^k_{\mathcal{Y}/S}\right)^{\mathrm{an}} \longrightarrow \widetilde{\Omega}^k_{\mathcal{Y}^{\mathrm{an}}/S}$ is an isomorphism.*
4. *The canonical map $R^i f_* \widetilde{\Omega}^k_{\mathcal{Y}/S} \longrightarrow R^i f^{\mathrm{an}}_* \widetilde{\Omega}^k_{\mathcal{Y}^{\mathrm{an}}/S}$ is an isomorphism.*

Proof. The question is local in $\mathcal{Y}$, so we may assume that $\mathcal{Y} = Y \times S$ is the trivial deformation. Then the resolution (4.2) is simply the pullback of the analogous resolution for Y along the flat morphism $Y \times S \longrightarrow Y$. This implies (1) and (2) by [Fri83, Prop 1.5]. As $(\Omega_{\mathcal{Y}/S})^{\mathrm{an}} \cong \Omega_{\mathcal{Y}^{\mathrm{an}}/S}$, (3) follows from (1) because analytification is exact by [SGA1, Exp XII, Prop 1.3.1] and compatible with taking the wedge product. Moreover, (3) implies (4) by [SGA1, Exp XII, Thm 4.2]. ☐

The following result is due to Deligne, see [Del68, Thm 5.5], for smooth morphisms. The same arguments prove a little more general statement in our setting. For convenience we reproduce them here.

Theorem 4.8. *Let Y be a proper, simple normal crossing $\mathbb{C}$-variety, let $R, R' \in \mathrm{Art}_{\mathbb{C}}$, let $f : \mathcal{Y} \longrightarrow S = \operatorname{Spec} R$ be a locally trivial deformation of Y and consider a morphism $S' = \operatorname{Spec} R' \longrightarrow S$. Then the following holds.*

1. *The associated spectral sequence*

$$E_1^{p,q} = R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \Rightarrow R^{p+q} f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S} = H^{p+q}(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \tag{4.3}$$

   degenerates at E_1.
2. *The R-modules $R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ are free and compatible with arbitrary base change in the sense that for $\mathcal{Y}' = \mathcal{Y} \times_S S'$ the morphism*

$$R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \otimes_R R' \longrightarrow R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}'/S'}$$

   is an isomorphism.

The analogous statements hold if $f : \mathcal{Y} \longrightarrow S$ is replaced by a deformation $\mathcal{X} \longrightarrow S$ of a compact Kähler manifold X.

Proof. We argue as in [Del68], Théorème 5.5 for the morphism $f : \mathcal{Y} \longrightarrow S$. By [Del68, (3.5.1)] a complex K of R-modules satisfies

$$\lg_R(H^n(K)) \leq \lg(R) \dim_{\mathbb{C}}(H^n(K \otimes^{\mathbb{L}}_R \mathbb{C}))$$

and $H^n(K)$ is a free R-module if equality holds. Here lg denotes the length of a module. To apply this to the E_1-term of the spectral sequence (4.3) we need [EGAIII], Théorème (6.10.5) saying that there is a bounded below complex L of free R-modules and an isomorphism of ∂-functors $R^q f_*\left(\widetilde{\Omega}^p_{\mathcal{Y}/S} \otimes f^*Q\right) \longrightarrow H^q(L \otimes Q)$ in the bounded complex Q of quasi-coherent R-modules. Here we use that $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is flat over R. Let $\bar{f} : Y \longrightarrow \operatorname{Spec}\mathbb{C}$ be the restriction of f to the central fiber. We will compare the spectral sequence (4.3) with the spectral sequence of $\bar{f}$. Again by [Del68, (3.5.1)] we have

$$\begin{aligned} \operatorname{lg}_R(R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}) &= \operatorname{lg}_R(H^q(L)) \\ &\leq \operatorname{lg}(R) \dim_{\mathbb{C}}(H^q(L \otimes_R \mathbb{C})) \\ &= \operatorname{lg}(R) \dim_{\mathbb{C}}(R^q \bar{f}_* \widetilde{\Omega}^p_{Y/\mathbb{C}}) \end{aligned} \tag{4.4}$$

and $R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$ is a free R-module if equality holds. We have

$$\begin{aligned} \operatorname{lg}(R^n f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}) &\leq \sum_{p+q=n} \operatorname{lg}_R(R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}) \\ &\leq \operatorname{lg}(R) \sum_{p+q=n} \dim_{\mathbb{C}}(R^q \bar{f}_* \widetilde{\Omega}^p_{Y/\mathbb{C}}) \\ &= \operatorname{lg}(R) \dim_{\mathbb{C}}(R^n \bar{f}_* \widetilde{\Omega}^\bullet_{Y/\mathbb{C}}), \end{aligned}$$

where the first inequality comes from the existence of the spectral sequence, the second inequality is (4.4) and the last equality comes from the degeneration of the spectral sequence for Y, which is [Fri83, Prop 1.5]. But Lemma 4.7 (2) implies that $\operatorname{lg}(R^n f_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/S}) = \operatorname{lg}(R) \dim_{\mathbb{C}}(R^n \bar{f}_* \widetilde{\Omega}^\bullet_{Y/\mathbb{C}})$, so we have equality everywhere. Hence (1) and the first assertion of (2) follows. The second assertion of (2) follows from the first by [EGAIII, (7.8.5)].

The Kähler case works literally as above, we only have to replace the reference to [EGAIII, Thm 6.10.5] by [BS77, Ch 3, Thm 4.1] and the reference to [EGAIII, 7.8.5] by [BS77, Ch 3, Cor 3.10]. The rest of the proof of Theorem 4.8 goes through if we note that the spectral sequence associated with $\Omega^\bullet_X$ degenerates as X is a compact Kähler manifold. ∎

4.9. Pure Hodge structures on smooth families. Let $f : \mathcal{Y} \longrightarrow S$ be a smooth and proper morphism of complex spaces where $S = \operatorname{Spec} R$ for $R \in \operatorname{Art}_{\mathbb{C}}$. We are going to put a pure Hodge structure over R on $H^k(Y, \underline{R}_Y)$ if $Y = \mathcal{Y}^{\mathrm{red}}$ is a Kähler manifold. From the decreasing filtration $F^p\Omega^\bullet_{\mathcal{Y}/S} := \Omega^{\geq p}_{\mathcal{Y}/S}$ we obtain the *Hodge filtration* $F^pH^k(Y, \underline{R}_Y)$ on $H^k(Y, \underline{R}_Y)$ by setting

$$F^pR^kf_*\Omega^\bullet_{\mathcal{Y}/S} := \operatorname{im}\left(R^kf_*F^p\Omega^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\Omega^\bullet_{\mathcal{Y}/S}\right) \tag{4.5}$$

and using the isomorphisms $H^k(Y, \underline{R}_Y) \longrightarrow R^kf_*\Omega^\bullet_{\mathcal{Y}/S}$ from [Del68, Lem 5.5.3].

LEMMA 4.10. *Let $f : \mathcal{Y} \longrightarrow S = \operatorname{Spec} R$ be a smooth and proper morphism of complex spaces where $R \in \operatorname{Art}_{\mathbb{C}}$. Assume that $Y := \mathcal{Y} \times_S \mathbb{C}$ is a Kähler manifold. Then $\mathcal{H}^k(\mathcal{Y}) := \left(H^k(Y, \mathbb{R}), F^pH^k(Y, \underline{R}_Y)\right)$ is a pure Hodge structure of weight k over R, whose central fiber is the usual Hodge structure on $H^k(Y, \mathbb{R})$. Moreover, the canonical morphism $R^kf_*F^p\Omega^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\Omega^\bullet_{\mathcal{Y}/S}$ is injective, so that*

$R^k f_* F^p \Omega^\bullet_{\mathcal{Y}/S} \cong F^p H^k(Y, \underline{R}_Y)$. *The association* $\mathcal{Y} \mapsto \mathcal{H}^k(\mathcal{Y})$ *is functorial on the category of smooth and proper complex spaces over* S *with Kähler central fiber.*

Proof. The filtration defined in (4.5) is the one, whose graded objects are found on E_∞ of the spectral sequence (4.3). By [Del68, Thm 5.5] we have $E_\infty = E_1$, so $\mathrm{Gr}^p_F R^k f_* \Omega^\bullet_{\mathcal{Y}/S} = R^{k-p} f_* \Omega^p_{\mathcal{Y}/S} = R^{k-p} f_* \mathrm{Gr}^p_F \Omega^\bullet_{\mathcal{Y}/S}$. The same theorem tells us that $R^{k-p} f_* \Omega^p_{\mathcal{Y}/S}$ is free. Therefore using

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^k f_* F^{p+1}\Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & R^k f_* F^p \Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & R^k f_* \mathrm{Gr}^p_F \Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^{p+1} R^k f_* \Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & F^p R^k f_* \Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & \mathrm{Gr}^p_F R^k f_* \Omega^\bullet_{\mathcal{Y}/S} & \longrightarrow & 0 \end{array}$$

we find inductively that $R^k f_* F^p \Omega^\bullet_{\mathcal{Y}/S} \cong F^p R^k f_* \Omega^\bullet_{\mathcal{Y}/S}$ and that these are free submodules. Again by [Del68, Thm 5.5], all graded objects are compatible with base change and therefore restrict to a pure Hodge structure on the central fiber. The functoriality statement is clear. ∎

Corollary 4.11. *There is a natural isomorphism* $R^{k-p} f_* \Omega^p_{\mathcal{Y}/S} \longrightarrow \mathrm{Gr}^p_F H^k(Y, R)$.

Proof. Consider the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^k f_* \Omega^{\geq p+1}_{\mathcal{Y}/S} & \longrightarrow & R^k f_* \Omega^{\geq p}_{\mathcal{Y}/S} & \longrightarrow & R^{k-p} f_* \Omega^p_{\mathcal{Y}/S} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \exists ? & & \\ 0 & \longrightarrow & F^{p+1} H^k(Y, R) & \longrightarrow & F^p H^k(Y, R) & \longrightarrow & \mathrm{Gr}^p_F H^k(Y, R) & \longrightarrow & 0 \end{array} \tag{4.6}$$

where the first two vertical maps are isomorphisms by Lemma 4.10. These isomorphisms imply that the upper sequence is exact on the left. As it is part of the long exact sequence associated with the sequence

$$0 \longrightarrow \Omega^{\geq p+1}_{\mathcal{Y}/S} \longrightarrow \Omega^{\geq p}_{\mathcal{Y}/S} \longrightarrow \Omega^p_{\mathcal{Y}/S}[-p] \longrightarrow 0$$

of complexes, injectivity at the $(k+1)$-st direct image yields surjectivity at the k-th, hence exactness of the upper sequence. Therefore, the morphism $R^{k-p} f_* \Omega^p_{\mathcal{Y}/S} \longrightarrow \mathrm{Gr}^p_F H^k(Y, R)$ exists and by the five-lemma it is an isomorphism. ∎

Corollary 4.12. *There is a natural isomorphism, functorial in* $\mathcal{Y}$*:*

$$\left(R^{k-p} f_* \Omega^p_{\mathcal{Y}/S}\right)_{\mathrm{wl}} \xrightarrow{\cong} H^{p,q}(Y, R) := F^p_{\mathrm{wl}} \cap \overline{F^q_{\mathrm{wl}}} \subset H^k(Y, \mathbb{R}) \otimes R,$$

Proof. This is deduced directly by applying Weil restriction to the diagram (4.6) and using Lemma 3.7. ∎

Recall that a module homomorphism has constant rank if and only if its cokernel is free by [Eis95, Prop 20.8].

Proposition 4.13. *Let* $f : \mathcal{Y} \longrightarrow S$, $g : \mathcal{X} \longrightarrow S$ *be proper and smooth over* $S = \operatorname{Spec} R$, $R \in \mathrm{Art}_{\mathbb{C}}$ *and let* $i : \mathcal{Y} \longrightarrow \mathcal{X}$ *be an* S*-morphism. Then the induced morphisms* $i^* : R^q g_* \Omega^p_{\mathcal{X}/S} \longrightarrow R^q f_* \Omega^p_{\mathcal{Y}/S}$ *have constant rank.*

Proof. By Lemma 4.10 we know that the morphism i induces a morphism $\mathcal{H}^k(\mathcal{X}) \longrightarrow \mathcal{H}^k(\mathcal{Y})$ between the pure Hodge structures over R associated with $\mathcal{X}$ and $\mathcal{Y}$. Taking Weil restrictions this gives a morphism $\mathcal{H}^k(\mathcal{X})_{\mathrm{wl}} \longrightarrow \mathcal{H}^k(\mathcal{Y})_{\mathrm{wl}}$ of Hodge-Weil structures by Lemma 3.6. Let $i^{p,q} : H^{p,q}(X) \longrightarrow H^{p,q}(Y)$ be the induced map. By Corollary 4.12 the diagram

$$\begin{tikzcd}
\left(R^{k-p}f_*\Omega^p_{\mathcal{Y}/S}\right)_{\mathrm{wl}} \arrow[r, "i^*_{\mathrm{wl}}"] \arrow[d, "\cong"] & \left(R^{k-p}g_*\Omega^p_{\mathcal{X}/S}\right)_{\mathrm{wl}} \arrow[r] \arrow[d, "\cong"] & \operatorname{coker} i^*_{\mathrm{wl}} \arrow[r] \arrow[d] & 0 \\
H^{p,q}(Y) \arrow[r, "i^{p,q}"] & H^{p,q}(X) \arrow[r] & \operatorname{coker} i^{p,q} \arrow[r] & 0
\end{tikzcd}$$

commutes and the first two vertical maps are isomorphisms. Therefore, also the third vertical map is an isomorphism. We know that $\operatorname{coker} i^{p,q}$ is free by Lemma 3.9, hence so is $\operatorname{coker} i^*_{\mathrm{wl}}$. Now the claim follows from Lemma 2.5, as $\operatorname{coker} i^*_{\mathrm{wl}} = (\operatorname{coker} i^*)_{\mathrm{wl}}$ by Lemma 2.4. ∎

Proposition 4.13 together with Lemma 3.9 can be seen as a formalization of the following argument: If S is the base manifold of a small deformation and $t \in S$, the maps $H^q(X_t, \Omega^p_{X_t}) \longrightarrow H^q(Y_t, \Omega^p_{Y_t})$, the rank of which is semi-continuous in t, add up to the topological map $H^i(X_t, \mathbb{C}) \longrightarrow H^i(Y_t, \mathbb{C})$ by the Hodge decomposition. The rank of the latter is independent of t and by semi-continuity the summands also have constant rank.

4.14. Mixed Hodge structures on normal crossing families. Let Y be a simple normal crossing $\mathbb{C}$-variety and $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$. By Lemma 4.7 (2) there is a quasi-isomorphism $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \simeq \mathfrak{s}((a_\bullet)_*\Omega^\bullet_{\mathcal{Y}^\bullet/S})$, where $\mathfrak{s}(\cdot)$ denotes the single complex associated with a double complex. We define filtrations $W_{-m}\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \mathfrak{s}((a_{\geq m})_*\Omega^\bullet_{\mathcal{Y}^{\geq m}/S})$ and $F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \widetilde{\Omega}^{\geq p}_{\mathcal{Y}/S}$. These give rise to filtrations $F^pH^k(Y, R)$ and $W_mH^k(Y, R)$ on $H^k(Y, R)$ if we put

$$W_mR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \operatorname{im}\left(R^kf_*W_{m-k}\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}\right) \tag{4.7}$$

and

$$F^pR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} := \operatorname{im}\left(R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}\right) \tag{4.8}$$

and use the isomorphisms $H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \longrightarrow R^kf^{\mathrm{an}}_*\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from Lemma 4.7 (2) and $R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf^{\mathrm{an}}_*\widetilde{\Omega}^\bullet_{\mathcal{Y}^{\mathrm{an}}/S}$ from Lemma 4.7 (4).

THEOREM 4.15. *Let Y be a proper simple normal crossing variety over $\mathbb{C}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$. Then*

$$\mathcal{H}^k(\mathcal{Y}) = (H^k(Y^{\mathrm{an}}, \mathbb{R}), W_mH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}), F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})) \tag{4.9}$$

is a mixed Hodge structure over R. Moreover, $R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is injective so that $F^pH^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}}) \cong R^kf_*\widetilde{\Omega}^{\geq p}_{\mathcal{Y}/S}$.

Proof. Literally as in the pure case, see Lemma 4.10 and Corollary 4.11, one shows that the R-modules $\mathrm{Gr}^p_FR^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ are free and isomorphic to $R^kf_*\mathrm{Gr}^p_F\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} =$

$R^kf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ and that $R^kf_*F^p\widetilde{\Omega}^\bullet_{\mathcal{Y}/S} \longrightarrow R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is injective, using Theorem 4.8. To verify that (4.9) is a mixed Hodge structure over R, we have to show that the graded objects $\mathrm{Gr}^W_m\mathrm{Gr}^p_F H^k(Y^{\mathrm{an}}, \underline{R}_{Y^{\mathrm{an}}})$ are free R-modules, or equivalently that the $\mathrm{Gr}^W_m\mathrm{Gr}^p_F R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ are free R-modules, and that the central fiber is a mixed Hodge structure in the ordinary sense.

The free R-module $R^kf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ is the abutment of the spectral sequence

$$E_1^{k,m} = R^mf_*\, a_{k*}\Omega^p_{\mathcal{Y}^k/S} \Rightarrow R^{k+m}f_*\widetilde{\Omega}^p_{\mathcal{Y}/S} \tag{4.10}$$

induced by the resolution (4.2) for fixed p. The filtration defined in (4.7) induces a weight filtration $\mathrm{Gr}^p_F R^kf_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ in the obvious way and the graded objects with respect to this filtration are the E_∞ terms of the spectral sequence (4.10). By [Del68, Thm 5.5] the R-modules $E_1^{k,m}$ are free and compatible with base change. Moreover, the differential d_1 on $E_1^{k,m}$ is given by the semi-simplicial differential $\delta : R^mf_*a_{k*}\Omega^p_{\mathcal{Y}^k/S} \longrightarrow R^mf_*a_{k*}\Omega^p_{\mathcal{Y}^{k+1}/S}$.

This morphism has constant rank by Proposition 4.13 and consequently, its cohomology $E_2^{k,m}$ is free, too, as one easily verifies. Therefore, $E_2^{k,m}$ is compatible with base change. In the case $R = \mathbb{C}$ the spectral sequence is known to degenerate at E_2, see [PS08, Thms 3.12, 3.18]. As all E_2-terms of (4.10) are compatible with base change we have for all n that

$$\begin{aligned}\sum_{k+m=n} \lg_R\left(E_2^{k,m}\right) &= \lg_R(R) \sum_{k+m=n} \dim_{\mathbb{C}}\left(E_2^{k,m}\otimes\mathbb{C}\right)\\ &= \lg_R(R)\dim_{\mathbb{C}}\left(R^nf_*\widetilde{\Omega}^p_{Y/\mathbb{C}}\right)\\ &= \lg_R\left(R^nf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}\right).\end{aligned}$$

Thus, the spectral sequence 4.10 also degenerates at E_2 and the R-modules $E_\infty^{k,m} = \mathrm{Gr}^W_m R^{k+m}f_*\widetilde{\Omega}^p_{\mathcal{Y}/S} = \mathrm{Gr}^W_m\mathrm{Gr}^p_F R^{k+m}f_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ coincide with the free R-modules $E_2^{k,m}$. Again, as all graded objects are compatible with base change, $\mathcal{H}$ restricts to a mixed Hodge structure on the central fiber, which is the usual mixed Hodge structure on Y. □

Let us isolate an observation from the proof of the previous lemma.

COROLLARY 4.16. *Let Y be a proper simple normal crossing variety over $\mathbb{C}$ and let $f : \mathcal{Y} \longrightarrow S$ be a locally trivial deformation of Y over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$. Then the spectral sequence* (4.10) *degenerates at E_2.* □

THEOREM 4.17. *Let $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$, let Y be a proper simple normal crossing $\mathbb{C}$-variety and let $g : \mathcal{X} \longrightarrow S$ and $f : \mathcal{Y} \longrightarrow S$ be proper, algebraic S-schemes. Assume that $\mathcal{Y} \longrightarrow S$ is a locally trivial deformation of Y and that $\mathcal{X} \longrightarrow S$ is smooth. Let $i : \mathcal{Y} \longrightarrow \mathcal{X}$ be an S-morphism. Then for all p, q the morphism $i^* : R^qg_*\Omega^p_{\mathcal{X}/S} \longrightarrow R^qf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ has constant rank.*

Proof. Let $\ldots \Rrightarrow \mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0 \to \mathcal{Y}$ be the semi-simplicial resolution of $\mathcal{Y}$ over S from Lemma 4.5. This means in particular that $\mathcal{Y}^0$ is a locally trivial deformation of the normalization. By Theorem 4.8 the R-modules $R^qg_*\Omega^p_{\mathcal{X}/S}$, $R^qf_*\widetilde{\Omega}^p_{\mathcal{Y}/S}$ and $R^qf_*\Omega^p_{\mathcal{Y}^k/S}$ are free and compatible with base change. By Corollary 4.16 we know that the spectral sequences (4.10) degenerate at E_2 for each p. As $E_2^{0,q} = \ker\left(R^q f_* \Omega^p_{\mathcal{Y}^0/S} \longrightarrow R^q f_* \Omega^p_{\mathcal{Y}^1/S}\right)$, this implies that the first row in

$$
\begin{tikzcd}
0 \arrow[r] & W_{p+q-1} R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \arrow[r] & R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S} \arrow[r, "\eta"] & R^q f_* \Omega^p_{\mathcal{Y}^0/S} \arrow[r, "\delta"] & R^q f_* \Omega^p_{\mathcal{Y}^1/S} \\
& & R^q g_* \Omega^p_{\mathcal{X}/S} \arrow[u, "i^*"] \arrow[ur, "\varphi"'] & &
\end{tikzcd}
\tag{4.11}
$$

is exact. Here $\operatorname{im} i^*$ does not intersect $W_{p+q-1} R^q f_* \widetilde{\Omega}^p_{\mathcal{Y}/S}$, as it does not on the central fiber. This last claim can be shown using a standard argument involving Deligne's weak splitting on the central fiber, see e.g. [PS08, Ex 3.3 and Lem-Def 3.4]. Moreover, φ has constant rank by Proposition 4.13. Also η has constant rank as δ has constant rank by Proposition 4.13 and hence $\operatorname{coker} \eta = \ker \delta$ is free. As $\operatorname{im} i^* \cap \ker \eta = 0$, also $\operatorname{coker} i^*$ is free completing the proof. □

4.18. Vista. It seems obvious that the results of this article are only a master example of what is true in general. We believe that the statements of Theorem 4.15 and Theorem 4.17 should hold true mutatis mutandis for locally trivial deformations of arbitrary varieties over an Artinian base scheme. The theory should also yield applications when we drop the local triviality hypothesis and consider smoothings of normal crossing varieties as in [Fri83, KN] in the infinitesimal setting. In the situation of [Le11] the step from locally trivial deformations to smoothings should correspond to replacing $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ by a logarithmic version. This will be the content of a forthcoming work.

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