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$U(n)$-INVARIANT KÄHLER METRICS WITH NONNEGATIVE QUADRATIC BISECTIONAL CURVATURE*

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Abstract. By perturbing the complete $U(n)$-invariant metrics with positive bisectional curvature constructed by Wu-Zheng [10], we obtain complete $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$, $n \geq 3$, which have nonnegative quadratic bisectional curvature ($\mathbf{QB} \geq 0$) everywhere, and which do not have nonnegative orthogonal bisectional curvature and do not have nonnegative Ricci curvature at some points. We prove that $\mathbf{QB} \geq 0$ is preserved under the Kähler-Ricci flow for complete $U(n)$-invariant solution with bounded curvature. We prove that $\mathrm{Ric} \geq 0$ is also preserved under an additional assumption.

Key words. $U(n)$-invariant Kähler metrics, quadratic bisectional curvature, Kähler-Ricci flow.

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1. Introduction.

Let (M^n, g) be a Kähler manifold of complex dimension n and let $o \in M$. M is said to have nonnegative bisectional curvature (abbreviated as $\mathbf{B} \geq 0$) at o if for any $X, Y \in T_o^{(1,0)}(M)$, $R(X, \bar{X}, Y, \bar{Y}) \geq 0$. M is said to have nonnegative *orthogonal* bisectional curvature (abbreviated as $\mathbf{B}^\perp \geq 0$) at o if $R(X, \bar{X}, Y, \bar{Y}) \geq 0$ for all unitary pairs $X, Y \in T_o^{(1,0)}(M)$. M is said to have *nonnegative quadratic orthogonal bisectional curvature* at o (abbreviated as $\mathbf{QB} \geq 0$) if for any unitary frame e_i at o and real numbers ξ_i we have

$$\sum_{i,j} R_{i\bar{i}j\bar{j}}(\xi^i - \xi^j)^2 \geq 0. \tag{1.1}$$

Here $R_{i\bar{i}j\bar{j}} = R(e_i, \bar{e}_i, e_j, \bar{e}_j)$. $\mathbf{B} > 0$ and $\mathbf{B}^\perp > 0$ are defined similarly. It is obvious that $\mathbf{B} \geq 0 \Rightarrow \mathbf{B}^\perp \geq 0 \Rightarrow \mathbf{QB} \geq 0$. Note that in dimension $n = 2$, the conditions $\mathbf{B}^\perp \geq 0$ and $\mathbf{QB} \geq 0$ are the same. We will say that M satisfies $\mathbf{B} \geq 0$ (respectively $\mathbf{B}^\perp \geq 0, \mathbf{QB} \geq 0$) provided that M has $\mathbf{B} \geq 0$ (respectively $\mathbf{B}^\perp \geq 0, \mathbf{QB} \geq 0$) at every point.

Even though the condition $\mathbf{QB} \geq 0$ is weaker than the condition that bisectional curvature is nonnegative, Kähler manifolds with $\mathbf{QB} \geq 0$ still have many interesting properties. In fact the condition $\mathbf{QB} \geq 0$ appeared implicitly in [3] and it was proved that a real harmonic (1,1) form on a compact Kähler manifold with this curvature condition must be parallel. In [9] (see also [11, 1]), it was proved on a compact Kähler manifold with $\mathbf{QB} \geq 0$ any numerically effective (nef) line bundle admits a smooth metric with nonnegative curvature. In [1], it was proved that any Kähler manifold with $\mathbf{QB} \geq 0$ must have nonnegative scalar curvature and must be flat if the scalar curvature is zero and if the manifold has complex dimension at least three. It was also proved that an irreducible compact Kähler manifold with $\mathbf{QB} \geq 0$ has positive first Chern class. It is then a question whether there are Kähler manifolds satisfying $\mathbf{QB} \geq 0$ but not $\mathbf{B} \geq 0$ or even $\mathbf{B}^\perp \geq 0$. In [4] Li-Wu-Zheng are able to construct the

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first example of compact Kähler manifold which satisfies $\mathbf{QB} \geq 0$ but not $\mathbf{B}^{\perp} \geq 0$. The example is a Kähler C-space. In [2], necessary and sufficient conditions on any Kähler C-space with $b_2 = 1$ satisfies $\mathbf{QB} \geq 0$ are obtained by Chau and the second author. In particular, many more examples of compact Kähler manifolds satisfying $\mathbf{QB} \geq 0$ but not $\mathbf{B}^{\perp} \geq 0$ are given.

There is a problem on complete noncompact Kähler manifold on solving the Poincaré-Lelong equation. Namely, given a closed real (1,1) form ρ, one would like to find a potential function u so that $\sqrt{-1}\partial\bar{\partial}u = \rho$. Previous works assume that the bisectional curvature is nonnegative and decay suitably, see references in [6]. However, the result is still true if we only assume $\mathbf{QB} \geq 0$ and $\mathrm{Ric} \geq 0$. So it is also interesting to see if there are noncompact examples with $\mathbf{QB} \geq 0$ but not $\mathbf{B}^{\perp} \geq 0$.

There are not very many examples of complete noncompact Kähler manifolds with nonnegative bisectional curvature. In [10] H. Wu and Zheng systematically studied $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature and are able to give many examples. In this paper, by perturbing a metric constructed in [10] we are going to construct examples of complete noncompact Kähler manifolds which satisfy $\mathbf{QB} \geq 0$ but not $\mathbf{B}^{\perp} \geq 0$. Each such metric with positive bisectional curvature in [10] is determined up to scaling by a smooth function ξ on $[0, \infty)$ such that $\xi(0) = 0$, $0 < \xi < \infty$ on $(0, \infty)$ and $\xi' > 0$. We will show that given such a ξ, one can find $R > 1$, $\bar{\xi}$ such that $\bar{\xi} = \xi$ outside $[R-1, R+1]$ and $0 < \bar{\xi} < 1$ on $(0, \infty)$, such that the $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ $(n \geq 3)$ determined by $\bar{\xi}$ satisfies $\mathbf{QB} \geq 0$ but not $\mathbf{B}^{\perp} \geq 0$. See Theorem 4.1 for more details. Actually, the metrics also do not satisfy $\mathrm{Ric} \geq 0$.

Motivated by the generalized Hartshorne conjecture, see [12, p.218], Li-Wu-Zheng [4] conjectured that very irreducible compact Kähler manifold with $\mathbf{QB}^{\perp} \geq 0$ must be biholomorphic to a Kähler C-space. It is natural to see if one might use Kähler-Ricci flow. However, it is unclear if the condition $\mathbf{QB} \geq 0$ is preserved under the Kähler-Ricci flow. In this work, we will prove that this curvature condition is preserved for $U(n)$-invariant solution of the Kähler-Ricci flow provided the curvature is bounded. With an additional condition, we will prove that the condition $\mathrm{Ric} \geq 0$ is also preserved.

The paper is organized as follows. In §2, we will review the construction of H. Wu and Zheng [10] on $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$. In §3, we will give characterization of $\mathbf{B}^{\perp} \geq 0$ and $\mathbf{QB} \geq 0$ for $U(n)$-invariant Kähler metrics. In §4, we will construct the examples mentioned above. In §5, we will discuss preservation of some curvature conditions including $\mathbf{QB} \geq 0$ and $\mathrm{Ric} \geq 0$ for the $U(n)$-invariant solution of the Kähler-Ricci flow.

2. The construction of Wu-Zheng. In this work, we always assume that $n \geq 3$. We first recall the construction of $U(n)$-invariant Kähler metrics on $\mathbb{C}^n$ by H. Wu and Zheng [10]. In the standard coordinates of $\mathbb{C}^n$, the general form of an $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ is given by:

$$g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j \tag{2.1}$$

where $r = |z|^2$, f is a smooth function on $[0, \infty)$. Let $h = (rf)'$. The following is in [10].

LEMMA 2.1. *g is complete Kähler metric if and only if $f > 0, h > 0$ and*

$$\int_0^\infty \frac{\sqrt{h}}{\sqrt{r}} dr = \infty.$$

If $h > 0$, then $\xi = -\frac{rh'}{h}$ is a smooth function on $[0,\infty)$ with $\xi(0) = 0$. On the other hand, if ξ is a smooth positive function on $[0,\infty)$ with $\xi(0) = 0$. Define $h(r) = h(0)\exp(-\int_0^r \frac{\xi(s)}{s} ds)$ and $f(r) = \frac{1}{r}\int_0^r h(s)ds$ with $h(0) > 0$. Then (2.1) define a $U(n)$-invariant Kähler metric on $\mathbb{C}^n$. The following is also from [10].

THEOREM 2.1.
(i) If $0 < \xi < 1$ on $(0,\infty)$, then g is complete.
(ii) g is complete and has positive bisectional curvature if and only if $\xi' > 0$ and $0 < \xi < 1$ on $(0,\infty)$.
(iii) Every complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ with positive bisectional curvature is given by a smooth function ξ in (ii).

The curvature tensor of a $U(n)$-invariant Kähler metric under the unitary frame

$$\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \cdots, e_n = \frac{1}{\sqrt{f}}\partial_{z_n}\} \tag{2.2}$$

at the point $z = (z_1, 0, \cdots, 0)$ is given as follows, see [10]:

$$\begin{aligned} A :=& R_{1\bar{1}1\bar{1}} = \frac{\xi'}{h};\\ B :=& R_{1\bar{1}i\bar{i}} = \frac{1}{(rf)^2}\left[rh - (1-\xi)\int_0^r hds\right],\ i \geq 2;\\ C :=& R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \frac{2}{r^2 f^2}\left(\int_0^r hds - rh\right),\ i \neq j; i,j \geq 2. \end{aligned} \tag{2.3}$$

All other components are zero, except those obtained by the symmetric properties of R.

3. Characterizations of $\mathbf{B}^\perp \geq 0$ and $\mathbf{QB} \geq 0$. In this section, we will give some characterizations of a $U(n)$-invariant Kähler metric to satisfy $\mathbf{B}^\perp \geq 0$ and $\mathbf{QB} \geq 0$.

THEOREM 3.1. *A $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ satisfies $\mathbf{B}^\perp \geq 0$ if and only if $A + C \geq 0$, $B \geq 0$ and $C \geq 0$.*

Proof. Since the metric is invariant under the $U(n)$ action, one just need to check the sign of the curvature at the point $p = (z_1, 0, \cdots, 0)$. Let e_i be the unitary frame of p as in (2.2). We shall compute $R(X, \bar{X}, Y, \bar{Y})$ for any given unitary pairs $X, Y \in T_p^{(1,0)}(M)$ first. As in [10], by applying a $U(n)$ action which preserves the point p, we may assume that

$$X = x_1 e_1 + x_2 e_2,\ Y = y_1 e_1 + y_2 e_2 + y_3 e_3.$$

Then

$$\begin{aligned} R(X, \bar{X}, Y, \bar{Y}) =& A|x_1|^2|y_1|^2 + B\big(|x_1|^2|y_2|^2 + |x_1|^2|y_3|^2 + |x_2|^2|y_1|^2\\ &+ x_2\bar{x_1}y_1\bar{y_2} + x_1\bar{x_2}y_2\bar{y_1}\big) + C(|x_2|^2|y_2|^2 + \frac{1}{2}|x_2|^2|y_3|^2). \end{aligned}$$

Since $X \perp Y$ which implies that $x_1\bar{y_1} + x_2\bar{y_2} = 0$, we have

$$R(X,\bar{X},Y,\bar{Y}) = (A+C)|x_1|^2|y_1|^2 + \frac{C}{2}|x_2|^2|y_3|^2 + B\left(|x_1|^2|y_2|^2 + |x_2|^2|y_1|^2 - 2|x_1|^2|y_1|^2 + |x_1|^2|y_3|^2\right).$$

Suppose $A+C \geq 0$, $B \geq 0$ and $C \geq 0$. Denote

$$\alpha = |x_1|^2|y_2|^2 + |x_2|^2|y_1|^2 - 2|x_1|^2|y_1|^2.$$

To prove that $\mathbf{B}^\perp \geq 0$, it is sufficient to show that $\alpha \geq 0$.

If $y_1 \neq 0$, then

$$|y_1|^2\alpha = |x_2|^2(|y_2|^2 - |y_1|^2)^2 \geq 0,$$

because $x_1\bar{y}_1 + x_2\bar{y}_2 = 0$. Hence $\alpha \geq 0$, and the coefficient of B is non-negative. Similarly, if $y_2 \neq 0$, then

$$|y_2|^2\alpha = |x_1|^2(|y_2|^2 - |y_1|^2)^2 \geq 0$$

and the coefficient of B is also non-negative. If $y_1 = y_2 = 0$, then the coefficient of B is just $|x_1|^2|y_3|^2$ which is nonnegative. Thus, we conclude that if $A+C \geq 0$, $B \geq 0$ and $C \geq 0$ then $\mathbf{B}^\perp \geq 0$.

Conversely, suppose $\mathbf{B}^\perp \geq 0$, then $B \geq 0$ and $C \geq 0$ since $B = R_{1\bar{1}i\bar{i}}$ and $C = 2R_{i\bar{i}j\bar{j}}$ for any $2 \leq i \neq j \leq n$, . Take $x_1 = y_1 = y_2 = -x_2 = \frac{1}{\sqrt{2}}$ and $y_3 = 0$ which implies that X, Y form a unitary pair. Then we get $A+C \geq 0$. □

Next we want to characterize $U(n)$-invariant Kähler metric with $\mathbf{QB} \geq 0$ in terms of A, B, C. By [2], a Kähler manifold M satisfies $\mathbf{QB} \geq 0$ if and only if the symmetric form $G - F$ is positive semi-definite on the space $\Omega^{1,1}_{\mathbb{R}}(M)$ of real (1,1) forms at any $p \in M$, where

$$F(\rho,\sigma) = \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}\rho^{i\bar{l}}\sigma^{k\bar{j}} = \sum_{i,j,k,l} R_{i\bar{l}k\bar{j}}\rho^{i\bar{l}}\sigma^{k\bar{j}},$$

and

$$G(\rho,\sigma) = \frac{1}{2}(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}})\rho^{i\bar{l}}\sigma^{k\bar{j}}.$$

Here $\rho^{i\bar{l}}, \sigma^{k\bar{j}}$ are the local components of ρ, σ with indices raised.

Theorem 3.2. *A $U(n)$-invariant Kähler metric satisfies* $\mathbf{QB} \geq 0$ *if and only if* $B \geq 0$, $A + (n-2)B + \frac{n}{2}C \geq 0$ *and* $B + \frac{n-1}{2}C \geq 0$.

Proof. As before, it is sufficient to prove that the theorem is true at the point $p = (z_1, 0, \cdots, 0)$. Now in the unitary frame $\{\frac{1}{\sqrt{h}}\partial_{z_1}, \frac{1}{\sqrt{f}}\partial_{z_2}, \cdots, \frac{1}{\sqrt{f}}\partial_{z_n}\}$ at the point $p = (z_1, 0, \cdots, 0)$, the Ricci curvature is diagonalized with

$$\begin{cases} R_{1\bar{1}} = & A + (n-1)B; \\ R_{i\bar{i}} = & B + C + \frac{n-2}{2}C = B + \frac{n}{2}C, \text{if } i \geq 2. \end{cases} \tag{3.1}$$

So

$$G(\rho,\sigma)=\sum_{i,k}R_{i\bar{i}}\rho^{i\bar{k}}\sigma^{k\bar{i}}.$$

$$\begin{aligned}F(\rho,\sigma)=&\sum_{i,j}R_{i\bar{i}j\bar{j}}\rho^{i\bar{i}}\sigma^{j\bar{j}}+\sum_{i\neq l,\text{or }\ k\neq j}R_{i\bar{l}k\bar{j}}\rho^{i\bar{l}}\sigma^{k\bar{j}}\\ =&\sum_{i,j}R_{i\bar{i}j\bar{j}}\rho^{i\bar{i}}\sigma^{j\bar{j}}+\sum_{i\neq k}R_{i\bar{k}k\bar{i}}\rho^{i\bar{k}}\sigma^{k\bar{i}}\\ =&\sum_{i,j}R_{i\bar{i}j\bar{j}}\rho^{i\bar{i}}\sigma^{j\bar{j}}+\sum_{i\neq j}R_{i\bar{i}j\bar{j}}\rho^{i\bar{j}}\sigma^{j\bar{i}}\end{aligned}$$

with $\rho^{i\bar{j}}=\overline{\rho^{j\bar{i}}}=\rho_{i\bar{j}}$. Note that $a_i:=\rho_{i\bar{i}}$ is real. Then

$$\begin{aligned}&G(\rho,\rho)-F(\rho,\rho)\\ =&(A+(n-1)B)\sum_{k\geq 1}|\rho_{1\bar{k}}|^2+(B+\frac{n}{2}C)\sum_{i\geq 2,k\geq 1}|\rho_{i\bar{k}}|^2\\ &-Aa_1^2-2Ba_1\sum_{i\geq 2}a_i-C\sum_{i\geq 2}a_i^2-\frac{C}{2}\sum_{2\leq i\neq j\leq n}a_ia_j\\ &-2B\sum_{i\geq 2}|\rho_{1\bar{i}}|^2-\frac{C}{2}\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2\\ =&(n-1)Ba_1^2+\left(B+(\frac{n}{2}-1)C\right)\sum_{i\geq 2}a_i^2-2Ba_1\sum_{i\geq 2}a_i\\ &-\frac{C}{2}\sum_{2\leq i\neq j\leq n}a_ia_j+\left(A+(n-2)B+\frac{n}{2}C\right)\sum_{i\geq 2}|\rho_{1\bar{i}}|^2\\ &+\left(B+(\frac{n}{2}-\frac{1}{2})C\right)\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2\\ =&B\left((n-1)a_1^2+\sum_{i\geq 2}a_i^2-2a_1\sum_{i\geq 2}a_i\right)+C\left((\frac{n}{2}-1)\sum_{i\geq 2}a_i^2-\frac{1}{2}\sum_{2\leq i\neq j\leq n}a_ia_j\right)\\ &+\left(A+(n-2)B+\frac{n}{2}C\right)\sum_{i\geq 2}|\rho_{1\bar{i}}|^2+\left(B+(\frac{n}{2}-\frac{1}{2})C\right)\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2\\ =&B\left(\sum_{i\geq 2}(a_1-a_i)^2\right)+\frac{1}{2}C\left(\sum_{2\leq i<j\leq n}(a_i-a_j)^2\right)\\ &+\left(A+(n-2)B+\frac{n}{2}C\right)\sum_{i\geq 2}|\rho_{1\bar{i}}|^2+\left(B+\frac{n-1}{2}C\right)\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2.\end{aligned}$$

Denote $a_1 - a_i$ by x_i, $i \geq 2$. Then

$$
\begin{aligned}
&G(\rho,\rho) - F(\rho,\rho)\\
=&B\sum_{i\geq 2} x_i^2 + \frac{1}{2}C\left(\sum_{2\leq i<j\leq n}(x_i-x_j)^2\right)\\
&+\left(A+(n-2)B+\frac{n}{2}C\right)\sum_{i\geq 2}|\rho_{1\bar{i}}|^2 + \left(B+\frac{n-1}{2}C\right)\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2\\
=&\frac{1}{n-1}B\left((n-1)\sum_{i\geq 2}x_i^2 - \sum_{2\leq i<j\leq n}(x_i-x_j)^2\right)\\
&+\frac{1}{n-1}\left(B+\frac{n-1}{2}C\right)\left(\sum_{2\leq i<j\leq n}(x_i-x_j)^2\right)\\
&+\left(A+(n-2)B+\frac{n}{2}C\right)\sum_{i\geq 2}|\rho_{1\bar{i}}|^2 + \left(B+\frac{n-1}{2}C\right)\sum_{2\leq i\neq j\leq n}|\rho_{i\bar{j}}|^2.
\end{aligned}
\tag{3.2}
$$

Hence if $B \geq 0$, $A + (n-2)B + \frac{n}{2}C \geq 0$ and $B + \frac{n-1}{2}C \geq 0$, then $\mathbf{QB} \geq 0$.

If we choose suitable $\rho_{i\bar{j}}$, we can conclude that $\mathbf{QB} \geq 0$ then $B \geq 0$, $A + (n-2)B + \frac{n}{2}C \geq 0$ and $B + \frac{n-1}{2}C \geq 0$. This is also a consequence of Proposition 3.1 in the following. □

To get a better understanding of the conditions in the more general setting, we have the following proposition. The first part is due to Niu [7], and the second part is an immediate consequence of the definition of $\mathbf{QB} \geq 0$.

Proposition 3.1. *Let (M^n, g) be a Kähler manifold with complex dimension n. Suppose $\mathbf{QB} \geq 0$ at a point p. Then for any unitary frame $e_1, \ldots, e_n$ at p, we have*

(i) $R_{i\bar{i}} + R_{j\bar{j}} - 2R_{i\bar{i}j\bar{j}} \geq 0$ for all $i \neq j$; and

(ii) $R_{i\bar{i}} - R_{i\bar{i}i\bar{i}} \geq 0$.

Before we prove the proposition, let us consider the $U(n)$-invariant Kähler metric. With the above notations:

$$
\begin{cases}
R_{1\bar{1}} + R_{j\bar{j}} - 2R_{1\bar{1}j\bar{j}} = A + (n-2)B + \frac{n}{2}C, & \text{for } j \geq 2;\\
R_{i\bar{i}} + R_{j\bar{j}} - 2R_{i\bar{i}j\bar{j}} = 2\left(B + \frac{n-1}{2}C\right), & \text{for } 2 \leq i \neq j;\\
R_{1\bar{1}} - R_{1\bar{1}1\bar{1}} = (n-1)B; & \\
R_{i\bar{i}} - R_{i\bar{i}i\bar{i}} = B + \frac{n-2}{2}C, & \text{for } i \geq 2.
\end{cases}
$$

Hence the necessity part of Theorem 3.2 follows from the proposition. Hence the theorem means that for $U(n)$-invariant metric, the conditions (i) and (ii) are also sufficient for $\mathbf{QB} \geq 0$.

Proof. [Proof of the Proposition] Suppose $\mathbf{QB} \geq 0$. In a unitary frame,

$$
F(\rho,\rho) - G(\rho,\rho) = \sum_{i,j} R_{i\bar{i}j\bar{j}}\rho_{i\bar{i}}\rho_{j\bar{j}} + \sum_{i\neq l,\text{or } k\neq j} R_{i\bar{l}k\bar{j}}\rho_{l\bar{i}}\rho_{j\bar{k}} - \sum_{i,k} R_{i\bar{i}}\rho_{k\bar{i}}\rho_{i\bar{k}}. \tag{3.3}
$$

where $\bar{\rho}_{i\bar{j}} = \rho_{j\bar{i}}$. Let $\rho_{i\bar{j}} = 0$ for all i, j except $\rho_{1\bar{2}}$ and $\rho_{2\bar{1}}$. Then

$$
\begin{aligned}
0 \geq& F(\rho,\rho) - G(\rho,\rho)\\
=&\left(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}} - R_{2\bar{2}}\right)|\rho_{1\bar{2}}|^2 + R_{2\bar{1}2\bar{1}}\rho_{1\bar{2}}^2 + R_{1\bar{2}1\bar{2}}\rho_{2\bar{1}}^2\\
=&\left(2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}} - R_{2\bar{2}}\right)|\rho_{1\bar{2}}|^2 + 2\Re\left(R_{2\bar{1}2\bar{1}}\rho_{1\bar{2}}^2\right).
\end{aligned}
\tag{3.4}
$$

Replace $\rho_{1\bar 2}$ by $\sqrt{-1}\rho_{1\bar 2}$, then we have

$$(2R_{1\bar 1 2\bar 2} - R_{1\bar 1} - R_{2\bar 2})\,|\rho_{1\bar 2}|^2 - 2\Re\left(R_{2\bar 1 2\bar 1}\rho_{1\bar 2}^2\right) \le 0.$$

Hence we have

$$R_{1\bar 1} + R_{2\bar 2} - 2R_{1\bar 1 2\bar 2} \ge 0.$$

This proves (i).

Now take $\rho_{1\bar 1} \ne 0$, and all other $\rho_{i\bar j}$ are zeros, we have

$$0 \ge F(\rho,\rho) - G(\rho,\rho) = (R_{1\bar 1 1\bar 1} - R_{1\bar 1})\,|\rho_{1\bar 1}|^2.$$

From this (ii) follows. □

4. $U(n)$-invariant metric with $\mathbf{QB} \ge 0$. In this section, we will perturb a complete $U(n)$-invariant Kähler metric with positive bisectional curvature to obtain a $U(n)$-invariant Kähler metric which satisfies $\mathbf{QB} \ge 0$ and does not satisfy $\mathbf{B}^\perp \ge 0$.

As mentioned in section 2, a complete $U(n)$-invariant Kähler metric with positive bisectional curvature is determined up to scaling by a smooth function ξ on $[0,\infty)$ with $\xi(0) = 0$, $\xi' > 0$ and $0 < \xi < 1$. We normalize the metric so that

$$h(r) = \exp\left(-\int_0^r \frac{\xi(t)}{t}dt\right).$$

Fix a smooth cutoff function ϕ on $\mathbb{R}$ such that
(i) $0 \le \phi \le c_0$;
(ii) $\operatorname{supp}\phi \subset [-1,1]$
(iii) $\phi'(0) = 1$ and $|\phi'| \le 1$.
We have the following:

Theorem 4.1. *For $n \ge 3$, for any $r_0 > 0$ there is $R > r_0$ and $\beta > 0$ such that $\bar\xi(r) = \xi(r) - \beta\phi(r-R)$ will give a complete $U(n)$-invariant Kähler metric which satisfies $\mathbf{QB} \ge 0$, and does not satisfy $\mathbf{B}^\perp \ge 0$ or $\mathrm{Ric} \ge 0$.*

Let $a = \lim_{r\to\infty}\xi(r)$. Then $0 < a \le 1$. By Theorem 2.1, if β is chosen so that $\beta c_0 < a$ and R is chosen large enough, then $\bar\xi$ will give a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$.

Let

$$\begin{aligned}\bar h(r) &= \exp\left(-\int_0^r \frac{\bar\xi(t)}{t}dt\right);\\ r\bar f(r) &= \int_0^r \bar h ds.\end{aligned} \tag{4.1}$$

In the unitary frame $\{\frac{1}{\sqrt{\bar h}}\partial_{z_1}, \frac{1}{\sqrt{\bar f}}\partial_{z_2}, \cdots, \frac{1}{\sqrt{\bar f}}\partial_{z_n}\}$ at a point $z = (z_1, 0, \cdots, 0)$, the curvature tensor of the perturbed metric are given by

$$\begin{aligned}\bar A &= \bar R_{1\bar 1 1\bar 1} = \frac{\bar\xi'}{\bar h};\\ \bar B &= \bar R_{1\bar 1 i\bar i} = \frac{1}{(r\bar f)^2}\left[r\bar h - (1-\bar\xi)\int_0^r \bar h\right],\ i \ge 2;\\ \bar C &= \bar R_{i\bar i i\bar i} = 2\bar R_{i\bar i j\bar j} = \frac{2}{r^2\bar f^2}\left(\int_0^r \bar h ds - r\bar h\right),\ i \ne j, i, j \ge 2.\end{aligned} \tag{4.2}$$

In order to prove the theorem, we need to obtain some estimates, which are rather elementary and some of them may already be obtained in [10].

LEMMA 4.1. *Let* ξ *be as above. Suppose* $\lim_{r\to\infty}\xi = a$, $0 < a \le 1$. *We have the following:*

(i) $\lim_{r\to\infty} h(r) = 0$, *and for any* r_0, $\lim_{r\to\infty}\frac{h(r+r_0)}{h(r)} = 1$.

(ii) $\lim_{r\to\infty}\int_0^r h = \infty$ *and for any* r_0,

$$\lim_{r\to\infty}\frac{\int_0^{r+r_0} h}{\int_0^r h} = 1.$$

(iii) $(rh + (\xi - 1)\int_0^r h)' > 0$ *for* $r > 0$.

(iv)

$$\lim_{r\to\infty}\frac{rh}{\int_0^r h} = 1 - a.$$

(v) *For any* $\epsilon > 0$, *and for any* $r_0 > 0$, *there is* $R > r_0$ *such that*

$$\xi'(R) - \epsilon h(R)C(R) < 0.$$

(vi) $\lim_{r\to\infty} h(r)C(r) = 0$.

(vii) *For all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $R \ge 3$, $\delta \ge \eta \ge 0$ *is a smooth function with support in* $[R-1, R+1]$, *then for all* $r \ge 0$,

$$h(r) \le \bar h(r) \le (1+\epsilon)h(r); \quad \text{and} \quad \int_0^r h \le \int_0^r \bar h \le (1+\epsilon)\int_0^r h,$$

where $\bar h(r) = \exp\left(-\int_0^r \frac{\bar\xi(t)}{t}dt\right)$ *and* $\bar\xi = \xi - \eta$.

Proof. (i) Since $\lim_{r\to\infty}\xi = a > 0$, $\int_1^r \frac{\xi(t)}{t}dt \to \infty$ as $r\to\infty$. Hence $h(r)\to 0$ as $r\to\infty$. This proves the first assertion of (i). For fixed r_0 (which may be negative),

$$\frac{h(r+r_0)}{h(r)} = \exp\left(-\int_r^{r+r_0}\frac{\xi(t)}{t}dt\right).$$

Since for fixed r_0, $\int_r^{r+r_0}\frac{\xi(t)}{t}dt \to 0$ as $r\to\infty$, the second assertion of (i) is also true.

(ii) Since $\xi \le a \le 1$, $h(r) \ge h(1)r^{-1}$ for $r \ge 1$. Hence $\int_0^r h \to \infty$ as $r\to\infty$.

$$\frac{\int_0^{r+r_0} h}{\int_0^r h} = 1 + \frac{\int_r^{r+r_0} h}{\int_0^r h}.$$

Since $h' < 0$, and $h(0) = 1$, $h \le 1$. Since $\int_0^r h \to \infty$ as $r\to\infty$, we have

$$\lim_{r\to\infty}\frac{\int_0^{r+r_0} h}{\int_0^r h} = 1.$$

(iii)

$$\left(rh + (\xi-1)\int_0^r h\right)' = rh' + h + \xi'\int_0^r h + (\xi-1)h = \xi'\int_0^r h > 0$$

for $r > 0$, because $\xi = -\frac{rh'}{h} > 0$.

(iv) By (ii) and the l'Hospital rule, we have

$$\lim_{r\to\infty} \frac{rh}{\int_0^r h} = \lim_{r\to\infty} \frac{rh' + h}{h} = 1 - a.$$

(v) Suppose (v) is not true. Then there is $r_0 > 0$ and $\epsilon > 0$ such that $\xi'(r) \geq \epsilon h(r)C(r)$ for all $r \geq r_0$. By (iv)

$$\lim_{r\to\infty} C(r) \int_0^r h = \lim_{r\to\infty} 2\left(1 - \frac{rh}{\int_0^r h}\right) = 2a > 0.$$

Hence there exists $r_1 > r_0$ such that for all $r \geq r_1$, we have

$$C(r) \geq \frac{a}{\int_0^r h}.$$

Then for all $r \geq r_1$, we have

$$\xi'(r) \geq a\epsilon \frac{h}{\int_0^r h}.$$

Integrate the above inequality, we have

$$\xi(r) - \xi(r_1) \geq a\epsilon \log\left(\frac{\int_0^r h}{\int_0^{r_1} h}\right).$$

This is impossible because ξ is bounded and $\int_0^r h \to \infty$ as $r \to \infty$.

(vi)

$$h(r)C(r) = \frac{2h(r)}{\int_0^r h}\left(1 - \frac{rh}{\int_0^r h}\right) \leq \frac{2}{r}\frac{rh}{\int_0^r h}.$$

By (iv), we conclude that $h(r)C(r) \to 0$ as $r \to \infty$.

(vii) It is sufficient to show the first part.

$$\bar{h}(r) = h(r)e^{\int_0^r \frac{\eta(t)}{t}dt} \leq h(r)e^{\int_{R-1}^{R+1} \frac{\eta(t)}{t}dt} \leq h(r)e^{2\delta\frac{1}{R-1}} \leq h(r)e^{\delta}.$$

We take $\delta = \ln(1+\epsilon)$, then we have $\bar{h}(r) \leq (1+\epsilon)h(r)$. Since $\eta \geq 0$, $h(r) \leq \bar{h}(r)$. ∎

LEMMA 4.2. *There is $2 \geq \alpha > 0$ such that for any $r_0 > 0$ there is $R > r_0$ satisfying the following: If $\bar{\xi}(r) = \xi(r) - \alpha h(R)C(R)\phi(r-R)$, then $\bar{\xi}$ determines a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ such that*

(i) $\bar{A}(R) + \bar{C}(R) < 0$;

(ii) $\bar{A} + \frac{n}{2}\bar{C} > 0$ on $[R-1, R+1]$; and

(iii) $\bar{A}(R) + (n-1)\bar{B}(R) < 0$.

Proof. Let $\epsilon > 0$ and $R \geq 3$ to be chosen later. Then by Lemma 4.1(vii), there is $a > \delta > 0$ independent of R, such that if $\bar{\xi}(r) = \xi(r) - \beta\phi(r-R)$ with $\beta > 0$ and $\beta c_0 < \delta$, then $\bar{\xi}$ determines a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$ such that for all r

$$h(r) \leq \bar{h}(r) \leq (1+\epsilon)h(r); \int_0^r h \leq \int_0^r \bar{h} \leq (1+\epsilon)\int_0^r \bar{h}. \tag{4.3}$$

Then for any $r_0 > 3$, there is $r_1 > r_0$ which is independent of R such that if $r > r_1$, then

$$\begin{aligned}
\bar{C}(r) =& \frac{2}{\int_0^r \bar{h}}(1 - \frac{r\bar{h}}{\int_0^r \bar{h}}) \\
\leq& \frac{2}{\int_0^r \bar{h}}(1 - \frac{rh}{(1+\epsilon)\int_0^r h}) \\
\leq& \frac{2}{\int_0^r \bar{h}}(1 - \frac{1-a-\epsilon}{1+\epsilon}) \\
\leq& \frac{2}{\int_0^r h}\frac{a+2\epsilon}{1+\epsilon}
\end{aligned}$$

where we have used Lemma 4.1(iv). On the other hand, by Lemma 4.1(iv), we may choose r_1 large enough so that if $r \geq r_1$, then

$$C(r) = \frac{2}{\int_0^r h}(1 - \frac{rh}{\int_0^r h}) \geq \frac{2}{\int_0^r h}(a - \epsilon). \tag{4.4}$$

Hence if $r > r_1$, and $\epsilon < a$, then

$$\bar{C}(r) \leq \frac{a+2\epsilon}{(a-\epsilon)(1+\epsilon)}C(r).$$

Suppose $R > r_1$, then

$$\begin{aligned}
\bar{A}(R) + \bar{C}(R) =& \frac{\xi'(R) - \beta}{\bar{h}(R)} + \bar{C}(R) \\
\leq& \frac{\xi'(R) - \beta}{\bar{h}(R)} + \frac{a+2\epsilon}{(a-\epsilon)(1+\epsilon)}C(R) \\
\leq& \frac{\xi'(R) - \beta}{(1+\epsilon)h(R)} + \frac{a+2\epsilon}{(a-\epsilon)(1+\epsilon)}C(R) \\
=& \frac{1}{(1+\epsilon)h(R)}\left(\xi'(R) - \beta + \frac{a+2\epsilon}{a-\epsilon}h(R)C(R)\right)
\end{aligned} \tag{4.5}$$

provided that $\xi'(R) - \beta < 0$. On the other hand, if $R > r_1$ as above, by Lemma 4.1(i) and (ii) we may choose a larger r_1 so that for $r \in [R-1, R+1]$, we have

$$\begin{aligned}
\bar{C}(r) =& \frac{2}{\int_0^r \bar{h}}(1 - \frac{r\bar{h}}{\int_0^r \bar{h}}) \\
\geq& \frac{2}{\int_0^r \bar{h}}(1 - \frac{(1+\epsilon)rh}{\int_0^r h}) \\
\geq& \frac{2}{\int_0^r \bar{h}}(a - 2\epsilon + a\epsilon - \epsilon^2) \\
\geq& \frac{2}{(1+\epsilon)\int_0^r h}(a - 2\epsilon + a\epsilon - \epsilon^2) \\
\geq& \frac{2}{(1+\epsilon)^2\int_0^R h}(a - 2\epsilon + a\epsilon - \epsilon^2)
\end{aligned} \tag{4.6}$$

provided $a-2\epsilon+a\epsilon-\epsilon^2>0$. We choose $\epsilon>0$ so that it satisfies this condition.

On the other hand,

$$C(R)=\frac{2}{\int_0^R h}(1-\frac{Rh}{\int_0^R h})\le\frac{2}{\int_0^R h}(a+\epsilon)$$

if r_1 is large enough depending on ϵ and $R>r_1$ by Lemma 4.1(iv). Hence if ϵ and r_1 satisfy the above conditions, then for $r\in[R-1,R+1]$

$$\bar{C}(r)\ge\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}C(R).$$

Therefore if $\epsilon>0$ satisfies $a>\epsilon$ and $a-2\epsilon+a\epsilon-\epsilon^2>0$, then we can find $r_1>r_0$ such that if $R>r_1$, then for $r\in[R-1,R+1]$,

$$\begin{aligned}\bar{A}(r)+\frac{n}{2}\bar{C}(r)\ge&\frac{\xi'(r)-\beta}{\bar{h}(r)}+\frac{n}{2}\bar{C}(r)\\ \ge&\frac{-\beta}{h(r)}+\frac{n}{2}\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}C(R)\\ \ge&\frac{-\beta}{(1-\epsilon)h(R)}+\frac{n}{2}\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}C(R)\\ =&\frac{1}{(1-\epsilon)h(R)}\left[-\beta+\frac{n}{2}(1-\epsilon)\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}h(R)C(R)\right].\end{aligned}\tag{4.7}$$

Choose $\epsilon>0$ which also satisfies:

$$\frac{n}{2}(1-\epsilon)\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}>\frac{a+2\epsilon}{a-\epsilon}.$$

This can be done because $n\ge3$. Let $\beta=\alpha h(R)C(R)$, where $2\ge\alpha>0$ is a fixed constant depending on ϵ, a and n such that

$$\frac{n}{2}(1-\epsilon)\frac{a-2\epsilon+a\epsilon-\epsilon^2}{(a+\epsilon)(1+\epsilon)^2}>\alpha>\frac{a+2\epsilon}{a-\epsilon}.$$

We can also choose R large enough, so that $\beta c_0<\delta$ by Lemma 4.1(vi). Hence by Lemma 4.1(v), (4.5) and (4.7), one can find $R>r_0$ which is large enough, so that

$$\bar{A}(R)+\bar{C}(R)<0$$

and

$$\bar{A}(r)+\frac{n}{2}\bar{C}(r)>0$$

for $r\in[R-1,R+1]$. Hence (i) and (ii) are satisfied. It remains to show that one can choose R so that (iii) is also satisfies. First note that

$$\begin{aligned}\bar{B}(R)=&\frac{1}{\int_0^R\bar{h}}\left(\frac{R\bar{h}(R)}{\int_0^R\bar{h}}-(1-\xi(R)+\alpha h(R)C(R)\phi(0))\right)\\ \le&\frac{1}{\int_0^R\bar{h}}\left((1+\epsilon)\cdot\frac{Rh(R)}{\int_0^R h}-(1-\xi(R)+\alpha h(R)C(R)\phi(0))\right).\end{aligned}$$

By Lemma 4.1(iv), given $\epsilon_1 > 0$ one can choose R large enough, so that

$$\bar{B}(R) \le \frac{\epsilon_1}{\int_0^R h}.$$

On the other hand by (4.4) and (4.3)

$$\bar{A}(R) \le -\bar{C}(R) \le -\frac{2(a-\epsilon)}{\int_0^R h}.$$

Hence

$$\bar{A}(R) + (n-1)\bar{B}(R) < 0$$

if R is chosen large enough. Hence (iii) is also satisfied. □

LEMMA 4.3. *Let ϕ as before and let $\bar{\xi}(r) = \xi(r) - \beta\phi(r-R)$ with $\beta > 0$ so that $\bar{\xi}(r) > 0$.*

(i) Suppose $\beta c_0 \le c_1$ and $\beta(R)\int_0^R h \to 0$ as $R \to +\infty$. Then there exists $r_0 > 0$ such that if $R > r_0$, then $\bar{B} > 0$ on $[0,\infty)$.

(ii) There exists $\delta > 0$ such that if $\beta c_0 < \delta$, then there exists $r_0 > 0$ such that if $R > r_0$, then $\bar{C} > 0$ on $[0,\infty)$.

Proof. Denote $\beta\phi(r-R)$ by $\eta(r)$. To prove (i), suppose $\beta c_0 \le c_1$. Then as in the proof of Lemma 4.1(vii), there is a constant c_2 depending only on c_1 such that if $r \ge 3$, then

$$h(r) \le \bar{h}(r) \le (1+c_2)h(r), \quad \int_0^r h \le \int_0^r \bar{h} \le (1+c_2)\int_0^r h.$$

If $r < R-1$, then $\bar{B}(r) = B(r) > 0$. If $R \ge 4$ and $r \ge R-1$, then

$$\begin{aligned}
(r\bar{f})^2\bar{B}(r) =& r\bar{h}(r) - (1-\bar{\xi}(r))\int_0^r \bar{h}(t)dt \\
=& \int_0^r \left[(t\bar{h}(t))' - (1-\bar{\xi}(r))\bar{h}(t)\right] dt \\
=& \int_0^r (\bar{\xi}(r) - \bar{\xi}(t))\bar{h}(t)dt \\
=& \int_0^r (\xi(r)-\xi(t))\bar{h}(t)dt - \int_{R-1}^r (\eta(r)-\eta(t))\bar{h}(t)dt - \eta(r)\int_0^{R-1} h(t)dt \\
\ge& \int_0^r (\xi(r)-\xi(t))h(t)dt - 2c_1(1+c_2)h(R-1) \\
&- \beta c_0 \int_0^{R-1} h(t)dt \quad (\text{since } \xi' > 0,\ h' < 0) \\
\ge& rh(r) + (\xi(r)-1)\int_0^r h(t)dt - 2c_1(1+c_2)h(R-1) - c_0\beta\int_0^R h(t)dt.
\end{aligned}$$

By Lemma 4.1(iii), there is a constant $c_3 > 0$ such that

$$rh(r) + (\xi(r)-1)\int_0^r h(t)dt \ge c_3,$$

for all $r \geq 1$. Hence if $R \geq 4$, we have

$$(r\bar{f})^2\bar{B}(r) \geq c_3 - 2c_1(1+c_2)h(R-1) - c_0\beta \int_0^R h(t)dt.$$

By Lemma 4.1(i), we know that $h(R-1) \to 0$ as $R \to \infty$. From this it is easy to see that $\bar{B} > 0$ if R is large.

To prove (ii), let $\epsilon > 0$ be such that $1-(1+\epsilon)(1-a+\epsilon) > 0$. Such ϵ exists because $a > 0$. By Lemma 4.1(vii) there exists $\delta > 0$ such that if $\beta c_0 < \delta$, then (4.3) is true for all r if $R \geq 3$. If $r < R-1$ then $\bar{C}(r) = C(r) > 0$. If $r \geq R-1$, then

$$\begin{aligned} r^2\bar{f}^2\bar{C} &= r\bar{f} - r\bar{h}(r) \\ &= \int_0^r \bar{h}(t)dt - r\bar{h}(r) \\ &\geq \int_0^r h - r\bar{h}(r) \\ &= \left(\int_0^r h\right)\left(1 - \frac{rh(r)}{\int_0^r h}\frac{\bar{h}(r)}{h(r)}\right) \\ &\geq \left(\int_0^r h\right)\left(1-(1+\epsilon)\frac{rh(r)}{\int_0^r h}\right) \\ &\geq \left(\int_0^r h\right)(1-(1+\epsilon)(1-a+\epsilon)) \\ &> 0. \end{aligned}$$

if R is large enough, where we have used Lemma 4.1(iv). This completes the proof of (ii). □

Proof of Theorem 4.1. Let α be as in Lemma 4.2. For any $r_0 > 0$, let $R > r_0$ be such that Lemma 4.2 is true. One can choose r_0 large enough so that the conclusions of Lemma 4.3 are true for $\beta = \alpha h(R)C(R)$, because $h(r)C(r) \to 0$, $h(r) \to 0$ as $r \to \infty$ and $c(r) \sim \frac{1}{\int_0^r h}$. Since $\bar{A}(r) > 0$ on $[0, R-1] \cup [R+1, \infty)$ and $\bar{B}(r), \bar{C}(r) > 0$ for all r, the metric determined by $\bar{\xi}$ satisfies $\mathbf{QB} \geq 0$ by Lemma 4.2(ii) and Theorem 3.2. Since $\bar{A}(R) + \bar{C}(R) < 0$, it does not satisfies $\mathbf{B}^{\perp} \geq 0$. Since $\bar{A}(R) + (n-1)\bar{B}(R) < 0$, it does not satisfy $\mathrm{Ric} \geq 0$. □

Remark 4.1. Using similar method one may also construct, in a more simple way, complete $U(n)$-invariant Kähler metrics with $\mathbf{B}^{\perp} \geq 0$ which does not satisfy $\mathbf{B} \geq 0$ at some points.

5. Preservation of $\mathbf{QB} \geq 0$ in Kähler-Ricci flow. In this section, we will prove that $\mathbf{QB} \geq 0$ is preserved under the Kähler-Ricci flow for complete $U(n)$-invariant metrics, provided the curvature is bounded. We will also prove that $\mathrm{Ric} \geq 0$ is preserved with an additional assumption.

Let g_0 be a complete $U(n)$-invariant Kähler metric on $\mathbb{C}^n$, and $g(t)$ is a complete $U(n)$-invariant solution of the Kähler-Ricci flow on $\mathbb{C}^n \times [0, T]$, $T > 0$ with $g(0) = g_0$. We have a time-dependent orthonormal moving frame $\{e_1(t) = \frac{1}{\sqrt{h(r,t)}}\partial_{z_1}, e_2(t) =$

$\frac{1}{\sqrt{f(r,t)}}\partial_{z_2}, \cdots, e_n(t) = \frac{1}{\sqrt{f(r,t)}}\partial_{z_n}\}$ at the point $z = (z_1, 0, \cdots, 0)$. Denote

$$\begin{aligned} A(z,t) =& \operatorname{Rm}_{g(t)}(e_1(t), \bar{e}_1(t), e_1(t), \bar{e}_1(t)),\\ B(z,t) =& \operatorname{Rm}_{g(t)}(e_1(t), \bar{e}_1(t), e_i(t), \bar{e}_i(t)),\\ C(z,t) =& \operatorname{Rm}_{g(t)}(e_i(t), \bar{e}_i(t), e_i(t), \bar{e}_i(t)) = 2\operatorname{Rm}_{g(t)}(e_i(t), \bar{e}_i(t), e_j(t), \bar{e}_j(t)), \end{aligned}$$

where $2 \le i \ne j \le n$. Direct computations show [13] :

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) A =& A^2 + 2(n-1)B^2,\\ \left(\frac{\partial}{\partial t} - \Delta\right) B =& -B^2 + AB + \frac{n}{2}BC,\\ \left(\frac{\partial}{\partial t} - \Delta\right) C =& 2B^2 + \frac{n}{2}C^2. \end{aligned} \tag{5.1}$$

THEOREM 5.1. *Assume $g(t), t \in [0,T]$ is a complete solution of the $U(n)$-invariant Kähler-Ricci flow on $\mathbb{C}^n$, and $\operatorname{Rm}(z,t)$ is uniformly bounded on $\mathbb{C}^n \times [0,T]$. Suppose $g(0)$ satisfies $\mathbf{QB} \ge 0$. Then $g(t)$ satisfies $\mathbf{QB} \ge 0$ for all $t \in [0,T]$.*

Proof. By Theorem 3.2, it is sufficient to prove that if $B \ge 0$, $E := A + (n-2)B + \frac{n}{2}C \ge 0$ and $E := B + \frac{n-1}{2}C \ge 0$ at $t = 0$, then B, E, F are also nonnegative for all $t \in [0,T]$. We have the following system for B, E, F

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) B = & B\left(-B + A + \frac{n}{2}C\right),\\ \left(\frac{\partial}{\partial t} - \Delta\right) E = & (A - \frac{n}{2}C)E + \frac{2n(n-2)}{n-1}BF + \frac{n^2}{2}C^2 + \frac{2n}{n-1}B^2\\ \left(\frac{\partial}{\partial t} - \Delta\right) F = & BE + \frac{n(n-1)}{4}C^2. \end{cases} \tag{5.2}$$

We may proceed as in [8, Chapter 3, section 8]. Since the curvature of $g(t)$ is bounded on $\mathbb{C}^n \times [0,T]$, by [5, Lemma 1.1], for all $a > 0$ and $c > 0$, there exists a smooth function ϕ and a positive constant b such that

$$e^{a(1+r_o(x))} \le \phi(x,t) \le e^{b(1+r_o(x))}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi > c\phi$$

on $\mathbb{C}^n \times [0,T]$, where $r_o(x)$ is the distance from a fixed point o with respect to the initial metric $g(0)$. Let ϕ be such a function with $a = 1$, and c to be determined later. Let $\epsilon > 0$ be any constant and let $B_1 = B + \epsilon\phi$, $E_1 = E + \epsilon\phi$, and $F_1 = F + \epsilon\phi$. We want to prove that B_1, E_1, F_1 are nonnegative in $\mathbb{C}^n \times [0,T]$. Suppose not, since B, E, F are bounded and they are nonnegative at $t = 0$, we conclude that there exist $(x_0, t_0) \in \mathbb{C}^n \times (0,T]$ such that

$$\min\{B_1(x_0,t_0), E_1(x_0,t_0), F_1(x_0,t_0)\} = 0$$

and $B_1, E_1, F_1 \ge 0$ on $\mathbb{C}^n \times [0,t_0]$.

Suppose $B_1(x_0,t_0)=0$, then at (x_0,t_0)

$$\begin{aligned} 0 \geq& \left(\frac{\partial}{\partial t}-\Delta\right) B_1 \\ >&B_1\left(-B+A+\frac{n}{2}C\right)+\epsilon\phi\left[c-\left(-B+A+\frac{n}{2}C\right)\right] \\ =&\epsilon\phi\left[c-\left(-B+A+\frac{n}{2}C\right)\right]. \end{aligned} \tag{5.3}$$

Similarly, suppose $E_1(x_0,t_0)=0$, then

$$\begin{aligned} 0 \geq& \left(\frac{\partial}{\partial t}-\Delta\right) E_1 \\ >&(A-\frac{n}{2}C)E+\frac{2n(n-2)}{n-1}BF+c\epsilon\phi \\ =&\frac{2n(n-2)}{n-1}BF+\epsilon\phi\left[c-\left(A-\frac{n}{2}C\right)\right] \\ \geq& \begin{cases} \epsilon\phi\left(c-(A-\frac{n}{2}C)\right), & \text{if } B(x_0,t_0)F(x_0,t_0)\geq 0; \\ \epsilon\phi\left(c-\left(A-\frac{n}{2}C\right)-\frac{2n(n-2)B}{n-1}\right), & \text{if } B(x_0,t_0)>0 \text{ and } F(x_0,t_0)<0; \\ \epsilon\phi\left(c-\left(A-\frac{n}{2}C\right)-\frac{2n(n-2)F}{n-1}\right), & \text{if } B(x_0,t_0)<0 \text{ and } F(x_0,t_0)>0. \end{cases} \end{aligned} \tag{5.4}$$

Here we have used the fact that $B_1(x_0,t_0)\geq 0$ and $F_1(x_0,t_0)\geq 0$.

Suppose $F_1(x_0,t_0)=0$, then

$$\begin{aligned} 0 \geq& \left(\frac{\partial}{\partial t}-\Delta\right) F_1 \\ >&BE+\epsilon c\phi \\ \geq& \begin{cases} \epsilon c\phi, & \text{if } B(x_0,t_0)E(x_0,t_0)\geq 0; \\ \epsilon\phi\left(c-B\right), & \text{if } B(x_0,t_0)>0 \text{ and } E(x_0,t_0)<0; \\ \epsilon\phi\left(c-E\right), & \text{if } B(x_0,t_0)<0 \text{ and } E(x_0,t_0)>0. \end{cases} \end{aligned} \tag{5.5}$$

Here we have used the fact that $B_1(x_0,t_0)\geq 0$ and $E_1(x_0,t_0)\geq 0$.

By (5.3)–(5.5), if we choose c such that

$$c>\sup_{\mathbb{C}^n\times[0,T]}\left(|-B+A+\frac{n}{2}C|+|A-\frac{n}{2}C|+\frac{2n(n-2)}{n-1}(|B|+|F|)+|E|\right),$$

which is independent of ϵ. Then we have a contradiction. Hence B_1,E_1,F_1 are nonnegative on $\mathbb{C}^n\times[0,T]$. Let $\epsilon\to 0$, we conclude the theorem is true. □

THEOREM 5.2. *Assume $g(t), t\in[0,T]$ is a complete solution of the $U(n)$-invariant Kähler-Ricci flow on $\mathbb{C}^n$, and $\mathrm{Rm}(z,t)$ is uniformly bounded on $\mathbb{C}^n\times[0,T]$. Suppose $g(0)$ satisfies $\mathrm{Ric}\geq 0$ and $B\geq 0$. Then $g(t)$ satisfies $\mathrm{Ric}\geq 0$ and $B\geq 0$ for all t.*

Proof. By (3.1), it is sufficient to prove that if $B\geq 0$, $G:=A+(n-1)B\geq 0$ and $H:=B+\frac{n}{2}C\geq 0$ at $t=0$, then these are still true for all $t\in[0,T]$.

$$\begin{cases} \left(\frac{\partial}{\partial t}-\Delta\right)B= & B\left(-B+A+\frac{n}{2}C\right), \\ \left(\frac{\partial}{\partial t}-\Delta\right)G= & AG+(n-1)BH, \\ \left(\frac{\partial}{\partial t}-\Delta\right)H= & BG+\frac{n}{2}CH. \end{cases} \tag{5.6}$$

Then we can proceed as in the proof of the previous theorem. □

REMARK 5.1. In the proofs of the above two theorems, we may prove that $B \geq 0$ is preserved first. Then the rest of the proofs are somewhat simpler.

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