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APPROXIMATE CONVERSE THEOREM*

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Abstract. We present an approximate converse theorem which measures how close a given set of irreducible admissible unramified unitary generic local representations of $GL(n)$ is to a genuine cuspidal representation. To get a formula for the measure, we introduce a quasi-Maass form on the generalized upper half plane for a given set of local representations. We also construct an annihilating operator which enables us to write down an explicit cuspidal automorphic function.

Key words. Automorphic representations, Hecke-Maass forms .

AMS subject classifications. 11M41, 11F03, 11F66.

1. Introduction. The spectral theory of non-holomorphic automorphic forms for the Poincaré upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ began with Maass in 1949. It is a highly non-trivial problem to show that there exist infinitely many even Maass forms of the Laplacian for $L^2\left(SL(2,\mathbb{Z})\backslash\mathbb{H}^2\right)$. The existence of infinitely many even Maass forms for $SL(2,\mathbb{Z})$ was first proved by Selberg [20] in 1956. He introduced the trace formula as a tool to obtain Weyl's law, which gives an asymptotic count for the number of Maass forms with Laplacian eigenvalue $|\lambda| \leqslant X$ as $X \to \infty$. Selberg's method was extended by Miller [18] to obtain Weyl's law for Maass forms for $SL(3,\mathbb{Z})$. In 2004, Müller [19] further extended Selberg's method to obtain Weyl's law for Maass forms for the congruence subgroups $\Gamma < SL(n,\mathbb{Z})$, $n \geqslant 2$.

Up to now, no one has found a single explicit example of a Maass form for $SL(n,\mathbb{Z})$, with $n \geqslant 2$, although Maass [16] discovered some examples for congruence subgroups $\Gamma < SL(2,\mathbb{Z})$ of finite index by using Hecke L-functions. In the 1970's a number of authors considered the problem of computing Maass forms for $SL(2,\mathbb{Z})$ numerically. The first notable algorithms for computing Maass forms on $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$ are due to Stark [21] and Hejhal [9]. In 2006, Booker, Strömbergsson and Venkatesh [3] computed the Laplace and many Hecke eigenvalues for the first few Maass forms on $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$ to over 1000 decimal places, based on Hejhal's algorithm. Moreover, they suggested a method of how to check the numerical computation rigorously and verified that Laplacian eigenvalues were correct up to 100 decimal places.

Recently, Lindenstrauss and Venkatesh [13] obtained Weyl's law for spherical cusp forms on $G(\mathbb{Z})\backslash G(\mathbb{R})/K_\infty$ where G is a split semisimple group over $\mathbb{Q}$ and $K_\infty \subset G$ is the maximal compact subgroup. In the Appendix [13], they explained a short constructive proof of the existence of cusp forms using Whittaker functions. This proof does not give Weyl's law, but it gives a very explicit method for constructing cuspidal functions, which was used in [3]. Moreover, this constructive method can be used to attack the following "approximate converse" problem suggested by Peter Sarnak, at the conference in 2008 on *Analytic number theory in higher rank groups*:

Given a positive number X, a set S of places and a representation π_v of $GL(n,\mathbb{Q}_v)$ for $v \in S$, give an algorithm to determine whether or not there is a global automorphic representation σ whose analytic conductor is at most X and whose local component at v is within ϵ-distance from π_v for each place $v \in S$.

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The main goal of this paper is to suggest the approximate converse theorem (Theorem 1.14) as an answer to this question for globally unramified cuspidal automorphic representations of $GL(n,\mathbb{A})$ where $\mathbb{A}$ denotes the ring of adeles over $\mathbb{Q}$.

1.1. Quasi-Maass forms and the annihilating operator. Let $n \geqslant 2$ be an integer. For a place $v \leqslant \infty$, let π_v be an irreducible admissible unramified unitary generic representation of $\mathbb{Q}_v^\times\backslash GL(n,\mathbb{Q}_v)$. Let $L(s,\pi_v)$ be the local L-function attached to π_v. Then there is the associated Satake (or Langlands) parameter $\ell_{\pi_v} = (\ell_{\pi_v,1},\ldots,\ell_{\pi_v,n}) \in \mathbb{C}^n$ with $\ell_{\pi_v,1}+\cdots+\ell_{\pi_v,n} = 0$ such that

$$L(s,\pi_v) = \begin{cases} \pi^{-\frac{ns}{2}} \prod_{j=1}^{n} \Gamma\left(\frac{s+\ell_{\pi_\infty,j}}{2}\right), & \text{if } v=\infty \\ \prod_{j=1}^{n} \left(1-p^{-\ell_{\pi_p,j}-s}\right)^{-1}, & \text{if } v=p<\infty \end{cases} \tag{1.1}$$

for $s \in \mathbb{C}$.

For $v = p$ a finite prime, we have

$$L(s,\pi_p) = \left(1-\lambda_p^{(1)}(\ell_{\pi_p})p^{-s}+\cdots+(-1)^j\lambda_p^{(j)}(\ell_{\pi_p})p^{-js}+\cdots+(-1)^n p^{-ns}\right)^{-1}$$

where

$$\lambda_p^{(j)}(\ell_{\pi_p}) = \sum_{1\leqslant r_1<\cdots<r_j\leqslant n} p^{-(\ell_{\pi_p,r_1}+\cdots+\ell_{\pi_p,r_j})}. \tag{1.2}$$

For $v = \infty$ the infinity prime, we have the Whittaker function $W_J(z;\ell_{\pi_\infty},\pm 1)$ of type ℓ_{π_∞} in (2.14) for $z \in \mathbb{H}^n \cong SL(n,\mathbb{R})/SO(n,\mathbb{R})$, the generalized upper half plane. It is a solution of differential equations given by Casimir operators $\mathcal{C}_n^{(j)}$ in (2.5) with eigenvalues $\lambda_\infty^{(j)}(\ell_{\pi_\infty}) \in \mathbb{C}$ for $j = 1,\ldots,n-1$ in (2.7). The Casimir operators generate the center of the universal enveloping algebra of the Lie algebra of $GL(n,\mathbb{R})$. In particular $\Delta_n = -\mathcal{C}_n^{(1)}$ denotes the Laplacian and

$$\lambda_n(\ell_{\pi_\infty}) := -\lambda_\infty^{(1)}(\ell_{\pi_\infty}) = \frac{n+1}{12} - \frac{1}{n(n-1)}\left(\ell_{\pi_\infty,1}^2+\cdots+\ell_{\pi_\infty,n}^2\right)$$

is the corresponding eigenvalue. The Whittaker functions were constructed by Jacquet [11].

The goal of this paper is to suggest a method to compare a set of local representations and a global cuspidal representation. In order to compare sets of local representations, we should define a "distance" between them. Roughly speaking, the "distance" can be defined by the difference between the coefficients of local L-functions.

Let M a set of places of $\mathbb{Q}$, including ∞. For each $v \in M$, let π_v be an irreducible, admissible, unitary, unramified generic representation of $\mathbb{Q}_v^\times\backslash GL(n,\mathbb{Q}_v)$. Let

$$\Pi_M := \{\pi_v \mid v \in M\}.$$

Definition 1.3 (Distances between sets of local representations). *Let M and M' be sets of places of $\mathbb{Q}$ including ∞ and Π_M, $\Pi'_{M'}$ as above. Let $S \subset M \cap M'$ be a finite subset including ∞. Define the distance between Π_M and $\Pi'_{M'}$ for S to be*

$$d_S(\Pi_M,\Pi'_{M'}) := \sum_{j=1}^{n-1}\left|\lambda_\infty^{(j)}(\ell_{\pi_\infty})-\lambda_\infty^{(j)}(\ell_{\pi'_\infty})\right|^2 + \sum_{\substack{q\in S,\\ \text{finite}}}\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\left|\lambda_q^{(j)}(\ell_{\pi_q})-\lambda_q^{(j)}(\ell_{\pi'_q})\right|^2 \tag{1.4}$$

for $\pi_v \in \Pi_M$ *and* $\pi'_v \in \Pi'_{M'}$.

We use the quasi-mode construction [3] for a given set of local representations Π_M to construct an automorphic function.

For any non-negative integers $k_1, \ldots, k_{n-1}$, let $S_{k_1,\ldots,k_{n-1}}(x_1, \ldots, x_n)$ be a Schur polynomial as in (2.17). For $\pi_p \in \Pi_M$ for a prime p, define

$$A_{\Pi_M}(p^{k_1}, \ldots, p^{k_{n-1}}) := S_{k_1,\ldots,k_{n-1}}(p^{-\ell_{\pi_p,1}}, \ldots, p^{-\ell_{\pi_p,n}}).$$

For any positive integers $m_1, \ldots, m_{n-1}$, we construct $A_{\Pi_M}(m_1, \ldots, m_{n-1})$ satisfying

$$A_{\Pi_M}(m_1, \ldots, m_{n-1}) \cdot A_{\Pi_{M_1}}(m'_1, \ldots, m'_{n-1}) = A_{\Pi_M}(m_1 m'_1, \ldots, m_{n-1} m'_{n-1})$$

if $m_1 \cdots m_{n-1}$ and $m'_1 \cdots m'_{n-1}$ are relative prime to each other. Set $A_{\Pi_M}(m_1, \ldots, m_{n-1}) = 0$ if there exists a prime $q \notin M$ such that $q \mid m_1 \cdots m_{n-1}$.

By combining $W_J(*; \ell_{\pi_\infty}, \pm 1)$, the Whittaker function of type ℓ_{π_∞}, and complex numbers $A_{\Pi_M}(m_1, \ldots, m_{n-1})$ for $m_1, \ldots, m_{n-1} \in \mathbb{Z}$, we construct a function for $z \in \mathbb{H}^n$, which is essentially a Whittaker-Fourier expansion.

Definition 1.5 (Quasi-Maass form). *Let M be a set of places over $\mathbb{Q}$ including ∞. For $z \in \mathbb{H}^n$, define*

$$(1.6) \quad F_{\Pi_M}(z) = \sum_{\gamma \in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A_{\Pi_M}(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}}$$
$$\times W_J\left(\begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z; \ell_{\pi_\infty}, \frac{m_{n-1}}{|m_{n-1}|} \right),$$

Then F_{Π_M} is called a quasi-Maass form of Π_M.

A quasi-Maass form F_{Π_M} is a function on $\mathbb{H}^n$ which lies in the restricted tensor product of local representations $\pi_v \in \Pi_M$. By definition, we can easily observe that F_{Π_M} is an eigenfunction of the Casimir operators $\mathcal{C}_n^{(j)}$ with eigenvalues $\lambda_\infty^{(j)}(\ell_{\pi_\infty})$ for $\pi_\infty \in \Pi_M$, for $j = 1, \ldots, n-1$. It is also an eigenfunction of the Hecke operators. In particular, for each $p \in M$, the $\lambda_p^{(j)}(\ell_{\pi_p})$ for $\pi_p \in \Pi_M$ are eigenvalues of F_{Π_M} of Hecke operators $T_p^{(j)}$ for $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ given in Definition 2.19.

A Hecke-Maass form is a smooth function in $L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ which is an eigenfunction of the Casimir operators and the Hecke operators simultaneously. Every Hecke-Maass form is a quasi-Maass form but not vice versa. For an arbitrary set of local representations, the quasi-Maass form usually is not automorphic for $SL(n,\mathbb{Z})$. Fix a fundamental domain $\mathfrak{F}^n \cong SL(n,\mathbb{Z})\backslash\mathbb{H}^n$ (described in Remark 2.27), and define the automorphic lifting of the quasi-Maass form as follows.

Definition 1.7 (Automorphic lifting). *Define an automorphic lifting*

$$\widetilde{F}_{\Pi_M}(z) := F_{\Pi_M}(\gamma z),$$

for any $z \in \mathbb{H}^n$ and a unique $\gamma \in SL(n,\mathbb{Z})$ such that $\gamma z \in \mathfrak{F}^n$.

Let

$$\widetilde{\mathfrak{F}}^n = \bigcup_{\substack{\gamma\in SL(n-1,\mathbb{Z}),\\ c_1,\dots,c_{n-1}\in\mathbb{Z}}} \begin{pmatrix} & & & c_1 \\ & \gamma & & \vdots \\ & & & c_{n-1} \\ 0 & \dots & 0 & 1 \end{pmatrix} \mathfrak{F}^n, \tag{1.8}$$

then $\widetilde{F}_{\Pi_M}(z) = F_{\Pi_M}(z)$ for $z \in \widetilde{\mathfrak{F}}^n$. The function $\widetilde{F}_{\Pi_M}$ is automorphic for $SL(n,\mathbb{Z})$ and square-integrable. But it is neither smooth nor cuspidal in general. To construct a cuspidal function from $\widetilde{F}_{\Pi_M}$, we use an operator whose image is cuspidal, defined in [13].

The approach in [13] is based on the observation that there are strong relations between the spectrum of the Eisenstein series at different places. From this observation, a convolution operator was constructed, which annihilates the spectrum of the Eisenstein series. So the image of the operator is purely cuspidal. This convolution operator was used to obtain Weyl's law [13] and also was used to give a short and elementary proof which shows the existence of infinitely many Maass forms.

For $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$, this operator was given explicitly in [13] and it was used in [3] to check the numerical computation rigorously. For $n \geqslant 3$, although the operator was defined in a more general case, it is quite complicated and it is not easy to describe this operator explicitly. In this paper, we give an explicit construction of the convolution operator whose image is purely cuspidal. Moreover, the operator annihilates the self-dual spectrum. The construction goes as follows.

Fix a prime p. For $j = 1, 2$, set $\ell_j = (\ell_{j,1}, \dots, \ell_{j,n}) \in \mathbb{C}^n$, $\ell_{j,1} + \cdots + \ell_{j,n} = 0$. Let

$$\widehat{\natural}_p^n(\ell_1, \ell_2) := \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \prod_{1\leqslant j_1<\cdots<j_k\leqslant n} \left(1 - p^{-(\ell_{1,i_1}+\cdots+\ell_{1,i_k})-(\ell_{2,j_1}+\cdots+\ell_{2,j_k})}\right). \tag{1.9}$$

In Lemma 2.22, we use a Paley-Wiener type theorem [10] for $\widehat{\natural}_p^n$ and define the annihilating operator $\natural_p^n$ to be a certain polynomial in convolution operators and Hecke operators at a prime p. Then quasi-Maass forms are eigenfunctions of the operator $\natural_p^n$. For a given set of local representation Π_M, we have $\natural_p^n F_{\Pi_M} = \widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p}) \cdot F_{\Pi_M}$ for $\pi_\infty, \pi_p \in \Pi_M$.

As in [13], we prove the following theorem.

Theorem 1.10. *Let $n \geqslant 2$. The space of the image of the annihilating operator $\natural_p^n$ on smooth functions in $L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ is cuspidal and infinite dimensional. So there are infinitely many Hecke-Maass forms in $L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ which are not self-dual.*

We apply the annihilating operator $\natural_p^n$ to $\widetilde{F}_{\Pi_M}$ to construct a non-trivial cuspidal function. Since $\natural_p^n$ is a polynomial in convolution operators associated to some compactly supported distributions, we need to make the function $\widetilde{F}_{\Pi_M}$ to be smooth.

For any $g \in SL(n,\mathbb{R})$, by Polar decomposition, there exist $\xi_1, \xi_2 \in SO(n,\mathbb{R})$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$, $a_1 + \cdots + a_n = 0$ such that $g = \xi_1 \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{pmatrix} \xi_2$. As in [12], we define a polar height $\sigma(g) \geqslant 0$ to be

$$\sigma(g) := \sqrt{a_1^2 + \cdots + a_n^2}, \qquad \text{for any } g \in SL(n,\mathbb{R}). \tag{1.11}$$

Then $\sigma(g^{-1}) = \sigma(g)$ and $\sigma(g_1 g_2) \leqslant \sigma(g_1) + \sigma(g_2)$ for any $g_1, g_2 \in SL(n,\mathbb{R})$. For any $\delta > 0$, define

$$B_\delta := \{g \in SL(n,\mathbb{R}) \mid \sigma(g) < \delta\}\,. \tag{1.12}$$

For $\delta > 0$, let H_δ be a standard bump function with $\mathrm{supp}(H_\delta) \subset B_\delta$. Then $\widetilde{F}_{\Pi_M} * H_\delta(g) = \int_{SL(n,\mathbb{R})} \widetilde{F}_{\Pi_M}(gh^{-1}) H_\delta(h)\, dh$ is a smooth function. By Lemma 2.20, if

$$0 < \delta \leqslant \ln\left(\frac{n(n+6)}{8}\left(\sum_{j=1}^{n}\left|\ell_{\pi_\infty,j} + \frac{n-2j+1}{2}\right|\right)^{-1} + 1\right), \tag{1.13}$$

then $\widetilde{F}_{\Pi_M} * H_\delta$ is non-trivial. So $\natural_p^n(\widetilde{F}_{\Pi_M} * H_\delta)$ is a cuspidal function by Theorem 1.10.

The quasi-Maass form F_{Π_M}, and its automorphic and cuspidal liftings play important roles in the approximate converse theorem.

1.2. Approximate Converse Theorem. In this section we present an approximate converse theorem which measures how close a given set of irreducible admissible unramified unitary generic local representations of $GL(n)$ is to a global cuspidal representation.

For a finite prime q and for $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$, let $G_q^{(j)}$ be a finite set of matrices which generate the Hecke operator $T_q^{(j)}$ defined in Definition 2.19 by left translations. Let $\sharp(T_q^{(j)})$ be the number of elements in $G_q^{(j)}$. For a subset $V \subset \mathbb{H}^n$, let

$$T_q^{(j)} V := \bigcup_{\gamma \in G_q^{(j)}} \{\gamma z \mid z \in V\}$$

and

$$(T_q^{(j)})^{-1} V := \bigcup_{\gamma \in G_q^{(j)}} \{z \in \mathbb{H}^n \mid \gamma z \in V\}\,.$$

Theorem 1.14 (Approximate Converse Theorem). *Let M be a set of places of $\mathbb{Q}$ including ∞ properly and $S \subset M$ be a finite subset including ∞. Assume that $\widehat{\natural_p^n}(\ell_{\pi_\infty}, \ell_{\pi_p}) \neq 0$ for some $p \in M$ and $\pi_\infty, \pi_p \in \Pi_M$.*

For δ given in (1.13), there exists an unramified cuspidal representation π of $\mathbb{A}^\times \backslash GL(n,\mathbb{A})$ such that

$$d_S(\pi, \Pi_M) < \sup_{z \in B_2}\left|F_{\Pi_M}(z) - \widetilde{F}_{\Pi_M}(z)\right|^2 \times \frac{4\left(p^{-\frac{n^2-1}{2(n^2+1)}} + p^{\frac{n^2-1}{2(n^2+1)}}\right)^{n2^{n-1}} \cdot \mathrm{Vol}(B_1)\cdot(A_\infty + A_{S,\mathrm{finite}})}{\left|\widehat{\natural_p^n}(\ell_{\pi_\infty}, \ell_{\pi_p})\right|^2 \cdot \int_{e^{4(2^{n-1}\ln p + \delta)}}^{\infty} \cdots \int_{e^{4(2^{n-1}\ln p + \delta)}}^{\infty} |W_J(y;\ell_{\pi_\infty},1)|^2\, d^*y}, \tag{1.15}$$

where

$$A_\infty = \sum_{j=1}^{n-1}\left(\int_{SL(n,\mathbb{R})}\left|\left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H_\delta(g)\right|\, dg\right)^2,$$

$$A_{S,\text{finite}} = \begin{cases} 0, & \textit{if } S = \{\infty\}\ , \\ e^{\frac{n(n+6)}{4}\delta} \sum\limits_{\substack{q\in S,\\ \textit{finite}}} \sum\limits_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sharp T_q^{(j)}\right)^2, & \textit{otherwise,} \end{cases}$$

$$B_1 = \begin{cases} \left(\mathbb{H}^n - \widetilde{\mathfrak{F}}^n\right) \cdot B_\delta \cap \mathfrak{F}^n, & \textit{if } S = \infty, \\ \left(T_{q_{\max}}^{\lfloor \frac{n}{2} \rfloor}\right)^{-1} \left(\mathbb{H}^n - \widetilde{\mathfrak{F}}^n\right) \cap \mathfrak{F}^n, & \textit{otherwise,} \end{cases}$$

and

$$B_2 = \begin{cases} \mathfrak{F}^n \cdot B_\delta - \mathfrak{F}^n, & \textit{if } S = \{\infty\}\,, \\ T_{q_{\max}}^{\lfloor \frac{n}{2} \rfloor} \mathfrak{F}^n - \mathfrak{F}^n, & \textit{otherwise.} \end{cases}$$

Here $q_{\max} = \max\left(S - \{\infty\}\right)$.

Remark 1.16.

1. *If the right hand side of (1.15) is sufficiently small for sufficiently large S, then by Remark 8 [4], the genuine cuspidal representation π can be uniquely determined.*
2. *For an unramified cuspidal representation $\sigma \cong \otimes_v' \sigma_v$ of $\mathbb{A}^\times \backslash GL(n, \mathbb{A})$, define an analytic conductor $\mathcal{C}(\sigma) := \prod_{j=1}^{n} \left(1 + |\ell_{\sigma_\infty, j}|\right)$ as in [4]. Fix $Q \geqslant 2$. By [4], for any unramified cuspidal representation $\sigma \cong \otimes_v' \sigma_v$ with $\mathcal{C}(\sigma) \leqslant Q$, there exists a prime $p \ll \log Q$ such that $\left|\widehat{\natural}_p^n \left(\ell_{\sigma_\infty}, \ell_{\sigma_p}\right)\right|$ is sufficiently large.*

There are two important ingredients in the proof of Theorem 1.14: the annihilating operator $\natural_p^n$ and the spectral decomposition of $L^2\left(SL(n, \mathbb{Z})\backslash \mathbb{H}^n\right)$. For a given set of local representations Π_M, we can construct a cuspidal function explicitly, by applying $\natural_p^n$ to the automorphic lifting of a quasi-Maass form F_{Π_M}. The procedure is described in the previous section. Since the function is cuspidal, it is generated by Hecke-Maass forms for $SL(n, \mathbb{Z})$. We can get (1.15) by applying the Casimir operators and Hecke operators and compare the eigenvalues.

One could try to generalize this theorem by relaxing the globally unramified condition.

Another particular question attracts our attention. How close can a given positive real number get to a Laplace eigenvalue of an actual Maass form? The following is a sample case of Theorem 1.14, which can be regarded as an answer to this question.

Theorem 1.17. *Let M be a set of places of $\mathbb{Q}$ including ∞ properly. Assume that $\widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p}) \neq 0$ for at least one prime $p \in M$ and $\pi_\infty, \pi_p \in \Pi_M$. Then the Laplace eigenvalue $\lambda_n(\ell_{\pi_\infty}) \in \mathbb{C}$ of F_{Π_M} satisfies the following: for any δ given in* (1.13)*, there* *exists an eigenvalue λ of the Laplacian of a Maass form for $SL(n,\mathbb{Z})$ such that*

$$|\lambda - \lambda_n(\ell_{\pi_\infty})|^2 < \sup_{z\in\mathfrak{F}_n\cdot B_\delta} \left|\widetilde{F}_{\Pi_M}(z) - F_{\Pi_M}(z)\right|^2$$

$$\times \frac{\left(p^{-\frac{n^2-1}{2(n^2+1)}} + p^{\frac{n^2-1}{2(n^2+1)}}\right)^{n2^{n-1}} \cdot \mathrm{Vol}\left(\left((\mathbb{H}^n - \widetilde{\mathfrak{F}}^n)\cdot B_\delta\right)\cap \mathfrak{F}^n\right)}{\left|\widehat{\natural}_p^n(\ell_{\pi_\infty},\ell_{\pi_p})\right|^2 \cdot \int\limits_{e^{4(2^{n-1}\ln p+\delta)}}^{\infty} \cdots \int\limits_{e^{4(2^{n-1}\ln p+\delta)}}^{\infty} |W_J(y;\ell_{\pi_\infty},1)|^2 \, d^*y}$$

$$\times \left[\frac{6(1+e^{2\delta})}{\delta^4} C_\delta \cdot \mathrm{Vol}(B_\delta) + 2\lambda_n(\ell_{\pi_\infty})\right]^2$$

where $C_\delta = \left(\int_{B_\delta} e^{-\frac{1}{1-(\delta^{-1}\sigma(g))^2}}\, dg\right)^{-1}$.

We choose a “good” bump function H_δ to prove this theorem.

Recently, Booker and his student Bian computed the Laplace and Hecke eigenvalues for Maass forms on $SL(3,\mathbb{Z})\backslash\mathbb{H}^3$ [1], [2]. Moreover, Mezhericher presented an algorithm for evaluating a (quasi-) Maass form for $SL(3,\mathbb{Z})$ in his thesis [17]. We expect that we might use the approximate converse theorem to certify Bian’s computations.

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2. Quasi-Maass forms and the Annihilating operator. We review basic facts about the Hecke-Maass forms in the first two sections. The main reference is [6].

In the introduction, we define quasi-Maass forms corresponding to the given set of local representations. By applying the annihilating operator $\natural_p^n$, it is possible to write down a cuspidal function explicitly. We study about the quasi-Maass forms and the annihilating operator in §2.3.

2.1. Preliminaries. Let $n \geqslant 2$ be an integer. Let

$$A^0(n,\mathbb{R}) = \left\{ \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n}\end{pmatrix} \;\middle|\; \begin{matrix} a_1,\dots,a_n \in \mathbb{R}, \\ a_1+\cdots+a_n = 0\end{matrix} \right\} \subset SL(n,\mathbb{R})$$

then

$$\mathfrak{a}(n) = \{a = (a_1,\dots,a_n) \in \mathbb{R}^n \mid a_1+\cdots+a_n = 0\}$$

is isomorphic to the Lie algebra of $A^0(n,\mathbb{R})$. For each $a = (a_1,\dots,a_n) \in \mathfrak{a}(n)$, define

$$\exp a = \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n}\end{pmatrix} \in A^0(n,\mathbb{R}).$$

Let

$$N(n,\mathbb{R}) := \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \middle| \; x_{i,j} \in \mathbb{R}, \quad \text{for } 1 \leqslant i < j \leqslant n \right\}.$$

Define the generalized upper half plane $\mathbb{H}^n$ to be a set of matrices $z = xy \in SL(n,\mathbb{R})$ such that

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in N(n,\mathbb{R}) \tag{2.1}$$

and

$$y = a_{y_1,\dots,y_{n-1}} := \left(\prod_{j=1}^{n-1} y_j^{n-j} \right)^{-\frac{1}{n}} \cdot \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & y_1 \cdots y_{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in A^0(n,\mathbb{R}) \tag{2.2}$$

for $y_1, \dots, y_{n-1} > 0$. By the Iwasawa decomposition, any $g \in SL(n,\mathbb{R})$ can be written uniquely as $g = xy\xi$ with $x \in N(n,\mathbb{R})$, $y \in A_0(n,\mathbb{R})$ and $\xi \in SO(n,\mathbb{R})$. So the generalized upper half plane $\mathbb{H}^n$ can be identified with the quotient $SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

Define $\mathrm{Iw}_Y(g) \in \mathfrak{a}(n)$ to be

$$\exp(\mathrm{Iw}_Y(g)) = a_{y_1,\dots,y_{n-1}}. \tag{2.3}$$

LEMMA 2.4. *For any $g \in SL(n,\mathbb{R})$, let $a_{y_1,\dots,y_{n-1}} = \exp(\mathrm{Iw}_Y(g))$ for $y_1, \dots, y_{n-1} > 0$. Then we have*

$$e^{-2\sigma(g)} \leqslant y_1 \leqslant e^{2\sigma(g)}$$

and

$$e^{-4\sigma(g)} \leqslant y_j \leqslant e^{4\sigma(g)}, \text{ (for } j = 2, \dots, n-1).$$

Proof of Lemma 2.4. By the Iwasawa decomposition and the polar decomposition, we have $g = z\xi_{\mathrm{Iw}} = \xi_1 \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{pmatrix} \xi_2$, for $\xi_{\mathrm{Iw}}, \xi_1, \xi_2 \in SO(n,\mathbb{R})$ and $(a_1, \dots, a_n) \in \mathfrak{a}(n)$, where $z = xa_{y_1,\dots,y_{n-1}} \in \mathbb{H}^n$ for $x \in N(n,\mathbb{R})$ and $a_{y_1,\dots,y_{n-1}} = \exp(\mathrm{Iw}_Y(g))$. Then

$$z \cdot {}^t z = x \cdot a_{y_1,\dots,y_{n-1}}^2 \cdot {}^t x = k_1 \cdot \begin{pmatrix} e^{2a_1} & & \\ & \ddots & \\ & & e^{2a_n} \end{pmatrix} \cdot {}^t k_1.$$

Let $y_n' = \prod_{j=1}^{n-1} y_j^{-\frac{2(n-j)}{n}}$ and $y_j' = y_n' \cdot (y_1 \cdots y_{n-j})^2$ for $j = 1, \ldots, n-1$. Comparing the diagonal parts, we have $e^{-2\sigma(g)} \leqslant y_j' \leqslant e^{2\sigma(g)}$, for $j = 1, \ldots, n$, since $\sigma(g^{-1}) = \sigma(g)$. Therefore we have

$$e^{-4\sigma(g)} \leqslant y_j' \cdot y_n'^{-1} = (y_1 \cdots y_{n-j})^2 \leqslant e^{4\sigma(g)}$$

for $j = 1, \ldots, n-1$. ∎

Let $\mathfrak{a}^*(n) = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}$ be the dual space of $\mathfrak{a}(n)$ and $\mathfrak{a}^*_{\mathbb{C}}(n) = \mathfrak{a}^*(n) \otimes_{\mathbb{R}} \mathbb{C}$. For $\alpha, \alpha' \in \mathfrak{a}(n)$ or $\mathfrak{a}^*_{\mathbb{C}}(n)$, let $\langle \alpha, \alpha' \rangle = \sum_{j=1}^n \alpha_j \alpha_j'$. Then $\langle , \rangle$ is a (complex) symmetric bilinear form and it is positive definite on $\mathfrak{a}^*(n)$. For $\alpha \in \mathfrak{a}^*(n)$, we put $||\alpha|| = \sqrt{\langle \alpha, \alpha \rangle}$. Let W_n denote the Weyl group of $(\mathbb{R}^\times \cap SL(n,\mathbb{R}))\backslash SL(n,\mathbb{R})$, consisting of all $n \times n$ matrices in $SL(n,\mathbb{Z}) \cap SO(n,\mathbb{R})$ which have exactly one ± 1 in each row and column. The Weyl group W_n acts on $\mathfrak{a}(n)$ and $\mathfrak{a}^*_{\mathbb{C}}(n)$ as a permutation group.

Let $\mathfrak{gl}(n,\mathbb{R})$ be the Lie algebra of $GL(n,\mathbb{R})$ with the Lie bracket $[,]$ given by $[X,Y] = XY - YX$ for $X, Y \in \mathfrak{gl}(n,\mathbb{R})$. The universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ can be realized as an algebra of differential operators D_X acting on smooth functions $f : GL(n,\mathbb{R}) \to \mathbb{C}$. The action is given by

$$D_X f(g) := \frac{\partial}{\partial t} f(g \exp(tX)) \Big|_{t=0}$$

for $X \in \mathfrak{gl}(n,\mathbb{R})$ where $\exp(tX) = \sum_{n=0}^\infty \frac{(tX)^n}{n!}$. For $1 \leqslant i,\ j \leqslant n$, let $E_{i,j} \in \mathfrak{gl}(n,\mathbb{R})$ be the matrix with 1 at the (i,j)th entry and 0 elsewhere. Let $D_{i,j} = D_{E_{i,j}}$ for $1 \leqslant i,\ j \leqslant n$. For $j = 1, \ldots, n-1$, we define Casimir operators $\mathcal{C}_n^{(j)}$ given by

$$\mathcal{C}_n^{(j)} = \frac{(n-j-1)!}{n!} \sum_{i_1=1}^n \cdots \sum_{i_j=1}^n D_{i_1,i_2} \circ D_{i_2,i_3} \circ \cdots \circ D_{i_{j+1},i_1} \tag{2.5}$$

where $\circ$ is the composition of differential operators. Let $\Delta_n := -\mathcal{C}_n^{(1)}$ be the Laplace operator. Let $\mathcal{Z}^n$ be the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$. It is well known that $\mathcal{Z}^n \cong \mathbb{R}\left[\mathcal{C}_n^{(1)}, \ldots, \mathcal{C}_n^{(n-1)}\right]$.

There is a standard procedure to construct simultaneous eigenfunctions of all differential operators of $D \in \mathcal{Z}^n$. For $\ell = (\ell_1, \ldots, \ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$, define

$$\varphi_\ell(g) := \prod_{j=1}^{n-1} y_j^{\sum_{k=1}^{n-j}\left(\ell_k + \frac{n-2k+1}{2}\right)}, \tag{2.6}$$

where $\exp(\mathrm{Iw}_Y(g)) = a_{y_1,\ldots,y_{n-1}}$ for $g \in SL(n,\mathbb{R})$. Then φ_ℓ is a simultaneous eigenfunction of $\mathcal{Z}^n$. For $j = 1, \ldots, n-1$, define $\lambda_\infty^{(j)}(\ell) \in \mathbb{C}$ to be the eigenvalue of $\mathcal{C}_n^{(j)}$ for φ_ℓ, such that

$$\mathcal{C}_n^{(j)} \varphi_\ell(g) = \lambda_\infty^{(j)}(\ell) \cdot \varphi_\ell(g), \quad (\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)). \tag{2.7}$$

Lemma 2.8. *Let $n \geqslant 2$ and $\ell = (\ell_1, \ldots, \ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$. The Laplace eigenvalue is*

$$\lambda_n(\ell) = -\lambda_\infty^{(1)}(\ell) = \frac{n+1}{12} - \frac{1}{n(n-1)}(\ell_1^2 + \cdots + \ell_n^2). \tag{2.9}$$

Proof of Lemma 2.8. For any $y \in A^0(n,\mathbb{R})$, consider $\Delta_n \varphi_\ell(y)$. Then

$$\Delta_n \varphi_\ell(y) = -\frac{1}{n(n-1)} \left\{ \sum_{j=1}^{n} D_{j,j} \circ D_{j,j} \varphi_\ell(y) + \sum_{1 \leqslant i < j \leqslant n} \left(D_{i,j} \circ D_{j,i} + D_{j,i} \circ D_{i,j}\right) \varphi_\ell(y) \right\}.$$

For any $x \in N(n,\mathbb{R})$, we have $\varphi_\ell(yx) = \varphi_\ell(y)$. So $D_{j,i}\varphi_\ell(y) = 0$ for $1 \leqslant i < j \leqslant n$. For $1 \leqslant i < j \leqslant n$, we have

$$D_{i,j} \circ D_{j,i} + D_{j,i} \circ D_{i,j} = 2D_{i,j} \circ D_{j,i} + D_{j,j} - D_{i,i}.$$

Therefore

$$\begin{aligned} \Delta_n \varphi_\ell(y) &= -\frac{1}{n(n-1)} \left\{ \sum_{j=1}^{n} D_{j,j}^2 \varphi_\ell(y) + \sum_{1 \leqslant i < j \leqslant n} (D_{j,j} - D_{i,i}) \varphi_\ell(y) \right\} \\ &= -\frac{1}{n(n-1)} \left\{ \sum_{j=1}^{n} (D_{j,j}^2 - (n-2j+1) D_{j,j}) \varphi_\ell(y) \right\} \end{aligned}$$

$$= -\frac{1}{n(n-1)} \sum_{j=1}^{n} \left\{ \left(\frac{n-2j+1}{2} + \ell_j \right)^2 - (n-2j+1) \left(\frac{n-2j+1}{2} + \ell_j \right) \right\} \varphi_\ell(y). \quad \square$$

For $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$, we define the spherical function of type ℓ:

$$\beta_\ell(g) := \int_{SO(n,\mathbb{R})} \varphi_\ell(\xi g)\, d\xi \tag{2.10}$$

for $g \in SL(n,\mathbb{R})$. Here $d\xi$ is the normalized Haar measrue $d\xi$ on $SO(n,\mathbb{R})$. Then the spherical function β_ℓ satisfies the followings:

- $\beta_\ell(\xi_1 g \xi_2) = \beta_\ell(g)$ for any $\xi_1,\ \xi_2 \in SO(n,\mathbb{R})$
- β_ℓ is an eigenfunction of the Casimir operators $\mathcal{C}_n^{(j)}$ with an eigenvalue $\lambda_\infty^{(j)}(\ell)$ for $j = 1, \ldots, n-1$
- $\beta_\ell(1) = 1$

We identify $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ with the right $SO(n,\mathbb{R})$-invariant subspace of $L^2\left(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})\right)$. If k is a bi-$SO(n,\mathbb{R})$-invariant compactly supported smooth function on $SL(n,\mathbb{R})$, it gives rise to a convolution operator $f \to f * k$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ given by

$$f * k(g) = \int_{SL(n,\mathbb{R})} f(gh^{-1}) k(h)\, dh.$$

More generally, one can consider convolution with compactly supported distributions instead of functions. In this case, the convolution operator is well defined only on suitable smooth functions f, for example on $C^\infty\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$.

Let k be a bi-$SO(n,\mathbb{R})$-invariant compactly supported smooth function on $SL(n,\mathbb{R})$. For any $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$, the spherical function β_ℓ is an eigenfunction of the corresponding convolution operator. The spherical transform $\widehat{k}(\ell) \in \mathbb{C}$ is defined to be the corresponding eigenvalue:

$$\left(\beta_\ell * k\right)(g) = \widehat{k}(\ell) \cdot \beta_\ell(g) \tag{2.11}$$

for $g \in SL(n,\mathbb{R})$. The inverse of the spherical transform is given explicitly in terms of the Plancherel measure μ_{Planch} on $\mathfrak{a}^*_{\mathbb{C}}(n)$ by

$$k(g) = \int_{i\mathfrak{a}^*(n)} \widehat{k}(\alpha) \beta_\alpha(g)\, d\mu_{\mathrm{Planch}}(\alpha) \quad ([10]).$$

2.2. Automorphic functions. The group $SL(n,\mathbb{Z})$ acts on $\mathbb{H}^n$ discretely. Fix $a,b \geqslant 0$. We define the Siegel set $\Sigma_{a,b} \subset \mathbb{H}^n$ to be the set of all matrices of the form

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot a_{y_1,\ldots,y_{n-1}}$$

with $|x_{i,j}| \leqslant b$ for $1 \leqslant i < j \leqslant n$ and $y_i > a$ for $1 \leqslant i \leqslant n-1$. Let $\Sigma_a := \Sigma_{a,\infty}$.

The Sigel set $\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$ is a good approximation of a fundamental domain: $\mathbb{H}^n = \bigcup_{\gamma \in SL(n,\mathbb{Z})} \gamma \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$.

An automorphic function for $SL(n,\mathbb{Z})$ is a function $f : \mathbb{H}^n \to \mathbb{C}$ such that $f(\gamma z) = f(z)$ for any $\gamma \in SL(n,\mathbb{Z})$ and $z \in \mathbb{H}^n$. Consider $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ to be the space of automorphic functions $f : \mathbb{H}^n \to \mathbb{C}$ satisfying

$$||f||_2^2 := \int_{SL(n,\mathbb{Z})\backslash\mathbb{H}^n} |f(z)|^2 \; d^*z < \infty$$

where $d^*z = d^*x \; d^*y$ is the left invariant $GL(n,\mathbb{R})$-measure on $\mathbb{H}^n$. Here $d^*x = \prod_{1 \leqslant i<j \leqslant n} dx_{i,j}$, and $d^*y = \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k$. For $f_1, \; f_2 \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, define the inner product

$$\langle f_1, f_2 \rangle := \int_{SL(n,\mathbb{Z})\backslash\mathbb{H}^n} f_1(z)\overline{f_2(z)} \; d^*z.$$

Let R be a commutative ring with identity 1. For positive integers $n \geqslant 2$ and $1 \leqslant n_1, \ldots, n_r \leqslant n$ with $n_1 + \cdots + n_r = n$, define

$$P_{n_1,\ldots,n_r}(n,R) := \left\{ \begin{pmatrix} A_1 & & * \\ & \ddots & \\ & & A_r \end{pmatrix} \in SL(n,R) \;\middle|\; \begin{matrix} A_i \in SL(n_i,R), \\ 1 \leqslant i \leqslant r \end{matrix} \right\}$$

to be the standard parabolic subgroup of $SL(n,R)$ associated to $(n_1,\ldots,n_r)$. The integer r is termed the rank of the parabolic subgroup $P_{n_1,\ldots,n_r}(n,R)$. Define

$$M_{n_1,\ldots,n_r}(n,R) := \left\{ \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix} \;\middle|\; A_i \in SL(n_i,R), \; 1 \leqslant i \leqslant r \right\}$$

to be the standard Levi subgroup of $P_{n_1,\ldots,n_r}(n,R)$. Define

$$N_{n_1,\ldots,n_r}(n,R) := \left\{ \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \in SL(n,R) \right\}$$

where I_k is the $k \times k$ identity matrix for an integer $k \geqslant 1$, to be the unipotent radical of $P_{n_1,\ldots,n_r}(n,R)$.

The automorphic function f for $SL(n,\mathbb{Z})$ is cuspidal if

$$\int\limits_{(SL(n,\mathbb{Z})\cap N_{n_1,\ldots,n_r}(n,\mathbb{Z}))\backslash N_{n_1,\ldots,n_r}(n,\mathbb{R})} f(uz)\, d^*u = 0$$

for any partition $n_1 + \cdots + n_r = n$ and $r \geqslant 1$. Let $L^2_{\text{cusp}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ denote the space of automorphic cuspidal functions in $L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$.

For each positive integer $N \geqslant 1$, define

$$G_N := \left\{ \begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \,\middle|\, \begin{array}{l} c_1\cdots c_n = N,\ c_1,\ldots,c_n > 0 \\ 0 \leqslant c_{i,j} < c_i \ (1 \leqslant i < j \leqslant n) \end{array} \right\}. \tag{2.12}$$

Let $f : \mathbb{H}^n \to \mathbb{C}$ be a function. For each integer $N \geqslant 1$, we define a Hecke operator

$$T_N f(z) := \frac{1}{N^{\frac{n-1}{2}}} \sum_{\gamma\in G_N} f(\gamma z). \tag{2.13}$$

Clearly T_1 is the identity operator. If $f \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ then $T_N f \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$. For $n = 2$, the Hecke operators are self-adjoint with respect to the inner product. For $n \geqslant 3$, the Hecke operator is no longer self-adjoint, but the adjoint operator is again a Hecke operator and the Hecke operator commutes with its adjoint, so it is a normal operator.

If a smooth function $f \in L_{\text{cusp}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$ is a simultaneous eigenfunction of Casimir operators $\mathcal{C}_n^{(j)}$ for $j = 1,\ldots,n-1$ and Hecke operators T_N for any $N \geqslant 1$, then f is called a Hecke-Maass form.

If f is a Hecke-Maass form, then there exists $\ell_\infty(f) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ such that

$$\mathcal{C}_n^{(j)} f = \lambda_\infty^{(j)}(\ell_\infty(f)) \cdot f.$$

For $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$ and $\epsilon = \pm 1$, define

$$W_J(z;\ell,\epsilon) := \int_{N(n,\mathbb{R})} \varphi_\ell\left(\begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2} \rfloor} \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \cdot \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ & 1 & \cdots & u_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} z \right) \times e^{-2\pi i(\epsilon\cdot u_{1,2} - u_{2,3} - \cdots - u_{n-1,n})}\, d^*u. \tag{2.14}$$

to be Jacquet's Whittaker function of type ℓ. Then $\mathcal{C}_n^{(j)} W_J(z;\ell,\epsilon) = \lambda_\infty^{(j)}(\ell)\cdot W_J(z;\ell,\epsilon)$ for $j = 1,\ldots,n-1$. For any $u = \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ & 1 & \cdots & u_{2,n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \in N(n,\mathbb{R})$, we have

$$W_J(uz;\ell,\epsilon) = e^{2\pi i(-\epsilon\cdot u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})} \cdot W_J(z;\ell,\epsilon)$$

for any $z \in \mathbb{H}^n$. Moreover,

$$\int\limits_{\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}} |W(z;\ell,\epsilon)|^2\, d^*z < \infty.$$

By (9.1.2) [6], every Hecke-Maass form f has a Fourier-Whittaker expansion of the form

$$(2.15)\quad f(z) = \sum_{\gamma \in N_{n-1}\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A_f(m_1,\ldots,m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \times W_J\left(\begin{pmatrix} m_1\cdots|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1\end{pmatrix} \cdot \begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z; \ell_\infty(f), \frac{m_{n-1}}{|m_{n-1}|}\right)$$

where $A_f(m_1,\ldots,m_{n-1}) \in \mathbb{C}$. We assume that $A_f(1,\ldots,1) = 1$. Then $T_N f = A_f(N,1,\ldots,1)\cdot f$ for any integer $N \geqslant 1$ and $A_f(m_1,\ldots,m_{n-1})$ satisfies the following (multiplicative) relation [6]: for $(m_1,\ldots,m_{n-1}) \in \mathbb{Z}^{n-1}$, and an integer $m \geqslant 1$, we have

$$(2.16)\quad A_f(m,1,\ldots,1)A_f(m_1,\ldots,m_{n-1}) = \sum_{\substack{\prod_{j=1}^n c_j = m,\\ c_1|m_1,\ldots,c_{n-1}|m_{n-1}}} A_f\left(\frac{m_1c_n}{c_1},\ldots,\frac{m_jc_{j-1}}{c_j},\ldots,\frac{m_{n-1}c_{n-2}}{c_{n-1}}\right)$$

and here $c_1,\ldots,c_n > 0$. Since Hecke operators are normal, we have $A_f(m_{n-1},\ldots,m_1) = \overline{A_f(m_1,\ldots,m_{n-1})}$.

For any non-negative integers $k_1,\ldots,k_{n-1}$, let

$$(2.17)\quad S_{k_1,\ldots,k_{n-1}}(x_1,\ldots,x_n) := \left|\begin{pmatrix} x^{k_1+\cdots+k_{n-1}+n-1} & \ldots & x_n^{k_1+\cdots+k_{n-1}+n-1} \\ x_1^{k_1+\cdots+k_{n-2}+n-2} & \ldots & x_n^{k_1+\cdots+k_{n-2}+n-2} \\ \vdots & \vdots & \vdots \\ x_1^{k_1+1} & \ldots & x_n^{k_1+1} \\ 1 & \ldots & 1\end{pmatrix}\right| \cdot \left|\begin{pmatrix} x_1^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \vdots \\ x_1 & \ldots & x_n \\ 1 & \ldots & 1\end{pmatrix}\right|^{-1}$$

be a Schur polynomial.

Since f is an eigenfunction of Hecke operators, there exist $\ell_p(f) = (\ell_{p,1}(f),\ldots,\ell_{p,n}(f)) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ for any finite prime p, such that

$$(2.18)\qquad A_f(p^{k_1},\ldots,p^{k_{n-1}}) = S_{k_1,\ldots,k_{n-1}}(p^{-\ell_{p,1}(f)},\ldots,p^{-\ell_{p,n}(f)}).$$

Definition 2.19. *Let $n \geqslant 2$ be an integer and fix a prime p. For $j = 1,\ldots,n-1$, define*

$$T_p^{(j)} := \sum_{k=0}^{j-1} (-1)^k T_{p^{k+1}} T_p^{(j-k-1)}$$

where $T_{p^r}^{(1)} = T_{p^r}$ for any integer $r \geqslant 0$ and $T_p^{(0)}$ is an identity operator.

By the multiplicative relations (2.16), we have

$$T_p^{(j)} f = A_f(\underbrace{1,\ldots,1,p}_{j},1,\ldots,1)\cdot f, \quad (\text{ for } j = 1,\ldots,n-1)$$

for any prime p. Then

$$\lambda_p^{(j)}(\ell_p(f)) = A_f(\underbrace{1,\ldots,1,p}_{j},1,\ldots,1)$$

for $j = 1,\ldots,n-1$.

Let $n \geqslant 2$ be an integer. If a Maass form f satisfies

$$f(z) = \widetilde{f}(z) := f\left(w \cdot {}^t(z^{-1}) \cdot w\right), \quad w = \begin{pmatrix} & & & (-1)^{\lfloor \frac{n}{2} \rfloor} \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

then f is called a self-dual Maass form.

2.3. Quasi-Maass forms and the annihilating operator. Let M be a set of places over $\mathbb{Q}$ including ∞ and Π_M be a set of local representations as given in the introduction. We construct a quasi-Maass form F_{Π_M} of Π_M on $\mathbb{H}^n$, which lies in the restricted tensor product of local representations $\pi_v \in \Pi_M$ in Definition 1.5. Then for $j = 1,\ldots,n-1$, we have $\mathcal{C}_n^{(j)} F_{\Pi_M} = \lambda_\infty^{(j)}(\ell_{\pi_\infty}) \cdot F_{\Pi_M}$ and $T_q^{(j)} F_{\Pi_M} = \lambda_q^{(j)}(\ell_{\pi_q}) \cdot F_{\Pi_M}$.

To define the automorphic lifting of quasi-Maass forms, we fix a fundamental domain for $SL(n,\mathbb{Z})\backslash\mathbb{H}^n$. We define $\mathfrak{F}^n$ to be the susbset of the Siegel set $\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$, which contains $\Sigma_{1,\frac{1}{2}}$, satisfying:

- for any $z \in \mathbb{H}^n$, there exists $\gamma \in SL(n,\mathbb{Z})$ such that $\gamma z \in \mathfrak{F}^n$;
- for any $z \in \mathfrak{F}^n$, $\gamma z \notin \mathfrak{F}^n$ for any $I_n \neq \gamma \in SL(n,\mathbb{Z})$ where I_n is the $n \times n$ identity matrix.

Then $\mathfrak{F}^n$ becomes a fundamental domain for $SL(n,\mathbb{Z})$.

In Definition 1.7, we defined the automorphic lifting $\widetilde{F}_{\Pi_M}$ of a quasi-Maass form F_{Π_M} with respect to the fixed fundamental domain $\mathfrak{F}^n$ and $\widetilde{F}_{\Pi_M} \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$.

We get a smooth automorphic lifting of a quasi-Maass form F_{Π_M} defined via $\widetilde{F}_{\Pi_M} * \kappa$ for any nonzero smooth compactly supported bi-$SO(n,\mathbb{R})$-invariant function κ on $SL(n,\mathbb{R})$.

For $\delta > 0$, if a smooth compactly supported bi-$SO(n,\mathbb{R})$-invariant function H_δ satisfies the following conditions:

- $\mathrm{supp}(H) \subset B_\delta$;
- $H_\delta(g) = H_\delta(g^{-1})$ for any $g \in SL(n,\mathbb{R})$;
- $H_\delta(g) \geqslant 0$;
- $\int_{SL(n,\mathbb{R})} H_\delta(g)\, dg = 1$

then it is called the standard bump function. By Lemma 2.4, for $z \in \Sigma_{e^{4\delta}}$, we have

$$\widetilde{F}_{\Pi_M} * H_\delta(z) = F_{\Pi_M} * H_\delta(z) = \widehat{H}_\delta(\ell_{\pi_\infty}) \cdot F_{\Pi_M}(z).$$

By the following lemma, it is always possible to choose $\delta > 0$ such that $\widetilde{F}_{\Pi_M} * H_\delta$ is non-trivial.

LEMMA 2.20. *For any $\ell = (\ell_1,\ldots,\ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ satisfying $|\mathrm{Re}(\ell_j)| < \frac{1}{2}$, let*

$$\mathrm{LB}_\delta(\ell) := 1 - \frac{4\left(e^{\frac{n(n+6)}{4}\cdot\delta} - 1\right)}{n(n+6)} \sum_{j=1}^{n} \left|\ell_j + \frac{n-2j+1}{2}\right| \tag{2.21}$$

for $\delta > 0$, where $\lambda_n(\ell)$ is the eigenvalue of the Laplacian Δ_n for φ_ℓ as in Lemma 2.8. Choose $\delta > 0$ such that $LB_\delta(\ell) > 0$. Then

$$\left|\widehat{H}_\delta(\ell)\right| > \mathrm{LB}_\delta(\ell).$$

Proof. [Proof of Lemma 2.20] For $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$, we have

$$\left|\widehat{H}_\delta(\ell) - 1\right| \leqslant \int_{SL(n,\mathbb{R})} H_\delta(g)\ |\varphi_\ell(g) - 1|\ dg \leqslant \sup_{g \in B_\delta} |\varphi_\ell(g) - 1|\,.$$

So, $\left|\widehat{H}_\delta(\ell)\right| \geqslant 1 - \sup\limits_{g\in B_\delta} |\varphi_\ell(g) - 1|$. For $g \in SL(n,\mathbb{R})$, we have $\exp(\mathrm{Iw}_Y(g)) = (y'_1, \ldots, y'_n) \in \mathfrak{a}(n)$ and $\varphi_\ell(g) = e^{\left(\sum_{j=1}^n \left(\ell_j + \frac{n-2j+1}{2}\right)\cdot \ln y'_j\right)}$. For any $a, b \in \mathbb{R}$, we have $e^{a+ib} - 1 = (a+ib)\int_1^e x^{a+ib-1}\,dx$. So

$$|\varphi_\ell(g) - 1| \leqslant \left|\sum_{j=1}^n \left(\ell_j + \frac{n-2j+1}{2}\right)\cdot \ln y'_j\right| \cdot \frac{e^{\sum_{j=1}^n \left(\mathrm{Re}(\ell_j) + \frac{n-2j+1}{2}\right)\cdot \ln y'_j} - 1}{\sum_{j=1}^n \left(\mathrm{Re}(\ell_j) + \frac{n-2j+1}{2}\right)\cdot \ln y'_j},$$

if $\sum_{j=1}^n \left(\mathrm{Re}(\ell_j) + \frac{n-2j+1}{2}\right)\cdot \ln y'_j \neq 0$. Otherwise,

$$|\varphi_\ell(g) - 1| \leqslant \left|\sum_{j=1}^n \left(\ell_j + \frac{n-2j+1}{2}\right)\cdot \ln y'_j\right|.$$

By Lemma 2.4, for $g \in B_\delta$, we have $-\delta \leqslant \ln y'_j \leqslant \delta$, for any $j = 1, \ldots, n$. So,

$$\left|\sum_{j=1}^n \left(\ell_j + \frac{n-2j+1}{2}\right)\cdot \ln y'_j\right| \leqslant \delta \cdot \sum_{j=1}^n \left|\ell_j + \frac{n-2j+1}{2}\right|$$

by Cauchy-Schwartz inequality. Since $|\mathrm{Re}(\ell_j)| < \frac{1}{2}$, we have

$$\left|\sum_{j=1}^n \left(\mathrm{Re}(\ell_j) + \frac{n-2j+1}{2}\right)\ln y'_j\right| \leqslant \delta \cdot \sum_{j=1}^n \left|\mathrm{Re}(\ell_j) + \frac{n-2j+1}{2}\right| \leqslant \delta \cdot \frac{n}{2}\left(\frac{n}{2} + 3\right).$$

Since $\lim\limits_{t\to 0} \frac{e^t - 1}{t} = 1$ and $\frac{e^t-1}{t}$ is increasing, we obtain the lemma. □

Take δ as given in (1.13). Since π_∞ is irreducible admissible unramified unitary generic representation of $\mathbb{R}^\times\backslash GL(n,\mathbb{R})$, we have $|\mathrm{Re}(\ell_{\pi_\infty,j})| < \frac{1}{2}$ for $j = 1, \ldots, n$ [7]. So $\left|\widehat{H}_\delta(\ell_{\pi_\infty})\right| > \frac{1}{2}$ and $\widetilde{F}_{\Pi_M} * H_\delta$ is a non-trivial, smooth automorphic function on $SL(n,\mathbb{Z})\backslash\mathbb{H}^n$.

In (1.9), we defined $\widehat{\natural}^n_p(\ell_1, \ell_2)$ for $\ell_1, \ell_2 \in \mathfrak{a}^*_{\mathbb{C}}(n)$. In the following lemma, we apply a Paley-Wiener type theorem to $\widehat{\natural}^n_p$ and construct the annihilating operator $\natural^n_p$ explicitly as a polynomial in convolution operators and in Hecke operators.

Lemma 2.22. *Let $n \geqslant 2$ be an integer and fix a prime p. There exists an operator denoted $\natural^n_p$, which is a polynomial in convolution operators (associated to some compactly supported bi-$SO(n,\mathbb{R})$-distributions) and in Hecke operators at p, satisfying*

$$\natural^n_p f(z) = \widehat{\natural}^n_p\left(\ell_\infty(f), \ell_p(f)\right)\cdot f(z), \qquad (z \in \mathbb{H}^n).$$

Here f is a smooth function on $\mathbb{H}^n$ which is an eigenfunction of $\mathcal{Z}^n$ of type $\ell_\infty(f)$ and also an eigenfunction of Hecke operators at p, with eigenvalues as in (2.18) for $\ell_p(f)$.

REMARK 2.23. *Before proving Lemma 2.22, we give an example of $\natural_p^n$ for the cases $n = 2$ and $n = 3$.*

1. *For $n = 2$, we have*

$$\natural_p^2 = T_{p^2} + T_p^2 - 2T_p\mathcal{L}_\kappa + 1$$

where $\mathcal{L}_\kappa$ is the convolution operator associated to the distribution κ such that $\widehat{\kappa}(\ell) = p^{\ell_1} + p^{\ell_2}$ for any $\ell = (\ell_1, \ell_2) \in \mathfrak{a}_\mathbb{C}^*(2)$. This operator satisfies $\natural_p^2 = \aleph \circ \aleph$ for the operator $\aleph$ constructed in §2, [13].

2. *Let $n = 3$. For $j = 1, 2, 3$, define the compactly supported bi-$SO(3,\mathbb{R})$-distributions $\kappa_{\pm j}$ such that $\widehat{\kappa_1}(\ell) = p^{\ell_1}+p^{\ell_2}+p^{\ell_3}$, $\widehat{\kappa_{-1}}(\ell) = p^{-\ell_1}+p^{-\ell_2}+p^{-\ell_3}$, $\widehat{\kappa_2}(\ell) = -\widehat{\kappa_{-1}}(\ell)^2 + 3\widehat{\kappa_1}(\ell)$, $\widehat{\kappa_{-2}}(\ell) = \widehat{\kappa_1}(\ell)^2 - 3\widehat{\kappa_{-1}}(\ell)$, $\widehat{\kappa_3(\ell)} = -\widehat{\kappa_2}(\ell)\cdot\widehat{\kappa_1}(\ell)$, and $\widehat{\kappa_{-3}}(\ell) = -\widehat{\kappa_{-2}}(\ell)\cdot\widehat{\kappa_{-1}}(\ell)$, for any $\ell = (\ell_1, \ell_2, \ell_3) \in \mathfrak{a}_\mathbb{C}^*(3)$. Then*

$$\natural_p^3 = T_p\mathcal{L}_{\kappa_3} + T_p^2\mathcal{L}_{\kappa_2} - T_p^3 - T_p(T_p^{(2)})^2\mathcal{L}_{\kappa_1} + T_p^2T_p^{(2)}\mathcal{L}_{\kappa_{-1}} + (T_p^{(2)})^2\mathcal{L}_{\kappa_{-2}} + (T_p^{(2)})^3 + T_p^{(2)}\mathcal{L}_{\kappa_{-3}}.$$

Proof of Lemma 2.22. Let W_n be the Weyl group of $SL(n,\mathbb{R})$. For any w_1, $w_2 \in W_n$, we have $\widehat{\natural_p^n}(w_1.\ell_1, w_2.\ell_2) = \widehat{\natural_p^n}(\ell_1, \ell_2)$, and $\widehat{\natural_p^n}(\ell_1, \ell_2)$ is holomorphic in $\ell_1, \ell_2 \in \mathfrak{a}_\mathbb{C}^*(n)$.

For $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ and $1 \leqslant r \leqslant d_k(n) = \frac{n!}{k!(n-k)!}$, consider homogeneous degree $k \cdot r$ symmetric polynomials $B_{r,k}$ in n variables, defined by

$$\prod_{1\leqslant j_1<\cdots<j_k\leqslant n} \left(1 - xp^{-(\alpha_{j_1}+\cdots+\alpha_{j_k})}\right) = 1 - B_{1,k}(\alpha)x + \cdots + (-1)^r B_{r,k}(\alpha)x^r + \cdots + (-1)^{d_k(n)}x^{d_k(n)}$$

for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathfrak{a}_\mathbb{C}^*(n)$. By [6], Hecke eigenvalues can be described in Schur polynomials in n variables, and they are a linear basis for the space of homogeneous symmetric polynomials in n variables. So $B_{r,k}(\alpha)$ is an eigenvalue of a linear combination of Hecke operators at p.

By using an analogous of the Paley-Wiener theorem [10] for distributions, we show that there exist compactly supported bi-$SO(n,\mathbb{R})$-invariant distributions whose spherical transform is $B_{r,k}(\alpha)$.

For $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, define homogeneous symmetric polynomials $a_{j,k}$ and $b_{j,k}$ by

$$\prod_{1\leqslant j_1<\cdots<j_k\leqslant n}\ \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \left(1 - p^{-(\ell_{1,i_1}+\cdots+\ell_{1,i_k})-(\ell_{2,j_1}+\cdots+\ell_{2,j_k})}\right) = \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \left(\sum_{r=0}^{d_k(n)} B_{r,k}(\ell_1)p^{-r(\ell_{2,i_1}+\cdots+\ell_{2,i_k})}\right) = \sum_{j=0}^{\widetilde{d}_k(n)} a_{j,k}(\ell_1)\cdot b_{j,k}(\ell_2)$$

for $\ell_1 = (\ell_{1,1}, \ldots, \ell_{1,n})$, $\ell_2 = (\ell_{2,1}, \ldots, \ell_{2,n}) \in \mathfrak{a}_\mathbb{C}^*(n)$ and some positive integer $\widetilde{d}_k(n)$. Then $a_{j,k}(\ell_1)$ is a polynomial in $B_{r,k}(\ell_1)$. By symmetry, $b_{j,k}(\ell_2)$ is also a polynomial in $B_{r,k}(\ell_2)$. So, there exist compactly supported bi-$SO(n,\mathbb{R})$-invariant distributions $\kappa_j^{(k)}$ whose spherical transform is $a_{j,k}(\ell_1)$. For each $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ and $1 \leqslant j \leqslant \widetilde{d}_k(n)$, let $\mathcal{L}_{\kappa_j^{(k)}}$ be the convolution operator associated to the distribution $\kappa_j^{(k)}$.

Moreover, there exist Hecke operators $S_j^{(k)}$ such that

$$S_j^{(k)} f = b_{j,k}(\ell_p(f)) \cdot f$$

where f is an eigenfunction of Hecke operators with parameter $\ell_p(f) \in \mathfrak{a}_{\mathbb{C}}^*(n)$.

Therefore

$$\natural_p^n = \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{j=0}^{\widetilde{d}_k(n)} S_j^{(k)} \mathcal{L}_{\kappa_j^{(k)}} \right) \tag{2.24}$$

and

$$\natural_p^n f = \widehat{\natural}_p^n(\ell_\infty(f), \ell_p(f)) \cdot f$$

where f is an eigenfunction of Casimir operators and the Hecke operators. ∎

Since we use distributions to define the annihilating operator $\natural_p^n$, the operator is well defined in the space of smooth functions. For $\delta > 0$, let H_δ be a standard bump function and we define the operator $\natural_p^n H_\delta$ to be

$$\natural_p^n H_\delta f = \natural_p^n (f * H_\delta)$$

for a function $f : \mathbb{H}^n \to \mathbb{C}$ which makes the integral convergent. Then

$$\widehat{\natural_p^n H_\delta}(\ell_1, \ell_2) = \widehat{\natural}_p^n(\ell_1, \ell_2) \cdot \widehat{H}_\delta(\ell_1), \qquad \text{(for any } \ell_1, \ell_2 \in \mathfrak{a}_{\mathbb{C}}^*(n)).$$

By the Paley-Wiener Theorem [10] and Lemma 2.22, the operator $\natural_p^n H_\delta$ is a polynomial in convolution operators (associated to bi-$SO(n,\mathbb{R})$-invariant, compactly supported smooth functions), and in Hecke operators at the prime p.

Lemma 2.25. *Let M be a set of places over $\mathbb{Q}$ including ∞. Let F_{Π_M} be a quasi-Maass form for Π_M and $\widetilde{F}_{\Pi_M}$ be an automorphic lifting of F_{Π_M}. Assume that $T > 4(2^{n-1} \ln p + \delta)$ for a given $\delta > 0$. Then for any $z \in \Sigma_{e^T}$,*

$$\natural_p^n H_\delta \widetilde{F}_{\Pi_M}(z) = \widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p}) \cdot \widehat{H}_\delta(\ell_{\pi_\infty}) \cdot F_{\Pi_M}(z).$$

Proof of Lemma 2.25. Let κ be a compactly supported function with support in $B_b = \{z \in \mathbb{H}^n \mid \sigma(z) \leqslant b\}$ for some $b > 0$. For $t > e^{4b}$, for any $z \in \Sigma_t$, assume that $\sigma(zh^{-1}) \leqslant b$ for some $h \in SL(n,\mathbb{R})$. Let $\exp(\mathrm{Iw}_Y(z)) = a_{y_1,\ldots,y_{n-1}}$ and $\exp(\mathrm{Iw}_Y(h)) = a_{v_1,\ldots,v_{n-1}}$. Then, by Lemma 2.4, for $j = 1, \ldots, n-1$, we have $e^{-4b} \leqslant y_j v_j^{-1} \leqslant e^{4b}$. So, $v_j \geqslant y_j \cdot e^{-4b} \geqslant t \cdot e^{-4b} > 1$. Then $\widetilde{F}_{\Pi_M}(h) = F_{\Pi_M}(h)$ because $\Sigma_1 \subset \widetilde{\mathfrak{F}}^n$. So for $z \in \Sigma_t$, we have

$$\widetilde{F}_{\Pi_M} * \kappa(z) = \int_{z \cdot B_b} \widetilde{F}_{\Pi_M}(h) \kappa(zh^{-1})\, dh = \widehat{\kappa}(\alpha_\infty) \cdot F_{\Pi_M}(z).$$

For integers $c_1, \ldots, c_n \geqslant 1$ and $c_{i,j} \geqslant 0$ for $1 \leqslant i < j \leqslant n$, and $z \in \mathbb{H}^n$, we have

$$\begin{pmatrix} c_1 & c_{1,2} & \ldots & c_{1,n} \\ & c_2 & \ldots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} z = x' \cdot a_{\frac{c_{n-1}}{c_n} y_1, \ldots, \frac{c_{n-j}}{c_n} y_j, \ldots, \frac{c_1}{c_n} y_{n-1}} \tag{2.26}$$

for $x' \in N(n, \mathbb{R})$ and $\exp(\mathrm{Iw}_Y(z)) = a_{y_1, \ldots, y_{n-1}}$. So, if $y_j / c_n \geqslant 1$ for $1 \leqslant j \leqslant n-1$, then $\begin{pmatrix} c_1 & c_{1,2} & \ldots & c_{1,n} \\ & c_2 & \ldots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} z \in \Sigma_1$.

As in (2.24), $\natural_p^n$ is a polynomial in Hecke operators and convolution operators associated with compactly supported distributions $\kappa_j^{(k)}$ for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ and $j = 0, \ldots, \frac{n!}{k!(n-k)!}$. Moreover, the spherical transform $\left| \widehat{\kappa}_j^{(k)}(\ell) \right| \ll e^{\frac{n!}{k!(n-k)!} \ln p \|\mathrm{Re}(\ell)\|}$ for any $\ell \in \mathfrak{a}_{\mathbb{C}}^*(n)$ and the implied constant depends on n and k. So, the operator $\natural_p^n H_\delta$ is a polynomial in Hecke operators and convolution operators associated with compactly supported functions which have support in B_b for $b \leqslant 2^{n-1} \ln p + \delta$, by Paley-Wiener's theorem [10].

Since Hecke operators are generated by left translations as in (2.26), and the largest possible c_n is $p^{2^{n-1}}$ for $\natural_p^n$. Therefore, for any $z \in \Sigma_{e^T}$ for $T > 4(2^{n-1} \ln p + \delta)$, we obtain the lemma. ☐

So if $\widehat{\natural_p^n}(\ell_{\pi_\infty}, \ell_{\pi_p}) \neq 0$, then $\natural_p^n \left(\widetilde{F}_{\Pi_M} * H_\delta \right)(z)$ is non trivial for δ satisfying $\mathrm{LB}_\delta(\ell_{\pi_\infty}) > 0$. By Theorem 1.10, we show that $\natural_p^n \left(\widetilde{F}_{\Pi_M} * H_\delta \right)(z)$ is a smooth cuspidal automorphic function on $SL(n, \mathbb{Z}) \backslash \mathbb{H}^n$. We prove Theorem 1.10 in the next section.

Remark 2.27. *By §2 in [8], we get the following explicit description of the fundamental domain $\mathfrak{F}^n$. Let $n \geqslant 2$ be an integer and $\overline{\mathfrak{F}^n}$ be the closure of the fundamental domain $\mathfrak{F}^n$.*

1. *For $n = 2$, the closure of the fundamental domain $\overline{\mathfrak{F}^n}$ is the set of $z = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in \mathbb{H}^2$ for $x, y \in \mathbb{R}$ and $y > 0$ satisfying $x^2 + y^2 \geqslant 1$ and $|x| \leqslant \frac{1}{2}$.*
2. *For $n > 2$, the closure of the fundamental domain $\overline{\mathfrak{F}^n}$ is the set of*

$$z = \begin{pmatrix} & & & x_1 \\ & I_{n-1} & & \vdots \\ & & & x_{n-1} \\ 0 & \ldots & 0 & 1 \end{pmatrix} y_1^{-\frac{n-1}{n}} \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

for $x_1, \ldots, x_{n-1} \in \mathbb{R}$ and $y_1 > 0$ satisfying the following conditions:
(a) $z' \in \overline{\mathfrak{F}^{n-1}}$;

(b) for any $\begin{pmatrix} & & b_1 \\ & * & \vdots \\ & & b_{n-1} \\ c_1 \ldots c_{n-1} & & a \end{pmatrix} \in SL(n,\mathbb{Z})$*, we have*

$$(a + c_1x_1 + \cdots + c_{n-1}x_{n-1})^2 + y_1^2 \left(\prod_{j=2}^{n-1} y_j^{n-j}\right)^{\frac{2}{n-1}} \times \begin{pmatrix} c_1 & \ldots & c_{n-1}\end{pmatrix} z' \, {}^t z' \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1}\end{pmatrix} \geqslant 1;$$

(c) $|x_j| \leqslant \frac{1}{2}$ *for* $j = 1, \ldots, n-1$.

2.4. Proof of Theorem 1.10. As suggested in the Appendix [13], we prove Theorem 1.10 for the annihilating operator $\natural_p^n$.

Remark 2.28. *Let* $\ell_1 = (\ell_{1,1}, \ldots, \ell_{1,n})$, $\ell_2 = (\ell_{2,1}, \ldots, \ell_{2,n}) \in \mathfrak{a}_{\mathbb{C}}^*(n)$. *By definition, we have* $\widehat{\natural_p^n}(\ell_1, \ell_2) = 0$, *whenever* $(\ell_{1,i_1} + \cdots + \ell_{1,i_r}) + (\ell_{2,j_1} + \cdots + \ell_{2,j_r}) = 0$ *for any* $1 \leqslant r \leqslant n$. *By [13], it can be proved that the image of the annihilating operator* $\natural_p^n$ *is cuspidal. Here we give an explicit proof.*

For $\delta > 0$, let H_δ be a standard bump function. Then the operator $\natural_p^n H_\delta$ can be defined for the functions in $L^2(\mathbb{H}^n)$ and $\widehat{\natural_p^n H_\delta}(\ell_1, \ell_2) = \widehat{\natural_p^n}(\ell_1, \ell_2) \cdot \widehat{H}_\delta(\ell_1)$ for any $\ell_1,\ \ell_2 \in \mathfrak{a}_{\mathbb{C}}^*(n)$.

The Langlands spectral decomposition states that

$$L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n) = L^2_{\text{cont.}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n) \oplus L^2_{\text{resi.}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n) \oplus L^2_{\text{cusp}}(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$$

where L^2_{cusp} denote the space of Maass forms, $L^2_{\text{resi.}}$ consists of iterated residues of Eisenstein series and $L^2_{\text{cont.}}$ is the space spanned by integrals of Eisenstein series. For any $f \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$, there exists $f_{\text{cont.}} \in L^2_{\text{cont.}}$, $f_{\text{resi.}} \in L^2_{\text{resi.}}$ and $f_{\text{cusp}} \in L^2_{\text{cusp}}$ such that

$$f(z) = f_{\text{cont.}}(z) + f_{\text{resi.}}(z) + f_{\text{cusp}}(z).$$

Our goal is to show that $\natural_p^n H_\delta f_{\text{cont.}} = \natural_p^n H_\delta f_{\text{resi.}} \equiv 0$, therefore $\natural_p^n H_\delta f(z) = \natural_p^n H_\delta f_{\text{cusp}}(z)$. We should show that for any Eisenstein series E, $\natural_p^n E = 0$ for any prime p.

Review some facts on Eisensteins series on $SL(n,\mathbb{Z})\backslash\mathbb{H}^n$ [6]. Let $n \geqslant 2$ be an integer. For each partition $n = n_1 + \cdots + n_r$ with rank $1 \leqslant r \leqslant n$, we have the factorization $P_{n_1,\ldots,n_r}(n,\mathbb{R}) = N_{n_1,\ldots,n_r}(n,\mathbb{R}) \cdot M_{n_1,\ldots,n_r}(n,\mathbb{R})$. It follows that for any $g \in P_{n_1,\ldots,n_r}(n,\mathbb{R})$, we have

$$g \in N_{n_1,\ldots,n_r}(n,\mathbb{R}) \cdot \begin{pmatrix} \mathfrak{m}_{n_1}(g) & 0 & \ldots & 0 \\ & \mathfrak{m}_{n_2}(g) & \ldots & 0 \\ & & \ddots & \vdots \\ & & & \mathfrak{m}_{n_r}(g) \end{pmatrix},$$

where $\mathfrak{m}_{n_i}(g) \in SL(n_i,\mathbb{R})$ for $i = 1, \ldots, r$.

Let $n \geqslant 2$ be an integer and fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r \leqslant n$. For each $i = 1, \ldots r$, let ϕ_i be either a Maass form for $SL(n_i,\mathbb{Z})\backslash\mathbb{H}^{n_i}$ of type

$\ell_\infty(\phi_i) \in \mathfrak{a}^*_{\mathbb{C}}(n_i)$ or a constant with $\ell_\infty(\phi_i) = (0, \ldots, 0)$. For $t = (t_1, \ldots, t_r) \in \mathbb{C}^r$ with $n_1 t_1 + \cdots + n_r t_r = 0$, define a function

$$\varphi_{n_1,\ldots,n_r}(\cdot\,; t; \phi_1, \ldots, \phi_r) : P_{n_1,\ldots,n_r}(n, \mathbb{R}) \to \mathbb{C}$$

by the formula

$$\varphi_{n_1,\ldots,n_r}(g; t; \phi_1, \ldots, \phi_r) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_{n_i}(g)) \cdot |\det(\mathfrak{m}_{n_i}(g))|^{t_i}$$

for $g \in P_{n_1,\ldots,n_r}(n, \mathbb{R})$. We can check that $\varphi_{n_1,\ldots,n_r}(g; t; \phi_1, \ldots, \phi_r) = \varphi_{n_1,\ldots,n_r}(z; t; \phi_1, \ldots, \phi_r)$ for $g = z\xi$ with $z \in \mathbb{H}^n$ and $\xi \in SO(n, \mathbb{R})$. Define the Eisenstein series by the infinite series

$$\begin{aligned} E(z) &= E_{n_1,\ldots,n_r}(z; t; \phi_1, \ldots, \phi_r) \\ &:= \sum_{\gamma \in P_{n_1,\ldots,n_r}(n,\mathbb{Z}) \cap SL(n,\mathbb{Z}) \backslash SL(n,\mathbb{Z})} \varphi_{n_1,\ldots,n_r}(\gamma z; t; \phi_1, \ldots, \phi_r) \end{aligned} \tag{2.29}$$

for $z \in \mathbb{H}^n$. Then the Eisenstein series E is an eigenfunction of $\mathcal{Z}^n$ of type $\ell_\infty(E)$. The Eisenstein series is also an eigenfunction of Hecke operators with a parameter $\ell_p(E)$ for any prime p, if $\phi_1, \ldots, \phi_r$ are Hecke eigenfunctions. The Eisenstein series are not contained in $L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$, but they generate the continuous and residual spectrum in $L^2(SL(n, \mathbb{Z}) \backslash \mathbb{H}^n)$.

The following lemma shows that an Eisenstein series is controlled by few parameters for the archimedean.

LEMMA 2.30. *Let $n \geqslant 2$ be an integer. Fix a partition $n = n_1 + \cdots + n_r$ with $1 \leqslant n_1, \ldots, n_r < n$. For each $i = 1, \ldots, r$, let ϕ_i be either a Hecke-Maass form for $SL(n_i, \mathbb{Z}) \backslash \mathbb{H}^{n_i}$ of type $\ell_\infty(\phi_i) \in \mathfrak{a}^*_{\mathbb{C}}(n_i)$ or a constant with $\ell_\infty(\phi_i) = (0, \ldots, 0)$. Let $t = (t_1, \ldots, n_r) \in \mathbb{C}^r$ with $n_1 t_1 + \cdots + n_r t_r = 0$. Let $E(z) := E_{n_1,\ldots,n_r}(z; t; \phi_1, \ldots, \phi_r)$ be an Eisenstein series (2.29). Let $\eta_1 = 0$ and $\eta_i = n_1 + \cdots + n_{i-1}$ for $i = 2, \ldots, r$. Then for $i = 1, \ldots, r$ and $\eta_i + 1 \leqslant j \leqslant \eta_i + n_i$, we have*

$$\ell_{v,j}(E) = (-1)^\delta \left(\frac{n_i - n}{2} + t_i + \eta_i \right) + \ell_{v,j-\eta_i}(\phi_i) \tag{2.31}$$

where $\delta = \begin{cases} 0, & \text{if } v = \infty; \\ 1, & \text{if } v < \infty\,. \end{cases}$

Proof of Lemma 2.30. By Proposition 10.9.1 [6], the Eisenstein series $E(z)$ is an eigenfunction of Casimir operators of type $\ell_\infty(E)$.

For an integer $N \geqslant 1$, let $A_{\phi_i}(N) \in \mathbb{C}$ be the Hecke eigenvalue of T_N for ϕ_i for $i = 1, \ldots, r$. Then by Proposition 10.9.3 [6], the Eisenstein series $E(z)$ is an eigenfunction of the Hecke operators T_{p^k} (for any $k \geqslant 0$ and prime p) with eigenvalues

$$A_E(p^k) = p^{-\frac{k(n-1)}{2}} \sum_{\substack{k_1 + \cdots + k_r = k, \\ 0 \leqslant k_j \in \mathbb{Z}}} \prod_{j=1}^{r} \left(A_{\phi_j}(p^{k_j}) \cdot p^{k_j \left(\frac{n_j - 1}{2} + t_j + \eta_j \right)} \right).$$

By using the multiplicative relations (2.16), we get the formula (2.31). □

By the lemma above, for any Eisenstein series E, we have

$$\widehat{\mathfrak{h}}^n_p(\ell_\infty(E), \ell_p(E)) = 0$$

for any prime p. So $\natural_p^n E = 0$ for any prime p. Moreover, for any constant $C \in \mathbb{C}$, we have $\natural_p^n C = 0$ for any prime p. Since the invariant integral operators and Hecke operators preserve the space of cuspidal functions, we have $\natural_p^n H_\delta f = \natural_p^n H_\delta f_{\text{cusp}}$. Therefore the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is cuspidal.

Assume that f is a self-dual Hecke-Maass form for $SL(n,\mathbb{Z})$. By definition, we get $\ell_\infty(f) = -\ell_\infty(f)$ and $\ell_p(f) = -\ell_p(f)$ for any prime p, up to permutation. So $\widehat{\natural}_p^n(\ell_\infty(f), \ell_p(f)) = 0$ and $\natural_p^n f = 0$ for any prime p. Therefore the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is generated by non self-dual Hecke-Maass forms.

We already show that the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is non-trivial in Lemma 2.25. So it remains to prove that the image is infinite dimensional.

Take $\alpha_\infty = (\alpha_{\infty,1}, \ldots, \alpha_{\infty,n})$, $\alpha_p = (\alpha_{p,1}, \ldots, \alpha_{p,n}) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ such that $\widehat{\natural}_p^n(\alpha_\infty, \alpha_p) \neq 0$. Assume that $\operatorname{Re}(\alpha_{v,j}) = 0$ for $1 \leqslant j \leqslant n$, for $v = \infty$, p. Take δ as given in (1.13), then we have $\left|\widehat{H}_\delta(\alpha_\infty)\right| > \frac{1}{2}$ by Lemma 2.20.

As in Definition 1.5, we construct a quasi-Maass form F for $\{\alpha_\infty, \alpha_p\}$ such that

$$F(z) = \sum_{\gamma \in N(n-1,\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{\epsilon = \pm 1} \sum_{k_1, \ldots, k_{n-1} \geqslant 0} \frac{A_F\left(p^{k_1}, \ldots, p^{k_{n-1}}\right)}{p^{\frac{1}{2}\sum_{j=1}^{n-1} k_j (n-j) j}} \times W_J\left(\begin{pmatrix} p^{k_1+\cdots+k_{n-1}} & & & \\ & p^{k_1+\cdots+k_{n-2}} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z; \alpha_\infty, \epsilon\right)$$

where $A_F(p^{k_1}, \ldots, p^{k_{n-1}}) = S_{k_1,\ldots,k_{n-1}}(p^{-\alpha_{p,1}}, \ldots, p^{-\alpha_{p,n}})$. Then

$$\natural_p^n H_\delta F(z) = \widehat{\natural}_p^n(\alpha_\infty, \alpha_p) \cdot \widehat{H}_\delta(\alpha_\infty) \cdot F(z)$$

for $z \in \mathbb{H}^n$.

Let $\widetilde{F}$ be the automorphic lifting of F as Definition 1.7. Then $\natural_p^n H_\delta \widetilde{F} \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is smooth and cuspidal as we show above. By Lemma 2.25, $\natural_p^n H_\delta \widetilde{F}$ is non-trivial since $\widehat{H}_\delta(\alpha_\infty) \cdot \widehat{\natural}_p^n(\alpha_\infty, \alpha_p) \neq 0$.

Assume that the space of the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is finite dimensional. Let

$$\natural_p^n \mathcal{U} := \left\{ u_j, \quad \text{a Hecke-Maass form of type } \mu_j \in \mathfrak{a}_{\mathbb{C}}^*(n) \;\middle|\; \begin{array}{c} \natural_p^n u_j \neq 0, \\ \text{and } \|u_j\|_2^2 = 1 \end{array} \right\}.$$

This finite set $\natural_p^n \mathcal{U}$ is not empty and it is an orthonormal basis of the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$. Let B be the number of elements of $\natural_p^n \mathcal{U}$. Then $\natural_p^n \mathcal{U} = \{u_1, \ldots, u_B\}$ and there exist $c_1, \ldots, c_B \in \mathbb{C}$ such that

$$\natural_p^n H_\delta \widetilde{F}(z) = \sum_{j=1}^{B} c_j u_j(z). \tag{2.32}$$

Here $c_j \neq 0$ for at least one $j = 1, \ldots, B$. Assume that $c_1 \neq 0$.

Let $T > 4(2^{n-1} \ln p + \delta)$. For any $z \in \Sigma_{e^T}$, we have

$$\left(\mathcal{C}_n^{(i)} - \lambda_\infty^{(i)}(\alpha_\infty)\right) \natural_p^n H_\delta \widetilde{F}(z) = \sum_{j=1}^{B} c_j \left(\mathcal{C}_n^{(i)} - \lambda_\infty^{(i)}(\alpha_\infty)\right) u_j(z) = 0$$

for any $i = 1, \ldots, n-1$. Since B is a finite positive integer, it is possible to assume that $\alpha_\infty \neq \mu_j$ (up to permutations) for any $j = 1, \ldots, B$. Then there exists $i = 1, \ldots, n-1$ such that $\lambda_\infty^{(i)}(\mu_1) - \lambda_\infty^{(i)}(\alpha_\infty) \neq 0$. So there exists $c_2', \ldots, c_B' \in \mathbb{C}$ such that $u_1(z) = \sum_{j=2}^{B} c_j' \cdot u_j(z)$ for $z \in \Sigma_{e^T}$. So in a similar manner, we deduce that there exists $1 \leqslant j \leqslant B$ such that $u_j(z) = 0$ for $z \in \Sigma_{e^T}$. This gives a contradiction.

Therefore $\natural_p^n \mathcal{U}$ should be an infinite set. It follows that the image of $\natural_p^n H_\delta$ on $L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is infinite dimensional.

3. Proof.

3.1. Proof of Theorem 1.14. Let M be a set of places over $\mathbb{Q}$ including ∞. For a given set of local representations Π_M, we construct the quasi-Maass form $F_{\Pi_M}(z)$ for Π_M and its automorphic lifting $\widetilde{F}_{\Pi_M}(z)$ with respect to the fixed fundamental domain $\mathfrak{F}^n$ as in Definition 1.5 and Definition 1.7 respectively. For each local representation $\pi_v \in \Pi_M$, we have the Satake (or Langlands) parameter $\ell_{\pi_v} = (\ell_{\pi_v,1}, \ldots, \ell_{\pi_v,n}) \in \mathfrak{a}_{\mathbb{C}}^*(n)$ as in (1.1). By Lemma 2.20, for a given δ, the standard bump function H_δ satisfies $\left|\widehat{H}_\delta(\ell_{\pi_\infty})\right| > \frac{1}{2}$. By Theorem 1.10,

$$\natural_p^n H_\delta \widetilde{F}_{\Pi_M} = \natural_p^n \left(\widetilde{F}_{\Pi_M} * H_\delta\right) \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$$

and $\natural_p^n H_\delta \widetilde{F}_{\Pi_M}$ is non-trivial since $\widehat{\natural_p^n}(\ell_{\pi_\infty}, \ell_{\pi_p}) \cdot \widehat{H}_\delta(\ell_{\pi_\infty}) \neq 0$.

The key idea to prove Theorem 1.14 is applying the following lemma to the cuspidal function $\natural_p^n H_\delta \widetilde{F}_{\Pi_M}(z)$. This lemma is a generalization of Lemma 1 [3].

Lemma 3.1. *Let $n \geqslant 2$ be an integer. Let M be a set of places of $\mathbb{Q}$ including ∞ and Π_M be a set of local representations as in the introduction. Let $S \subset M$ be a finite subset including ∞. If there exists a non-zero smooth function $f \in L^2\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, which is cuspidal, such that*

$$\sum_{j=1}^{n-1} \left|\left| \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) f \right|\right|_2^2 + \sum_{\substack{q \in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left|\left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) f \right|\right|_2^2 < \epsilon \cdot ||f||_2^2 \tag{3.2}$$

for $\pi_v \in \Pi_M$, for some $\epsilon > 0$, then there exists an unramified cuspidal automorphic representation σ of $\mathbb{A}^\times\backslash GL(n,\mathbb{A})$ such that $d_S(\sigma, \Pi_M) < \epsilon$.

Proof of Lemma 3.1. By the spectral decomposition, the space $L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$ is spanned by Hecke-Maass forms $u_i(z)$ with $||u_i||_2^2 = 1$ for $i = 1, 2, \ldots$. For each u_i, there exists an unramified cuspidal automorphic representation σ_i of $\mathbb{A}^\times\backslash GL(n,\mathbb{A})$ such that u_j is the Hecke-Maass form for σ_j. Then, for any $f \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right)$, we have $f(z) = \sum_{i=1}^{\infty} \langle f, u_i \rangle \cdot u_i(z)$.

For $\epsilon > 0$, let

$$\mathcal{U}_\epsilon(\Pi_M) := \{u_i \mid d_S(\sigma_i, \Pi_M) < \epsilon\}$$

and define

$$\text{Pr}_\epsilon f(z) := \sum_{u_i \in \mathcal{U}_\epsilon(\Pi_M)} \langle f, u_i \rangle \cdot u_i(z) \in L^2_{\text{cusp}}\left(SL(n,\mathbb{Z})\backslash\mathbb{H}^n\right).$$

Assume that f is smooth and satisfies (3.2). Then

$$||\mathrm{Pr}_\epsilon f||_2^2 = ||f||_2^2 - \sum_{u_i \notin \mathcal{U}_\epsilon(\Pi_M)} |\langle f, u_i\rangle|^2$$
$$\geqslant ||f||_2^2 - \frac{1}{\epsilon} \cdot \left\{ \sum_{j=1}^{n-1} || \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) f||_2^2 + \sum_{\substack{q \in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} || \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) f||_2^2 \right\}$$
$$> 0.$$

Therefore $\mathcal{U}_\epsilon(\Pi_M) \neq \varnothing$. ∎

We are going to construct a formula for ϵ satisfying

$$\frac{1}{||H_\delta \natural_p^n \widetilde{F}_{\Pi_M}||_2^2} \cdot \left\{ \sum_{j=1}^{n-1} || \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H_\delta \natural_p^n \widetilde{F}_{\Pi_M}||_2^2 \right.$$
$$\left. + \sum_{\substack{q \in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} || \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) H_\delta \natural_p^n \widetilde{F}_{\Pi_M}||_2^2 \right\} < \epsilon.$$

Then by Lemma 3.1, there exists an unramified cuspidal representation π such that $d_S(\pi, \Pi_M) < \epsilon$.

The following lemma gives the lower bound for $||H_\delta \natural_p^n \widetilde{F}_{\Pi_M}||_2^2$.

Lemma 3.3. *Let $n \geqslant 2$ be an integer. For a cuspidal function $f \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$, for $(m_1, \ldots, m_{n-1}) \in \mathbb{Z}^n$, let*

$$W_f(z; m_1, \ldots, m_{n-2}, m_{n-1})$$
$$:= \int_{(N(n,\mathbb{R}) \cap SL(n,\mathbb{Z}))\backslash N(n,\mathbb{R})} f(uz)\, d^{-2\pi i(m_1 u_{n-1,n} + \cdots + m_{n-2}u_{2,3} + m_{n-1}u_{1,2})}\, d^*u$$

where $u = \begin{pmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ & \ddots & & \vdots \\ & & 1 & u_{n-1,n} \\ & & & 1 \end{pmatrix} \in N(n,\mathbb{R})$, *for $z \in \mathbb{H}^n$. Then, for $T \geqslant 1$, we have*

$$||f||_2^2 > \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \int_T^\infty \cdots \int_T^\infty |W_f(y; m_1, \ldots, m_{n-2}, m_{n-1})|^2\, d^*y$$

where $y = a_{y_1,\ldots,y_{n-1}} \in A^0(n,\mathbb{R})$.

Proof of Lemma 3.3. We follow the argument in §5.3, [6]. For $j = 1, \ldots, n-1$, let

$$u_{n-j+1} := \begin{pmatrix} I_{n-j} & \begin{matrix} u_{1,n-j+1} \\ \vdots \\ u_{n-j,n-j+1} \end{matrix} & 0_{n-j \times j-1} \\ 0_{j \times n-j} & & I_j \end{pmatrix} \in N(n,\mathbb{R})$$

where $u_{1,n-j+1}, \ldots, u_{n-j,n-j+1} \in \mathbb{R}$, I_a is the $a \times a$ identity matrix and $0_{a\times b}$ is an $a \times b$ matrix with 0 for every entry.

Let $z = xy \in \mathbb{H}^n$ for $x = \begin{pmatrix} 1 & x_{1,2} & \ldots & x_{1,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \\ & & & 1 \end{pmatrix} \in N(n,\mathbb{R})$ and $y = a_{y_1,\ldots,y_{n-1}} \in A_0(n,\mathbb{R})$. Fix $j = 1, \ldots, n-1$. For $m_1, \ldots, m_j \in \mathbb{Z}$, define

$$f_j(z; m_1, \ldots, m_j) := \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f\left(u_n \cdot u_{n-1} \cdots u_{n-j+1} \cdot z\right) \times e^{-2\pi i (m_1 u_{n-1,n} + \cdots + m_j u_{n-j,n-j+1})} \, d^* u_n \cdots d^* u_{n-j+1},$$

where $d^* u_{n-j+1} = \prod_{k=1}^{n-j} du_{k,n-j+1}$. Then for $j = n-1$, we have

$$f_{n-1}(z; m_1, \ldots, m_{n-1}) = W_f(z; m_1, \ldots, m_{n-1}).$$

Let $f_0(z) := f(z)$ with $z \in \mathbb{H}^n$. By the proof of Theorem 5.3.2, [6], we can prove the followings.

- For $j = 1, \ldots, n-1$, we have

$$f_j(z; m_1, \ldots, m_j) = \int_{\mathbb{Z}\backslash\mathbb{R}} \cdots \int_{\mathbb{Z}\backslash\mathbb{R}} f_{j-1}(u_{n-j+1} z; m_1, \ldots, m_{j-1}) e^{-2\pi i m_j u_{n-j,n-j+1}} \, d^* u_{n-j+1}.$$

- Fix $j = 1, \ldots, n-2$. For positive integers $m_1, \ldots, m_{j-1}$, we have

$$f_{j-1}(z; m_1, \ldots, m_{j-1}) = \sum_{m_j=1}^{\infty} \sum_{\gamma_{n-j} \in P_{n-j-1,1}(\mathbb{Z})\backslash SL(n-j,\mathbb{Z})} f_j\left(\begin{pmatrix} \gamma_{n-j} & \\ & I_j \end{pmatrix} z; m_1, \ldots, m_{j-1}, m_j \right).$$

- For positive integers $m_1, \ldots, m_{n-2}$, we have

$$f_{n-2}(z; m_1, \ldots, m_{n-2}) = \sum_{0 \neq m_{n-1} \in \mathbb{Z}} W_f\left(z; m_1, \ldots, m_{n-2}, m_{n-1}\right).$$

Since the Siegel set $\Sigma_{1,\frac{1}{2}} \subset \mathfrak{F}^n$,

$$||f||_2^2 = \int_{\mathfrak{F}^n} |f(z)|^2 \, d^* z \; > \; \int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \, d^* z.$$

Then

$$\begin{aligned} &\int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \, d^* z \\ &= \int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1=1}^{\infty} \sum_{\gamma_{n-1} \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \overline{f(z)} e^{2\pi i m_1 (\gamma_{n-1,1} x_{1,n} + \cdots \gamma_{n-1,n-1} x_{n-1,n})} \\ &\qquad \times f_1\left(\begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \cdot y_1^{-\frac{n-1}{n}} \cdot \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}; m_1 \right) d^* z, \end{aligned}$$

where $\gamma_{n-1} = \begin{pmatrix} & * & \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-1} \end{pmatrix} \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})$. Here $z' \in \mathbb{H}^{n-1}$. For a positive integer m_1 and $\gamma_{n-1} = \begin{pmatrix} & * & \\ \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-1} \end{pmatrix} \in P_{n-2,1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})$, it follows that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{f(z)} e^{2\pi i m_1 (\gamma_{n-1,1}x_{1,n} + \cdots \gamma_{n-1,n-1}x_{n-1,n})} \prod_{k=1}^{n-1} dx_k$$

$$= \overline{f_1\left(\begin{pmatrix} \gamma_{n-1} & \\ & 1 \end{pmatrix} \cdot y_1^{-\frac{n-1}{n}} \cdot \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}\right)}$$

So,

$$\int_1^{\infty}\int_1^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(z)|^2 \; d^*z$$

$$\geqslant \sum_{m_1=1}^{\infty} \int_1^{\infty} \cdots \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f_1\left(y_1^{-\frac{n-1}{n}} \cdot \begin{pmatrix} & & & 0 \\ & y_1 z' & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}; m_1 \right)\right|^2 \prod_{1\leqslant i<j\leqslant n-1} dx_{i,j} \; d^*y.$$

After continuing this process inductively for $n-1$ steps, we finally obtain the lemma. ◻

Let $T > 4(2^{n-1}\ln p + \delta)$. By Lemma 2.25, for any $z \in \Sigma_{e^T,\frac{1}{2}} \subset \mathfrak{F}^n$, we have $H_\delta \natural_p^n \widetilde{F}_{\Pi_M}(z) = \widehat{H}_\delta(\ell_{\pi_\infty}) \cdot \widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p}) \cdot F_{\Pi_M}(z)$. Then for $z \in \Sigma_{e^T,\frac{1}{2}}$, we get

$W_{\natural_p^n H_\delta \widetilde{F}_{\Pi_M}}(z; 1, \ldots, 1) = \widehat{H}_\delta(\ell_{\pi_\infty}) \cdot \widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p}) \cdot W_J(y; \ell_{\pi_\infty}, 1) e^{2\pi i(x_{n-2,n-1} + x_{n-2,n-1} + \cdots + x_{1,2})}$.

Therefore by Lemma 3.3, we have

$$||H_\delta \natural_p^n \widetilde{F}_{\Pi_M}||_2^2 > \frac{1}{4}\left|\widehat{\natural}_p^n(\ell_{\pi_\infty}, \ell_{\pi_p})\right|^2 \cdot \int_{p^T}^{\infty} \cdots \int_{p^T}^{\infty} |W_J(y; \ell_{\pi_\infty}, 1)|^2 \; d^*y$$

for $y = a_{y_1,\ldots,y_{n-1}} \in A^0(n, \mathbb{R})$.

LEMMA 3.4. *Let $n \geqslant 2$ be an integer and p be a prime. Then*

$$||\natural_p^n f||_2^2 \leqslant \left(p^{-\frac{n^2-1}{2(n^2+1)}} + p^{\frac{n^2-1}{2(n^2+1)}}\right)^{n2^{n-1}} \cdot ||f||_2^2 \tag{3.5}$$

for any smooth function $f \in L^2(SL(n,\mathbb{Z})\backslash \mathbb{H}^n)$.

Proof. Since $\natural_p^n$ annihilates the continuous part, we should focus on the cuspidal functions. For any smooth cuspidal function $f \in L^2(SL(n,\mathbb{Z})\backslash\mathbb{H}^n)$, we have $f(z) = \sum_{j=1}^{\infty} \langle f, u_j\rangle u_j(z)$, where $u_j(z)$ are Hecke-Maass forms for $SL(n,\mathbb{Z})$ and $||u_j||_2^2 = 1$. So

$$||\natural_p^n f||_2^2 \leqslant \sup_j \left|\widehat{\natural}_p^n(\ell_\infty(u_j), \ell_p(u_j))\right|^2 \cdot \|f\|_2^2.$$

For each $j \geqslant 1$, let $\ell_1 := \ell_\infty(u_j)$ and $\ell_2 := \ell_p(u_j)$. By [14] and [15], we have $|\mathrm{Re}(\ell_{1,i})|, |\mathrm{Re}(\ell_{2,i})| \leqslant \frac{1}{2} - \frac{1}{n^2+1}$ for $i = 1, \ldots, n$. Then for any $1 \leqslant j_1 < \cdots < j_k$ and $1 \leqslant i_1 < \cdots < i_k$ for $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, we have

$$\left| p^{-\frac{\ell_{1,i_1}+\cdots+\ell_{1,i_k}+\ell_{2,j_1}+\cdots+\ell_{2,j_k}}{2}} - p^{\frac{\ell_{1,i_1}+\cdots+\ell_{1,i_k}+\ell_{2,j_1}+\cdots+\ell_{2,j_k}}{2}} \right| \leqslant p^{-k\cdot\frac{n^2-1}{2(n^2+1)}} + p^{k\cdot\frac{n^2-1}{2(n^2+1)}}.$$

So, by (1.9), we obtain (3.5). ☐

Lemma 3.6. *Let $n \geqslant 2$ be an integer. For $\delta > 0$ let H_δ be the standard bump function. For any $\ell = (\ell_1, \ldots, \ell_n) \in \mathfrak{a}^*_{\mathbb{C}}(n)$ with $|\mathrm{Re}(\ell_j)| < \frac{1}{2}$, we have*

$$\left|\widehat{H}_\delta(\ell)\right| < e^{\frac{n(n+6)}{4}\delta}.$$

Proof of Lemma 3.6. For $\ell \in \mathfrak{a}^*_{\mathbb{C}}(n)$, we have

$$\left|\widehat{H}_\delta(\ell)\right| = \left|\int_{SL(n,\mathbb{R})} H_\delta(g)\cdot\varphi_\ell(g)\, dg\right| \leqslant \sup_{g\in B_\delta} |\varphi_\ell(g)|$$

since $H_\delta(g) \geqslant 0$ and $\int_{B_\delta} H_\delta(g)\, dg = 1$. As in the proof of Lemma 2.20, we have $|\varphi_\ell(g)| < e^{\frac{n(n+6)}{4}\delta}$, for any $g \in B_\delta$. ☐

The following lemma finally gives (1.15).

Lemma 3.7. *Let A_∞, $A_{S,\mathrm{finite}}$, B_1 and B_2 be as in Theorem 1.14. Then we have*

$$\sum_{j=1}^{n-1} \left|\left| \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) \natural_p^n H_\delta \widetilde{F}_{\Pi_M} \right|\right|_2^2 + \sum_{\substack{q\in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \left|\left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) \natural_p^n H_\delta \widetilde{F}_{\Pi_M} \right|\right|_2^2$$

$$< \sup_{z\in B_2} \left|\widetilde{F}_{\Pi_M}(z) - F_{\Pi_M}(z)\right|^2 \left(p^{-\frac{n^2-1}{2(n^2+1)}} + p^{\frac{n^2-1}{2(n^2+1)}}\right)^{n2^{n-1}} \mathrm{Vol}(B_1)\cdot(A_\infty + A_{S,\mathrm{finite}}).$$

Proof of Lemma 3.7. Since the operator $\natural_p^n$ commutes with the invariant differential operators $\mathcal{C}_n^{(j)}$ and Hecke operators $T_q^{(j)}$ for $j = 1, \ldots, n-1$, we have

$$\sum_{j=1}^{n-1} \left|\left| \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) \natural_p^n H_\delta \widetilde{F}_{\Pi_M} \right|\right|_2^2 + \sum_{\substack{q\in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \left|\left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) \natural_p^n H_\delta \widetilde{F}_{\Pi_M} \right|\right|_2^2$$

$$< \left(p^{-\frac{n^2-1}{2(n^2+1)}} + p^{\frac{n^2-1}{2(n^2+1)}}\right)^{n2^{n-1}}$$

$$\times \left\{ \sum_{j=1}^{n-1} \left|\left| \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) \widetilde{F}_{\Pi_M} * H_\delta \right|\right|_2^2 + \sum_{\substack{q\in S,\\ \text{finite}}} \sum_{j=1}^{\lfloor \frac{n}{2}\rfloor} \left|\left| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) \widetilde{F}_{\Pi_M} * H_\delta \right|\right|_2^2 \right\}$$

by Lemma 3.4.

Consider the case when $v = \infty$. Since $\left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) F_{\Pi_M} * H_\delta \equiv 0$, it follows that

$$\left|\left| \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) \widetilde{F}_{\Pi_M} * H_\delta \right|\right|_2^2 = \left|\left| \left(\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right) * \left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H_\delta \right|\right|_2^2.$$

So,

$$
\begin{aligned}
&|| \left(\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right) * \left(\left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H_\delta\right) ||_2^2 \\
&\leqslant \int_{\mathfrak{F}^n} \sup_{g\in B_\delta} \left|\left(\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right)(zg^{-1})\right|^2 \cdot \left(\int_{SL(n,\mathbb{R})} \left|\left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H(g)\right| \, dg\right)^2 \, d^*z \\
&\qquad \leqslant \sup_{\mathfrak{F}^n\cdot B_\delta - \mathfrak{F}^n} \left|\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right|^2 \cdot \mathrm{Vol}\left(\left(\mathbb{H}^n - \widetilde{\mathfrak{F}}^n\right)\cdot B_\delta \cap \mathfrak{F}^n\right) \\
&\qquad\qquad \times \left(\int_{SL(n,\mathbb{R})} \left|\left(\mathcal{C}_n^{(j)} - \lambda_\infty^{(j)}(\ell_{\pi_\infty})\right) H_\delta(g)\right| \, dg\right)^2 .
\end{aligned}
$$

Consider the case when $v = q < \infty$ and $q \in S$. Since the Hecke operators commute with the convolution operator, we have

$$
|| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right)\left(\widetilde{F}_{\Pi_M} * H_\delta\right) ||_2^2 \leqslant e^{\frac{n(n+6)}{4}\delta} \cdot || \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) \widetilde{F}_{\Pi_M} ||_2^2
$$

by Lemma 3.6. Since $T_q^{(j)} F_{\Pi_M} = \lambda_q^{(j)}(\ell_{\pi_q})\cdot F_{\Pi_M}$ and $\widetilde{F}_{\Pi_M}(z) - F_{\Pi_M}(z) = 0$ for $z \in \widetilde{\mathfrak{F}}^n$, we have

$$
\begin{aligned}
&|| \left(T_q^{(j)} - \lambda_q^{(j)}(\ell_{\pi_q})\right) \widetilde{F}_{\Pi_M} ||_2^2 = \int_{\mathfrak{F}^n} \left|T_q^{(j)}\left(\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right)(z)\right|^2 \, d^*z \\
&\qquad \leqslant \sup_{T_q^{(j)}\mathfrak{F}^n - \mathfrak{F}^n} \left|\widetilde{F}_{\Pi_M} - F_{\Pi_M}\right|^2 \cdot \mathrm{Vol}\left((T_q^{(j)})^{-1}\left(\mathbb{H}^n - \widetilde{\mathfrak{F}}^n\right)\cap \mathfrak{F}^n\right) \cdot \left(\sharp T_q^{(j)}\right)^2 .
\end{aligned}
$$

□

3.2. Proof of Theorem 1.17. For $\delta > 0$, let C_δ be as in Theorem 1.17 and

$$
H_\delta(g) := \begin{cases} C_\delta \cdot e^{-\frac{1}{1-(\delta^{-1}\sigma(g))^2}}, & \text{if } \sigma(g) < \delta \\ 0, & \text{otherwise} \end{cases} \tag{3.8}
$$

for $g \in SL(n,\mathbb{R})$. Then H_δ is a standard bump function.

For $j = 1, \ldots, n$, let $D_j = D_{j,j}$. For $g = \xi_1 \exp(a)\xi_2$ for $\xi_1, \xi_2 \in SO(2,\mathbb{R})$ and $a = (a_1, \ldots, a_n) \in \mathfrak{a}(n)$ with $a_1 > \cdots > a_n$, by Theorem 4.1, VII, §4 in [12], we have

$$
\begin{aligned}
\Delta_n H_\delta(g) &= \Delta_n H_\delta(\exp a) \\
&= -\frac{1}{n(n-1)}\left\{\sum_{j=1}^{n-1} \frac{1}{j^2+j}\left(\sum_{i=1}^{j}(D_i - D_{j+1})\right)^2 \right. \\
&\qquad \left. + \sum_{1\leqslant i<j\leqslant n} \coth(a_i - a_j)\,(D_i - D_j)\right\} H_\delta(\exp a).
\end{aligned}
$$

So we have $|\Delta_n H_\delta(\exp a)| \leqslant \frac{3(1+e^{2\delta})}{\delta^4} C_\delta$, for $g \in B_\delta$.

Therefore

$$\begin{aligned}(3.9)\quad \int_{SL(n,\mathbb{R})} &\left|(\Delta_n - \lambda_\infty(\pi_\infty))\, H_\delta(g)\right| \, dg \\ &\leqslant \int_{SL(n,\mathbb{R})} \left|\Delta_n H_\delta(g)\right| \, dg + \int_{SL(n,\mathbb{R})} \left|\lambda_n(\ell_{\pi_\infty}) H_\delta(g)\right| \, dg \\ &\leqslant \frac{3(1+e^{2\delta})}{\delta^4} C_\delta \cdot \mathrm{Vol}(B_\delta) + \left|\lambda_n(\ell_{\pi_\infty})\right| .\end{aligned}$$

By applying (3.9) to Theorem 1.14, we prove Theorem 1.17.

REFERENCES

[1] C. Bian, *Computing $GL(3)$ automorphic forms*, Bull. London Math. Soc., 42:5 (2010), pp. 827–842.

[2] A. Booker, *Uncovering a new L-function*, Notices of the AMS, 55 (2008), pp. 1088–1094.

[3] A. R. Booker, A. Strömbergsson, and A. Venkatesh, *Effective computation of Maass cusp forms*, Int. Math. Res. Not., (2006), Art. ID 71281, 34.

[4] F. Brumley, *Effective multiplicity on on GL_N and narrow zero-free regions for Rankin-Selberg L-functions*, American Journal of Mathematics, 128 (2006), pp. 1455–1474.

[5] D. Flath, *Decomposition of representations into tensor products*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I. (1979), pp. 170–183.

[6] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n,\mathbb{R})$*, no. 99, Cambridge studies in advanced mathematics, 2006.

[7] D. Goldfeld and J. Hundley, *Automorphic representations and L-functions for the general linear group*, no. 129, 130, Cambridge studies in advanced mathematics, 2011.

[8] D. Grenier, *On the shape of fundamental domains in $GL(n,\mathbb{R})$*, Pacific Journal of Math., 160 (1993), pp. 53–65.

[9] D. A. Hejhal, *On eigenfunctions of the Laplacian for Hecke triangle groups. In Emerging applications of number theory (Minneapolis, MN, 1996)*, IMA Vol. Math. Appl., 109 (1999), pp. 291–315.

[10] S. Helgasson, *Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions*, no. 83, American Mathematical Society, 2000.

[11] H. Jacquet, *Functions de Whittaker associates aux groupes de Chevalley (French)*, Bull.Soc. Math. France, 95 (1967), pp. 243–309.

[12] J. Jorgenson and S. Lang, *Spherical Inversion on $SL_n(\mathbb{R})$*, Springer Monographs in Mathematics, Springer-Verlag New York, Inc., 2001.

[13] E. Lindenstrauss and A. Venkatesh, *Existence and Weyl's law for Spherical Cusp forms*, GAFA, 17 (2007), pp. 220–251.

[14] W. Luo, Z. Rudnick, and P. Sarnak, *On Selberg's Eigenvalue Conjecture*, GAFA, 5 (1995), pp. 387–401.

[15] ———, *On the generalized Ramanujan conjecture for $GL(n)$*, Proc. Sym. Pure Math., II (1999), no. 66, pp. 301–311.

[16] H. Maass, *Über eine neue Art von nichatanalytischen automorphen funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Ann. Math., 122 (1949), pp. 141–83.

[17] B. Mezhericher, *Evaluating Jacquet's Whittaker functions and Maass forms for $SL(3,\mathbb{Z})$*, Ph.D. thesis, Columbia University (2008).

[18] S. D. Miller, *On the existence and temperedness of cusp forms for $SL_3(\mathbb{Z})$*, J. Reine Angew. Math., 533 (2001), pp. 127–169.

[19] W. Müller, *Weyl's law for the cuspidal spectrum of SL_n*, Ann. of Math., 165 (2007), pp. 275–333.

[20] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.), 20 (1956), pp. 47–87.

[21] H. Stark, *Fourier coefficients of Maass waveforms*, Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res. Horwood, Chichester (1984), pp. 263–169.