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A REFINEMENT OF GÜNTHER'S CANDLE INEQUALITY*

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Dedicated to our friends Sylvain Gallot and Albert Schwarz

Abstract. We analyze an upper bound on the curvature of a Riemannian manifold, using "$\sqrt{\mathrm{Ric}}$" curvature, which is in between a sectional curvature bound and a Ricci curvature bound. (A special case of $\sqrt{\mathrm{Ric}}$ curvature was previously discovered by Osserman and Sarnak for a different but related purpose.) We prove that our $\sqrt{\mathrm{Ric}}$ bound implies Günther's inequality on the candle function of a manifold, thus bringing that inequality closer in form to the complementary inequality due to Bishop.

Key words. Günther-Bishop Theorem, Riemannian manifold, Ricci curvature, candle function, volume bounds, curvature bounds.

AMS subject classifications. 53B20.

1. Introduction.

Two important relations between curvature and volume in differential geometry are Bishop's inequality [3, §11.10], which is an upper bound on the volume of a ball from a lower bound on Ricci curvature, and Günther's inequality [9], which is a lower bound on volume from an upper bound on *sectional* curvature. Bishop's inequality has a weaker hypothesis then Günther's inequality and can be interpreted as a stronger result. The asymmetry between these inequalities is a counterintuitive fact of Riemannian geometry.

In this article, we will partially remedy this asymmetry. We will define another curvature statistic, the root-Ricci function, denoted $\sqrt{\mathrm{Ric}}$, and we will establish a comparison theorem that is stronger than Günther's inequality[1]. $\sqrt{\mathrm{Ric}}$ is not a tensor because it involves square roots of sectional curvatures, but it shares other properties with Ricci curvature.

After the first version of this article was written, we learned that a special case of $\sqrt{\mathrm{Ric}}$ was previously defined by Osserman and Sarnak [15], for the different but related purpose of estimating the entropy of geodesic flow on a closed manifold. (See Section 3.1.) Although their specific results are different, there is a common motivation arising from volume growth in a symmetric space.

1.1. Growth of the complex hyperbolic plane.

Consider the geometry of the complex hyperbolic plane $\mathbb{CH}^2$. In this 4-manifold, the volume of a ball of radius r is

$$\mathrm{Vol}(B(r)) = \frac{\pi^2}{2}\sinh(r)^4 \sim \frac{\pi^2}{32}\exp(4r).$$

The corresponding sphere surface volume has a factor of $\sinh(2r)$ from the unique complex line containing a given geodesic γ, which has curvature -4, and two factors

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[1] We take the "ic" in the Ricci tensor Ric to mean taking a partial trace of the Riemann tensor R, but we take a square root first.

of $\sinh(r)$ from the totally real planes that contain γ, which have curvature -1. Günther's inequality and Bishop's inequality yield the estimates

$$\frac{\pi^2}{48}\exp(3\sqrt{2}r) \gtrsim \mathrm{Vol}(B(r)) \gtrsim \frac{\pi^2}{12}\exp(3r).$$

The true volume growth of balls in $\mathbb{C}\mathrm{H}^2$ (and in some other cases, see Section 3.1) is governed by the average of the square roots of the negatives of the sectional curvatures. This is how we define the $\sqrt{\mathrm{Ric}}$ function, for each tangent direction u at each point p in M.

1.2. Root-Ricci curvature. Let M be a Riemannian n-manifold with sectional curvature $K \le \rho$ for some constant $\rho \ge 0$; we will implicitly assume that $\rho \ge \kappa$. For any unit tangent vector $u \in UT_pM$ with $p \in M$, we define

$$\sqrt{\mathrm{Ric}}(\rho, u) \stackrel{\text{def}}{=} \mathrm{Tr}(\sqrt{\rho - R(\cdot, u, \cdot, u)}).$$

Here $R(u, v, w, x)$ is the Riemann curvature tensor expressed as a tetralinear form, and the square root is the positive square root of a positive semidefinite matrix or operator.

The formula for $\sqrt{\mathrm{Ric}}(\rho, u)$ might seem arcane at first glance. Regardless of its precise form, the formula is both local (i.e., a function of the Riemannian curvature) and also optimal in certain regimes. Any such formula is potentially interesting. One important, simpler case is $\rho = 0$, which applies only to non-positively curved manifolds:

$$\sqrt{\mathrm{Ric}}(0, u) = \mathrm{Tr}(\sqrt{-R(\cdot, u, \cdot, u)}).$$

In other words, $\sqrt{\mathrm{Ric}}(0, u)$ is the sum of the square roots of the sectional curvatures $-K(u, e_i)$, where (e_i) is a basis of $u^\perp$ that diagonalizes the Riemann curvature tensor. This special case was defined previously by Osserman and Sarnak [15] (Section 3.1), which in their notation would be written $-\sigma(u)$.

For example, when $M = \mathbb{C}\mathrm{H}^2$, one sectional curvature $K(u, e_i)$ is -4 and the other two are -1, so

$$\sqrt{\mathrm{Ric}}(0, u) = \sqrt{4} + \sqrt{1} + \sqrt{1} = 4,$$

which matches the asymptotics in Section 1.1.

In the general formula $\sqrt{\mathrm{Ric}}(\rho, u)$, the parameter ρ is important because it yields sharper bounds at shorter length scales. In particular, in the limit $\rho \to \infty$, $\sqrt{\mathrm{Ric}}(\rho, u)$ becomes equivalent to Ricci curvature. Section 2 discusses other ways in which $\sqrt{\mathrm{Ric}}$ fits the framework of classical Riemannian geometry. Our definition for general ρ was motivated by our proof of the refined Günther inequality, more precisely by equation (10). The energy (8) of a curve in a manifold can be viewed as linear in the curvature $R(\cdot, u, \cdot, u)$. We make a quadratic change of variables to another matrix A, to express the optimization problem as quadratic minimization with linear constraints; and we noticed an allowable extra parameter ρ in the quadratic change of variables.

Another way to look at root-Ricci curvature is that it is equivalent to an average curvature, like the normalized Ricci curvature $\mathrm{Ric}/(n-1)$, but after a reparameterization. By analogy, the L^p norm of a function, or the root-mean-square concept in statistics, is also an average of quantities that are modified by the function $f(x) = x^p$. In our case, we can obtain a type of average curvature which is equivalent to $\sqrt{\mathrm{Ric}}$ if we conjugate by $f(x) = \sqrt{\rho - x}$. Taking this viewpoint, we say that the manifold M is of $\sqrt{\mathrm{Ric}}$ *class* (ρ, κ) if $K \leq \rho$, and if also

$$\frac{\sqrt{\mathrm{Ric}}(\rho, u)}{n-1} \geq \sqrt{\rho - \kappa}$$

for all $u \in UTM$. This is the $\sqrt{\mathrm{Ric}}$ curvature analogue of the sectional curvature condition $K \leq \kappa$.

1.3. A general candle inequality. The best version of either Günther's or Bishop's inequality is not directly a bound on the volume of balls in M, but rather a bound on the logarithmic derivative of the *candle function* of M. Let $\gamma = \gamma_u$ be a geodesic curve in M that begins at $p = \gamma(0)$ with initial velocity $u \in UT_pM$. Then the candle function $s(\gamma, r)$ is by definition the Jacobian of the map $u \mapsto \gamma_u(r)$. In other words, it is defined by the equations

$$\mathrm{d}q = s(\gamma_u, r)\, \mathrm{d}u\, \mathrm{d}r \qquad q = \gamma_u(r) = \exp_p(ru),$$

where $\mathrm{d}q$ is Riemannian measure on M, $\mathrm{d}r$ is Lebesgue measure on $\mathbb{R}$, and $\mathrm{d}u$ is Riemannian measure on the sphere UT_pM. This terminology has the physical interpretation that if an observer is at the point q in M, and if a unit candle is at the point p, then $1/s(\gamma, r)$ is its apparent brightness[2].

The candle function $s_\kappa(r)$ of a geometry of constant curvature κ is given by

$$s_\kappa(r) = \begin{cases} \left(\frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}\right)^{n-1} & \kappa > 0 \\ r^{n-1} & \kappa = 0 \\ \left(\frac{\sinh(\sqrt{-\kappa} r)}{\sqrt{-\kappa}}\right)^{n-1} & \kappa < 0 \end{cases}.$$

THEOREM 1.1. *Let M be a Riemannian n-manifold is of $\sqrt{\mathrm{Ric}}$ class (ρ, κ) for some $\kappa \leq \rho \geq 0$. Then*

$$(\log s(\gamma, r))' \geq (\log s_\kappa(r))'$$

for every geodesic γ in M, when $2r\sqrt{\rho} \leq \pi$.

The prime denotes the derivative with respect to r.

When $\rho = 0$, the conclusion of Theorem 1.1 is identical to Günther's inequality for manifolds with $K \leq \kappa$, but the hypothesis is strictly weaker. When $\rho > 0$, the curvature hypothesis is weaker still, but the length restriction is stronger. The usual version of the inequality holds up to a distance of $\pi/\sqrt{\kappa}$. For our distance restriction, we replace κ with ρ and divide by 2.

The rest of this article is organized as follows. In Section 2 we give several relations between curvature bounds and volume comparisons. In Section 3 we list applications of Theorem 1.1, and we prove Theorem 1.1 in Section 4.

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[2]Certain distant objects in astronomy with known luminosity are called *standard candles* and are used to estimate astronomical distances.

2. Relations between conditions.

2.1. Candle conditions. We first mention two interesting properties of the candle function $s(\gamma, r)$:

1. $s(\gamma, r)$ vanishes when $\gamma(0)$ and $\gamma(r)$ are conjugate points.
2. The candle function is symmetric: If $\bar{\gamma}(t) = \gamma(r-t)$, then $s(\bar{\gamma}, r) = s(\gamma, r)$.

The second property is not trivial to prove, but it is a folklore fact in differential geometry [19][Lem. 5] (and a standard principle in optics).

Say that a manifold M is Candle(κ) if the inequality

$$s(\gamma, r) \geq s_\kappa(r)$$

holds for all γ, r; or LCD(κ), for *logarithmic candle derivative*[3], if the logarithmic condition

$$(\log s(\gamma, r))' \geq (\log s_\kappa(r))'$$

holds for all γ, r; or Ball(κ) if the volume inequality

$$\mathrm{Vol}(B(p, r)) \geq \mathrm{Vol}(B_\kappa(r))$$

holds for all p and r; here B_κ denotes a ball in the simply connected space of constant curvature κ. (If $\kappa > 0$, then the first two conditions are only meaningful up to the distance $\pi/\sqrt{\kappa}$ between conjugate points in the comparison geometry.) We also write Candle(κ, ℓ), LCD(κ, ℓ), and Ball(κ, ℓ) if the same conditions hold up to a distance of $r = \ell$.

The logarithmic derivative $(\log s(\gamma, r))'$ of the candle function has its own important geometric interpretation: it is the mean curvature of the geodesic sphere with radius r and center $p = \gamma(0)$ at the point $\gamma(r)$. So it also equals Δr, where Δ is the Laplace Beltrami operator, and r is the distance from any point to p. So if M is LCD(κ), then we obtain the comparison $\Delta r \geq \Delta_\kappa r_\kappa$, and the statement that spheres in M are more extrinsically curved than spheres in a space of constant curvature κ.

2.2. Curvature and volume comparisons. If $\kappa \leq \rho = 0$, then we can organize the comparison properties of an n-manifold M that we have mentioned as follows:

$$(1)\quad K \leq \kappa \implies \sqrt{\mathrm{Ric}}\text{ class } (0, \kappa) \implies \mathrm{LCD}(\kappa) \implies \mathrm{Candle}(\kappa) \implies \mathrm{Ball}(\kappa, \mathrm{inj}(M)),$$

where $\mathrm{inj}(M)$ is the injectivity radius of M. The first implication is elementary, while the second one is Theorem 1.1. The third and fourth implications are also elementary, given by integrating with respect to length r.

If $\kappa \leq \rho > 0$, then

$$K \leq \kappa \implies \sqrt{\mathrm{Ric}}\text{ class } (\rho, \kappa) \implies \mathrm{LCD}(\kappa, \frac{\pi}{2\sqrt{\rho}}) \implies \mathrm{Candle}(\kappa, \frac{\pi}{2\sqrt{\rho}}) \implies \mathrm{Ball}(\kappa, \ell),$$

where

$$\ell = \min(\mathrm{inj}(M), \frac{\pi}{2\sqrt{\rho}}).$$

[3] And not to be confused with liquid crystal displays.

Finally, for all $\ell > 0$,

$$\mathrm{Candle}(\kappa, \ell) \implies \mathrm{Ric} \le (n-1)\kappa g,$$

where g is the metric on M, because

$$s(\gamma, r) = r^{n-1} - \mathrm{Ric}(\gamma'(0)) r^n + O(r^{n+1}). \tag{2}$$

In particular, in two dimensions, all of the implications in (1) are equivalences.

2.3. Curvature bounds. The function $\sqrt{\mathrm{Ric}}(\rho)$ increases with ρ faster than

$$(n-1)\sqrt{\rho - \kappa}$$

in the sense that for all $\kappa \le \rho \le \rho'$,

$$\sqrt{\mathrm{Ric}} \text{ class } (\rho, \kappa) \implies \sqrt{\mathrm{Ric}} \text{ class } (\rho', \kappa).$$

In addition, the conjugate version of root-Ricci curvature converges to normalized Ricci curvature for large ρ:

$$\lim_{\rho \to \infty} \rho - \left(\frac{\sqrt{\mathrm{Ric}}(\rho, u)}{n-1} \right)^2 = \frac{\mathrm{Ric}(u,u)}{n-1} \qquad \forall u \in UTM.$$

The corresponding limit $\rho \to \infty$ in Theorem 1.1 has the interpretation that the upper bound looks more and more like a bound based on Ricci curvature at short distances. This is an optimal limit in the sense that Ricci curvature is the first non-trivial derivative of $s(\gamma, r)$ at $r = 0$ by (2). On the other hand, without the length restriction, the limit $\rho \to \infty$ is impossible. That limit would be exactly Günther's inequality with Ricci curvature, but such an inequality is not generally true.

Finally we can deduce a root-Ricci upper bound from a combination of sectional curvature and Ricci bounds. The concavity of the square root function implies that given the value of $\mathrm{Ric}(u,u)$, the weakest possible value of $\sqrt{\mathrm{Ric}}(\rho, u)$ is achieved when $R(\cdot, u, \cdot, u)$ has one small eigenvalue and all other eigenvalues equal. For all $\kappa \le \alpha \le \rho$, we then get a number $\beta = \beta(\kappa, \alpha, \rho)$, decreasing in α, such that

$$K \le \alpha \text{ and } \mathrm{Ric} \le \beta g \implies \sqrt{\mathrm{Ric}} \text{ class } (\rho, \kappa). \tag{3}$$

An explicit computation yields the optimal value

$$\beta = \rho + (n-2)\alpha - \left((n-1)\sqrt{\rho - \kappa} - (n-2)\sqrt{\rho - \alpha}\right)^2.$$

In particular,

$$\beta(\kappa, \rho, \rho) = (n-1)^2 \kappa - n(n-1)\rho$$
$$\beta(\kappa, \kappa, \rho) = (n-1)\kappa.$$

In order to deduce $\sqrt{\mathrm{Ric}}$ class (ρ, κ) from classical curvature upper bounds, we can therefore ask for the strong condition $K \le \kappa$ (which implies $\mathrm{Ric} \le (n-1)\kappa g$), or ask for the weaker $K \le \rho$ together with $\mathrm{Ric} \le \beta(\kappa, \rho, \rho) g$, or choose from a continuum of combined bounds on K and Ric. Moreover, the above calculation holds pointwise, so that in (3), α can be a function on UTM instead of a constant.

3. Applications. Most of the established applications of Günther's inequality are also applications of Theorem 1.1. The subtlety is that different applications use different criteria in the chain of implications (1). We give some examples. In general, let $\tilde{M}$ denote the universal cover of M.

3.1. Exponential growth of balls. One evident application of our result is to estimate the rate of growth of balls, as already given by (1). This is related to the volume entropy of a closed Riemannian manifold M, which is by definition

$$\mathrm{h}_{\mathrm{vol}}(M) \stackrel{\mathrm{def}}{=} \lim_{r\to+\infty} \frac{\log \mathrm{Vol}\, B_{\tilde{M}}(p,r)}{r}.$$

By abuse of notation, we will use this same volume entropy expression when $M = \tilde{M}$ is simply connected rather than closed. Since a hyperbolic space of curvature $\kappa < 0$ and dimension n has volume entropy $(n-1)\sqrt{-\kappa}$, Theorem 1.1 implies that when $K \leq 0$,

$$\mathrm{h}_{\mathrm{vol}}(M) \geq \alpha \stackrel{\mathrm{def}}{=} \inf_u \sqrt{\mathrm{Ric}}(0,u). \tag{4}$$

The estimate (4) is sharp for every rank one symmetric space. (Recall that the rank one symmetric spaces are the generalized hyperbolic spaces $\mathbb{R}\mathrm{H}^n$, $\mathbb{C}\mathrm{H}^n$, $\mathbb{H}\mathrm{H}^n$, and $\mathbb{O}\mathrm{H}^2$.) The reason is that the operator $R(\cdot,\gamma',\cdot,\gamma')$ is constant along any geodesic γ. So by the Jacobi field equation (Section 4), the volume of $B(p,r)$ has factors of $\sinh\sqrt{\lambda_k}r$ for each eigenvalue λ_k of $R(\cdot,\gamma',\cdot,\gamma')$. So we obtain the estimate

$$\mathrm{Vol}\, B(p,r) \propto \prod_k (\sinh\sqrt{\lambda_k}r) \sim \exp(\alpha r).$$

However, although (4) is a good estimate, it is superseded by the previous discovery of $\sqrt{\mathrm{Ric}}(0,u)$, for the specific purpose of estimating entropies. In addition to the volume entropy of M, the geodesic flow on M has a topological entropy $\mathrm{h}_{\mathrm{top}}(M)$ and a measure-theoretic entropy $\mathrm{h}_\mu(M)$ with respect to any invariant measure μ. Manning [12] showed that $\mathrm{h}_{\mathrm{top}}(M) \geq \mathrm{h}_{\mathrm{vol}}(M)$ for any closed M, with equality when M is nonpositively curved. Goodwyn [8] showed that $\mathrm{h}_{\mathrm{top}}(M) \geq \mathrm{h}_\mu(M)$ for any μ, with equality for the optimal choice of μ. (In fact he showed this for any dynamical system.)

With these background facts, Osserman and Sarnak [15] defined $\sqrt{\mathrm{Ric}}(0,u)$ and established that

$$\mathrm{h}_\mu(M) \geq \int_{UTM} \sqrt{\mathrm{Ric}}(0,u)d\mu(u) \tag{5}$$

when M is negatively curved, i.e., $K \leq \kappa < 0$, and μ is normalized Riemannian measure on UTM. This result was generalized to non-positive curvature by Ballmann and Wojtkowski [2].

This use of $\sqrt{\mathrm{Ric}}$ curvature concludes a topic that began with the Schwarz-Milnor theorem [14, 16] that if M is negatively curved, then $\pi_1(M)$ has exponential growth. Part of their result is that if M is compact, then $\pi_1(M)$ has exponential growth if and only if $\mathrm{h}_{\mathrm{vol}}(M) > 0$. So equation (5), together with Manning's theorem, shows that if M is compact and nonpositively curved, then either M is flat, or the growth of $\pi_1(M)$ is bounded below by (5).

Ballmann [1] also showed that a non-positively curved manifold M of finite volume satisfies the weak Tits alternative: either M is flat, or its fundamental group contains a non-abelian free group. This is qualitatively a much stronger version of the Schwarz-Milnor theorem, and even its extension due to Manning, Osserman, Sarnak, Ballmann, and Wojtkowski.

3.2. Isoperimetric inequalities. Yau [19, 5] established that if M is complete, simply connected, and has $K \leq \kappa < 0$, and $D \subseteq M$ is a domain, then D satisfies a linear isoperimetric inequality:

$$\operatorname{Vol}(\partial D) \geq (n-1)\sqrt{-\kappa}\operatorname{Vol}(D).$$

His proof only uses a weakening of condition LCD(κ), namely that

$$(\log s(\gamma, r))' \geq (n-1)\sqrt{-\kappa}.$$

So Theorem 1.1 yields Yau's inequality when M is of $\sqrt{\mathrm{R}}$ic class $(0, \kappa)$.

McKean [13] showed that the same weak LCD(κ) condition also implies a spectral gap

$$\lambda_0(\tilde{M}) \geq \frac{-\kappa(n-1)^2}{4}$$

for the first eigenvalue of the positive Laplace-Beltrami operator acting on $L^2(M)$. This spectral gap follows from a Poincaré inequality that is independently interesting:

$$\int_M f^2 \leq \frac{4}{-\kappa(n-1)^2}\int_M |\nabla f|^2$$

for all smooth, compactly supported functions f. McKean stated his result under the hypothesis $K \leq \kappa$; it has been generalized by Setti [17] and Borbély [4] to mixed sectional and Ricci bounds; Theorem 1.1 provides a further generalization. Note in particular that Borbély's result is optimal for complex hyperbolic spaces (and we get the same bound in this case), but we get better bounds for quaternionic and octonionic hyperbolic spaces.

Croke [6] establishes the isoperimetric inequality for a compact non-positively curved 4-manifold M with unique geodesics. In other words, if B is a Euclidean 4-ball with

$$\operatorname{Vol}(M) = \operatorname{Vol}(B),$$

then

$$\operatorname{Vol}(\partial M) \geq \operatorname{Vol}(\partial B).$$

His proof only uses the condition Candle(0), in fact only for maximal geodesics between boundary points[4]. So, Croke's theorem also holds if M is of $\sqrt{\mathrm{R}}$ic class $((\frac{\pi}{2L})^2, 0)$, where L is the maximal length of a geodesic; for any given L, this curvature bound is weaker than $K \leq 0$. It is a well-known conjecture that if M is n-dimensional and non-positively curved, then the isoperimetric inequality holds. The conjecture could be attributed to Weil [18], because his proof in dimension $n = 2$ initiated the

[4]We credit [6] as our original motivation for this article.

subject, but it is usually named after Cartan and Hadamard. More recently, Kleiner [10] established the case $n = 3$. We are led to ask whether the Cartan-Hadamard conjecture still holds for Candle(0) or LCD(0) manifolds.

In a forthcoming paper, we will partially generalize Croke's result to signed curvature bounds. In these generalizations, the main direct hypotheses are the Candle(κ) and LCD(κ) conditions, which are natural but not local. Theorem 1.1 provides important local conditions under which these hypotheses hold.

3.3. Almost non-positive curvature. As mentioned above, one strength of root-Ricci curvature estimates is that we can adjust the parameter ρ; however, most of the applications mentioned so far are in the non-positively curved case $\rho = 0$. It is therefore natural to ask to which extent manifolds with almost non-positive sectional curvature and negative root-Ricci curvature behave like negatively curved manifolds.

More precisely, suppose that M is compact, has diameter δ and satisfies both curvature bounds

$$K \leq \rho \quad \text{and} \quad \sqrt{\mathrm{Ric}}(\rho) \leq \kappa.$$

Say that M is *almost non-positively curved* if $0 < \rho \ll \delta^{-2}$, and that M is *strongly negatively root-Ricci curved* if $\kappa \ll -\delta^{-2}$. Under these assumptions, Theorem 1.1 shows that the balls in $\tilde{M}$ grow exponentially up to a large multiple of the diameter δ. We conjecture that if M is also compact, then $\pi_1(M)$ has exponential growth or equivalently that M has positive volume entropy.

In light of Ballmann's result that a non-positively curved manifold M of finite volume satisfies the weak Tits alternative, we ask whether a compact, almost-non-positively curved, strongly negatively root-Ricci curved manifold must contain a non-abelian free group in its fundamental group. We conjecture at the very least that an almost non-positively curved manifold with strongly negative root-Ricci curvature cannot be a torus. This would be an interesting complement to the result of Lohkamp [11] that every closed manifold of dimension $n \geq 3$ has a Ricci-negative metric.

4. The proof. In this section, we will prove Theorem 1.1. The basic idea is to analyze the energy functional that arises in a standard proof of Günther's inequality, with the aid of the change of variables $R = A^2 - \rho I$.

Using the Jacobi field model, Theorem 1.1 is really a result about linear ordinary differential equations. The normal bundle to the geodesic $\gamma(t)$ can be identified with $\mathbb{R}^{n-1}$ using parallel transport. Then an orthogonal vector field $y(t)$ along γ is a Jacobi field if it satisfies the differential equation

$$y'' = -R(t)y, \tag{6}$$

where

$$R(t) = R(\cdot, u(t), \cdot, u(t))$$

is the sectional curvature matrix and $u(t) = \gamma'(t)$ is the unit tangent to γ at time t. By the first Bianchi identity, $R(t)$ is a symmetric matrix. The candle function $s(r) = s(\gamma, r)$ is determined by a matrix solution

$$Y'' = -R(t)Y \qquad Y(0) = 0 \tag{7}$$

by the formula

$$s(r) = \frac{\det Y(r)}{\det Y'(0)}.$$

Its logarithmic derivative is given by

$$(\log s(r))' = \frac{s'(r)}{s(r)} = \frac{(\det Y)'(r)}{\det Y(r)}.$$

All invertible solutions $Y(r)$ to (7) are equivalent by right multiplication by a constant matrix, and yield the same value for $s(r)$ and its derivative. In particular, if we let $Y(r) = I$, then the logarithmic derivative simplifies to

$$(\log s(r))' = \mathrm{Tr}(Y'(r)).$$

Following a standard proof of Günther's inequality [7][Thm. 3.101], we define an energy functional whose minimum, remarkably, both enforces (7) and minimizes the objective $(\log s(r))'$. Namely, we assume Dirichlet boundary conditions

$$y(0) = 0 \qquad y(r) = v,$$

and we let

$$E(R, y) = \int_0^r (\langle y', y' \rangle - \langle y, Ry \rangle)\, \mathrm{d}t. \tag{8}$$

By a standard argument from calculus of variations, the critical points of $E(R, y)$ are exactly the solutions to (6) with the given boundary conditions.

We can repeat the same calculation with the matrix solution

$$Y(0) = 0 \qquad Y(r) = I,$$

with the analogous energy

$$E(R, Y) = \int_0^r (\langle Y', Y' \rangle - \langle Y, RY \rangle)\, \mathrm{d}t.$$

Here the inner product of two matrices is the Hilbert-Schmidt inner product

$$\langle A, B \rangle = \mathrm{Tr}(A^T B).$$

Moreover, if Y is a solution to (7), then $E(R, Y)$ simplifies to $(\log s(r))'$ by integration by parts:

$$\begin{aligned} E(R, Y) &= \int_0^r (\langle Y', Y' \rangle - \langle Y, RY \rangle)\, \mathrm{d}t \\ &= \langle Y(r), Y'(r) \rangle - \langle Y(0), Y'(0) \rangle - \int_0^r \langle Y, Y'' + RY \rangle\, \mathrm{d}t \\ &= \langle I, Y'(r) \rangle - 0 - 0 = \mathrm{Tr}(Y'(r)) = (\log s(r))'. \end{aligned}$$

Thus, our goal is to minimize $E(R, Y)$ with respect to both Y and R. We want to minimize with respect to Y in order to solve (7). Then for that Y, we want to minimize with respect to R to prove Theorem 1.1.

The following proposition tells us that (6) or (7) has a unique solution with Dirichlet boundary conditions, and that it is an energy minimum. Here and below, recall the matrix notation $A \leq B$ (which was already used for Ricci curvature in the introduction) to express the statement that $B - A$ is positive semidefinite.

Proposition 4.1. *If $R \leq \rho I$, and if y is continuous with an L^2 derivative, then $E(R,y)$ is a positive definite quadratic function of y when $\sqrt{\rho}r < \pi$, with the Dirichlet boundary conditions $y(0) = y(r) = 0$.*

Proof. Let

$$E(\rho, y) = E(\rho I, y) = \int_0^r (\langle y', y' \rangle - \rho \langle y, y \rangle) \, \mathrm{d}t$$

be the corresponding energy of the comparison case with constant curvature ρ. (Recall that the ultimate comparison is with constant curvature κ, but to get started we use ρ instead.) Then

$$E(\rho, y) \leq E(R, y),$$

so it suffices to show that $E(\rho, y)$ is positive definite. When $\rho = 0$, $E(\rho, y)$ is manifestly positive definite. Otherwise $E(\rho, y)$ is diagonalized in the basis of functions

$$y_k(t) = \sin(\frac{\pi k t}{r})$$

with $k \geq 1$. A direct calculation yields

$$E(\rho, y_k) = \frac{\pi^2 k^2 - r^2 \rho}{r} > 0,$$

as desired. ☐

Remark. There is also a geometric reason that the comparison case $E(\rho, y)$ is positive definite: When $\rho = 0$, a straight line segment in Euclidean space is a minimizing geodesic; when $\rho > 0$, the same is true of a geodesic arc of length $r < \pi/\sqrt{\rho}$ on a sphere with curvature $\sqrt{\rho}$. We give a direct calculation to stay in the spirit of ODEs.

Proposition 4.2. *Let ρ and $r < \pi/\sqrt{\rho}$ be fixed and suppose that $R \leq \rho I$. Then $s(r)$ and $(\log s(r))'$ are both bounded below.*

Proof. We will simply prove the usual Günther inequality. As in the proof of Proposition 4.1,

$$E(R, Y) \geq E(\rho, Y)$$

for all R and Y with $Y(0) = 0$ and $Y(r) = I$. For each fixed R, the minimum of the left side is $(\log s(r))'$. The minimum of the right side (which may occur for a different Y, but no matter) is $(\log s_\rho(r))'$, which is a positive number. We obtain the same conclusion for $s(r)$ by integration. ☐

Proposition 4.3. *Assume the hypotheses of Proposition 4.2. If R is L^∞, then the solution Y to (7) is bounded uniformly, i.e., with a bound that depends only on $||R||$ (and r and ρ). Also Y' is uniformly bounded and Lipschitz, and Y'' is uniformly bounded and L^∞.*

Proof. In this proposition and nowhere else, it is more convenient to assume the initial conditions

$$\hat{Y}(0) = 0 \qquad \hat{Y}'(0) = I$$

rather than Dirichlet boundary conditions. The fact that $\hat{Y}$ and its derivatives are uniformly bounded, with these initial conditions, is exactly Grönwall's inequality. To convert back to Dirichlet boundary conditions, we want to instead bound

$$Y(t) = \hat{Y}(t)\hat{Y}(r)^{-1}.$$

This follows from Proposition 4.2 by the formula

$$\hat{Y}(r)^{-1} = \operatorname{adj}(\hat{Y}(r)) \det(\hat{Y}(r))^{-1},$$

where adj denotes the adjugate of a matrix.

Finally, $Y''(t)$ is L^∞ and uniformly bounded because $Y(t)$ satisfies (7). Also $Y'(0) = \hat{Y}(r)^{-1}$ is uniformly bounded, so we can integrate to conclude that $Y'(t)$ is uniformly bounded and Lipschitz. □

To prove Theorem 1.1, we want to minimize $(\log s(r))'$ or $E(R, Y)$ over all R such that

$$R \leq \rho I \qquad \operatorname{Tr}(\sqrt{\rho I - R}) \geq \alpha \stackrel{\text{def}}{=} (n-1)\sqrt{\rho - \kappa}. \tag{9}$$

To better understand this minimization problem, we make a change of variables. Let $A(t)$ be a symmetric matrix such that

$$R(t) = \rho I - A(t)^2 \qquad \operatorname{Tr}(A(t)) \geq \alpha. \tag{10}$$

In order to know that every $R(t)$ is realized, we can let

$$A = \sqrt{\rho I - R}$$

be the positive square root of $\rho I - R$. Even if A is not positive semidefinite, $R(t)$ still satisfies (9). This simplifies the optimization problem: in the new variable A, the semidefinite hypothesis can be waived.

Now the energy function becomes:

$$\begin{aligned} E(A, Y) &= \int_0^r \big(\langle Y', Y'\rangle - \langle Y, (\rho - A^2)Y\rangle\big)\, dt \\ &= \int_0^r \big(\operatorname{Tr}((Y')^T Y') + \operatorname{Tr}(Y^T A^2 Y) - \rho \operatorname{Tr}(Y^T Y)\big)\, dt. \end{aligned}$$

Fix Y for the moment. Then as a function of A,

$$E(A) = \int_0^r \operatorname{Tr}(A^2 Y Y^T)\, dt + \text{constant}.$$

Since YY^T is symmetric and strictly positive definite, E is a positive-definite quadratic function of A, and we can directly solve for the minimum as

$$A = \frac{\alpha (YY^T)^{-1}}{\operatorname{Tr}((YY^T)^{-1})}. \tag{11}$$

Even though we waived the assumption that A is positive semidefinite, minimization restores it as a conclusion. Moreover,

$$\operatorname{Tr}(A) = \operatorname{Tr}(\sqrt{\rho I - R}) = \alpha. \tag{12}$$

Proposition 4.4. *With the hypotheses* (9), *and if* $r < \pi/\sqrt{\rho}$, *a minimum of* $(\log s(r))'$ *exists. Equivalently, a joint minimum of* $E(A, Y)$ *or* $E(R, Y)$ *exists.*

Proof. The above calculation lets us assume (12), which means that R is uniformly bounded. By Proposition 4.3, so is Y''. We can restrict to a set of pairs (R, Y'') of class L^∞, which is compact in the weak-* topology by the Banach-Alaoglou theorem. Equivalently, we can restrict to a uniformly bounded, uniformly Lipschitz set of pairs $(\int R, Y')$, which is compact in the uniform topology by the Arzela-Ascoli theorem. By integration by parts, we can write

$$E(R, Y) = \int_0^r (\langle Y', Y'\rangle - \langle Y, RY\rangle)\,\mathrm{d}t.$$
$$= [\langle Y, (\textstyle\int R)Y\rangle]_0^r + \int_0^r (\langle Y', Y'\rangle\,\mathrm{d}t + 2\langle Y', (\textstyle\int R)Y\rangle)\,\mathrm{d}t.$$

Thus the energy is continuous as a function of $\int R$ and Y' and has a minimum on a compact family. □

Proposition 4.4 reduces Theorem 1.1 to solving the following non-linear matrix ODE, which is obtained by combining (7) and (11):

$$Y'' = (A^2 - \rho)Y \qquad A = \frac{\alpha(YY^T)^{-1}}{\mathrm{Tr}((YY^T)^{-1})}$$
$$Y(0) = 0 \qquad Y(r) = I.$$

Proposition 4.4 tells us that this ODE has at least one solution; we will proceed by finding all solutions with the given boundary conditions. First, if we suppress the boundary condition $Y(r) = I$, the solutions $Y(t)$ are invariant under both left and right multiplication by $O(n-1)$. So we can write

$$Y(t) = U\hat{Y}(t)V,$$

where $\hat{Y}'(0)$ is diagonal with positive entries. In this case $\hat{A}(0)$ is also diagonal, and we obtain that $\hat{Y}(t)$ is diagonal for all t, and with positive entries because the entries cannot cross 0. Therefore $UV = I$, because the identity is the only diagonal orthogonal matrix with positive entries.

So we can assume that $Y = \hat{Y}$, with diagonal entries

$$\lambda_1(t), \lambda_2(t), \ldots, \lambda_{n-1}(t) > 0.$$

Each of these entries satisfies the same scalar ODE,

$$w'' = \beta(t)w^{-1} - \rho w \qquad w(0) = 0 \qquad w(r) = 1, \tag{13}$$

where

$$\beta(t) = \frac{\alpha}{\mathrm{Tr}((Y(t)Y(t)^T)^{-1})^2}.$$

We claim that if $w > 0$, then $w' > 0$ as well. If $\rho = 0$, then this is immediate. Otherwise, a positive solution $w(t)$ satisfies

$$w(t) > \frac{\sin(\sqrt{\rho}t)}{\sin(\sqrt{\rho}r)} \qquad w'(t) > \frac{\sqrt{\rho}\cos(\sqrt{\rho}t)}{\sin(\sqrt{\rho}r)},$$

because the right side is the solution to $w'' = -\rho w$ with the same boundary conditions. So we obtain that $w' > 0$ provided that

$$r < \frac{\pi}{2\sqrt{\rho}}.$$

(This is where we need half of the distance allowed in the usual form of Günther's inequality.)

To complete the proof, consider the phase diagram in the strip $[0,1] \times (0,\infty)$ of the positive solutions $(w(t), w'(t))$ to (13). If we let $x = w(t)$, then the total elapsed time to reach $x = 1$ is

$$r = \int_0^1 \frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{w'(w^{-1}(x))},$$

which is a positive integral. On the other hand, if w_1 and w_2 are two distinct solutions with

$$w_1(0) = w_2(0) = 0 \qquad w_1'(0) > w_2'(0),$$

then the solutions cannot intersect in the phase diagram; we must have

$$w_1'(w_1^{-1}(x)) > w_2'(w_2^{-1}(x)) > 0.$$

So two distinct, positive solutions to (13) cannot reach $w(t) = 1$ at the same time, which means with the given boundary conditions that there is only one solution. Thus, the diagonal entries $\lambda_k(t)$ of $Y(t)$ are all equal. In conclusion, Y, A, and R all are isotropic at the minimum of the logarithmic candle derivative $(\log s(r))'$. This additional property implies the estimate for $(\log s(r))'$ immediately. (Note that when R is isotropic, the hypothesis becomes equivalent to $K \leq \kappa$, the usual assumption of Günther's inequality.)

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