ASIAN J. MATH.
Vol. 19, No. 1, pp. 083–120, January 2015

004

ON HIGHER REGULATORS OF SIEGEL THREEFOLDS I: THE VANISHING ON THE BOUNDARY*

FRANCESCO LEMMA†

Abstract. In this paper, we prove the vanishing of a map between the absolute Hodge cohomology spaces of the boundaries of the Baily-Borel compactifications of the product of two modular curves on one side and of the Siegel threefold on the other side. As an application, we construct some 1-extensions of mixed Hodge structures between the trivial Hodge structure and the middle-degree interior cohomology of the Siegel threefold, which come from motivic cohomology. These are conjecturally related to non-critical values of the degree 4 L-function of some cuspidal automorphic representations of GSp_4.

Key words. Siegel threefolds, higher regulators, mixed Hodge modules.

AMS subject classifications. 19F27, 14D07.

1. Introduction. This work is motivated by Beilinson's conjectures on special values of L-functions. Given a pure motive H over $\mathbb{Q}$ and of weight $w \leq -3$, these conjectures relate the image of the Betti realization functor

$$\mathrm{Ext}^1_{\mathrm{MM}(\mathbb{Q})}(\mathbb{Q}(0), H) \xrightarrow{r_B} \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H_B)$$

where $\mathrm{MM}(\mathbb{Q})$ is the abelian category of mixed motives over $\mathbb{Q}$, where $\mathrm{MHS}^+_{\mathbb{R}}$ is the abelian category of real mixed $\mathbb{R}$-Hodge structures and H_B is the Betti realization of H, to the L-value $L(0, H)$ of the Hasse-Weil L-function of H. Even if the abelian category $\mathrm{MM}(\mathbb{Q})$ has not been discovered yet, Beilinson's regulator from motivic cohomology to absolute Hodge cohomology

$$H^{n+1}_{\mathcal{M}}(X, \mathbb{Q}(m)) \xrightarrow{r_{\mathcal{H}}} H^{n+1}_{\mathcal{H}}(X/\mathbb{R}, \mathbb{R}(m)),$$

defined for any variety X which is smooth and quasi-projective over $\mathbb{Q}$, can be seen as a substitute for r_H. In fact, when X is smooth and projective, for $2m \neq n+1$ the space $H^{n+1}_{\mathcal{M}}(X, \mathbb{Q}(m))$ is conjecturally isomorphic to $\mathrm{Ext}^1_{\mathrm{MM}(\mathbb{Q})}(\mathbb{Q}(0), H^n(X)(m))$ and the space $H^{n+1}_{\mathcal{H}}(X, \mathbb{R}(m))$ is isomorphic to $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^n_B(X)(m))$. Note that the pure motive $H^n(X)(m)$ has weight $w = n - 2m$ so that the case $2m = n+1$ is excluded by the assumption $w \leq -3$. The reader unfamiliar with this circle of ideas might consult the survey article [N] for explanations and a precise statement of the conjecture.

The most recent proof of (a weak form of) Beilinson's conjecture, due to Kings [K1], concerns the motive H of the intersection cohomology of some Hilbert modular surfaces. Like in Beilinson's work for elliptic modular forms, the first key step is to construct some elements in $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H_B)$ coming from motivic cohomology via Beilinson's Eisenstein symbol. The latter provides some non-trivial motivic cohomology classes over the product of the universal elliptic curve over the modular curves whose image under the regulator can be expressed in terms of real analytic Eisenstein series. Roughly speaking, Kings considers the embedding of the product

*Received November 22, 2012; accepted for publication October 11, 2013.
†Institut de mathématiques de Jussieu-Paris Rive Gauche, UMR 7586, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France (lemma@math.univ-paris-diderot.fr).

of two universal elliptic curves over a modular curve in the universal abelian surface over the Hilbert modular surface and maps the Eisenstein symbol in the motivic cohomology of the universal abelian surface via the Gysin morphism associated to the mentionned closed embedding (see the introduction of [K1] for more details). In order to show that the image under the regulator of these classes define some elements of $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H_B)$, he shows that this image vanishes on the boundary of the Baily-Borel compactification of the Hilbert modular surface (see [K1] 5.4).

The goal of this paper is to begin the study of Beilinson's conjecture for the motive H of the intersection cohomology of the Siegel threefolds, which are the Shimura varieties associated to the symplectic group GSp_4. Our main result is the construction of some elements in $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H_B)$ coming from motivic cohomology via some cup-products of Eisenstein symbols (see Thm. 6.8 for a precise statement).

Let us explain the main ideas of the present work: let $p, q \geq 0$ be two integers. Choose $k \geq k' \geq 0$ two integers satisfying the following conditions:

- $k + k' \equiv p + q \pmod 2$,
- If $0 \leq p < k'$ and $p < k - k'$ then $k - k' - p \leq q \leq k - k' + p$,
- If $0 \leq p < k'$ and $k - k' \leq p$ then $p - k + k' \leq q \leq p + k - k'$,
- If $k' \leq p \leq k$ and $k' < k - p$ then $k - k' - p \leq q \leq k + k' - p$,
- If $k' \leq p \leq k$ and $k - p \leq k'$ then $p - k + k' \leq q \leq k + k' - p$.

We have an embedding

$$\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \xrightarrow{\iota} \mathrm{GSp}_4$$

where the left hand side denotes the group of pairs of invertible matrices of size 2 with the same determinant. Actually the conditions on k and k' are equivalent to the following: denote $c = p + q + 6$ and let W be an irreducible algebraic representation of GSp_4 with highest weight $\lambda(k, k', c)$ with the conventions of section 2.1. Then we have

$$(Sym^p V_2 \boxtimes Sym^q V_2) \otimes det^{\otimes 3} \subset \iota^* W$$

where V_2 denotes the standard representation of GL_2. Now let E/M be the universal elliptic curve over a modular curve M and for any $n \geq 0$, let E^n be the n-th fold fiber product over M. The Eisenstein symbol is a non-trivial morphism

$$Eis^n : \mathcal{B}_n \longrightarrow H^{n+1}_{\mathcal{M}}(E^n, \mathbb{Q}(n+1))$$

whose composite $Eis^n_{\mathcal{H}}$ with Beilinson's regulator factors through $H^1_{\mathcal{H}}(M/\mathbb{R}, Sym^n V_2(1))$, which is a subspace of $H^{n+1}_{\mathcal{H}}(E^n/\mathbb{R}, \mathbb{R}(n+1))$ (see for example [K1] (5.3.4)). Here V_2 abusively denotes the variation of $\mathbb{R}$-Hodge structure on M associated to V_2. Let S be the Siegel threefold and denote by W the variation of $\mathbb{R}$-Hodge structure on S associated to W. The main result of this paper is the construction, for many of the choices of k and k' as above, of a natural $\mathbb{Q}$-linear map

$$Eis^{p,q}_{\mathcal{H}} : \mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^3_!(S, W)) \tag{1}$$

where $H^3_!(S, W)$ is the image of the cohomology with compact support in the cohomology without support, a pure Hodge structure of weight $-p - q - 3 \leq -3$. This can be done in three steps as follows.

• By taking the external cup-product of $Eis^p_{\mathcal{H}}$ and $Eis^q_{\mathcal{H}}$ we have a map

$$Eis^p_{\mathcal{H}} \sqcup Eis^q_{\mathcal{H}} : \mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow H^2_{\mathcal{H}}(M \times M/\mathbb{R}, (Sym^p V_2 \boxtimes Sym^q V_2)(2)).$$

Composing with the map induced by the inclusion $(Sym^p V_2 \boxtimes Sym^q V_2)(2) \subset \iota^* W(-1)$ and with the Gysin morphism associated to the codimension 1 embedding $M \times M \longrightarrow S$ induced by ι we obtain a map

$$\mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow H^4_{\mathcal{H}}(S/\mathbb{R}, W).$$

• Denote by j the open embedding of S into its Baily-Borel compactification and by i the complementary reduced closed embedding, so that we have a diagram

$$S \xrightarrow{\ j\ } S^* \xleftarrow{\ i\ } \partial S.$$

We will show (see Prop. 5.5 and 5.6) that one has an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^3_!(S, W)) \to H^4_{\mathcal{H}}(S/\mathbb{R}, W) \to H^2_{\mathcal{H}}(\partial S/\mathbb{R}, i^* j_* W)$$

for most of the choices of k and k' as above.

• Finally comes the main step of the construction, which is to show that the composite

$$\mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow H^4_{\mathcal{H}}(S/\mathbb{R}, W) \longrightarrow H^2_{\mathcal{H}}(\partial S/\mathbb{R}, i^* j_* W)$$

is the zero map. In fact this map fits into the following commutative diagram

$$\begin{array}{ccc}
\mathcal{B}_p \otimes \mathcal{B}_q & & \\
\downarrow & & \\
H^2_{\mathcal{H}}(M \times M/\mathbb{R}, (Sym^p V_2 \boxtimes Sym^q V_2)(2)) & \longrightarrow & H^1_{\mathcal{H}}(\partial(M \times M)/\mathbb{R}, i'^* j'_* (Sym^p V_2 \boxtimes Sym^q V_2)(2)) \\
\downarrow & & \downarrow \\
H^4_{\mathcal{H}}(S/\mathbb{R}, W) & \longrightarrow & H^2_{\mathcal{H}}(\partial S/\mathbb{R}, i^* j_* W)
\end{array}$$

where

$$M \times M \xrightarrow{\ j'\ } (M \times M)^* \xleftarrow{\ i'\ } \partial(M \times M)$$

denotes the boundary of the Baily-Borel compactification of $M \times M$ with the complementary reduced closed embedding of the boundary. What we really show is that the right hand vertical arrow above is zero for many choices of k and k' as above (see Thm. 6.6 for a precise statement). This provides us with the expected map (1). Roughly speaking, the assumption we make on k and k' are there to avoid the presence of weight zero Eisenstein cohomology in the Betti cohomology of S and to avoid the coincidence of weights in the Betti cohomologies of the boundaries of $M \times M$ and S.

The author announced a similar result some time ago ([Le1] Thm. 1), but his computations of higher direct images of variations of Hodge structure on the boundary contained an error. The present article shows that a slight variant of [Le1] Thm. 1 is true. Our proof heavily relies on the formalism of Grothendieck's 6 functors in the derived categories of mixed Hodge modules and on the main result of [BuW] allowing to identify the restriction to the boundary strata of the Baily-Borel compactification of higher direct images of variations of Hodge structure associated to algebraic representations of the group underlying the considered Shimura variety. It is probably unnecessary to claim that our situation is much more complicated than Kings', mainly because the boundary of the Baily-Borel compactification of Siegel threefolds in not only made of cusps, but of cusps and modular curves. We also would like to emphasize the fact that the proof is motivic in nature and one can expect that in a world where a full formalism of mixed motivic sheaves, weights and the motivic analogue of [BuW] were available, we could construct a map

$$Eis^{p,q}_{\mathcal{M}} : \mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow \mathrm{Ext}^1_{\mathrm{MM}(\mathbb{Q})}(\mathbb{Q}(0), H^3_!(S, W))$$

whose composite with the Betti realization functor would be our $Eis^{p,q}_{\mathcal{H}}$.

Let us briefly outline the contents of this paper. In the second part we fix some conventions and notations concerning the symplectic group GSp_4, state a branching formula which plays an important role in this work and review some important results on mixed Hodge modules and absolute Hodge cohomology. In the third part we determine the geometric setting we are interested in and define the map in absolute Hodge cohomology that we want to study. The fourth part concerns the computation of higher direct images of variations of Hodge structure, via the main result of [BuW] and a theorem of Kostant, in the Baily-Borel compactifications of the product of two modular curves and of the Siegel threefolds. The fifth part is dedicated to the study of the relations between the Ext^1 space we are interested in and the absolute Hodge cohomology of the boundary. Finally, in the last part, we show our main vanishing result and explain how it allows to construct some elements in $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^3_!(S, W))$ coming from motivic cohomology.

Thanks to the work of [L], [T] and [We] the L-function $L(s, H^3_!(S, W))$ associated to the l-adic avatars of $H^3_!(S, W)$ is known to coincide with the L-function $L(s, \pi)$ of some cuspidal automorphic representation π of GSp_4. In a forthcoming work, we will relate the 1-extensions constructed in the present article to the special value of this L-function predicted by Beilinson's conjecture.

Acknowledgements. It is a pleasure to thank Jörg Wildeshaus for constant support, Michel Duflo and Bruno Klingler for answering my questions and the anonymous referee for her/his very careful reading of the manuscript and for her/his comments.

2. Preliminaries.

2.1. The algebraic group $\mathbf{GSp}_4$ and its representations. Let I_2 be the identity matrix of size 2 and let

$$J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

The symplectic group GSp_4 is the reductive algebraic group over $\mathbb{Q}$ defined as

$$\mathrm{GSp}_4 = \{g \in \mathrm{GL}_{4/\mathbb{Q}} \,|\, {}^t gJg = \nu(g)J,\ \nu(g) \in \mathbb{G}_m\}.$$

Its derived group is $\mathrm{Sp}_4 = \mathrm{Ker}\,\nu$. We denote by $T \subset \mathrm{GSp}_4$ the diagonal maximal torus given by

$$T = \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \,|\, \alpha_1, \alpha_2, \nu \in \mathbb{G}_m\}$$

and by $B = TU$ the standard Borel subgroup of upper triangular matrices in GSp_4. We identify the group $X^*(T)$ of algebraic characters (we will also, as usual, say "weights") of T to the subgroup of $\mathbb{Z}^2 \oplus \mathbb{Z}$ of triples (k, k', c) such that $k + k' \equiv c \pmod 2$ via

$$\lambda(k, k', c) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \longmapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{c-k-k'}{2}}.$$

Write $\rho_1 = \lambda(1, -1, 0)$ for the short simple root and $\rho_2 = \lambda(0, 2, 0)$ for the long simple root. Then the set $R \subset X^*(T)$ of roots of T in GSp_4 is

$$R = \{\pm\rho_1, \pm\rho_2, \pm(\rho_1 + \rho_2), \pm(2\rho_1 + \rho_2)\}$$

and the subset $R^+ \subset R$ of positive roots with respect to B is

$$R^+ = \{\rho_1, \rho_2, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}.$$

The set of dominant, resp. regular, weights is the set of $\lambda(k, k', c)$ such that $k \geq k' \geq 0$, resp. $k > k' > 0$. For any dominant weight λ, there is an irreducible algebraic representation V_λ of GSp_4 of highest weight λ, unique up to isomorphism, and all isomorphism classes of irreducible algebraic representations of GSp_4 are obtained in this way. If W is irreducible with highest weight $\lambda(k, k', c)$ the contragredient of W has highest weight $\lambda(k, k', -c)$. In particular, the contragredient of an irreducible representation whose highest weight is regular has regular highest weight. The Weyl group W of (GSp_4, T) is defined as the normalizer of T in GSp_4 modulo its centralizer. It is a group of order 8 such that the images in W of the elements

$$s_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, s_2 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}$$

generate W. The Weyl group acts on $X^*(T)$ according to the rule

$$(w.\lambda)(t) = \lambda(w^{-1}tw)$$

and we have $s_1.\lambda(k, k', c) = \lambda(k', k, c)$ and $s_2.\lambda(k, k', c) = \lambda(k, -k', c)$ which means that s_1 corresponds to the reflection associated to the short simple root ρ_1 and s_2 to the one associated to the long simple root ρ_2.

Denote by $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ the group of pairs of invertible matrices with the same determinant. We have the embedding

$$\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \xrightarrow{\iota} \mathrm{GSp}_4 \tag{2}$$

defined by

$$\iota\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

Denote by $\pi_i : \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \longrightarrow \mathrm{GL}_2$ the i-th projection, for $i = 1, 2$. Given representations ρ_i of GL_2, we write $\rho_1 \boxtimes \rho_2$ for the representation of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ given by $\pi_1^*\rho_1 \otimes \pi_2^*\rho_2$. For any integer t and non-negative integers p, q, we denote by

$(Sym^pV_2 \boxtimes Sym^qV_2)(t)$ the irreducible representation $(Sym^pV_2 \boxtimes Sym^qV_2) \otimes det^{\otimes t}$ where det is the determinant character and V_2 is the standard representation of GL_2. Note that

$$(Sym^pV_2 \otimes det^{\otimes t}) \boxtimes (Sym^qV_2 \otimes det^{\otimes t'}) = (Sym^pV_2 \boxtimes Sym^qV_2)(t+t').$$

Proposition 2.1. *Let $k \geq k' \geq 0$, let c be an integer such that $c \equiv k+k' \pmod 2$ and let V_λ be an irreducible representation of GSp_4 of highest weight $\lambda = \lambda(k,k',c)$. Then we have the following branching rule*

$$\begin{aligned}\iota^*V_\lambda = &\bigoplus_{0\leq p<k',p<k-k'} \bigoplus_{a=0}^{p} \left(Sym^pV_2 \boxtimes Sym^{k-k'+p-2a}V_2\right)\left(\frac{c-k+k'-2p+2a}{2}\right)\\ &\oplus \bigoplus_{0\leq p<k',k-k'\leq p} \bigoplus_{a=0}^{k-k'} \left(Sym^pV_2 \boxtimes Sym^{k-k'+p-2a}V_2\right)\left(\frac{c-k+k'-2p+2a}{2}\right)\\ &\oplus \bigoplus_{k'\leq p\leq k,k'<k-p} \bigoplus_{a=0}^{k'} \left(Sym^pV_2 \boxtimes Sym^{k+k'-p-2a}V_2\right)\left(\frac{c-k-k'+2a}{2}\right)\\ &\oplus \bigoplus_{k'\leq p\leq k,k-p\leq k'} \bigoplus_{a=0}^{k-p} \left(Sym^pV_2 \boxtimes Sym^{k+k'-p-2a}V_2\right)\left(\frac{c-k-k'+2a}{2}\right).\end{aligned}$$

Proof. According to [WY] Thm 3.3, we have the following branching rule

$$\begin{aligned}\iota^*V_\lambda = &\bigoplus_{0\leq p<k'} \left(Sym^pV_2 \boxtimes (Sym^{k-k'}V_2 \otimes Sym^pV_2)\right)\left(\frac{c-k+k'-2p}{2}\right)\\ &\oplus \bigoplus_{k'\leq p\leq k} \left(Sym^pV_2 \boxtimes (Sym^{k-p}V_2 \otimes Sym^{k'}V_2)\right)\left(\frac{c-k-k'}{2}\right).\end{aligned}$$

Note that [WY] Thm 3.3 is a branching rule for the embedding

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \xrightarrow{\ \iota'\ } \mathrm{Sp}_4$$

and that the branching rule stated above can be easily deduced from the corresponding one for ι' by using the fact that the common center of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and of GSp_4 acts by the same character on the left and on the right. Now, according to [FH] Ex. 11.11, together with the same remark on the action of the center, we have

$$Sym^sV_2 \otimes Sym^tV_2 = \bigoplus_{a=0}^{t} Sym^{s+t-2a}V_2(a)$$

for any $s \geq t$. This implies the statement. □

The following is a trivial consequence of the previous proposition.

Corollary 2.2. *Let $p,q \geq 0$ be two integers. Let $k \geq k' \geq 0$ be two integers satisfying the following conditions:*

(i) $k+k' \equiv p+q \pmod 2$,

(ii) If $0 \leq p < k'$ and $p < k-k'$ then $k-k'-p \leq q \leq k-k'+p$,

(iii) If $0 \leq p < k'$ *and* $k - k' \leq p$ *then* $p - k + k' \leq q \leq p + k - k'$*,*
(iv) If $k' \leq p \leq k$ *and* $k' < k - p$ *then* $k - k' - p \leq q \leq k + k' - p$*,*
(v) If $k' \leq p \leq k$ *and* $k - p \leq k'$ *then* $p - k + k' \leq q \leq k + k' - p$*.*

Let $c = p + q + 6$ *and let* V_λ *be an irreducible algebraic representation of* GSp_4 *of highest weight* $\lambda = \lambda(k, k', c)$*. Then we have*

$$(Sym^p V_2 \boxtimes Sym^q V_2)(3) \subset \iota^* V_\lambda.$$

2.2. Mixed Hodge modules and absolute Hodge cohomology. As we already mentionned, the present work heavily relies on the formalism of mixed Hodge modules, which are the relative version of mixed Hodge structures and are the "archimedean" analogues of mixed l-adic perverse sheaves. In this section we collect the facts that we will need in the following about mixed Hodge modules and absolute Hodge cohomology and we also set some important conventions. A useful guide through this complicated theory can be found in [S1] and [HW1] A.

Let $A \subset \mathbb{R}$ be a subfield and $Sch(\mathbb{Q})$ be the category of quasi-projective $\mathbb{Q}$-schemes. For $X \in Sch(\mathbb{Q})$ we have the abelian category $\mathrm{MHM}_A(X/\mathbb{R})$ of real algebraic mixed A-Hodge modules ([HW1] Def A.2.4). Let $D^b_c(X^{an}_{\mathbb{C}}, A)$ be the bounded derived category of sheaves for the analytic topology of A-vector spaces with constructible cohomology objects, and consider $\mathrm{Perv}_A(X^{an}_{\mathbb{C}}) \subset D^b_c(X^{an}_{\mathbb{C}}, A)$ the subcategory of perverse sheaves for the autodual perversity on $X^{an}_{\mathbb{C}}$. The main result of [Be1] is that the natural functor $D^b(\mathrm{Perv}_A(X^{an}_{\mathbb{C}})) \longrightarrow D^b_c(X^{an}_{\mathbb{C}}, A)$ is an equivalence of categories. According to [S3] Thm. 0.1, there is a functor

$$rat : \mathrm{MHM}_A(X/\mathbb{R}) \longrightarrow \mathrm{Perv}_A(X^{an}_{\mathbb{C}})$$

which is faithful and exact. We will denote again by

$$rat : D^b(\mathrm{MHM}_A(X/\mathbb{R})) \longrightarrow D^b_c(X^{an}_{\mathbb{C}}, A)$$

the derived functor. For $M \in D^b(\mathrm{MHM}_A(X/\mathbb{R}))$ the cohomology objects $\mathcal{H}^i M$, which are objects of $\mathrm{MHM}_A(X/\mathbb{R})$, verify $rat(\mathcal{H}^i M) = {}^p\mathcal{H}^i rat(M)$ where ${}^p\mathcal{H}^i$ is the perverse cohomology functor.

Assume that X is smooth and purely of dimension d. Then for any local system V of A-vector spaces on $X^{an}_{\mathbb{C}}$, the complex $V[d]$ concentrated in degree $-d$ is an object of $\mathrm{Perv}_A(X^{an}_{\mathbb{C}})$. Denote by $\mathrm{MHM}_A(X/\mathbb{R})^s$ the full subcategory of $\mathrm{MHM}_A(X/\mathbb{R})$ of objects whose underlying perverse sheaf is such a shifted local system and by $\mathrm{Var}_A(X/\mathbb{R})$ the category of real admissible polarizable variations of mixed A-Hodge structure over X (see [HW1] Def. A.2.1 b) for the definition). There is an equivalence of categories

$$\mathrm{Var}_A(X/\mathbb{R}) \simeq \mathrm{MHM}_A(X/\mathbb{R})^s$$

according to [HW1] Def A.2.4 b). As a consequence we have an equivalence of abelian categories

$$\mathrm{MHM}_A(\mathrm{Spec}\,\mathbb{Q}/\mathbb{R}) \simeq \mathrm{MHS}^+_A$$

where the right hand side denotes the abelian category of mixed real A-Hodge structure (see [HW1] Lem. A.2.2). In the following we will only consider the case $A = \mathbb{R}$

and will simply write "variation of Hodge structure" instead of "real admissible polarizable variation of mixed $\mathbb{R}$-Hodge structure". We will make repeated use of the important following result:

THEOREM 2.3. [S2], [S3], [HW1] *Thm A.2.5. On the derived categories* $D^b(\mathrm{MHM}_A(X/\mathbb{R}))$ *we have the formalism of Grothendieck's* 6 *functors* $(f^*, f_*, f_!, f^!, \underline{\mathrm{Hom}}, \hat{\otimes})$ *and duality* $\mathbb{D}$. *Furthermore these functors commute with rat.*

REMARK 2.4. We have to deal with a shift of the index when viewing variations of mixed Hodge structure as a mixed Hodge module, which occurs either in the normalization of the embedding

$$\mathrm{Var}_A(X/\mathbb{R}) \longrightarrow D^b(\mathrm{MHM}_A(X/\mathbb{R}))$$

or in the numbering of the cohomology objects of functors induced by morphisms between schemes of different dimensions. Our convention will be the same as the one adopted in [BuW]: in this paper, a variation of mixed Hodge structure is a mixed Hodge module, via the identification explained above, and not a shift of a mixed Hodge module. In other words, when X is smooth and purely of dimension d, our embedding

$$\mathrm{Var}_A(X/\mathbb{R}) \longrightarrow D^b(\mathrm{MHM}_A(X/\mathbb{R}))$$

is characterized by the fact that for any object V of $\mathrm{Var}_A(X/\mathbb{R})$ the complex $rat(V)[-d]$ has a single non-trivial constituent in degree zero, which is a local system on $X^{an}_{\mathbb{C}}$. This implies that if X is an object of $Sch(\mathbb{Q})$, which is smooth and of pure dimension d, if $s : X \longrightarrow \mathrm{Spec}\,\mathbb{Q}$ is the structure morphism, and $A(n)$ is the Tate variation on X viewed as an object of $\mathrm{MHM}_A(X/\mathbb{R})$, we have $A(n) = s^*A(n)[d]$ in $\mathrm{MHM}_A(X/\mathbb{R})$ (see the remark following Def. A.1.2 in [HW1]).

Now for any $X \in Sch(\mathbb{Q})$ with structural morphism s, it follows from [HW1] Cor. A.1.7 c) that $\mathcal{H}^i s_* s^* A(0)$ is the i-th singular cohomology space of the topological space underlying $X^{an}_{\mathbb{C}}$ with coefficients in A and endowed with the mixed Hodge structure constructed by Deligne with the involution induced by the complex conjugation on $X(\mathbb{C})$. In particular, when X is smooth of pure dimension d and if $A(0)$ denotes the trivial Tate variation of Hodge structure on X, the i-th singular cohomology space is $\mathcal{H}^{i-d} s_* A(0)$. Hence, for any X and any $M \in D^b(\mathrm{MHM}_A(X/\mathbb{R}))$ we will call $\mathcal{H}^{i-d} s_* M$ the i-the singular cohomology space of X with coefficients in M and denote it by $H^i(X, M)$. Similarly, we have the compactly supported cohomology $H^i_c(X, M) = \mathcal{H}^{i-d} s_! M$. The i-th absolute Hodge cohomology space of X with coefficients in M is by definition

$$H^i_{\mathcal{H}}(X/\mathbb{R}, M) = \mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d], M[i]).$$

By adjunction we have

$$H^i_{\mathcal{H}}(X/\mathbb{R}, M) = \mathrm{Hom}_{D^b(\mathrm{MHS}^+_A)}(A(0), s_* M[i-d])$$

and as the abelian category MHS^+_A has cohomological dimension 1 for $A = \mathbb{R}$, the Leray spectral sequence reduces to the exact sequence

$$(3) \quad 0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^{i-1}(X, M)) \to H^i_{\mathcal{H}}(X, M) \to \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^i(X, M)) \longrightarrow 0$$

for all i and all $M \in D^b(\mathrm{MHM}_{\mathbb{R}}(X/\mathbb{R}))$. In the following, we will simply write $1(n)$ for the Tate Hodge structure $\mathbb{R}(n) \in \mathrm{MHS}_{\mathbb{R}}^+$, for any integer n, and write 1 for $1(0)$. Note that there is also a shift in the weights when one regards variations of Hodge structure as mixed Hodge modules.

THEOREM 2.5. ([S4] Thm. 2) *Let $X \in Sch(\mathbb{Q})$ smooth and purely d dimensional. Then a variation of Hodge structure of weight w is a mixed Hodge module of weight $w+d$ via the identification above.*

REMARK 2.6. We would like to warn the reader that the perverse t-structure gives rise to unusual shifts in the following situation that we will constantly use in this work: let

$$U \xrightarrow{\ j\ } X \xleftarrow{\ i\ } Y$$

be a diagram in $Sch(\mathbb{Q})$ where j is an open embedding and i is the complementary reduced closed embedding. Assume that U has the same dimension as X and that Y has codimension c in X. Let N be an object of $D^b(\mathrm{MHM}_{\mathbb{R}}(X/\mathbb{R}))$. According to [S3] (4.4.1) we have an exact triangle

$$j_!j^*N \longrightarrow N \longrightarrow i_*i^*N \xrightarrow{\ +\ }$$

in $D^b(\mathrm{MHM}_{\mathbb{R}}(X/\mathbb{R}))$. Now, taking an object M of $D^b(\mathrm{MHM}_{\mathbb{R}}(U/\mathbb{R}))$, and applying the above triangle to $N = j_*M$ we get the exact triangle

$$j_!M \longrightarrow j_*M \longrightarrow i_*i^*j_*M \xrightarrow{\ +\ } .$$

Let $s : X \longrightarrow \mathrm{Spec}\ \mathbb{Q}$ be the structure morphism. Applying the functor s_* to the previous exact triangle and taking cohomology gives us the long exact sequence of mixed Hodge structures

$$(4) \qquad \longrightarrow H^i_c(U,M) \longrightarrow H^i(U,M) \longrightarrow H^{i-c}(Y, i^*j_*M) \longrightarrow H^{i+1}(U,M) \longrightarrow .$$

Similarly, we get the restriction map on the level of absolute Hodge cohomology

$$H^i_{\mathcal{H}}(U,M) \longrightarrow H^{i-c}_{\mathcal{H}}(Y, i^*j_*M)$$

as follows: applying the functor $\mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d],\ [i])$, where d denotes the dimension of X, to the second morphism of the previous exact triangle we get the map

$$\mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d], j_*M[i]) \longrightarrow \mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d], i_*i^*j_*M[i]).$$

By adjunction we have

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d], j_*M[i]) &= \mathrm{Hom}_{D^b(\mathrm{MHM}_A(U/\mathbb{R}))}(j^*s^*A(0)[d], M[i]) \\ &= H^i_{\mathcal{H}}(U,M).\end{aligned}$$

Furthermore

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathrm{MHM}_A(X/\mathbb{R}))}(s^*A(0)[d], i_*i^*j_*M[i]) &= \mathrm{Hom}_{D^b(\mathrm{MHM}_A(Y/\mathbb{R}))}(i^*s^*A(0)[d], i^*j_*M[i]) \\ &= H^{i-c}_{\mathcal{H}}(Y, i^*j_*M).\end{aligned}$$

3. Geometric setting.

3.1. The Shimura varieties and their Baily-Borel compactifications. Denote as usual by $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ the Deligne torus and let $w : \mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbb{S}$ the cocharacter inducing on real points the inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$. In this paper, we follow the convention of [P1] 1.3 for the correspondence between algebraic representations of $\mathbb{S}$ and semisimple mixed Hodge structures. In particular, given such a representation (ρ, V), the weight k subquotient of V is the space where $\rho \circ w$ acts by multiplication by t^{-k}.

Recall that a (pure) Shimura datum is a pair $(G, \mathcal{H})$ where G is a reductive linear algebraic group over $\mathbb{Q}$ and $\mathcal{H}$ is a left homogeneous space under $G(\mathbb{R})$ which is the source of a $G(\mathbb{R})$-equivariant map $h : \mathcal{H} \longrightarrow \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, satisfying Deligne-Pink's axioms (see [P1] Def. 2.1). In this paper, we will be interested in the cases where $G = \mathbb{G}_m$, GL_2, $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and GSp_4. Let us recall how to associate to these groups some Shimura data.

• Case $G = \mathbb{G}_m$ (see [P1] Ex. 2.8): let $k : \mathbb{S} \longrightarrow \mathbb{G}_{m,\mathbb{R}}$ the morphism inducing on real points the norm $\mathbb{C}^\times \longrightarrow \mathbb{R}^\times$, $z \longmapsto z\overline{z}$ and let $\mathcal{H}_0$ the set of isomorphisms between $\mathbb{Z}$ and $\mathbb{Z}(1)$. Consider the unique transitive action of $\pi_0(\mathbb{G}_m(\mathbb{R}))$ on $\mathcal{H}_0$ and denote by h the constant map $h : \mathcal{H}_0 \longrightarrow \{k\} \subset \mathrm{Hom}(\mathbb{S}, \mathbb{G}_{m,\mathbb{R}})$. Then $(\mathbb{G}_m, \mathcal{H}_0)$ is a Shimura datum.

• Case $G = \mathrm{GL}_2$: let $h : \mathbb{S} \longrightarrow \mathrm{GL}_{2,\mathbb{R}}$ the morphism inducing on real points

$$x + iy \longrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

and let $\mathcal{H}_2 \subset \mathrm{Hom}(\mathbb{S}, \mathrm{GL}_{2,\mathbb{R}})$ be its $\mathrm{GL}_2(\mathbb{R})$-conjugacy class. Then $(\mathrm{GL}_2, \mathcal{H}_2)$ is a Shimura datum. Note that under the convention explained at the beginning of this section, the irreducible algebraic representation $Sym^k V_2(t)$ acquires a mixed Hodge structure via h, which is pure of weight $-k - 2t$.

• Case $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$: let $h' : \mathbb{S} \longrightarrow (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)_{\mathbb{R}}$ the morphism inducing on real points

$$x + iy \longrightarrow \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right)$$

and let $\mathcal{H}_2' \subset \mathrm{Hom}(\mathbb{S}, (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)_{\mathbb{R}})$ its $(\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{R})$-conjugacy class. Then $((\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)_{\mathbb{R}}, \mathcal{H}_2')$ is a Shimura datum.

• Case $G = \mathrm{GSp}_4$: denote by $\mathcal{H}_4 \subset \mathrm{Hom}(\mathbb{S}, \mathrm{GSp}_{4,\mathbb{R}})$ the $\mathrm{GSp}_4(\mathbb{R})$-conjugacy class of $\iota \circ h'$. Then $(\mathrm{GSp}_4, \mathcal{H}_4)$ is a Shimura datum and ι induces a morphism of Shimura data

$$(\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2, \mathcal{H}_2') \longrightarrow (\mathrm{GSp}_4, \mathcal{H}_4)$$

in the sense of [P1] Def. 2.3. As above, an algebraic representation W of GSp_4 of central character c acquires a mixed Hodge structure which is pure of weight $-c$.

In general, given a Shimura datum $(G, \mathcal{H})$ and a compact open subgroup $K \subset G(\mathbb{A}_f)$, the space $G(\mathbb{Q})\backslash(\mathcal{H} \times G(\mathbb{A}_f)/K)$ underlies a complex analytic space, which in turn is the analytification of a quasi-projective scheme $M_K(G,\mathcal{H})$ defined over a number field $E(G,\mathcal{H})$, called the reflex field of $(G,\mathcal{H})$. It is easy to see that in all the cases we are interested in, the reflex field is just $\mathbb{Q}$. The Shimura variety $M_K(G,\mathcal{H})$ of level K is smooth when K is neat (see [P1] 0.5 and 0.6 for a definition). In the following all compact open subgroups K as above will be assumed to be neat and we will not mention it anymore. For any $g \in G(\mathbb{A}_f)$ and $K' \subset gKg^{-1}$ another compact open subgroup, right multiplication by g induces an étale cover $g_{K'K} : M_{K'}(G,\mathcal{H}) \longrightarrow M_K(G,\mathcal{H})$ so that when the level varies, the Shimura varieties form a projective system $(M_K(G,\mathcal{H}))_K$ endowed with an action of $G(\mathbb{A}_f)$.

For any compact open subgroup $L \subset \mathrm{GSp}_4(\mathbb{A}_f)$ denote by S_L the associated Shimura variety. This is a quasi-projective smooth threefold defined over $\mathbb{Q}$. The embedding ι defined in (2) gives rise to a closed embedding on the level of Shimura varieties as follows: let K and K' be two compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$ such that $\det(K) = \det(K')$. Write $K_0 = \det(K) = \det(K') \subset \mathbb{G}_m(\mathbb{A}_f)$ and denote by F_{K_0} the finite abelian extension of $\mathbb{Q}$ corresponding to K_0 via class field theory. Then there exists a compact open subgroup $L \in \mathrm{GSp}_4(\mathbb{A}_f)$ and a closed embedding, abusively denoted by ι, of the fibered product over F_{K_0} of the modular curves of levels K and K':

$$M_K \times_{F_{K_0}} M_{K'} \xrightarrow{\ \iota\ } S_L \tag{5}$$

into the Siegel threefold S_L. This fact is proved in [P1] 3.8 b).

Now, we would like to briefly explain the structure of the boundary of the Baily-Borel compactifications of these Shimura varieties and how the above morphism extends to the compactifications. The boundary of the Baily-Borel compactification is stratified by Shimura varieties associated to standard, i.e. containing a fixed Borel, admissible parabolic subgroups of the underlying reductive group G. Furthermore, roughly speaking, the closure of a Shimura variety of the boundary in the boundary is its own Baily-Borel compactification. The latter will play a crucial role in this work. The reader can find a detailed description of the construction of the Baily-Borel compactification of a general (pure) Shimura variety in [BuW] 1. For the symplectic groups of arbitrary rank, a very careful presentation is given in [M] 1.

Let us just recall the structure of the standard admissible parabolic subgroups from [P1] 4.25 in the cases of interest.

- Case $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$: denote by $B_2 \subset \mathrm{GL}_2$ the standard Borel. The standard admissible parabolic subgroups of G are $Q'_0 = B_2 \times_{\mathbb{G}_m} B_2$, $Q'_1 = B_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and $Q''_1 = \mathrm{GL}_2 \times_{\mathbb{G}_m} B_2$.

- Case $G = \mathrm{GSp}_4$: the standard admissible parabolic subgroups of G are just the standard maximal parabolics. We have the Siegel parabolic

$$Q_0 = W_0 \rtimes (\mathbb{G}_m \times \mathrm{GL}_2) = \left\{ \begin{pmatrix} \alpha A & AM \\ & {}^t A^{-1} \end{pmatrix}, \alpha \in \mathbb{G}_m, A \in \mathrm{GL}_2, {}^t M = M \right\}$$

and the Klingen parabolic

$$Q_1 = W_1 \rtimes (\mathrm{GL}_2 \times \mathbb{G}_m) = \left\{ \begin{pmatrix} \alpha & u & v & w \\ & a & w & b \\ & & \beta & \\ & c & -u & d \end{pmatrix}, \alpha\beta = ad - bc \in \mathbb{G}_m \right\}.$$

Note that the unipotent radicals W_0 and W_1 are of dimension 3 and that W_0 is abelian whereas W_1 is not. Denote by $M_{0,h}$, resp. $M_{0,l}$, the factor $\mathbb{G}_m$, resp. the factor GL_2 of the Levi subgroup of Q_0 and by $M_{1,h}$, resp. $M_{1,l}$, the factor GL_2, resp. the factor $\mathbb{G}_m$, of the Levi subgroup of Q_1. The subgroup $M_{0,h} = \mathbb{G}_m$ of Q_0 of matrices of the shape

$$\begin{pmatrix} \alpha I_2 & \\ & I_2 \end{pmatrix} \tag{6}$$

will underly the strata of dimension 0 in the boundary whereas the subgroup $M_{1,h} = \mathrm{GL}_2$ of Q_1 of matrices of the shape

$$\begin{pmatrix} ad-bc & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} \tag{7}$$

will underly the strata of dimension 1 in the boundary.

Let us come back to the closed embedding (5). Denote by j' the open embedding of $M_K \times_{F_{K_0}} M_{K'}$ in its Baily-Borel compactification and by i' the complementary reduced closed embedding of the boundary. We have the diagram

$$M_K \times_{F_{K_0}} M_{K'} \xrightarrow{j'} (M_K \times_{F_{K_0}} M_{K'})^* \xleftarrow{i'} \partial(M_K \times_{F_{K_0}} M_{K'}).$$

Furthermore denote by i'_1 the open embedding of the strata of dimension 1 of the boundary of $\partial(M_K \times_{F_{K_0}} M_{K'})$ and by i'_0 the complementary reduced closed embedding of the strata of dimension 0. We have the diagram

$$\partial(M_K \times_{F_{K_0}} M_{K'})_1 \xrightarrow{i'_1} \partial(M_K \times_{F_{K_0}} M_{K'}) \xleftarrow{i'_0} \partial(M_K \times_{F_{K_0}} M_{K'})_0.$$

Similarly denote by j the open embedding of S_L in its Baily-Borel compactification and by i the complementary reduced closed embedding of the boundary. We have the diagram

$$S_L \xrightarrow{j} S_L^* \xleftarrow{i} \partial S_L.$$

Finally denote by i_1 the open embedding of the strata of dimension 1 of the boundary of S_L and by i_0 the complementary reduced closed embedding of the strata of dimension 0. We have the diagram

$$\partial S_{L,1} \xrightarrow{i_1} \partial S_L \xleftarrow{i_0} \partial S_{L,0}.$$

The following result is a very particular case of the functoriality of the canonical models of the Baily-Borel compactifications of pure Shimura varieties ([P1] 12.3.b).

Proposition 3.1. *We have a commutative diagram with cartesian squares*

$$\begin{array}{ccccc} M_K \times_{F_{K_0}} M_{K'} & \xrightarrow{j'} & (M_K \times_{F_{K_0}} M_{K'})^* & \xleftarrow{i'} & \partial(M_K \times_{F_{K_0}} M_{K'}) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ S_L & \xrightarrow{j} & S_L^* & \xleftarrow{i} & \partial S_L \end{array} \tag{8}$$

and a commutative diagram with cartesian squares

$$
\begin{array}{ccccc}
\partial(M_K \times_{F_{K_0}} M_{K'})_1 & \xrightarrow{i'_1} & \partial(M_K \times_{F_{K_0}} M_{K'}) & \xleftarrow{i'_0} & \partial(M_K \times_{F_{K_0}} M_{K'})_0 \\
{\scriptstyle q_1}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle q_0}\downarrow \\
\partial S_{L,1} & \xrightarrow{i_1} & \partial S_L & \xleftarrow{i_0} & \partial S_{L,0}.
\end{array}
\tag{9}
$$

We will also need the following result in the proof of our main theorem.

LEMMA 3.2. *The morphism q_d appearing in the above commutative diagram is the composite of an étale cover and of the inclusion of some connected components.*

Proof. To prove this lemma, we need to go a bit in more details into the construction of the Baily-Borel compactifications. The proof is the same for q_0 and q_1 so we only write it for q_1. We follow [P2] 3.7. In what follows, we denote by G the reductive group GSp_4, resp. $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$, and by U the compact open subgroup L, resp. $K \times K'$, of G. Denote also by Q the parabolic subgroup of G which we denoted by Q_1, resp. Q'_1 or Q''_1, in the case $G = \mathrm{GSp}_4$, resp. in the case $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. Let P be the normal subgroup of Q defined in [P1] 4.7. Then Q and P have the same unipotent radical W. Let $g \in G(\mathbb{A}_f)$. Write $U_{g,P} = gUg^{-1} \cap P(\mathbb{A}_f)$, $U_{W,g} = gUg^{-1} \cap W(\mathbb{A}_f)$ and $U_g = U_{g,P}/U_{g,W}$. Then we have a morphism

$$M_{U_g}(G', \mathcal{H}') \xrightarrow{i_g} M_U(G, \mathcal{H})^*$$

where G' is the Levi quotient of Q, where $M_{U_g}(G', \mathcal{H}')$ is the Shimura variety of level U_g associated to G' and $M_U(G, \mathcal{H})^*$ is the Baily-Borel compactification of the Shimura variety $M_U(G, \mathcal{H})$ of level U associated to G (see [P2] 3.7 for the definition of i_g). Then i_g is the composite of a finite, étale (because we assume that U is neat) map and of a locally closed embedding. When g varies in $G(\mathbb{A}_f)$ the union of the images of the i_g is what we denoted by $\partial S_{L,1}$, resp. $\partial(M_K \times_{F_{K_0}} M_{K'})_1$, in the case $G = \mathrm{GSp}_4$, resp. $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$.

In the case $G = \mathrm{GSp}_4$, resp. $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ we have $P = W_1 \rtimes M_{1,h} = W_1 \rtimes \mathrm{GL}_2$, resp. $P = \mathbb{G}_a \rtimes \mathrm{GL}_2$. Denote the latter by P', and by π, resp. π', the projection on the Levi quotient $P \longrightarrow \mathrm{M}_{1,h} = \mathrm{GL}_2$, resp. $P' :\longrightarrow \mathrm{GL}_2$. Then, for any $g \in (\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)(\mathbb{A}_f)$, the morphism q_1 is induced by the composite of the change of level induced by the inclusion

$$\pi'(g(K \times K')g^{-1} \cap P'(\mathbb{A}_f)) \subset \pi(\iota(g)L\iota(g)^{-1} \cap P(\mathbb{A}_f))$$

which is finite and étale (by our neatness assumption) and of the inclusion of some connected components. This proves the statement. ☐

3.2. The maps in absolute Hodge cohomology. The main technical result of this work will be the vanishing of a certain map between some absolute Hodge cohomology spaces of the boundaries of the Baily-Borel compactifications of the Shimura varieties we are interested in. In this section we define the map we want to study and explain its connection with the Gysin morphism. In what follows, we denote by μ the $\mathbb{R}$-linear tensor functor associating to an algebraic representation of the group underlying a given Shimura variety the corresponding variation of Hodge structure on the considered Shimura variety (see [BuW] 2).

Let us first recall the construction of the Gysin morphism in the situation we are interested in.

LEMMA 3.3. *Let F be an algebraic representation of* $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ *and let E be an algebraic representation of* GSp_4 *such that $F \subset \iota^* E$. Then, with the notations of Prop. 3.1,*

(i) We have a natural map

$$\iota_* \mu(F(-1)) \longrightarrow \mu(E)[1]$$

in $D^b(\mathrm{MHM}_{\mathbb{R}}(S_L/\mathbb{R}))$,

(ii) Applying the absolute Hodge cohomology functor, we get the natural map

$$H^2_{\mathcal{H}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}, \mu(F(-1))) \longrightarrow H^4_{\mathcal{H}}(S_L/\mathbb{R}, \mu(E))$$

called the Gysin morphism.

Proof. (i) Taking the contragredient of the inclusion of representations $F \subset \iota^* E$ we get the morphism $\iota^* E^\vee \longrightarrow F^\vee$ where the superscript $^\vee$ denotes the contragredient representation. Twisting by the determinant character we obtain the morphism $\iota^* E^\vee(3) \longrightarrow F^\vee(3)$. Applying the functor μ we get the morphism of variation of Hodge structures

$$(\iota^s)^* \mu(E^\vee(3)) \longrightarrow \mu(F^\vee(3)).$$

Here the symbol $(\iota^s)^*$ denotes the pull-back of variation of Hodge structures, which should not be confused with the pull-back

$$\iota^* : D^b(\mathrm{MHM}_{\mathbb{R}}(S_L/\mathbb{R})) \longrightarrow D^b(\mathrm{MHM}_{\mathbb{R}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}))$$

given by Saito's formalism (Thm. 2.3). Actually, as ι is of codimension 1, we have $(\iota^s)^* \mu(E^\vee(3)) = \iota^* \mu(E^\vee(3))[-1]$ (see [Bl] Prop. 2.3.12). So we have the morphism

$$\iota^* \mu(E^\vee(3))[-1] \longrightarrow \mu(F^\vee(3))$$

in the derived category $D^b(\mathrm{MHM}_{\mathbb{R}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}))$. Applying the contravariant functor $\mathbb{D}$ to this morphism we obtain $\mu(F(-1)) \longrightarrow \iota^! \mu(E)[1]$. Here we used the fact that for any variation of Hodge structure L on a smooth $X \in Sch(\mathbb{Q})$ which is purely d dimensionnal the dual variation $L^\vee$ coincides with $\mathbb{D}(L)(-d)$ (see [Bl] Lem. 2.3.7). By adjunction, we obtain the morphism

$$\iota_! \mu(F(-1)) = \iota_* \mu(F(-1)) \longrightarrow \mu(E)[1]$$

as claimed in the statement of the lemma. The statement (ii) follows trivially from (i) by applying the functor $M \longmapsto \mathrm{Hom}_{D^b(\mathrm{MHM}_A(S_L/\mathbb{R}))}(1[3], M[3])$ where s is the structure morphism $S_L \longrightarrow \mathrm{Spec}\,\mathbb{Q}$. □

LEMMA 3.4. *Let F be an algebraic representation of* $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ *and let E be an algebraic representation of* GSp_4 *such that $F \subset \iota^* E$. Then, with the notations of Prop. 3.1,* *(i) We have a morphism*

$$i_*q_*i'^*j'_*\mu(F(-1)) \longrightarrow i_*i^*j_*\mu(E)[1]$$

in $D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_L/\mathbb{R}))$ *which is part of the following commutative diagram*

$$\begin{array}{ccc} p_*j'_*\mu(F(-1)) & \longrightarrow & i_*q_*i'^*j'_*\mu(F(-1)) \\ \downarrow & & \downarrow \\ j_*\mu(E)[1] & \longrightarrow & i_*i^*j_*\mu(E)[1]. \end{array}$$

(ii) Applying the absolute Hodge cohomology functor, we get the natural map

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^*j'_*\mu(F(-1)))) \longrightarrow H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^*j_*\mu(E))$$

which is part of the following commutative diagram

$$\begin{array}{ccc} H^2_{\mathcal{H}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}, \mu(F(-1))) & \longrightarrow & H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^*j'_*\mu(F(-1))) \\ \downarrow & & \downarrow \\ H^4_{\mathcal{H}}(S_L/\mathbb{R}, \mu(E)) & \longrightarrow & H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^*j_*\mu(E)) \end{array}$$

where the left hand vertical map is the Gysin morphism.

Proof. (i) The left hand vertical morphism of the commutative diagram in the statement is nothing but the morphism obtained by applyling the functor j_* to the morphism $\iota_*\mu(F(-1)) \longrightarrow \mu(E)[1]$ of Lem. 3.3 (i) and using functoriality. Now, applying the functor i_*i^* to this morphism $p_*j'_*\mu(F(-1)) \longrightarrow j_*\mu(E)[1]$ and using that $i^*p_* = q_*i'^*$ by the proper base change theorem (see [S3] 4.4.3) we obtain the morphism we looked for. The commutative diagram is obtained via the adjunction morphism $1 \longrightarrow i_*i^*$. (ii) As before this statement is deduced trivially from (i) by applying the absolute Hodge cohomology functor $M \longmapsto \mathrm{Hom}_{D^b(\mathrm{MHM}_A(S_L^*/\mathbb{R}))}(1[3], M[3])$ where s is the structure morphism $S_L^* \longrightarrow \mathrm{Spec}\,\mathbb{Q}$. □

LEMMA 3.5. *Let F be an algebraic representation of* $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ *and let E be an algebraic representation of* GSp_4 *such that $F \subset \iota^*E$. Then, with the notations of Prop. 3.1, for any $d = 0, 1$,*

(i) We have a morphism

$$i_{d\,*}q_{d\,*}i'^*_d i'^*j'_*\mu(F(-1)) \longrightarrow i_{d\,*}i^*_d i^*j_*\mu(E)[1]$$

in $D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_L/\mathbb{R}))$ *which is part of the following commutative diagram*

$$\begin{array}{ccc} i_*q_*i'^*j'_*\mu(F(-1)) & \longrightarrow & i_{d\,*}q_{d\,*}i'^*_d i'^*j'_*\mu(F(-1)) \\ \downarrow & & \downarrow \\ i_*i^*j_*\mu(E)[1] & \longrightarrow & i_{d\,*}i^*_d i^*j_*\mu(E)[1]. \end{array}$$

(ii) Applying the absolute Hodge cohomology functor, we get the natural map

$$H^{1-(1-d)}_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_d/\mathbb{R}, i'^*_d i'^*j'_*\mu(F(-1)))) \longrightarrow H^{2-(1-d)}_{\mathcal{H}}(\partial S_{L,d}/\mathbb{R}, i^*_d i^*j_*\mu(E))$$

which is part of the following commutative diagram

$$\begin{array}{ccc} H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^* j'_* \mu(F(-1))) & \longrightarrow & H^{1-(1-d)}_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_d/\mathbb{R}, i'^*_d i'^* j'_* \mu(F(-1)))) \\ \downarrow & & \downarrow \\ H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* \mu(E) & \longrightarrow & H^{2-(1-d)}_{\mathcal{H}}(\partial S_{L,d}/\mathbb{R}, i^*_d i^* j_* \mu(E)) \end{array}$$

where the left hand vertical map if the right hand vertical map of the commutative diagram (ii) in the previous lemma.

Proof. This morphisms and these commutative diagrams are obtained as above applying the functor $i_{d*}i_d^*$ to the right hand vertical morphism obtained in the first diagram of the previous lemma, using the adjunction $1 \longrightarrow i_{d*}i_d^*$ and applying the absolute Hodge cohomology functor. □

4. Computation of higher direct images in the Baily-Borel compactifications. The results of this section are almost direct applications of the main result of [BuW], which is the Hodge theoretic version of [P2], and of a theorem of Kostant. Let us first recall the statements of these two theorems.

4.1. The theorems of Burgos-Wildeshaus and Kostant. Let us consider the Baily-Borel compactification j of a finite level Shimura variety $M(G, \mathcal{H})_K$, with underlying reductive group G and i the embedding of a boundary stratum $M(G_1, \mathcal{H})_K$ in the compactification. We have the following diagram

$$M(G, \mathcal{H})_K \xrightarrow{\ j\ } M(G, \mathcal{H})^*_K \xleftarrow{\ i\ } M(G_1, \mathcal{H})_K.$$

Here $M(G_1, \mathcal{H})_K$ has underlying reductive group G_1, a subgroup of the Levi M of a given admissible standard parabolic subgroup Q of G. Write N for the unipotent radical of Q. Following [BuW], we denote by μ the tensor functor associating to an algebraic representation of G, resp. G_1, the corresponding variation of Hodge structure on $M(G, \mathcal{H})_K$, resp. $M(G_1, \mathcal{H})_K$. Finally, let c be the codimension of $M(G_1, \mathcal{H})_K$ in $M(G, \mathcal{H})^*_K$.

THEOREM 4.1. [BuW] *Th. 2.6, 2.9. Let E be an algebraic representation of G. In the derived category $D^b(\mathrm{MHM}_{\mathbb{R}}(M(G_1, \mathcal{H})_K/\mathbb{R}))$ we have*

$$i^* j_* \mu(E) = \bigoplus_n \mathcal{H}^n i^* j_* \mu(E)[-n].$$

There exists a neat arithmetic subgroup $\overline{H}_C$ of $M/G_1(\mathbb{Q})$ such that

$$\mathcal{H}^n i^* j_* \mu(E) = \bigoplus_{p+q=n+c} \mu\left(H^p(\overline{H}_C, H^q(N, E))\right).$$

We have an isomorphism of variations of Hodge structure

$$Gr^W_k \mathcal{H}^n i^* j_* \mu(E) = \bigoplus_{p+q=n+c} \mu\left(H^p(\overline{H}_C, Gr^W_k H^q(N, E))\right).$$

REMARKS 4.2. (i) The Levi subgroup M of G_1 acts on the cohomology $H^q(N, E)$ via its action both on N and on E and so it acts on the group cohomology $H^p(\overline{H}_C, H^q(N, E))$. Then this last space is considered in the second statement as a representation of G_1 via the inclusion $G_1 \subset M$.

(ii) In the third statement, the Gr_k^W on the left denotes the k-th graded piece of the weight filtration of the variation of Hodge structure $\mathcal{H}^n i^* j_* \mu(E)$ and the Gr_k^W on the right denotes the k-th graded piece of the weight filtration on $H^q(N, E)$ coming from the fact that this space is endowed with an action of the group G_1 which underlies a Shimura datum (see [P1] Prop. 1.4 for details).

The following theorem of Kostant identifies the space $H^q(N, E)$ as a representation of M. In the cases we are interested in, the group G_1 is a direct factor of M, thus this theorem allows to identify $H^q(N, E)$ as a representation of G_1. This makes the weights occurring in the $\mathcal{H}^n i^* j_* \mu(E)$ explicitly computable.

Let us now introduce some notation necessary to state Kostant's theorem. Recall that for any unipotent group N and any representation E of N we have $H^q(N, E) = H^q(Lie\, N, E)$ where the right hand term denotes the Lie algebra cohomology. Write $\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{m}$ where $\mathfrak{q} = Lie\, Q$, $\mathfrak{n} = Lie\, N$ and $\mathfrak{m} = Lie\, M$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g} = Lie\, G$ corresponding to the fixed Borel and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ the set of positive roots. The set $\Delta(\mathfrak{n}, \mathfrak{h})$ of roots appearing in $\mathfrak{n}$ is contained in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Denote by ρ the half-sum of positive roots, by $W(\mathfrak{g}, \mathfrak{h})$ the Weyl group. For any $w \in W(\mathfrak{g}, \mathfrak{h})$ write

$$\Delta^+(w) = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \,|\, w^{-1}\alpha \notin \Delta^+(\mathfrak{g}, \mathfrak{h})\},$$

$$l(w) = |\Delta^+(w)|,$$

$$W' = \{w \in W(\mathfrak{g}, \mathfrak{h}) \,|\, \Delta^+(w) \subset \Delta(\mathfrak{n}, \mathfrak{h})\}.$$

THEOREM 4.3. *[V] Th. 3.2.3. Let E_λ be an irreducible representation of $\mathfrak{g}$ of highest weight λ. Then*

$$H^q(\mathfrak{n}, E_\lambda) \simeq \bigoplus_{\{w \in W' \,|\, l(w) = q\}} F_{w(\lambda+\rho)-\rho}$$

where F_μ is an irreducible representation of $\mathfrak{m}$ of highest weight μ.

Let us now come back to the diagrams of Prop. 3.1 and perform the computations to make the results of the two theorems above explicit.

4.2. The case of $\mathbf{GL_2} \times_{\mathbb{G}_\mathbf{m}} \mathbf{GL_2}$. We first consider the case of a single modular curve. The following result is well known and follows, for example, from a trivial application of Thm. 4.1 and Thm. 4.3.

LEMMA 4.4. *Consider the embedding j'' of a modular curve of level $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ into its Baily-Borel compactification and let i'' the complementary reduced closed embedding. So we have the diagram*

$$M_K \xrightarrow{\ j''\ } M_K^* \xleftarrow{\ i''\ } \partial M_K.$$

Then as variations of mixed Hodge structure on ∂M_K we have

$$\mathcal{H}^{-1} i''^* j''_* \mu\left(Sym^d V_2(t)\right) = 1(d+t),$$
$$\mathcal{H}^{0} i''^* j''_* \mu\left(Sym^d V_2(t)\right) = 1(t-1)$$

and $\mathcal{H}^n i''^* j''_* \mu\left(Sym^d V_2(t)\right) = 0$ for $n > 0$.

Let us consider now the algebraic irreducible representation $F = (Sym^pV_2 \boxtimes Sym^qV_2)(t)$ of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. Note that any irreducible algebraic representation of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ is isomorphic to such a representation, for suitable p, q and t. We will first identify the variations of Hodge structure $\mathcal{H}^2 i_0^* i_0'^* j'_* \mu(F)$ on the strata of dimension 0 of the boundary.

LEMMA 4.5. *Let F be the irreducible algebraic representation $(Sym^pV_2 \boxtimes Sym^qV_2)(t)$ of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. Then, as variation of mixed Hodge structures on $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ we have*

$$\begin{aligned} \mathcal{H}^{-2} i_0'^* i'_* j'_* \mu(F) &= 1(p+q+t), \\ \mathcal{H}^{-1} i_0'^* i'_* j'_* \mu(F) &= 1(p+t-1) \oplus 1(q+t-1), \\ \mathcal{H}^{0} i_0'^* i'_* j'_* \mu(F) &= 1\,(t-2) \end{aligned}$$

and $\mathcal{H}^n i_0'^* i'_* j'_* \mu(F) = 0$ for $n > 0$.

Proof. The standard admissible parabolic subgroup of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ corresponding to $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ is $Q'_0 = B_2 \times_{\mathbb{G}_m} B_2$, whose unipotent radical is simply the direct sum $\mathbb{G}_a \oplus \mathbb{G}_a$ of two copies of the additive group and the Levi M is the diagonal maximal torus $\mathbb{G}_m^3$. The reductive group underlying the Shimura variety $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ is

$$G_1 = \mathbb{G}_m = \left\{ \left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}, \begin{pmatrix} x & \\ & 1 \end{pmatrix} \right), x \in \mathbb{G}_m \right\}.$$

With the notations of Thm. 4.1, the group $\overline{H}_C$ is a neat arithmetic subgroup of $M/G_1(\mathbb{Q}) = \mathbb{G}_m^2(\mathbb{Q})$ thus is trivial. As a consequence, the second statement of Thm. 4.1 implies that

$$\begin{aligned} \mathcal{H}^{-2} i_0'^* i'_* j'_* \mu(F) &= \mu\left(H^0(\mathbb{G}_a \oplus \mathbb{G}_a, F)\right) \\ \mathcal{H}^{-1} i_0'^* i'_* j'_* \mu(F) &= \mu\left(H^1(\mathbb{G}_a \oplus \mathbb{G}_a, F)\right) \\ \mathcal{H}^{0} i_0^* i_0'^* j'_* \mu(F) &= \mu\left(H^2(\mathbb{G}_a \oplus \mathbb{G}_a, F)\right). \end{aligned}$$

Let us write the representation F as $(Sym^pV_2) \boxtimes (Sym^qV_2\,(t))$. Note that because the unipotent group $\mathbb{G}_a$ is of dimension 1, the space $H^2(\mathbb{G}_a, F)$ vanishes. Then, the Künneth formula [BoW] I.1 shows that

$$\begin{aligned} H^0(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= H^0(\mathbb{G}_a, Sym^pV_2) \otimes H^0(\mathbb{G}_a, Sym^qV_2\,(t)), \\ H^1(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= H^0(\mathbb{G}_a, Sym^pV_2) \otimes H^1(\mathbb{G}_a, Sym^qV_2\,(t)) \\ &\quad \oplus H^1(\mathbb{G}_a, Sym^pV_2) \otimes H^0(\mathbb{G}_a, Sym^qV_2\,(t)), \\ H^2(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= H^1(\mathbb{G}_a, Sym^pV_2) \otimes H^1(\mathbb{G}_a, Sym^qV_2\,(t)). \end{aligned}$$

Now either a trivial application of Thm. 4.3 or a direct computation shows that the cohomology space $H^0(\mathbb{G}_a, Sym^kV_2(t))$ is a one dimensional vector space on which G_1 acts via $x \mapsto x^{-(k+t)}$ and that $H^1(\mathbb{G}_a, Sym^kV_2\,(t))$ is a one dimensional vector space on which G_1 acts via $x \longmapsto x^{t-1}$. Thus, as mixed Hodge structures, we have

$$\begin{aligned} H^0(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= 1\,(p+q+t)\,, \\ H^1(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= 1\,(p+t-1) \oplus 1\,(q+t-1)\,, \\ H^2(\mathbb{G}_a \oplus \mathbb{G}_a, F) &= 1\,(t-2)\,. \qquad \square \end{aligned}$$

Let us turn our attention to what happens on the strata $\partial(M_K \times_{F_{K_0}} M_{K'})_1$ of dimension 1 of the boundary. Denote by ∂M_K, resp. $\partial M_{K'}$ the boundary of the Baily-Borel compactification of M_K, resp. $M_{K'}$. These are simply a finite disjoint union of cusps. Then the $\mathbb{Q}$-scheme $\partial(M_K \times_{F_{K_0}} M_{K'})_1$ is the disjoint union of $\partial M_K \times_{F_{K_0}} M_{K'} \cup M_K \times_{F_{K_0}} \partial M_{K'}$ where the left, resp. right, hand side corresponds to the standard admissible parabolic Q_1', resp. Q_1''. As a consequence, a variation of Hodge structure on $\partial(M_K \times_{F_{K_0}} M_{K'})_1$ is a pair (V_1, V_2) where V_1 is a variation of Hodge structure on $\partial M_K \times_{F_{K_0}} M_{K'}$ and V_2 is a variation of Hodge structure on $M_K \times_{F_{K_0}} \partial M_{K'}$. In the following, we will denote such a pair by the symbol $V_1 \boxplus V_2$.

Lemma 4.6. *Let F be the irreducible algebraic representation $(Sym^p V_2 \boxtimes Sym^q V_2)(t)$ of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. Then, as variation of Hodge structure on $\partial(M_K \times_{F_{K_0}} M_{K'})_1$ we have*

$$\mathcal{H}^{-1} i_1'^* i'^* j'_* \mu(F) = \mu\left(Sym^q V_2\,(p+t)\right) \boxplus \mu\left(Sym^p V_2\,(q+t)\right),$$
$$\mathcal{H}^{0} i_1'^* i'^* j'_* \mu(F) = \mu\left(Sym^q V_2\,(t-1)\right) \boxplus \mu\left(Sym^p V_2\,(t-1)\right)$$

and $\mathcal{H}^n i_1'^* i'^* j'_* \mu(F) = 0$ for $n > 2$.

Proof. Let us consider the case of $\partial M_K \times_{F_{K_0}} M_{K'} \subset \partial(M_K \times_{F_{K_0}} M_{K'})_1$. As we said, the corresponding standard admissible parabolic subgroup of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ is the group $Q_1' = B_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$, whose unipotent radical is simply $\mathbb{G}_a$ and whose Levi is $M = T_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ where T_2 is the standard diagonal maximal torus of GL_2. The reductive group G_1 underlying the Shimura variety $\partial M_K \times_{F_{K_0}} M_{K'}$ is $\mathbb{G}_m \times_{\mathbb{G}_m} \mathrm{GL}_2 = \mathrm{GL}_2$ which is regarded as a subgroup of Q_1' by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{pmatrix} ad-bc & \\ & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

The group $\overline{H}_C$ is a neat arithmetic subgroup of $M/G_1(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q})$, thus is trivial. As a consequence, the second statement of Thm. 4.1 implies that

$$\mathcal{H}^{n-1} i_1'^* i'^* j'_* \mu(F) = \mu\left(H^n(\mathbb{G}_a, F)\right).$$

Here again, the Künneth formula and a trivial application of Thm. 4.3 shows that as representations of G_1 we have

$$\begin{aligned} H^0(\mathbb{G}_a, F) &= H^0(\mathbb{G}_a, Sym^p V_2) \otimes H^0\left(0, Sym^q V_2\,(t)\right) \\ &= det^{\otimes p} \otimes Sym^q V_2\,(t) \\ &= Sym^q V_2\,(p+t). \end{aligned}$$

Similarly we have

$$\begin{aligned} H^1(\mathbb{G}_a, F) &= H^1(\mathbb{G}_a, Sym^p V_2) \otimes H^0\left(0, Sym^q V_2\,(t)\right) \\ &= det^{\otimes -1} \otimes Sym^q V_2\,(t) \\ &= Sym^q V_2\,(t-1). \end{aligned}$$

The conclusion follows by interchanging p and q. □

Remark 4.7. Note that the computation of the action of $\mathbb{G}_m$ on the group cohomology $H^n(\mathbb{G}_a, Sym^d V_2(t))$ in the proofs of the previous lemmas coincides with the ones that can be found in [Ha2] (2.3.4).

4.3. The case of GSp_4. In this subsection, we fix an irreducible representation E of GSp_4 of highest weight $\lambda(k, k', c)$ (see section 2.1 for the definition of $\lambda(k, k', c)$). The reader may find helpful to draw a picture of the C_2 root system in order to follow the computations of the action of the Weyl group on the weights. Let us first compute the weights that may occur in the strata of dimension 0 of the boundary, i.e. the ones corresponding to the Siegel parabolic Q_0, in the degrees that are of interest to us.

LEMMA 4.8. *Let E be an irreducible algebraic representation of* GSp_4 *of highest weight $\lambda(k, k', c)$. Assume that $\lambda(k, k', c)$ is regular, i.e. that $k > k' > 0$, then*

(ii) The variation of Hodge structure $\mathcal{H}^0 i_0^* i^* j_* \mu(E)$ is pure of weight $-(c - (k - k' + 4))$,

(iii) The variation of Hodge structure $\mathcal{H}^1 i_0^* i^* j_* \mu(E)$ is pure of weight $-(c - (k + k' + 6))$.

Proof. Let $\mathfrak{h} = Lie\, T$, $\mathfrak{q}_0 = Lie\, Q_0 = \mathfrak{w}_0 \oplus \mathfrak{m}_0$ where $\mathfrak{w}_0 = Lie\, W_0$ and $\mathfrak{m}_0 = Lie\, M_0$ and M_0 is the Levi subgroup of Q_0. We use the notations of Thm. 4.3. As the unipotent radical W_0 is abelian, the set $\Delta(\mathfrak{w}_0, \mathfrak{h})$ contains two long roots. As a consequence we have $\Delta(\mathfrak{w}_0, \mathfrak{h}) = \{\rho_2, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}$. Recall that here we denote by $\rho_1 = \lambda(1, -1, 0)$ the short simple root and $\rho_2 = \lambda(0, 2, 0)$ the long simple root. Denote by s_ρ the reflection whose axis is orthogonal to the root ρ. We see that

$$W' = \{w_0, w_1, w_2, w_3\}$$

where

$$\begin{aligned} w_0 &= Id, \\ w_1 &= s_{\rho_2}, \\ w_2 &= s_{\rho_1+\rho_2} s_{\rho_2}, \\ w_3 &= s_{\rho_1+\rho_2}. \end{aligned}$$

The length is given by $l(w_i) = i$. We have

$$\begin{aligned} w_2.\lambda(k, k', c) &= \lambda(k', -k, c), \\ w_3.\lambda(k, k', c) &= \lambda(-k', -k, c) \end{aligned}$$

hence

$$\begin{aligned} w_2.(\lambda(k, k', c) + \rho) - \rho &= \lambda(k' - 1, -k - 3, c), \\ w_3.(\lambda(k, k', c) + \rho) - \rho &= \lambda(-k' - 3, -k - 3, c). \end{aligned}$$

Recall that the weight $\lambda(k, k', c)$ is defined by

$$\lambda(k, k', c) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \longmapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{c-k-k'}{2}}$$

and that the group $M_{0,h} = \mathbb{G}_m$ underlying the Shimura varieties of dimension 0 in the boundary is regarded as a subgroup of GSp_4 via

$$\alpha \longmapsto \begin{pmatrix} \alpha I_2 & \\ & I_2 \end{pmatrix}.$$

Thus, the restriction of $\lambda(k,k',c)$ to $M_{0,h} = \mathbb{G}_m$ is $\alpha \longmapsto \alpha^k \alpha^{k'} \alpha^{\frac{c-k-k'}{2}} = \alpha^{\frac{c+k+k'}{2}}$. Then, Thm. 4.3 shows that $M_{0,h}$ acts on $H^2(W_0,E)$ via $\alpha \mapsto \alpha^{\frac{c-(k-k'+4)}{2}}$, and on $H^3(W_0,E)$ via $\alpha \mapsto \alpha^{\frac{c-(k+k'+6)}{2}}$. According to our convention explained at the beginning of section 3.1 and the definition of the Shimura datum $(\mathbb{G}_m, \mathcal{H}_0)$ in section 3.1, the mixed Hodge structure $H^2(W_0,E)$ is pure of weight $-(c-(k-k'+4)$ and the mixed Hodge structure $H^3(W_0,E)$ is pure of weight $-(c-(k+k'+6))$. The group $\overline{H}_C$ occurring in the statement of Thm. 4.1 is a neat arithmetic subgroup of $M_{0,l}(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$ and the spaces $H^n(W_0,E)$ underly irreducible representations of $M_{0,l} = \mathrm{GL}_2$ whose highest weight is regular because of our assumption on the regularity of $\lambda(k,k',c)$ (this can be easily checked by hand in this specific case and [Sch2] Lem. 4.9 proves this fact in great generality). As the Poincaré upper half-plane is contractible the group cohomology $H^0(\overline{H}_C, H^n(W_0,E))$ coincides with the 0-th Betti cohomology of the modular curve of level $\overline{H}_C$ with values in the local system associated to $H^n(W_0,E)$, hence is zero according to Thm. 5.1. Similarly, we have $H^2(\overline{H}_C, H^n(W_0,E)) = 0$ thanks to Cor. 5.3. Furthermore as W_0 has dimension 3, we have $H^n(W_0,E) = 0$ for $n > 3$. As a consequence, the second statement of Thm. 4.1 shows that

$$\mathcal{H}^0 i_0^* i^* j_* \mu(E) = \mu(H^1(\overline{H}_C, H^2(W_0,E))),$$
$$\mathcal{H}^1 i_0^* i^* j_* \mu(E) = \mu(H^1(\overline{H}_C, H^3(W_0,E)))$$

and the lemma is proved. □

Remark 4.9. Actually, we can say a bit more: as the variation of Hodge structures $\mathcal{H}^0 i_0^* i^* j_* \mu(E)$ and $\mathcal{H}^1 i_0^* i^* j_* \mu(E)$ are associated via μ to an algebraic representation of the group $\mathbb{G}_m$ underlying the Shimura variety $\partial S_{L,0}$, we know that these variations are direct sums of Tate variations $1(n)$ of weights $-2n$ given by the lemma.

In the next lemma, we identify the variations of Hodge structure $\mathcal{H}^n i_1^* i^* j_* \mu(E)$ over the strata of dimension 1 of ∂S_L. In the proof of the lemma we denote by $\lambda(d,c)$ the algebraic character of the diagonal maximal torus of $M_{1,h} = \mathrm{GL}_2$ sending $\mathrm{diag}(\alpha_1, \alpha_1^{-1}\nu)$ to $\alpha_1^d \nu^{\frac{c-d}{2}}$ for all pairs of integers (d,c) such that $d \equiv c \,(\mathrm{mod}\, 2)$. When $d \geq 0$, the character $\lambda(d,c)$ is the highest weight of the irreducible representation $Sym^d V_2\left(\frac{c-d}{2}\right)$ of $M_{1,h}$.

Lemma 4.10. *As variations of Hodge structure on $\partial S_{L,1}$ we have*

$$\mathcal{H}^{-2} i_1^* i^* j_* \mu(E) = \mu\left(Sym^{k'} V_2\left(\frac{c+k-k'}{2}\right)\right),$$
$$\mathcal{H}^{-1} i_1^* i^* j_* \mu(E) = \mu\left(Sym^{k+1} V_2\left(\frac{c+k'-k-2}{2}\right)\right),$$
$$\mathcal{H}^{0} i_1^* i^* j_* \mu(E) = \mu\left(Sym^{k+1} V_2\left(\frac{c-k-k'-4}{2}\right)\right),$$
$$\mathcal{H}^{1} i_1^* i^* j_* \mu(E) = \mu\left(Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)\right)$$

and $\mathcal{H}^n i_1^* i^* j_* \mu(E) = 0$ for $n > 3$.

Proof. Let $\mathfrak{q}_1 = Lie\, Q_1 = \mathfrak{w}_1 \oplus \mathfrak{m}_1$ where $\mathfrak{w}_1 = Lie\, W_1$ and $\mathfrak{m}_1 = Lie\, M_1$ and M_1 is the Levi subgroup of Q_1. We use the notations of Thm. 4.3. In this case, the unipotent radical W_1 is not abelian so $\Delta(\mathfrak{w}_1, \mathfrak{h}) = \{\rho_1, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}$. Denote again by s_ρ the reflection whose axis is orthogonal to the root ρ. We see that

$$W' = \{w_0, w_1, w_2, w_3\}$$

where

$$\begin{aligned} w_0 &= Id, \\ w_1 &= s_{\rho_1}, \\ w_2 &= s_{2\rho_1+\rho_2} s_{\rho_1}, \\ w_3 &= s_{2\rho_1+\rho_2}. \end{aligned}$$

The length is given by $l(w_i) = i$. We have

$$\begin{aligned} w_1.\lambda(k, k', c) &= \lambda(k', k, c), \\ w_2.\lambda(k, k', c) &= \lambda(-k', k, c), \\ w_3.\lambda(k, k', c) &= \lambda(-k, k', c) \end{aligned}$$

thus

$$\begin{aligned} w_1.(\lambda(k, k', c) + \rho) - \rho &= \lambda(k' - 1, k + 1, c), \\ w_2.(\lambda(k, k', c) + \rho) - \rho &= \lambda(-k' - 3, k + 1, c), \\ w_3.(\lambda(k, k', c) + \rho) - \rho &= \lambda(-k - 4, k', c). \end{aligned}$$

Recall that the weight $\lambda(k, k', c)$ is defined by

$$\lambda(k, k', c) : \operatorname{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \longmapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{c-k-k'}{2}}$$

and that the group $M_{1,h} = \mathrm{GL}_2$ underlying the Shimura varieties of dimension 1 in the boundary is regarded as a subgroup of GSp_4 via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} ad - bc & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix}.$$

As a consequence, the restriction of $\lambda(k, k', c)$ to the diagonal maximal torus of $M_{1,h} = \mathrm{GL}_2$ is given by

$$\operatorname{diag}(\alpha, \alpha^{-1}\mu) \longrightarrow \mu^k \alpha^{k'} \mu^{\frac{c-k-k'}{2}} = \alpha^{k'} \mu^{\frac{c+k-k'}{2}}.$$

This is by definition the weight $\lambda(k', c + k)$ which is the highest weight of the irreducible representation $Sym^{k'} V_2 \left(\frac{c+k-k'}{2}\right)$. Note that Thm. 4.3 describes the cohomology $H^n(W_1, E)$ as a representation of $M_{1,l} \times M_{1,h}$. In this case $M_{1,l}$ is the multiplicative group $\mathbb{G}_m$, whose irreducible algebraic representations are just characters. Furthermore an irreducible representation of $M_{1,l} \times M_{1,h}$ is a tensor product of an irreducible representation of $M_{1,l}$ and of an irreducible representation of $M_{1,h}$. As a consequence according to Thm. 4.3, we have

$$H^0(W_1, E) = Sym^{k'} V_2 \left(\frac{c+k-k'}{2} \right),$$
$$H^1(W_1, E) = Sym^{k+1} V_2 \left(\frac{c+k'-k-2}{2} \right),$$
$$H^2(W_1, E) = Sym^{k+1} V_2 \left(\frac{c-k-k'-4}{2} \right),$$
$$H^3(W_1, E) = Sym^{k'} V_2 \left(\frac{c-k-k'-4}{2} \right)$$

as representations of $M_{1,h} = \mathrm{GL}_2$. For dimension reasons, we have $H^n(W_1, E)$ for $n > 3$. Finally, as the group $\overline{H}_C$ is a neat arithmetic subgroup of $M_{1,l}(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q})$, it is trivial. So now the proof follows from the second statement of Thm. 4.1. □

REMARK 4.11. Note that the computations given in the proofs of Lem. 4.8 and 4.10 coincide with the ones given in [Sch1] Table 2.3.3.

5. Interior and boundary cohomologies. Let p, q be two non-negative integers. Choose $k \geq k' \geq 0$ two integers satisfying the following conditions:

- $k + k' \equiv p + q \pmod 2$,
- If $0 \leq p < k'$ and $p < k - k'$ then $k - k' - p \leq q \leq k - k' + p$,
- If $0 \leq p < k'$ and $k - k' \leq p$ then $p - k + k' \leq q \leq p + k - k'$,
- If $k' \leq p \leq k$ and $k' < k - p$ then $k - k' - p \leq q \leq k + k' - p$,
- If $k' \leq p \leq k$ and $k - p \leq k'$ then $p - k + k' \leq q \leq k + k' - p$.

According to Cor. 2.2, the conditions imply the following: write $c = p + q + 6$ and denote for the rest of the paper W an irreducible algebraic representation of GSp_4 of highest weight $\lambda(k, k', c)$ (see section 2.1 for a definition). Then we have

$$(Sym^p V_2 \boxtimes Sym^q V_2)(3) \subset \iota^* W.$$

In this section and the following we will need several times the next result in the case where G is GSp_4 or GL_2:

THEOREM 5.1. *[Sa1] Thm. 5. Let G be a reductive algebraic group defined over $\mathbb{Q}$ and let D its associated symmetric space, i.e. $D = G(\mathbb{R})/K_\infty A_G$ where K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ and A_G is the identity component of the $\mathbb{R}$-valued points of a maximal $\mathbb{Q}$-split torus in the center of G. Let $\Gamma \subset G(\mathbb{Q})$ be a neat arithmetic subgroup, let $X = \Gamma \backslash D$ and let E be the local system on X associated to an irreducible algebraic representation of G.*

If D is a Hermitian or equal-rank symmetric space and E is of regular highest weight, then $H^i(X, E) = 0$ for all $i < \frac{1}{2} \dim X$.

REMARKS 5.2. (i) This result is announced without a detailed proof in [Sa1] and proved in the preprint [Sa2]. However, in the case of $G = \mathrm{GSp}_4$ a published proof can be found in [TU] Cor. A.1. In the case of GL_2, one can argue as follows, as Bruno Klingler explained to me: according to Borel density theorem [Bo], any arithmetic subgroup Γ of SL_2 is Zariski dense. Now let $v \in H^0(\Gamma, E)$ where E is a non-trivial irreducible algebraic representation of SL_2. Then the stabilizer of v in SL_2 is a closed algebraic subgroup which contains Γ. Hence v is stable under SL_2 which implies that $v = 0$.

(ii) Recall that the weight $\lambda(d,c)$, resp. $\lambda(k,k',c)$, of the diagonal maximal torus of GL_2, resp. GSp_4 is said regular if $d>0$, resp. if $k>k'>0$. In other words, the irreducible algebraic representation $Sym^d V_2(t)$ has regular highest weight if and only if $d>0$. The weights $\lambda(d,c)$, resp. $\lambda(k,k',c)$ have been defined just above the statement of Lem. 4.10, resp. section 2.1.

Corollary 5.3. *Let K, resp. L, be a neat arithmetic subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, resp. $\mathrm{GSp}_4(\mathbb{A}_f)$, and let M_K, resp. S_L, be the modular curve of level K, resp. the Siegel threefold of level L. Let k be a positive integer, let t be an arbitrary integer and let E be an irreducible algebraic representation of GSp_4 whose highest weight is regular.*

Then, the spaces $H_c^2(M_K, Sym^k V_2(t))$, $H_c^0(M_K, Sym^k V_2(t))$ and $H^2(M_K, Sym^k V_2(t))$ are zero and the space $H_c^i(S_L, E)$ is zero for any $i>3$.

Proof. As the arguments are exactly the same for the vanishing of $H_c^2(M_K, Sym^k V_2(t))$ and $H_c^i(S_L,E)$, we only write down a proof of the latter. Let $i>3$. The fact that L is neat implies that S_L is smooth. So, by Poincaré duality we know that the vector space $H_c^i(S_L,E)$ is dual to $H^{6-i}(S_L,E^\vee)$, where $E^\vee$ denotes the contragredient of E. Let $\lambda(k,k',c)$ be the highest weight of E. We have $k>k'>0$ by assumption. Then $E^\vee$ has highest weight $\lambda(k,k',-c)$, which is regular. As $i>3$, Thm. [Sa1] implies that $H^{6-i}(S_L,E^\vee)=0$, hence $H_c^i(S_L,E)=0$. Let us show that $H_c^0(M_K, Sym^k V_2(t))=0$. Let j'' the Baily-Borel compactification of M_K and i'' the complementary reduced closed embedding. We have an exact sequence

$$H^{-2}(\partial M_K, i''^* j''_* Sym^k V_2(t)) \longrightarrow H_c^0(M_K, Sym^k V_2(t)) \longrightarrow H^0(M_K, Sym^k V_2(t))$$

according to (4). Lem. 4.4 shows that the complex $i''^* j''_* Sym^k V_2(t)$ is concentrated in degrees -1 and 0 so that $H^{-2}(\partial M_K, i''^* j''_* Sym^k V_2(t))=0$. Hence $H_c^0(M_K, Sym^k V_2(t))$ is a subspace of $H^0(M_K, Sym^k V_2(t))$. As a consequence the vanishing of $H_c^0(M_K, Sym^k V_2(t))$ is implied by Thm. 5.1. To show that $H^2(M_K, Sym^k V_2(t))=0$ one uses Poincaré duality and the vanishing of $H_c^0(M_K, Sym^k V_2(t))$ as explained in the first part of the proof. □

The following lemma will be useful to us.

Lemma 5.4. *Let $K\subset \mathrm{GL}_2(\mathbb{A}_f)$ be a compact open subgroup. Then the mixed Hodge structure $H^1(M_K, Sym^k V_2(t))$ has weight zero if and only if $t=1$ or $k+2t=1$.*

Proof. Let $H_!^1(M_K, Sym^k V_2(t)) = \mathrm{Im}(H_c^1(M_K, Sym^k V_2(t)) \longrightarrow H^1(M_K, Sym^k V_2(t))$. The variation of Hodge structure $Sym^k V_2(t)$ is pure of weight $-k-2t$ (see section 3.1) so the mixed Hodge structure $H_!^1(M_K, Sym^k V_2(t))$ is pure of weight $-k-2t+1$. Let j'' the Baily-Borel compactification of M_K and i'' the complementary reduced closed embedding. The long exact sequence (4) gives rise to the exact sequence of mixed Hodge structures

$$0 \to H_!^1(M_K, Sym^k V_2(t)) \to H^1(M_K, Sym^k V_2(t)) \to H^0(\partial M_K, i''^* j''_* Sym^k V_2(t)).$$

As ∂M_K is of dimension zero we have

$$H^0(\partial M_K, i''^* j''_* Sym^k V_2(t)) = H^0(\partial M_K, \mathcal{H}^0 i''^* j''_* Sym^k V_2(t))$$

which equals $H^0(\partial M_K, 1(t-1))$ according to Lem. 4.4. This is a pure Hodge structure of weight $-2(t-1)$, so the conclusion follows. □

Denote by $H^3_!(S,W)$ the interior cohomology, which is the image of the cohomology with compact support $H^3_c(S,W)$ in the cohomology $H^3(S,W)$. As the central character of W is c, the variation of Hodge structure on S associated to W is of weight $-c$ according to Pink's convention recalled at the beginning of section 3.1. So $H^3_!(S,W)$ is a pure Hodge structure of weight $-c+3=-p-q-3\leq -3$ and the "motive" $H^3_!(S,W)$ is subject to Beilinson's conjecture as sketched in the introduction.

PROPOSITION 5.5. *Assume $k>k'>0$. If we are in the cases (i), (ii) of Cor. 2.2 where $0\leq p<k'$, assume that $k'\neq p+q+2$. Then we have an exact sequence of $\mathbb{R}$-vector spaces*

$$0\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^3_!(S_L,W))\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^3(S_L,W))\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^1(\partial S_L,i^*j_*W)).$$

Proof. Our assumption $k>k'>0$ means that the highest weight of W is regular. Hence, according to Cor. 5.3 we have $H^4_c(S_L,W)=0$. As a consequence, the long exact sequence (4) gives rise to the exact sequence of mixed Hodge structures

$$0\to H^3_!(S_L,W)\to H^3(S_L,W)\to H^1(\partial S_L,i^*j_*W)\to 0.$$

By application of the functor $M\mapsto \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1,M)$ we obtain the exact sequence

$$\mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^1(\partial S_L,i^*j_*W))\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^3_!(S_L,W))$$

$$\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^3(S_L,W))\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1,H^1(\partial S_L,i^*j_*W).$$

So it is enough to show that the mixed Hodge structure $H^1(\partial S_L,i^*j_*W)$ has no weight zero. To this end, consider the exact sequence

$$H^1_c(\partial S_{L,1},i_1^*i^*j_*W)\to H^1(\partial S_L,i^*j_*W)\to H^0(\partial S_{L,0},i_0^*i^*j_*W)$$

obtained by taking the cohomology of the exact triangle

$$i_{1\,!}i_1^!i^*j_*W=i_{1\,!}i_1^*i^*j_*W\longrightarrow i^*j_*W\longrightarrow i_{0\,*}i_0^*i^*j_*W\stackrel{+}{\longrightarrow}.$$

As $\partial S_{L,0}$ has dimension 0, we have $H^0(\partial S_{L,0},i_0^*i^*j_*W)=H^0(\partial S_{L,0},\mathcal{H}^0i_0^*i^*j_*W)$, whose weight is

$$-(c-(k-k'+4))=k-k'-p-q-2$$

according to Lem. 4.8 (i). Let us check that this is non-zero by definition of k and k'. There are four cases:

(i) $0\leq p<k'$, $p<k-k'$ and $k-k'-p\leq q\leq k-k'+p$: then $-q\leq -(k-k')+p$ so $k-k'-p-q-2\leq -2<0$,

(ii) $0\leq p<k'$, $k-k'\leq p$ and $p-k+k'\leq q\leq p+k-k'$: then $-p\leq -(k-k')$ so $k-k'-p-q-2\leq -q-2\leq -2<0$,

(iii) $k'\leq p\leq k$, $k'<k-p$ and $k-k'-p\leq q\leq k+k'-p$: then $-q\leq -(k-k')+p$ so $k-k'-p-q-2\leq -2<0$,

(iv) $k' \leq p \leq k$, $k-p \leq k'$ and $p-k+k' \leq q \leq k+k'-p$: then $-p \leq -(k-k')$ so $k-k'-p-q-2 \leq -q-2 \leq -2 < 0$.

As a consequence $H^0(\partial S_{L,0}, i_0^* i^* j_* W) = H^0(\partial S_{L,0}, \mathcal{H}^0 i_0^* i^* j_* W)$ has no weight zero.

Let us consider now the mixed Hodge structure $H_c^1(\partial S_{L,1}, i_1^* i^* j_* W)$. According to Thm. 4.1 we have

$$H_c^1(\partial S_{L,1}, i_1^* i^* j_* W) = \bigoplus_{n=-2}^{1} H_c^{1-n}(\partial S_{L,1}, \mathcal{H}^n i_1^* i^* j_* W).$$

The fact that $\partial S_{L,1}$ is one dimensional implies that $H_c^3(\partial S_{L,1}, \mathcal{H}^{-2} i_1^* i^* j_* W) = 0$. Furthermore according to Lem. 4.10 we have

$$\mathcal{H}^{-1} i_1^* i^* j_* W = Sym^{k+1} V_2 \left(\frac{c+k'-k-2}{2} \right),$$
$$\mathcal{H}^0 i_1^* i^* j_* W = Sym^{k'} V_2 \left(\frac{c-k-k'-4}{2} \right).$$

As $k+1 > 0$, resp. $k' > 0$ by assumption, we have $H_c^2(\partial S_{L,1}, \mathcal{H}^{-1} i_1^* i^* j_* W) = 0$, resp. $H_c^0(\partial S_{L,1}, \mathcal{H}^{-1} i_1^* i^* j_* W) = 0$ according to Cor. 5.3. Thus we need to show that $H_c^1(\partial S_{L,1}, \mathcal{H}^0 i_1^* i^* j_* W)$ has no weight zero. Our Lem. 4.10 shows that

$$\mathcal{H}^0 i_1^* i^* j_* W = Sym^{k+1} V_2 \left(\frac{c-k-k'-4}{2} \right).$$

Now the mixed Hodge structure

$$H_c^1(\partial S_{L,1}, \mathcal{H}^0 i_1^* i^* j_* W) = H_c^1 \left(\partial S_{L,1}, Sym^{k+1} V_2 \left(\frac{c-k-k'-4}{2} \right) \right)$$

is Poincaré dual to

$$H^1 \left(\partial S_{L,1}, \left(Sym^{k+1} V_2\right)^\vee \left(1 - \frac{c-k-k'-4}{2} \right) \right)$$

where $\left(Sym^{k+1} V_2\right)^\vee = \left(Sym^{k+1} V_2\right)(-k-1)$ is the contragredient to $Sym^{k+1} V_2$. Hence we want to show that

$$H^1 \left(\partial S_{L,1}, Sym^{k+1} V_2 \left(\frac{k'-k-c+4}{2} \right) \right)$$

has no weight zero. To this end let us check that we can apply Lem. 5.4 i.e. let us check that $\frac{k'-k-c+4}{2} \neq 1$ and $(k+1) + 2.\frac{k'-k-c+4}{2} \neq 1$. Note that the equality $\frac{k'-k-c+4}{2} \neq 1$ is equivalent to $k'-k-p-q-2 \neq 2$ and that we checked above that $k'-k-p-q-2 < 0$ in all possible cases. So it only remains to check that $(k+1) + 2.\frac{k'-k-c+4}{2} \neq 1$. This is equivalent to $k'-p-q-2 \neq 0$. As above, there are four cases:

(i), (ii) In these cases we assumed $k'-p-q-2 \neq 0$, (iii), (iv) In these cases $-p \le -k'$ so $k'-p-q-2 \le -q-2 \le -2 < 0$.

Thus $H^1_c(\partial S_{L,1}, i_1^* i^* j_* W)$ has no weight zero. □

PROPOSITION 5.6. *Assume $k > k' > 0$, $k+k' \neq p+q$, $k-k'-p-q \neq 2$ and $k-p-q \neq 1$. Then we have canonical isomorphisms of $\mathbb{R}$-vector spaces*

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3(S_L, W)) = H^4_{\mathcal{H}}(S_L/\mathbb{R}, W),$$
$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1(\partial S_L, i^* j_* W) = H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W).$$

Proof. The exact sequence (3) shows that our statement is equivalent to

$$\mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^4(S_L, W)) = \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^2(\partial S_L, i^* j_* W)) = 0.$$

As above, via Cor. 5.3, we have $H^4_c(S_L, W)) = 0$ so that $H^4(S_L, W))$ is a sub-mixed Hodge structure of $H^2(\partial S_L, i^* j_* W)$. As a consequence it is enough to show that $H^2(\partial S_L, i^* j_* W)$ has no weight zero. Taking the cohomology of the exact triangle

$$i_{1\,!} i_1^! i^* j_* W = i_{1\,!} i_1^* i^* j_* W \longrightarrow i^* j_* W \longrightarrow i_{0\,*} i_0^* i^* j_* W \xrightarrow{+}$$

we obtain the exact sequence of mixed Hodge structures

$$H^2_c(\partial S_{L,1}, i_1^* i^* j_* W) \longrightarrow H^2(\partial S_L, i^* j_* W) \longrightarrow H^1(\partial S_{L,0}, i_0^* i^* j_* W).$$

As $\partial S_{L,0}$ is zero-dimensional, we have $H^1(\partial S_{L,0}, i_0^* i^* j_* W) = H^0(\partial S_{L,0}, \mathcal{H}^1 i_0^* i^* j_* W)$ and according to Lem. 4.8, the variation of Hodge structure $\mathcal{H}^1 i_0^* i^* j_* W$ has weight

$$-(c-(k+k'+6)) = k+k'-p-q.$$

By assumption this weight is non-zero thus $H^1(\partial S_{L,0}, i_0^* i^* j_* W)$ is of non-zero weight.

Let us turn our attention to the mixed Hodge structure $H^2_c(\partial S_{L,1}, i_1^* i^* j_* W)$. For the same reasons given in the proof of the previous proposition we have

$$H^2_c(\partial S_{L,1}, i_1^* i^* j_* W) = H^1_c(\partial S_{L,1}, \mathcal{H}^1 i_1^* i^* j_* W) = H^1_c\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)\right)$$

where the second equality follows from Lem. 4.10. This mixed Hodge structure is Poincaré dual to

$$H^1\left(\partial S_{L,1}, \left(Sym^{k'} V_2\right)^\vee \left(1 - \frac{c-k-k'-4}{2}\right)\right)$$

where $\left(Sym^{k'} V_2\right)^\vee = Sym^{k'} V_2(-k')$ is the contragredient representation. Hence we have to show that

$$H^1\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{k-k'-c+6}{2}\right)\right)$$

has no weight zero. To this end we check the assumtions of Lem. 5.4: we have

$$\frac{k-k'-c+6}{2} = \frac{k-k'-p-q}{2} \neq 1,$$
$$k' + 2.\frac{k-k'-c+6}{2} = k-p-q \neq 1$$

by assumption. Hence Lem. 5.4 shows that

$$H^1\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{k-k'-c+6}{2}\right)\right)$$

has no weight zero and the proof is complete. □

6. Main results. Consider the integers p, q, k and k' satisfying the assumptions stated at the beginning of the previous section. Denote again by W an irreducible algebraic representation of GSp_4 of highest weight $\lambda(k, k', c)$ where $c = p + q + 6$ (see section 2.1 for the definition of $\lambda(k, k', c)$). In this section we are going to prove the vanishing of the right hand vertical map

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \longrightarrow H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W)$$

in the commutative diagram of Lem. 3.4 (ii) for $F = (Sym^p V_2 \boxtimes Sym^q V_2)(3)$ and $E = W$, under some assumptions on k and k'. This will be done in three steps: we show the vanishing of the corresponding map on the level of strata of dimension 1 and 0 and then deduce the vanishing of the above map by considering some long exact sequences. Let us show as a first step that the right hand vertical arrow

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_1/\mathbb{R}, i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \longrightarrow H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W)$$

in the commutative diagram of Lem 3.5 (ii) for $d = 1$, $F = (Sym^p V_2 \boxtimes Sym^q V_2)(3)$ and $E = W$ is zero for most of the choices of k and k' as above. We start with a preliminary lemma.

LEMMA 6.1. *Assume $k' > 0$, $k + k' \neq p + q$ and $k \neq p + q + 1$. Then*

$$H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) = H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^0 i_1^* i^* j_* W).$$

Proof. The first statement of Thm. 4.1 together with the last statement of Lem. 4.10 imply that

$$H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) = \bigoplus_{n=-2}^{1} H^{2-n}_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^n i_1^* i^* j_* W)$$

thus we need to show that

$$H^4_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^{-2} i_1^* i^* j_* W) = H^3_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^{-1} i_1^* i^* j_* W) = H^1_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^1 i_1^* i^* j_* W) = 0.$$

In the case of the first absolute Hodge cohomology space, the exact sequence (3) is

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3(\partial S_{L,1}, \mathcal{H}^{-2} i_1^* i^* j_* W)) \to H^4_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^{-2} i_1^* i^* j_* W)$$

$$\to \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^4(\partial S_{L,1}, \mathcal{H}^{-2} i_1^* i^* j_* W)) \to 0.$$

By the second statement of Thm. 4.1 the mixed Hodge module $\mathcal{H}^{-2} i_1^* i^* j_* W$ is in fact a variation of mixed Hodge structure over the curve $\partial S_{L,1}$, so it has no singular cohomology in degrees 3 and 4. This shows that $H^4_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^{-2} i_1^* i^* j_* W) = 0$. Similarly, we have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^2(\partial S_{L,1}, \mathcal{H}^{-1} i_1^* i^* j_* W)) \to H^3_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^{-1} i_1^* i^* j_* W)$$

$$\to \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3(\partial S_{L,1}, \mathcal{H}^{-1} i_1^* i^* j_* W)) \to 0$$

and for the same reason as above the right hand space is zero. The left hand space vanishes too because the variation of Hodge structure $\mathcal{H}^{-1} i_1^* i^* j_* W$ is associated to a representation of regular highest weight (see Lem. 4.10) and because of Cor. 5.3.

Let us turn our attention to $H^1_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^1 i_1^* i^* j_* W)$. According to Lem. 4.10 we have $\mathcal{H}^1 i_1^* i^* j_* W = Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)$. Our assumption $k' > 0$ says that the highest weight of $Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)$ is regular so that $H^0\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)\right)$ vanishes according to Thm. 5.1. This implies that

$$H^1_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^1 i_1^* i^* j_* W) = \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}\left(1, H^1\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)\right)\right).$$

The mixed Hodge structure $H^1\left(\partial S_{L,1}, Sym^{k'} V_2\left(\frac{c-k-k'-4}{2}\right)\right)$ has no weight zero because of our assumptions and Lem. 5.4: indeed, we have

$$\frac{c-k-k'-4}{2} = \frac{p+q-k-k'+2}{2} \neq 1,$$
$$k' + 2.\frac{c-k-k'-4}{2} = p+q-k+2 \neq 1$$

by assumption. As a consequence $H^1_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, \mathcal{H}^1 i_1^* i^* j_* W)$. □

At this point, the proof of the following proposition is a bit tedious but not difficult.

Proposition 6.2. *Let W be an irreducible algebraic representation of* GSp_4 *of highest weight $\lambda(k, k', c)$. Assume as in the previous lemma that $k' > 0$, $k + k' \neq p + q$ and $k \neq p + q + 1$. Furthermore, assume $p + q - k - k' < 2$ and $k' \neq p, q$. Then the right hand vertical map*

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_1/\mathbb{R}, i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \longrightarrow H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W)$$

appearing in the commutative diagram of Lem. 3.5 (ii) for $F = (Sym^p V_2 \boxtimes Sym^q V_2)(3)$ and $E = W$ is zero.

Proof. By construction, this map is obtained by applying the absolute Hodge cohomology functor to the morphism

$$i_{1\,*} q_{1\,*} i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) \longrightarrow i_1^* i^* j_* W[1]$$

appearing in Lem. 3.5 (i). Thanks to Lem. 6.1 it is enough to show that the morphism

$$q_{1\,*} i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) \longrightarrow \mathcal{H}^0 i_1^* i^* j_* W[1]$$

is zero. According to Lem. 4.10 we have

$$\mathcal{H}^0 i_1^* i^* j_* W = Sym^{k+1} V_2\left(\frac{p+q-k-k'+2}{2}\right).$$

By the first statement of Thm. 4.1 and Lem. 4.6 we have

$$\begin{aligned} & i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) \\ &= \mathcal{H}^{-1} i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[1] \oplus \mathcal{H}^0 i_1'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[0] \\ &= (Sym^q V_2\,(p+2) \boxplus Sym^p V_2\,(q+2))[1] \oplus (Sym^q V_2\,(1) \boxplus Sym^p V_2\,(1))[0]. \end{aligned}$$

According to Lem. 3.2, the morphism q_1 is the composite of an étale cover and of the inclusion of some connected components. This implies $q_{1\,*} = q_{1\,!}$ and $q_1^! = q_1^*$.

Furthermore we have the following identity: $q_1^* Sym^d V_2(t) = Sym^d V_2(t) \boxplus Sym^d V_2(t)$ for any d and t. By adjunction we deduce

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,1}/\mathbb{R}))}(q_{1\,*}(Sym^q V_2\,(p+2) \boxplus Sym^p V_2(q+2))[1], \mathcal{H}^0 i_1^* i^* j_* W[1])\\
&= \mathrm{Hom}\left(Sym^q V_2\,(p+2) \boxplus Sym^p V_2(q+2), q_1^* Sym^{k+1} V_2\,(r)\right)\\
&= \mathrm{Hom}\left(Sym^q V_2\,(p+2) \boxplus Sym^p V_2(q+2), Sym^{k+1} V_2\,(r) \boxplus Sym^{k+1} V_2\,(r)\right)
\end{aligned}$$

where the last two Hom spaces are taken in the category $D^b(\mathrm{MHM}_{\mathbb{R}}(\partial(M_K \times_{F_{K_0}} M_{K'})_1/\mathbb{R}))$ and where we wrote $r = \frac{p+q-k-k'+2}{2}$. Let us show that this space vanishes for weight reasons. Recall that the variation of Hodge structure $Sym^k V_2(t)$ is pure of weight $-k-2t$. Hence the variation of Hodge structure $Sym^q V_2\,(p+2)$, resp. $Sym^p V_2(q+2)$, has weight $-q-2p-4$, resp. $-p-2q-4$. The variation of Hodge structure $Sym^{k+1} V_2\,(r)$ has weight

$$-(k+1) - 2r = k' - p - q - 3.$$

But $-q-2p-4 = k'-p-q-3$ is equivalent to $k' = -p-1$ which is impossible because k' and p are non-negative integers. Similarly $-p-2q-4 \neq k'-p-q-3$. As a consequence

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,1}/\mathbb{R}))}(q_{1\,*}(Sym^q V_2\,(p+2) \boxplus Sym^p V_2(q+2))[1], \mathcal{H}^0 i_1^* i^* j_* W[1])\\
&= \mathrm{Hom}\left(Sym^q V_2\,(p+2) \boxplus Sym^p V_2(q+2), Sym^{k+1} V_2\,(r) \boxplus Sym^{k+1} V_2\,(r)\right) = 0.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,1}/\mathbb{R}))}\left(q_{1\,*}(Sym^q V_2\,(1) \boxplus Sym^p V_2\,(1))[0], \mathcal{H}^0 i_1^* i^* j_* W[1]\right)\\
&= \mathrm{Hom}\left(Sym^q V_2\,(1) \boxplus Sym^p V_2\,(1)\,, (Sym^{k+1} V_2\,(r) \boxplus Sym^{k+1} V_2\,(r))[1]\right)\\
&= \mathrm{Hom}\left(Sym^q V_2 \boxplus Sym^p V_2, (Sym^{k+1} V_2(r-1) \boxplus Sym^{k+1} V_2(r-1))[1]\right)\\
&= \mathrm{Hom}\left(1, (Sym^{k+1} V_2(r-1) \otimes (Sym^q V_2)^\vee \boxplus Sym^{k+1} V_2(r-1) \otimes (Sym^p V_2)^\vee)[1]\right)\\
&= \mathrm{Hom}\left(1, (Sym^{k+1} V_2 \otimes Sym^q V_2)(r-q-1) \boxplus (Sym^{k+1} V_2 \otimes Sym^p V_2)(r-p-1)[1]\right)
\end{aligned}$$

where all Hom spaces but the first are in the category $D^b(\mathrm{MHM}_{\mathbb{R}}(\partial(M_K \times_{F_{K_0}} M_{K'})_1/\mathbb{R}))$ and where the superscript $^\vee$ denotes the dual variation. In the last equality, we used the identity $V_2^\vee = V_2(-1)$ between algebraic representations of GL_2. The last Hom space above is by definition

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_1/\mathbb{R}, (Sym^{k+1} V_2 \otimes Sym^q V_2)(r-q-1) \boxplus (Sym^{k+1} V_2 \otimes Sym^p V_2)(r-p-1)).$$

Note that by definition of k we have $k \geq q$ in all cases (see the conditions stated at the beginning of section 5.). Thus, according to [FH] Ex. 11.11, we have the isotypical decomposition

$$(Sym^{k+1} V_2 \otimes Sym^q V_2)(r-q-1) = \bigoplus_{a=0}^{q} Sym^{k+q+1-2a} V_2(r-q-1+a)$$

as representations of GL_2. Consider a fixed piece $Sym^{k+q+1-2a} V_2(r-q-1+a)$ in the above decomposition, which is regarded as a variation of mixed Hodge structure on the modular curve $M = M_K \times_{F_{K_0}} \partial M_{K'}$. We have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^0(M, Sym^{k+q+1-2a} V_2(r-q-1+a)) \to H^1_{\mathcal{H}}(M/\mathbb{R}, Sym^{k+q+1-2a} V_2(r-q-1+a))$$

$$\to \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1(M, Sym^{k+q+1-2a}V_2(r-q-1+a)) \longrightarrow 0.$$

As $k+q+1-2a \geq k-q+1 > 0$, the singular cohomology $H^0(M, Sym^{k+q+1-2a}V_2(r-q-1+a)$ vanishes for all a thanks to Thm. 5.1. Let us explain why the mixed Hodge structure $H^1(M, Sym^{k+q+1-2a}V_2(r-q-1+a)$ has no weight zero: we just have to check that the conditions of Lem. 5.4 are verified for all a. But

$$r-q-1+a = \frac{p-q-k-k'+2a}{2} \leq \frac{p+q-k-k'}{2} < 1,$$
$$k+q+1-2a+2(r-q-1+a) = p-k'+1 \neq 1$$

by assumption. So, it follows from Lem. 5.4 that

$$\mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1(M, Sym^{k+q+1-2a}V_2(r-q-1+a)) = 0$$

for all $0 \leq a \leq q$ thus $H^1_{\mathcal{H}}(M/\mathbb{R}, Sym^{k+q+1-2a}V_2(r-q-1+a)) = 0$. Using the assumption $k' \neq q$, one proves exactly in the same way that

$$H^1_{\mathcal{H}}(M'/\mathbb{R}, (Sym^{k+1}V_2 \otimes Sym^p V_2)(r-p-1)) = 0$$

where $M' = \partial M_K \times_{F_{K_0}} M_{K'}$ is the other connected component of $\partial(M_K \times_{F_{K_0}} M_{K'})_1$. As a consequence, we have

$$\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,1}/\mathbb{R}))}(q_{1\,*}(Sym^q V_2\,(1) \boxplus Sym^p V_2\,(1))[0], \mathcal{H}^2 i_1^* i^* j_* W[1]) = 0$$

and the proof is complete. □

Let us turn our attention to the strata of dimension 0.

Lemma 6.3. *We have*

$$\begin{aligned} &H^0_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \\ &= H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, \mathcal{H}^{-1} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \\ &\quad \oplus H^0_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, \mathcal{H}^0 i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \end{aligned}$$

and

$$H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i^* j_* W) = H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, \mathcal{H}^0 i_0^* i^* j_* W) \oplus H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, \mathcal{H}^1 i_0^* i^* j_* W).$$

Proof. According to the first statement of Thm. 4.1 and to Lem. 4.5 we have

$$H^0_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2))$$

$$= \bigoplus_{n=-2}^{0} H^{-n}_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, \mathcal{H}^n i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)).$$

The exact sequence (3) and the fact that $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ is of dimension 0 shows that

$$H^2_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, \mathcal{H}^{-2} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) = 0.$$

So the first statement is proven. The second statement is proven the same way. □

Proposition 6.4. *Let W be an irreducible algebraic representation of* GSp_4 *of highest weight $\lambda(k,k',c)$. Assume that $k+k' \neq p+q$, that $\frac{k-k'-p-q-2}{2}$ is even and that the disjoint union of cusps $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ is totally real. Then the right hand vertical arrow*

$$H^0_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})_0/\mathbb{R}, i_0'^* * i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \longrightarrow H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i^* j_* W)$$

appearing in the commutative diagram of Lem. 3.5 (ii) for $F = (Sym^p V_2 \boxtimes Sym^q V_2)(3)$ and $E = W$ is zero.

Remark 6.5. See Rem. 6.7 for an explanation of the fact that the set of levels such that $\partial(M_K \times_{F_{K_0}} M_{K'})_0$ is totally real is non-empty, and even infinite.

Proof. By construction this map is induced by applying the absolute Hodge cohomology functor to the morphism

$$i_{0\,*} q_{0\,*} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) \longrightarrow i_{0\,*} i_0^* i^* j_* W[1]$$

of Lem. 3.5 (i). Thanks to Lem. 6.3, it is enough to show that the morphism

$$q_{0\,*}\mathcal{H}^{-1} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[1] \oplus q_{0\,*}\mathcal{H}^0 i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[0] \longrightarrow \mathcal{H}^0 i_0^* i^* j_* W[1] \oplus \mathcal{H}^1 i_0^* i^* j_* W[2]$$

is zero. Note that thanks to Lem. 4.8 and our assumption $k+k' \neq p+q$, the variation of Hodge structure $\mathcal{H}^1 i_0^* i^* j_* W$ has non-zero weight. This implies

$$H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, \mathcal{H}^1 i_0^* i^* j_* W) = \mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^0(\partial S_{L,0}, \mathcal{H}^1 i_0^* i^* j_* W) = 0$$

so the variation of Hodge structure $\mathcal{H}^1 i_0^* i^* j_* W$ does not contribute to the target of the morphism we are studying (see Lem. 6.3). As a consequence it is enough to show that

$$q_{0\,*}\mathcal{H}^{-1} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[1] \oplus q_{0\,*}\mathcal{H}^0 i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)[0] \longrightarrow \mathcal{H}^0 i_0^* i^* j_* W[1]$$

vanishes. By Lem. 4.5 we have

$$\mathcal{H}^{-1} i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) = 1(p+1) \oplus 1(q+1),$$
$$\mathcal{H}^0 i_0'^* i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2) = 1(0).$$

By properness of q_0 (see Lem. 3.2), the mixed Hodge module $q_{0\,*}1(p+1) = q_{0\,*}1(p+1)$ is pure of weight $-2(p+1)$ ([S1] Cor. 1.8). Similarly $q_{0\,*}1(q+1)$ is pure of weight $-2(q+1)$. According to Lem. 4.8 the only weight occurring in $\mathcal{H}^0 i_0^* i^* j_* W$ is $-(c-(k-k'+4))$. Recall that $c = p+q+6$ so that $-(c-(k-k'+4)) = k-k'-p-q-2$, which is non-negative by assumption. As both $-2(p+1)$ and $-2(q+1)$ are negative, we have

$$\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,0}/\mathbb{R}))}\left(q_{0\,*}(1(p+1) \oplus 1(q+1)), \mathcal{H}^0 i_0^* i^* j_* W\right) = 0.$$

It remains to show that

$$\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,0}/\mathbb{R}))}\left(q_{0\,*}1(0), \mathcal{H}^0 i_0^* i^* j_* W[1]\right) = 0.$$

By adjunction we have

$$\begin{aligned}&\mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial S_{L,0}/\mathbb{R}))}\left(q_{0\,*}1(0), \mathcal{H}^0 i_0^* i^* j_* W[1]\right)\\ &= \mathrm{Hom}_{D^b(\mathrm{MHM}_{\mathbb{R}}(\partial (M_K\times_{F_{K_0}} M_{K'})_0/\mathbb{R}))}\left(1(0), q_0^*\mathcal{H}^0 i_0^* i^* j_* W[1]\right).\end{aligned}$$

The Hom space above is nothing but

$$\begin{aligned}&H^1_{\mathcal{H}}(\partial(M_K\times_{F_{K_0}} M_{K'})_0/\mathbb{R}, q_0^*\mathcal{H}^0 i_0^* i^* j_* W)\\ &= \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^0(\partial(M_K\times_{F_{K_0}} M_{K'})_0, q_0^*\mathcal{H}^0 i_0^* i^* j_* W)).\end{aligned}$$

But the Tate variation of Hodge structure $q_0^*\mathcal{H}^0 i_0^* i^* j_* W$ has weight

$$-(c-(k-k'+4)) = k-k'-p-q-2$$

according to Lem. 4.8. But we assumed that $\frac{k-k'-p-q-2}{2}$ is even and that $\partial(M_K\times_{F_{K_0}} M_{K'})_0$ is totally real. Hence the above space is zero according to [HW1] Cor. A.2.12. This proves that the map

$$H^0_{\mathcal{H}}(\partial(M_K\times_{F_{K_0}} M_{K'})_0/\mathbb{R}, i_0'^* * i'^* j'_*(Sym^p V_2\boxtimes Sym^q V_2)(2)) \longrightarrow H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i^* j_* W)$$

is the zero map. □

We can now combine the two previous vanishing results to prove the main theorem for the construction of our 1-extensions. The proof relies on the fact that the boundary ∂S_L of the Baily-Borel compactification of S_L can be seen as the Baily-Borel compactification of $\partial S_{L,1}$.

THEOREM 6.6. *Let W be an irreducible algebraic representation of GSp_4 of highest weight $\lambda(k,k',c)$. Assume $k>k'>0$, $k+k'\neq p+q$, $k+k'-p-q+2>0$, $k\neq p+q+1$ in order to apply Prop. 6.2 and assume that $\frac{k-k'-p-q-2}{2}$ is even and that $\partial(M_K\times_{F_{K_0}} M_{K'})_0$ is totally real in order to apply Prop. 6.4. Furthermore, assume that $\frac{p+q+k-k'}{2}$ is even and the boundary of the Baily-Borel compactification of $\partial S_{L,1}$ is totally real. Then the right hand vertical arrow*

$$H^1_{\mathcal{H}}(\partial(M_K\times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^* j'_*(Sym^p V_2\boxtimes Sym^q V_2)(2)) \longrightarrow H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W)$$

in Lem. 3.4 for $F=(Sym^p V_2\boxtimes Sym^q V_2)(3)$ and $E=W$ is zero.

REMARK 6.7. Let $N\geq 3$ be an integer. Denote by $L(N)\subset \mathrm{GSp}_4(\mathbb{A}_f)$ the principal congruence subgroup of level N defined as the kernel of $\pi_N : \mathrm{GSp}_4(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$. Then $\partial S_{L(N),1}$ is a disjoint union of modular curves of principal level N, i.e. of Shimura varieties of the shape $M_{K(N)}$ where $K(N)$ is the kernel of $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\partial\partial S_{L(N),1}$ is the disjoint union of spectra of the cyclotomic field $\mathbb{Q}(\mu_N)$ (see [M] 1.2 p. 8). Let $L'(N)=\pi_N^{-1}(\pm I)$ where $I\in \mathrm{GSp}_4(\mathbb{Z}/N\mathbb{Z})$ is the identity matrix. Note that like $L(N)$, the subgroup $L'(N)$ is normal in $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$. Then, it follows easily that $\partial\partial S_{L(N)',1}$ is a disjoint union of spectra of the maximal totally real subfield $\mathbb{Q}(\mu_N)^+\subset \mathbb{Q}(\mu_N)$.

Proof. We have a morphism of exact triangles

$$\begin{array}{ccccccc} i_{1\,!}i_1^* i^* j_* W & \longrightarrow & i^* j_* W & \longrightarrow & i_{0\,*} i_0^* i^* j_* W & \xrightarrow{+} \\ \| & & \downarrow & & \downarrow & \\ i_{1\,!}i_1^* i^* j_* W & \longrightarrow & i_{1\,*} i_1^* i^* j_* W & \longrightarrow & i_{0\,*} i_0^* i_{1\,*} i_1^* i^* j_* W & \xrightarrow{+} \end{array}$$

in $D^b(\mathrm{MHM}_A(\partial S_L/\mathbb{R}))$ (see [BBD] 1.4.7.1). Applying the absolute Hodge cohomology functor we get the commutative diagrams with exact lines

$$\begin{array}{ccccc} H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) & \xrightarrow{\psi_1} & H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W) & \xrightarrow{\phi_0} & H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i^* j_* W) \\ \| & & \downarrow \phi_1 & & \downarrow \\ H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) & \xrightarrow{\psi_1'} & H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) & \longrightarrow & H^1_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W) \end{array}$$

$$\begin{array}{ccccc} H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i^* j_* W) & \xrightarrow{\psi_0} & H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) & \xrightarrow{\psi_1} & H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W) \\ \phi_0 \downarrow & & \| & & \downarrow \phi_1 \\ H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W) & \xrightarrow{\psi_0'} & H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W) & \xrightarrow{\psi_1'} & H^2_{\mathcal{H}}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W). \end{array}$$

Let v belonging to the image of

$$H^1_{\mathcal{H}}(\partial(M_K \times_{F_{K_0}} M_{K'})/\mathbb{R}, i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \longrightarrow H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W).$$

Let us first look at the first commutative diagram above. According to Prop. 6.4 and to the commutative diagram of Lem. 3.5 (ii) for $d = 0$, we have $\phi_0(v) = 0$. Hence $v = \psi_1(w)$ for some $w \in H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W)$. According to Prop. 6.2 and to the commutative diagram of Lem. 3.5 (ii) for $d = 1$, we have $\psi_1'(w) = \phi_1(v) = 0$. As a consequence we have $w = \psi_0'(u)$ for some $u \in H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W)$. Hence, it is enough to show that ψ_0' is the zero map.

We can argue as follows: the variety $\partial S_{L,1}$ is a disjoint union of modular curves so we can consider its Baily-Borel compactification $j_1 : \partial S_{L,1} \longrightarrow (\partial S_{L,1})^*$, with complementary closed reduced embedding $\iota_1 : \partial\partial S_{L,1} \longrightarrow (\partial S_{L,1})^*$. We have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \partial S_{L,1} & \xrightarrow{j_1} & (\partial S_{L,1})^* & \xleftarrow{\iota_1} & \partial\partial S_{L,1} \\ \| & & r \downarrow & & r' \downarrow \\ \partial S_{L,1} & \xrightarrow{i_1} & \partial S_L & \xleftarrow{i_0} & \partial S_{L,0} \end{array}$$

where r is finite (see [P1] 7.6). By functoriality and the proper base change theorem ([S3] 4.4.3) we have

$$i_0^* i_{1\,*} i_1^* i^* j_* W = i_0^* r_* j_{1\,*} i_1^* i^* j_* W = r'_* \iota_1^* j_{1\,*} i_1^* i^* j_* W$$

and by adjunction

$$\begin{aligned} H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W) &= H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, r'_* \iota_1^* j_{1\,*} i_1^* i^* j_* W) \\ &= H^0_{\mathcal{H}}(\partial\partial S_{L,1}/\mathbb{R}, \iota_1^* j_{1\,*} i_1^* i^* j_* W). \end{aligned}$$

Now applying twice Thm. 4.1 we see that

$$\iota_1^* j_{1\,*} i_1^* i^* j_* W = \bigoplus_s \bigoplus_t \mathcal{H}^s \iota_1^* j_{1\,*} \mathcal{H}^t i_1^* i^* j_* W[-(s+t)].$$

By (3) we have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^{-1}(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)) \to H^0_{\mathcal{H}}(\partial\partial S_{L,1}/\mathbb{R}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)$$

$$\to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^0(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)) \to 0$$

and as $\partial\partial S_{L,1}$ is of dimension 0 we have

$$\begin{aligned} & H^{-1}(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W) \\ = {} & H^0(\partial\partial S_{L,1}, \mathcal{H}^{-1}\iota_1^* j_{1\,*} \mathcal{H}^0 i_1^* i^* j_* W) \oplus H^0(\partial\partial S_{L,1}, \mathcal{H}^0\iota_1^* j_{1\,*} \mathcal{H}^{-1} i_1^* i^* j_* W), \\ & H^0(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W) \\ = {} & H^0(\partial\partial S_{L,1}, \mathcal{H}^{-1}\iota_1^* j_{1\,*} \mathcal{H}^1 i_1^* i^* j_* W) \oplus H^0(\partial\partial S_{L,1}, \mathcal{H}^0\iota_1^* j_{1\,*} \mathcal{H}^0 i_1^* i^* j_* W). \end{aligned}$$

All these Tate mixed Hodge structures are explicitly computable via a successive application of Lem. 4.10 and 4.4. We find that

$$\begin{aligned} \mathcal{H}^{-1}\iota_1^* j_{1\,*} \mathcal{H}^0 i_1^* i^* j_* W &= \mathcal{H}^0 \iota_1^* j_{1\,*}\left(Sym^{k+1}V_2\left(\frac{c-k-k'-4}{2}\right)\right) = 1\left(\frac{c+k-k'-2}{2}\right), \\ \mathcal{H}^0\iota_1^* j_{1\,*} \mathcal{H}^{-1} i_1^* i^* j_* W &= \mathcal{H}^1 \iota_1^* j_{1\,*}\left(Sym^{k+1}V_2\left(\frac{c+k'-k-2}{2}\right)\right) = 1\left(\frac{c+k'-k-4}{2}\right), \\ \mathcal{H}^{-1}\iota_1^* j_{1\,*} \mathcal{H}^1 i_1^* i^* j_* W &= \mathcal{H}^0 \iota_1^* j_{1\,*}\left(Sym^{k'}V_2\left(\frac{c-k-k'-4}{2}\right)\right) = 1\left(\frac{c-k+k'-4}{2}\right), \\ \mathcal{H}^0\iota_1^* j_{1\,*} \mathcal{H}^0 i_1^* i^* j_* W &= \mathcal{H}^1 \iota_1^* j_{1\,*}\left(Sym^{k+1}V_2\left(\frac{c-k-k'-4}{2}\right)\right) = 1\left(\frac{c-k-k'-6}{2}\right). \end{aligned}$$

By assumption $c-k+k'-4 = p+q-k+k'+2$ and $c-k-k'-6 = p+q-k-k'$ are non-zero so

$$\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^0(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)) = 0.$$

As a consequence we have

$$\begin{aligned} H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W) &= H^0_{\mathcal{H}}(\partial\partial S_{L,1}/\mathbb{R}, \iota_1^* j_{1\,*} i_1^* i^* j_* W) \\ &= \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^{-1}(\partial\partial S_{L,1}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)). \end{aligned}$$

Furthermore, by our assumption $k > k' > 0$ and Lem. 4.10 the variations of Hodge structure $\mathcal{H}^n i_1^* i^* j_* W$ are associated to a representation whose highest weight is regular. It follows from Cor. 5.3 that $\partial S_{L,1}$ has Betti cohomology with compact support with coefficients in such a variation of Hodge structure only in degree 1. As a consequence we have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^1_c(\partial S_{L,1}, \mathcal{H}^0 i_1^* i^* j_* W)) \to H^2_{\mathcal{H},c}(\partial S_{L,1}/\mathbb{R}, i_1^* i^* j_* W)$$

$$\to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}(1, H^1_c(\partial S_{L,1}, \mathcal{H}^1 i_1^* i^* j_* W)) \to 0.$$

According to Lem. 4.10 we have $\mathcal{H}^1 i_1^* i^* j_* W = Sym^{k'}V_2\left(\frac{c-k-k'-4}{2}\right)$. The mixed Hodge structure

$$H^1_c(\partial S_{L,1}, \mathcal{H}^1 i_1^* i^* j_* W) = H^1_c\left(\partial S_{L,1}, Sym^{k'}V_2\left(\frac{c-k-k'-4}{2}\right)\right)$$

is Poincaré dual to

$$\begin{aligned} & H^1\left(\partial S_{L,1}, Sym^{k'}V_2\left(1 - \frac{c-k-k'-4}{2} - k'\right)\right) \\ = {} & H^1\left(\partial S_{L,1}, Sym^{k'}V_2\left(\frac{k-k'-p-q}{2}\right)\right). \end{aligned}$$

Our assumptions imply $\frac{k-k'-p-q}{2} \neq 1$ and $k'+2.\frac{k-k'-p-q}{2} = k-p-q \neq 1$ so that Lem. 5.4 shows that $H^1\left(\partial S_{L,1}, Sym^{k'}V_2\left(\frac{k-k'-p-q}{2}\right)\right)$ has no weight zero. This implies that the right hand term of the above exact sequence vanishes. As a consequence, the map ψ'_0 is the composite of the projection

$$H^0_{\mathcal{H}}(\partial S_{L,0}/\mathbb{R}, i_0^* i_{1\,*} i_1^* i^* j_* W) = H^0_{\mathcal{H}}(\partial\partial S_{L,1}/\mathbb{R}, \iota_1^* j_{1\,*} i_1^* i^* j_* W)$$

$$\longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^0(\partial\partial S_{L,1}, \mathcal{H}^{-1}\iota_1^* j_{1\,*}\mathcal{H}^0 i_1^* i^* j_* W))$$

and of the map

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^0(\partial\partial S_{L,1}, \mathcal{H}^{-1}\iota_1^* j_{1\,*}\mathcal{H}^0 i_1^* i^* j_* W)) \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1_c(\partial S_{L,1}, \mathcal{H}^0 i_1^* i^* j_* W)).$$

Hence, to show that ψ'_0 is the zero map, it is enough to show that

$$\begin{aligned}&\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^0(\partial\partial S_{L,1}, \mathcal{H}^{-1}\iota_1^* j_{1\,*}\mathcal{H}^0 i_1^* i^* j_* W))\\ &= \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}\left(1, H^0\left(\partial\partial S_{L,1}, 1\left(\frac{p+q+k-k'+4}{2}\right)\right)\right)\end{aligned}$$

is zero. But this follows from our parity assumption and the assumption that $\partial\partial S_{L,1}$ is totally real according to [HW1] Cor. A.2.12. □

As we explained in the introduction, we have a $\mathbb{Q}$-linear map

$$Eis^{p,q}_{\mathcal{H}} : \mathcal{B}_p \otimes \mathcal{B}_q \to H^2_{\mathcal{H}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}, (Sym^p V_2 \boxtimes Sym^q V_2)(2)) \to H^4_{\mathcal{H}}(S_L/\mathbb{R}, W)$$

where $\mathcal{B}_p$, resp. $\mathcal{B}_q$, is the source of the Eisenstein symbol of weight p, resp. q, and where the map $H^2_{\mathcal{H}}(M_K \times_{F_{K_0}} M_{K'}/\mathbb{R}, (Sym^p V_2 \boxtimes Sym^q V_2)(2)) \to H^4_{\mathcal{H}}(S_L/\mathbb{R}, W)$ is the Gysin morphism. Our final result is now a trivial consequence of what we proved so far.

Theorem 6.8. *Let W be an irreducible algebraic representation of* GSp_4 *of highest weight $\lambda(k, k', c)$. Keep the assumptions of the previous theorem. Assume that $k - k' - p - q - 2 \neq 0$ and if we are in the cases (i), (ii) of Cor. 2.2 where $0 \leq p < k'$ assume that $k' \neq p + q + 2$ as in Prop. 5.5. Then the above map is the composite of a map*

$$Eis^{p,q}_{\mathcal{H}} : \mathcal{B}_p \otimes \mathcal{B}_q \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3_!(S_L, W))$$

and of the inclusion $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3_!(S_L, W)) \subset H^4_{\mathcal{H}}(S_L/\mathbb{R}, W)$.

Proof. According to the commutative diagram of Lem. 3.4 (ii), the composite map

$$\mathcal{B}_p \otimes \mathcal{B}_q \xrightarrow{Eis^{p,q}_{\mathcal{H}}} H^4_{\mathcal{H}}(S_L/\mathbb{R}, W) \longrightarrow H^4_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W)$$

fits into the following commutative diagram

$$\begin{array}{ccc}\mathcal{B}_p \otimes \mathcal{B}_q & & \\ \downarrow & & \\ H^2_{\mathcal{H}}(M \times M, (Sym^p V_2 \boxtimes Sym^q V_2)(2)) & \longrightarrow & H^1_{\mathcal{H}}(\partial(M \times M), i'^* j'_*(Sym^p V_2 \boxtimes Sym^q V_2)(2)) \\ \downarrow & & \downarrow \\ H^4_{\mathcal{H}}(S, W) & \longrightarrow & H^2_{\mathcal{H}}(\partial S, i^* j_* W).\end{array}$$

According to Thm. 6.6 the right hand vertical map of this diagram is zero. So the map $Eis^{p,q}_{\mathcal{H}}$ factors through the kernel of the lower horizontal map of the diagram. Now according to Prop. 5.6, we have

$$H^4_{\mathcal{H}}(S_L/\mathbb{R}, W) = \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3(S_L, W)),$$
$$H^2_{\mathcal{H}}(\partial S_L/\mathbb{R}, i^* j_* W) = \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1(\partial S_L, i^* j_* W)$$

and according to Prop. 5.5 we have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3_!(S_L, W)) \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3(S_L, W)) \to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^1(\partial S_L, i^* j_* W)).$$

This implies that the kernel of

$$H^4_{\mathcal{H}}(S, W) \longrightarrow H^2_{\mathcal{H}}(\partial S, i^* j_* W)$$

is $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(1, H^3_!(S_L, W))$. Hence the statement is proven. □

REFERENCES

[BBD] A. A. Beilinson, J. Bernstein, and P. Deligne, *Analyse et topologie sur les espaces singuliers I*, Astérisque, 100 (1982), pp. 1–172.

[Be1] A. A. Beilinson, *On the derived category of perverse sheaves*, LNM vol. 1289, Springer, Berlin (1987), pp. 27–41.

[Bl] D. Blottière, *Réalisation de Hodge du polylogarithme d'un schéma abélien et dégénérescence des classes d'Eisenstein des familles modulaires de Hilbert-Blumenthal*, thèse de l'Univ. Paris 13, (2006), available at `http://www.dblottiere.org/recherche/These.pdf`.

[Bo] A. Borel, *Density properties of certain subgroups of semisimple Lie groups without compact factors*, Annals of Math., 72 (1960), pp. 179–188.

[BoW] A. Borel and N. R. Wallach, *Continous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies 94, Princeton University Press, Princeton N.J., (1980).

[BuW] J. I. Burgos and J. Wildeshaus, *Hodge modules on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Ann. Sci Ecole Norm. Sup. (4), 37:3 (2004), pp. 363–413.

[FH] W. Fulton and J. Harris, *Representation theory. A first course*, GTM 129, Springer-Verlag, New-York, (1991).

[Ha2] G. Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math., 89:1 (1987), pp. 37–118.

[HW1] A. Huber and J. Wildeshaus, *Classical motivic polylogarithm according to Beilinson and Deligne*, Doc. Math., 3 (1998), pp. 27–133.

[HW2] A. Huber and J. Wildeshaus, *Correction to the paper: "Classical motivic polylogarithm according to Beilinson and Deligne"*, Doc. Math., 3 (1998), pp. 297–299.

[K1] G. Kings, *Higher regulatos, Hilbert modular surfaces and special values of L-functions*, Duke Math. J., 92:1 (1998), pp. 61–127.

[K2] G. Kings, *K-theory elements of the polylogarithm of abelian schemes*, J. Reine Angew. Math., 517 (1999), pp. 103–116.

[L] G. Laumon, *Fonctions zetas des variétés de Siegel de dimension 3*, Astérisque, 302 (2005), pp. 1–66.

[Le1] F. Lemma, *Régulateurs supérieurs, périodes et valeurs spéciales de la fonction L de degré 4 de* GSp_4, C. R. Math. Acad. Sci. Paris, 346 (2008), no. 19-20, pp. 1023–1028.

[M] S. Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc., 21:1 (2008), pp. 23–61.

[N] J. Nekovar, *Beilinson's conjectures*, Motives, Proc. Sympos. Pure Math. 55, Part 1, Amer. Math Soc., Providence, (1994), pp. 537–570.

[P1] R. Pink, *Arithmetical compactifications of mixed Shimura varieties*, Bonner Mathematische Schriften 209, (1990), available at `http://www.math.ethz.ch/~pink/dissertation.html`.

[P2] R. Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann., 292:2 (1992), pp. 197–240.

[S1] M. Saito, *Introduction to mixed Hodge modules. Actes du Colloque de théorie de Hodge (Luminy, 1987)*, Astérisque, 179-180 (1989), pp. 145–162.

[S2] M. Saito, *Extensions of mixed Hodge modules*, Comp. Math., 74:2 (1990), pp. 209–234.

[S3] M. Saito, *Mixed Hodge modules*, Publ. RIMS, Kyoto Univ., 26 (1990), pp. 221–333.

[S4] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci, 24:6 (1988), pp. 849–995.

[Sa1] L. Saper, *$\mathcal{L}$-modules and a conjecture of Rapoport and Goresky-Mac Pherson*, Astérisque, 298 (2005), pp. 319–334.

[Sa2] L. Saper, *$\mathcal{L}$-modules and micro-support*, available at `http://arxiv.org/abs/math.RT/0112251`.

[Sch1] J. Schwermer, *On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties*, Forum Math., 7:1 (1995), pp. 1–28.

[Sch2] J. Schwermer, *Eisenstein series and cohomology of arithmetic groups: the generic case*, Invent. Math., 116:1-3 (1994), pp. 481–511.

[T] R. Taylor, *On the l-adic cohomology of Siegel threefolds*, Inv. Math., 114:2 (1993), pp. 289–310.

[TU] J. Tilouine and E. Urban, *Several-variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations*, Ann. Sci. de l'ENS, 4-è série, 32:4 (1999), pp. 499–574.

[V] D. Vogan, *Representations of real reductive groups*, Prog. in Math. 15, Birkhäuser, (1981).

[WY] N. Wallach and O. Yacobi, *A multiplicity formula for tensor products of* SL_2 *modules and an explicit* Sp_{2n} *to* $\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2$ *branching formula*, Symmetry in mathematics and physics, Contemp. Math. 490, Amer. Math. Soc., Providence, RI, (2009), pp. 151–155.

[We] R. Weissauer, *Four dimensional Galois representations*, Astérisque, 302 (2005), pp. 67–150.