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ON THE ESTIMATE OF FIRST POSITIVE EIGENVALUE OF A SUBLAPLACIAN IN A PSEUDOHERMITIAN MANIFOLD*

YEN-WEN FAN† AND TING-JUNG KUO‡

Abstract. In this paper, we first obtain a CR version of Yau's gradient estimate for eigenfunctions of a sublaplacian. Second, by using CR analogue of Li-Yau's eigenvalue estimate, we are able to obtain a lower bound of the first positive eigenvalue in a pseudohermitian manifold of nonvanishing pseudohermitian torsion and nonpositive lower bound on pseudohermitian Ricci curvature.

Key words. CR gradient estimate, CR diameter, eigenvalue estimate, sublaplacian.

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1. Introduction. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold (see the Appendix A for basic notions in pseudohermitian geometry). More precisely, we first recall some notions as in Appendix. Let M be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_{\mathbb{R}} \xi = 2n$. A CR structure J compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition (see Appendix). A CR structure J can extend to $\mathbb{C}\otimes\xi$ and decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to eigenvalues i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ and $\xi = \ker\theta$. Such a choice determines a unique real vector field T transverse to ξ which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T, \cdot) = 0$.

Let $\{T, Z_\alpha, Z_{\bar\alpha}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha} = \overline{Z_\alpha} \in T_{0,1}$. The pseudohermitian Ricci curvature tensor $R_{\alpha\bar\beta}$ and the torsion tensor $A_{\alpha\beta}$ are defined on $T_{1,0}$ by

$$Ric(X, Y) = R_{\alpha\bar\beta}X^\alpha Y^{\bar\beta}$$

and

$$Tor(X, Y) = i\sum_{\alpha,\beta}(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta} - A_{\alpha\beta}X^\alpha Y^\beta).$$

Here $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$, $R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}$ and $R_\gamma{}^\gamma{}_{\alpha\bar\beta}$ is the pseudohermitian curvature tensor.

Greenleaf ([Gr]) proved the pseudohermitian analogue of Lichnerowicz's Theorem for the first positive eigenvalue λ_1 of the sublaplacian Δ_b (see the definition in Appendix A) in a closed pseudohermitian $(2n+1)$-manifold with $n \geq 3$. More precisely, under a condition on the pseudohermitian Ricci curvature and the torsion tensor

$$[Ric - \frac{n+1}{2}Tor](Z, Z) \geq k\langle Z, Z\rangle, \tag{1.1}$$

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†Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan R.O.C. (d98221003@ntu.edu.tw). Research supported in part by the NSC of Taiwan.
‡Taida Institute For Mathematical Sciences (TIMS), National Taiwan University, Taipei 10617, Taiwan, R.O.C (tjkuo@ntu.edu.tw). Research supported in part by TIMS.

for all $Z \in T_{1,0}$ and for some positive constant k. Then

$$\lambda_1 \geq \frac{nk}{n+1}$$

for $n \geq 3$. In [LL], Li and Luk proved the same result for the cases $n = 1$ and $n = 2$. However, in the case $n = 1$, they need an extra condition on a covariant derivative of the pseudohermitian torsion. Recently, it was proved by Chiu ([C]) that if (M^3, J, θ) is a closed pseudohermitian 3-manifold of nonnegative CR Paneitz operator P_0 with

$$[Ric - Tor](Z, Z) \geq k \langle Z, Z\rangle ,$$

for all $Z \in T_{1,0}$ and for some positive constant k. Then

$$\lambda_1 \geq \frac{k}{2}.$$

However, for a nonpositive constant k, the estimate of Lichnerowicz becomes trivial in this case. In the paper of S.-C. Chang and H.-L. Chiu ([CC2]), they are able to show that if (M^3, J, θ) is a closed pseudohermitian 3-manifold of vanishing torsion with

$$Ric\,(Z, Z) \geq -k_0 \langle Z, Z\rangle_{L_\theta}$$

for all $Z \in T_{1,0}$ and some nonnegative constant k_0, then

$$\lambda_1 \geq \frac{\left(1 + \sqrt{1 + 2k_0 d^2}\right)}{6d^2} e^{-\left(1+\sqrt{1+2k_0 d^2}\right)}. \tag{1.2}$$

Here d is the CR diameter as in (A.4).

In this paper, we first obtain a CR version of Yau's gradient estimate for eigenfunctions of a sublaplacian as in Theorem 1.1 ([Y], [CKL] and [CKT]). Then by using Li-Yau eigenvalue estimate ([LY2]), we are able to generalize the lower bound (1.2) of first positive eigenvalue λ_1 to a closed pseudohermitian $(2n + 1)$-manifold of nonvanishing pseudohermitian torsion as in Theorem 1.2.

Theorem 1.1. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n + 1)$-manifold. Suppose that*

$$\left(Ric - (n-2)\,Tor\right)(Z,\ Z) \geq -2k_0 \langle Z,\ Z\rangle_{L_\theta} \tag{1.3}$$

for all $Z \in T_{1,0}$ and some nonnegative constant k_0. If $u\,(x)$ is an eigenfunction of Δ_b on M with respect to λ (i.e. $\Delta_b u = -\lambda u$). Then for any $\ell > 0$ such that $(u + \ell) > 0$, we have

$$\frac{|\nabla_b u|^2}{(u+\ell)^2} + \frac{1}{H}\frac{u_0^2}{(u+\ell)^2} \leq Q + \frac{\ell}{(\ell - 1)}\lambda G_n.$$

Here $k_1 := \max_M \{|A_{\alpha\beta}|, \ |A_{\alpha\beta,\bar{\alpha}}|\}$ *and*

$$H(k_0, k_1, \ell, \lambda) = \{2(n+1)(n+3)^2 + 26n + 2\left[(n+3)^2+1\right]\frac{1}{k_1+k_0}\}k_1 + 2\left[(n+3)^2+1\right]k_0 + \left(\frac{2(n+1)\left[(n+3)^2+1\right]+6}{n}\right)\frac{\ell}{(\ell-1)}\lambda.$$

$$G_n = \frac{2(n+3)^4 n + 2(n+3)^4 + 3n(n+3)^2 + 8(n+3)^2 + 3(n+3)}{3n}. \tag{1.4}$$

$$Q(k_0, k_1, n) = \frac{(n+3)^2}{3}\{2(n+1)(n+3)^2 + 28n + 2n\left[(n+3)^2+1\right]\frac{1}{k_1+k_0}\}k_1 + \frac{(n+3)^2}{3}\left[2(n+3)^2+3\right]k_0.$$

As a consequence of Theorem 1.1, we have the following first eigenvalue estimate:

THEOREM 1.2. *Let* $(M, \ J, \ \theta)$ *be a closed pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$(Ric - (n-2)Tor)(Z, Z) \geq -2k_0 \langle Z, \ Z\rangle_{L_\theta}$$

for all $Z \in T_{1,0}$ *where* $k_0 \geq 0$. *Then*

$$\lambda_1 \geq \frac{2}{d^2 G_n}\left[1+\sqrt{1+2Qd^2}\right]e^{-\left(1+\sqrt{1+2Qd^2}\right)}$$

where G_n, Q *are as in Theorem 1.1.*

We briefly describe the methods used in our proofs. In Section 2, we first derive the CR version of Bochner-type estimate. In Section 3, It contains the crucial steps. By using the CR version of Yau's gradient estimate ([Y], [CKT], [CKL]), we are able to derive the gradient estimate for the eigenfunction of a sublaplacian. As a consequence ([LY2]), we have the lower bound estimate for the first positive eigenvalue. Finally, for the completeness, we introduce some basic material of pseudohermitian manifold as in Appendix A.

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2. The CR Bochner-Type estimate. Now we recall the Bochner formula from A. Greenleaf ([Gr]) and also ([CC2]) and derive some key Lemmas in a closed pseudohermitian $(2n+1)$-manifold (M, J, θ).

LEMMA 2.1. *For a real function* φ,

$$\Delta_b |\nabla_b \varphi|^2 = 2\left|\left(\nabla^H\right)^2\varphi\right|^2 + 2\langle \nabla_b\varphi, \ \nabla_b\Delta_b\varphi\rangle + (4Ric - 2(n-2)Tor)\left((\nabla_b\varphi)_C, \ (\nabla_b\varphi)_C\right) + 4\langle J\nabla_b\varphi, \ \nabla_b\varphi_0\rangle, \tag{2.1}$$

where $(\nabla_b\varphi)_C = \varphi_{\bar{\alpha}} Z_\alpha$ *is the corresponding complex* $(1, \ 0)$*-vector of* $\nabla_b\varphi$.

LEMMA 2.2. *For a real function φ and any $\nu > 0$, we have*

$$\Delta_b \left|\nabla_b \varphi\right|^2 \geq 4 \left(\sum_{\alpha,\beta=1}^{n} \left|\varphi_{\alpha\beta}\right|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^{n} \left|\varphi_{\alpha\bar{\beta}}\right|^2 \right) + \frac{1}{n} \left(\Delta_b \varphi\right)^2$$
$$+ n\varphi_0^2 + 2 \left\langle \nabla_b \varphi,\ \nabla_b \Delta_b \varphi \right\rangle$$
$$+ \left(4Ric - 2\left(n-2\right) Tor - \frac{4}{\nu} \right) \left(\left(\nabla_b \varphi\right)_C,\ \left(\nabla_b \varphi\right)_C \right) - 2\nu \left|\nabla_b \varphi_0\right|^2,$$

where $\left(\nabla_b \varphi\right)_C = \varphi_{\bar{\alpha}} Z_\alpha$ is the corresponding complex $(1,\ 0)$-vector of $\nabla_b \varphi$.

Proof. Since

$$\begin{aligned}
|(\nabla^H)^2\varphi|^2 &= 2 \sum_{\alpha,\beta=1}^{n} \left(\varphi_{\alpha\beta}\varphi_{\overline{\alpha\beta}} + \varphi_{\alpha\overline{\beta}}\varphi_{\overline{\alpha}\beta}\right) \\
&= 2 \sum_{\alpha,\beta=1}^{n} \left(|\varphi_{\alpha\beta}|^2 + |\varphi_{\alpha\overline{\beta}}|^2\right) \\
&= 2 \left(\sum_{\alpha,\beta=1}^{n} |\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} |\varphi_{\alpha\overline{\beta}}|^2 + \sum_{\alpha=1}^{n} |\varphi_{\alpha\overline{\alpha}}|^2 \right)
\end{aligned}$$

and from the commutation relation (A.5)

$$\begin{aligned}
\sum_{\alpha=1}^{n} |\varphi_{\alpha\overline{\alpha}}|^2 &= \frac{1}{4} \sum_{\alpha=1}^{n} \left(|\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \varphi_0^2\right) \\
&= \frac{1}{4} \sum_{\alpha=1}^{n} |\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \frac{n}{4}\varphi_0^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
|(\nabla^H)^2\varphi|^2 &= 2 \left(\sum_{\alpha,\beta=1}^{n} |\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} |\varphi_{\alpha\overline{\beta}}|^2 \right) + \frac{1}{2} \sum_{\alpha=1}^{n} |\varphi_{\alpha\overline{\alpha}} + \varphi_{\overline{\alpha}\alpha}|^2 + \frac{n}{2}\varphi_0^2 \\
&\leq 2 \left(\sum_{\alpha,\beta=1}^{n} |\varphi_{\alpha\beta}|^2 + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n} |\varphi_{\alpha\overline{\beta}}|^2 \right) + \frac{1}{2n} \left(\Delta_b\varphi\right)^2 + \frac{n}{2}\varphi_0^2.
\end{aligned}$$

On the other hand, for all $\nu > 0$

$$\begin{aligned}
4 \left\langle J\nabla_b\varphi,\ \nabla_b\varphi_0 \right\rangle &\geq -4 \left|\nabla_b\varphi\right| \left|\nabla_b\varphi_0\right| \\
&\geq -\frac{2}{\nu} \left|\nabla_b\varphi\right|^2 - 2\nu \left|\nabla_b\varphi_0\right|^2.
\end{aligned}$$

Then the result follows easily from Lemma 2.1. ☐

Definition 2.3. ([GL]) Let (M, J, θ) be a pseudohermitian $(2n+1)$-manifold. We define the purely holomorphic second-order operator Q by

$$Q\varphi = 2i \sum_{\alpha,\beta=1}^{n} (A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta})_{,\alpha}\,.$$

By apply the commutation relations (A.5), one obtains

Lemma 2.4. *([GL], [CKL]) Let $\varphi(x)$ be a smooth function defined on M. Then*

$$\Delta_b\varphi_0 = (\Delta_b\varphi)_0 + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].$$

That is

$$2\,\mathrm{Im}\,Q\varphi = [\Delta_b,\ T]\,\varphi.$$

Proof. By direct computation and the commutation relation (A.5), we have

$$\begin{aligned}
\Delta_b\varphi_0 &= \varphi_{0\alpha\overline{\alpha}} + \varphi_{0\overline{\alpha}\alpha}\\
&= \left(\varphi_{\alpha 0} + A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}} + \text{conjugate}\\
&= \varphi_{\alpha 0\overline{\alpha}} + \left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}} + \text{conjugate}\\
&= \varphi_{\alpha\overline{\alpha}0} + \varphi_{\overline{\alpha}\alpha 0} + 2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}} + + \left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right]\\
&= (\Delta_b\varphi)_0 + 2\left[\left(A_{\alpha\beta}\varphi_{\overline{\beta}}\right)_{\overline{\alpha}} + + \left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha}\right].
\end{aligned}$$

This completes the proof. □

Let u be an eigenfunction of Δ_b with respect to λ Then

$$\Delta_b u = -\lambda u.$$

Since

$$\begin{aligned}
0 &= \int_M \Delta_b u d\mu\\
&= -\lambda_1 \int_M u d\mu,
\end{aligned}$$

u must change sign. Hence we may normalize u to satisfy $\underset{x\in M}{min u} = -1$ and $\underset{x\in M}{max u} \leq 1$. Let $f(x,\ell) = \ln(u+\ell)$ where we choose $\ell > 0$ such that $(u+\ell) \geq 1$ and without any misunderstanding we denote $f(x,\ell)$ by $f(x)$. Then

$$\Delta_b f(x) = -\left|\nabla_b f\right|^2 - \frac{\lambda u}{u+\ell}.$$

We define

$$V(\varphi) = \sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\right)_{\alpha} + A_{\alpha\beta}\varphi_{\bar{\beta}}\varphi_{\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}\varphi_{\beta}\varphi_{\alpha}\right].$$

LEMMA 2.5. *Let u be an eigenfunction with $f = \ln(u+\ell)$. Then*

$$\Delta_b f_0 = -2\langle \nabla_b f,\ \nabla_b f_0\rangle - \frac{\lambda \ell f_0}{(u+\ell)} + 2V(f).$$

Proof. From Lemma 2.4

$$\Delta_b f_0 = (\Delta_b f)_0 + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}\varphi_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta}\varphi_{\beta}\right)_{\alpha}\right].$$

Since

$$\Delta_b f = -|\nabla_b f|^2 - \frac{\lambda u}{u+\ell},$$

it follows from the commutation relation (A.5) that

$$\begin{aligned}
\Delta_b f_0 &= (\Delta_b f)_0 + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_{\beta}\right)_{\alpha}\right] \\
&= \left(-|\nabla_b f|^2 - \frac{\lambda u}{u+\ell}\right)_0 + 2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_{\beta}\right)_{\alpha}\right] \\
&= -2\langle \nabla_b f_0,\ \nabla_b f\rangle - \frac{\lambda \ell f_0}{(u+\ell)} \\
&\quad +2\sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta} f_{\bar\beta}\right)_{\bar\alpha} + \left(A_{\bar\alpha\bar\beta} f_{\beta}\right)_{\alpha} + A_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + A_{\bar\alpha\bar\beta} f_{\alpha} f_{\beta}\right].
\end{aligned}$$

□

3. The Proof of Main Theorem. In this section, first we derive CR version of Yau gradient estimate ([Y]) as in Theorem 1.1. Then by using the method of Li-Yau's eigenvalue estimate ([LY2]), we are able to derive the lower bound of the first positive eigenvalue as in Theorem 1.2.

In the following, we always assume $\min_M u(x) = -1$ and $\max_M u(x) \le 1$. Recall $f(x,\ell) = \ln(u+\ell)$ where we choose $\ell > 1$ such that $(u+\ell) > 0$ and denote $f(x,\ell)$ by $f(x)$. Then for $\Delta_b u = -\lambda u$

$$\Delta_b f(x) = -|\nabla_b f|^2 - \frac{\lambda u}{u+\ell}. \tag{3.1}$$

We define a function $F(x,\ t,\ b,\ \ell) : M \times [0,\ 1] \times (0,\ \infty) \times (1,\ \infty) \to \mathbb{R}$ by

$$F = t\left(|\nabla_b f(x,\ell)|^2 + bt f_0^2(x,\ell)\right).$$

PROPOSITION 3.1. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(2Ric - (n-2)Tor)(Z,\ Z) \ge -2k_0|Z|^2 \tag{3.2}$$

for all $Z \in T_{1,0}$, where k_0 is a nonnegative constant. Then

$$\Delta_b F \geq -2\langle \nabla_b f,\ \nabla_b F\rangle + t\left[4\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2 + \frac{1}{n}(\Delta_b f)^2 + \left(n - \frac{2bt\lambda\ell}{u+\ell}\right)f_0^2 - \left(2k_0 + \frac{2}{bt} + \frac{2\lambda\ell}{u+\ell}\right)|\nabla_b f|^2 + 4btf_0V(f)\right].$$

Proof. By CR Bochner inequality in Lemma 2.2 and the assumption (3.2), we have

$$\begin{aligned}\Delta_b F &= t\left(\Delta_b|\nabla_b f|^2 + bt\Delta_b f_0^2\right)\\ &\geq t\left[4\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2 + \frac{1}{n}(\Delta_b f)^2 + nf_0^2 + 2\langle\nabla_b f,\ \nabla_b\Delta_b f\rangle\right.\\ &\quad\left. + \left(-2k_0 - \frac{2}{\nu}\right)|\nabla_b f|^2 + (2bt - 2\nu)|\nabla_b f_0|^2 + 2btf_0\Delta_b f_0\right].\end{aligned} \tag{3.3}$$

Next, by Lemma 2.5 and (3.1),

$$\begin{aligned}&2\langle\nabla_b f,\ \nabla_b\Delta_b f\rangle + 2btf_0\Delta_b f_0\\ &= 2\left\langle\nabla_b f,\ \nabla_b\left(-|\nabla_b f|^2 - \frac{\lambda u}{u+\ell}\right)\right\rangle + 2btf_0\left(-2\langle\nabla_b f,\ \nabla_b f_0\rangle - \frac{\lambda\ell f_0}{(u+\ell)} + 2V(f)\right)\\ &= -2\left\langle\nabla_b f,\ \nabla_b\left(\frac{F}{t} - btf_0^2\right)\right\rangle - 2\left\langle\nabla_b f,\ \frac{\lambda\ell\nabla_b u}{(u+\ell)^2}\right\rangle - \frac{2bt\lambda\ell}{(u+\ell)}f_0^2\\ &\quad -4btf_0\langle\nabla_b f,\ \nabla_b f_0\rangle - 4btf_0V(f)\\ &= -\frac{2}{t}\langle\nabla_b f,\ \nabla_b F\rangle - \frac{2\lambda\ell}{u+\ell}|\nabla_b f|^2 - \frac{2bt\lambda\ell}{(u+\ell)}f_0^2 - 4btf_0V(f).\end{aligned} \tag{3.4}$$

Finally, substituting (3.5) into (3.4) and choosing $\nu = bt$, we obtain

$$\Delta_b F \geq -2\langle \nabla_b f,\ \nabla_b F\rangle + t\left[4\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{\alpha\bar{\beta}}\right|^2 + \frac{1}{n}(\Delta_b f)^2 + \left(n - \frac{2bt\lambda\ell}{u+\ell}\right)f_0^2 - \left(2k_0 + \frac{2}{bt} + \frac{2\lambda\ell}{u+\ell}\right)|\nabla_b f|^2 + 4btf_0V(f)\right].$$

This completes the proof. □

Proposition 3.2. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(2Ric - (n-2)Tor)(Z,\ Z) \geq -2k_0|Z|^2$$

and for all $Z \in T_{1,0}$, where k_0 is a nonnegative constant. Then for all $a < -1$

$$\begin{aligned}
\Delta_b F \geq & -2\langle \nabla_b f,\ \nabla_b F\rangle \\
& + t\left[4\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}|f_{\alpha\bar{\beta}}|^2 + \frac{1}{n}\left(\frac{F}{at} - \frac{\lambda u}{u+\ell}\right)^2 \right.\\
& \left. + \frac{1}{n}\left(\frac{1+a}{a}|\nabla_b f|^2 + \frac{bt}{a}f_0^2\right)^2\right] \\
& + t\left[n - 2bt\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{na(\ell-1)}\right) - \frac{2b}{na^2}F\right] f_0^2 + 4bt^2 f_0 V(f) \\
& + t\left[\frac{-2(1+a)}{na^2 t}F - 2k_0 - \frac{2}{bt} - 2\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{n(\ell-1)} + \frac{1}{na(\ell-1)}\right)\right]|\nabla_b f|^2.
\end{aligned}$$

Proof. First, for any $a < -1$, we have

$$\begin{aligned}
(\Delta_b f)^2 &= \left(-|\nabla_b f|^2 - \frac{\lambda u}{u+\ell}\right)^2 \\
&= \left(\frac{1}{at}F - \frac{1}{a}|\nabla_b f|^2 - \frac{1}{a}btf_0^2 - |\nabla_b f|^2 - \frac{\lambda u}{u+\ell}\right)^2 \\
&= \left(\frac{1}{at}F - \frac{\lambda u}{u+\ell} - \frac{a+1}{a}|\nabla_b f|^2 - \frac{1}{a}btf_0^2\right)^2 \\
&= \left(\frac{1}{at}F - \frac{\lambda u}{u+\ell}\right)^2 + \left(\frac{a+1}{a}|\nabla_b f|^2 + \frac{1}{a}btf_0^2\right)^2 \\
&\quad -2\left(\frac{1}{at}F - \frac{\lambda u}{u+\ell}\right)\left(\frac{a+1}{a}|\nabla_b f|^2 + \frac{1}{a}btf_0^2\right) \\
&= \left(\frac{1}{at}F - \frac{\lambda u}{u+\ell}\right)^2 + \left(\frac{a+1}{a}|\nabla_b f|^2 + \frac{1}{a}btf_0^2\right)^2 \\
&\quad - \frac{2(1+a)}{a^2 t}F|\nabla_b f|^2 - \frac{2b}{a^2}Ff_0^2 + \frac{2\lambda(1+a)u}{a(u+\ell)}|\nabla_b f|^2 + \frac{2bt\lambda u}{a(u+\ell)}f_0^2.
\end{aligned}$$

Then

(3.5)

$$\begin{aligned}
\Delta_b F \geq & -2\langle \nabla_b f,\ \nabla_b F\rangle \\
& + t\left[4\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}|f_{\alpha\bar{\beta}}|^2 + \frac{1}{n}\left(\frac{F}{at} - \frac{\lambda u}{u+\ell}\right)^2 \right.\\
& + \frac{1}{n}\left(\frac{1+a}{a}|\nabla_b f|^2 + \frac{bt}{a}f_0^2\right)^2 + \left(n - \frac{2bt\lambda\ell}{u+\ell} + \frac{2bt\lambda u}{na(u+\ell)} - \frac{2b}{na^2}F\right)f_0^2 \\
& \left. + \left(\frac{-2(1+a)}{na^2 t}F - 2k_0 - \frac{2}{bt} - \frac{2\lambda\ell}{u+\ell} + \frac{2\lambda(1+a)u}{na(u+\ell)}\right)|\nabla_b f|^2 + 4btf_0V(f)\right].
\end{aligned}$$

Second, since

$$0 < \ell - 1 \leq (u+\ell) \leq \ell + 1, \text{ and } \ a < -1$$

in (3.6), we have

$$\frac{-\ell}{u+\ell}+\frac{u}{na\left(u+\ell\right)}\geq-\left(\frac{\ell}{\ell-1}+\frac{1}{na\left(\ell-1\right)}\right)$$

and

$$\begin{aligned}\frac{-\ell}{u+\ell}+\left(\frac{1+a}{na}\right)\frac{u}{u+\ell}&=\frac{-\ell}{u+\ell}+\frac{u}{na\left(u+\ell\right)}+\frac{1}{n}\frac{u}{u+\ell}\\&\geq-\left(\frac{\ell}{\ell-1}+\frac{1}{n\left(\ell-1\right)}+\frac{1}{na\left(\ell-1\right)}\right).\end{aligned}$$

Then

$$\begin{aligned}\Delta_b F\geq&-2\left\langle\nabla_b f,\ \nabla_b F\right\rangle\\&+t\left[4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2+\frac{1}{n}\left(\frac{F}{at}-\frac{\lambda u}{u+\ell}\right)^2\right.\\&\left.+\frac{1}{n}\left(\frac{1+a}{a}\left|\nabla_b f\right|^2+\frac{bt}{a}f_0^2\right)^2\right]\\&+t\left[n-2bt\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na\left(\ell-1\right)}\right)-\frac{2b}{na^2}F\right]f_0^2+4bt^2f_0V\left(f\right)\\&+t\left[\frac{-2\left(1+a\right)}{na^2t}F-2k_0-\frac{2}{bt}-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n\left(\ell-1\right)}+\frac{1}{na\left(\ell-1\right)}\right)\right]\left|\nabla_b f\right|^2.\end{aligned}$$

This completes the proof. □

PROPOSITION 3.3. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$\left(Ric-(n-2)\,Tor\right)(Z,\ Z)\geq-2k_0\left\langle Z,\ Z\right\rangle_{L_\theta}$$

for all $Z\in T_{1,0}$, where $k_0\geq 0$. Then

(3.6)

$$\begin{aligned}\Delta_b F\geq&-2\left\langle\nabla_b f,\ \nabla_b F\right\rangle\\&+t\left[4\left(1-bk_1\right)\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2+4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2\right.\\&\left.+\frac{1}{n}\left(\frac{F}{at}-\frac{\lambda u}{u+\ell}\right)^2+\left(\frac{1+a}{a}\left|\nabla_b f\right|^2+\frac{bt}{a}f_0^2\right)^2\right]\\&+t\left[n-2bt\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na\left(\ell-1\right)}\right)-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F\right]f_0^2\\&+t\left[\frac{-2\left(1+a\right)}{na^2t}F-2k_0-2k_1n\left(b+1\right)-\frac{2}{bt}\right.\\&\left.-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n\left(\ell-1\right)}+\frac{1}{na\left(\ell-1\right)}\right)\right]\left|\nabla_b f\right|^2.\end{aligned}$$

Here $k_1:=\max_M\left\{\left|A_{\alpha\beta}\right|,\ \left|A_{\alpha\beta,\bar{\alpha}}\right|\right\}$.

Proof. Firstly, we recall from Proposition 3.2 that

$$
\begin{aligned}
(3.7)\quad \Delta_b F \geq & -2\left\langle \nabla_b f,\ \nabla_b F\right\rangle \\
& + t\left[4\sum_{\alpha,\beta=1}^{n}\left|f_{a\beta}\right|^2 + 4\sum_{\alpha,\beta=1,\alpha\neq\beta}^{n}\left|f_{a\bar{\beta}}\right|^2 + \frac{1}{n}\left(\frac{F}{at} - \frac{\lambda u}{u+\ell}\right)^2 \right. \\
& \left. + \frac{1}{n}\left(\frac{1+a}{a}\left|\nabla_b f\right|^2 + \frac{bt}{a}f_0^2\right)^2\right] \\
& + t\left[n - 2bt\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{na(\ell-1)}\right) - \frac{2b}{na^2}F\right]f_0^2 + 4bt^2 f_0 V(f) \\
& + t\left[\frac{-2(1+a)}{na^2 t}F - 2k_0 - \frac{2}{bt} - 2\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{n(\ell-1)} + \frac{1}{na(\ell-1)}\right)\right]\left|\nabla_b f\right|^2 .
\end{aligned}
$$

In view of (3.8), we need to estimate $4bt^2 f_0 V(f)$. Recall that

$$
V(f) = \sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha} + A_{\alpha\beta}f_{\bar{\beta}}f_{\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta}f_{\alpha}\right].
$$

Then

$$
\begin{aligned}
(3.8)\quad & 4bt^2 f_0 V(f) \\
& = 4bt^2 f_0 \sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}}\right)_{\bar{\alpha}} + \left(A_{\bar{\alpha}\bar{\beta}}f_{\beta}\right)_{\alpha} + A_{\alpha\beta}f_{\bar{\beta}}f_{\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta}f_{\alpha}\right] \\
& = 4bt^2 f_0 \sum_{\alpha,\beta=1}^{n}\left[\left(A_{\alpha\beta}f_{\bar{\beta}\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta\alpha}\right) + \left(A_{\alpha\beta,^{\bar{a}}}f_{\bar{\beta}} + A_{\bar{\alpha}\bar{\beta},\alpha}f_{\beta}\right)\right. \\
& \quad \left. + \left(A_{\alpha\beta}f_{\bar{\beta}}f_{\bar{\alpha}} + A_{\bar{\alpha}\bar{\beta}}f_{\beta}f_{\alpha}\right)\right] \\
& \geq -8bt^2 k_1 \sum_{\alpha,\beta=1}^{n}\left|f_0\right|\left|f_{\bar{\beta}\bar{\alpha}}\right| - 8bt^2 k_1 \sum_{\alpha,\beta=1}^{n}\left|f_0\right|\left|f_{\bar{\beta}}\right| - 8bt^2 k_1 \sum_{\alpha,\beta=1}^{n}\left|f_0\right|\left|f_{\bar{\alpha}}\right|\left|f_{\bar{\beta}}\right| .
\end{aligned}
$$

In (3.9), by Young's inequality and noting that $t \leq 1$, we have following estimates:

$$
\begin{aligned}
(3.9)\quad -8bt^2 k_1 \sum_{\alpha,\beta=1}^{n}\left|f_0\right|\left|f_{\bar{\beta}\bar{\alpha}}\right| & \geq \sum_{\alpha,\beta=1}^{n}\left(-4bt^2 k_1\left|f_{\bar{\beta}\bar{\alpha}}\right|^2 - 4bt^2 k_1 f_0^2\right) \\
& \geq -4btk_1 n^2 f_0^2 - 4btk_1 \sum_{\alpha,\beta=1}^{n}\left|f_{\bar{\beta}\bar{\alpha}}\right|^2
\end{aligned}
$$

and

$$
\begin{aligned}
(3.10)\quad -8bt^2 k_1 \sum_{\alpha,\beta=1}^{n}\left|f_0\right|\left|f_{\bar{\beta}}\right| & \geq \sum_{\alpha,\beta=1}^{n}\left(-4bt^2 k_1 f_0^2 - 4bt^2 k_1\left|f_{\bar{\beta}}\right|^2\right) \\
& = -4bt^2 k_1 n^2 f_0^2 - 4bt^2 k_1 n\sum_{\beta=1}^{n}\left|f_{\bar{\beta}}\right|^2 \\
& = -4bt^2 k_1 n^2 f_0^2 - 2bt^2 k_1 n\left|\nabla_b f\right|^2
\end{aligned}
$$

and

(3.11)

$$\begin{aligned}
-8bt^2k_1\sum_{\alpha,\beta=1}^{n}|f_0||f_{\bar\alpha}|\left|f_{\bar\beta}\right| &\geq \sum_{\alpha,\beta=1}^{n}-4bt^2k_1\left(|f_{\bar\alpha}|^2+\left|f_{\bar\beta}\right|^2\right)|f_0| \\
&=\sum_{\beta=1}^{n}-4bt^2k_1n\,|f_{\bar\alpha}|^2|f_0|+\sum_{\alpha=1}^{n}-4bt^2k_1n\left|f_{\bar\beta}\right|^2|f_0| \\
&=-4bt^2k_1n\,|\nabla_bf|^2|f_0| \\
&\geq -2b^2t^2k_1n\,|\nabla_bf|^2f_0^2-2t^2k_1n\,|\nabla_bf|^2 \\
&\geq -2b^2tk_1nFf_0^2-2tk_1n\,|\nabla_bf|^2.
\end{aligned}$$

Finally, substituting (3.10), (3.11), and (3.12) into (3.8), one obtains

$$\begin{aligned}
\Delta_bF \geq& -2\left\langle\nabla_bf,\ \nabla_bF\right\rangle \\
&+t\left[4(1-bk_1)\sum_{\alpha,\beta=1}^{n}|f_{a\beta}|^2+\frac{1}{n}\left(\frac{F}{at}-\frac{\lambda u}{u+\ell}\right)^2+\left(\frac{1+a}{a}|\nabla_bf|^2+\frac{bt}{a}f_0^2\right)^2\right] \\
&+t\left[n-2bt\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right)-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F\right]f_0^2 \\
&+t\left[\frac{-2(1+a)}{na^2t}F-2k_0-2k_1n(b+1)-\frac{2}{bt}\right. \\
&\left.-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]|\nabla_bf|^2.
\end{aligned}$$

This completes the proof. □

PROPOSITION 3.4. *Let $(M,\ J,\ \theta)$ be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(2Ric-(n-2)\,Tor)\,(Z,\ Z)\geq -2k_0\,|Z|^2$$

for all $Z\in T_{1,0}$, where $k_0\geq 0$. Let b, ℓ be fixed, and $p(t)$ be the maximal point of F. For each $t\in(0,1]$. Then at $(p(t),t)$ we have

(3.12)

$$\begin{aligned}
0\geq& t\left[4(1-bk_1)\sum_{\alpha,\beta=1}^{n}|f_{a\beta}|^2\right] \\
&+t\left[n-2b\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right)-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F\right]f_0^2 \\
&+\left[\frac{-2(1+a)}{na^2}F-2k_0-2k_1n(b+1)-\frac{2}{b}\right. \\
&\left.-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]|\nabla_bf|^2.
\end{aligned}$$

Here $k_1:=\max_M\{|A_{\alpha\beta}|,\ \ |A_{\alpha\beta,\bar\alpha}|\}$.

Proof. Since $F(p(t),\ t,\ b,\ \ell) = \max_{x\in M} F(x,\ t,\ b,\ \ell)$, at a critical point $(p(t),\ t)$ of $F(x,\ t,\ b,\ \ell)$, we have

$$\nabla_b F(p(t),\ t,\ b,\ \ell) = 0. \tag{3.13}$$

On the other hand, since $(p(t),\ t)$ is a maximum point of F, we can apply the maximum principle at $(p(t),\ t)$. Then we have

$$\Delta_b F(p(t),\ t,\ b,\ \ell) \leq 0. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.7), and again noting that $t \leq 1$, one obtains

$$\begin{aligned} 0 \geq\ & t\left[4(1-bk_1)\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2\right] \\ & +t\left[n-2b\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right)-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F\right]f_0^2 \\ & +\left[\frac{-2(1+a)}{na^2}F-2k_0-2k_1n(b+1)-\frac{2}{b}\right. \\ & \left.-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]|\nabla_b f|^2. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.1. We claim that at $t = 1$, there exists a small constant $H = H(k_0, k_1, \ell, n) > 0$ such that for any $0 < b \leq \frac{1}{H}$

$$F(p(1), 1, b, \ell) < \frac{na^2}{-(1+a)}\left[k_0+k_1n(b+1)+\frac{1}{b}+\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right].$$

Here $(1+a) < 0$ for some a to be chosen later (say $1+a = -\frac{3}{n}$).

We prove it by contradiction. Suppose not, that is

$$\begin{aligned} F(p(1),\ 1,\ b,\ \ell) \geq \frac{na^2}{-(1+a)}&\left[k_0+k_1n(b+1)+\frac{1}{b}\right. \\ &\left.+\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]. \end{aligned}$$

Since $F(p(t),\ t,\ b,\ \ell)$ is continuous in the variable t and $F(p(0),\ 0,\ b,\ \ell) = 0$, by Intermediate-value theorem there exists a $t_0 \in (0,\ 1]$ such that

$$\begin{aligned} & F(p(t_0),\ t_0,\ b,\ \ell) \\ &= \frac{na^2}{-(1+a)}\left[k_0+k_1n(b+1)+\frac{1}{b}+\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]. \end{aligned} \tag{3.15}$$

Now we apply (3.13) at the point $(p(t_0), t_0)$, then

$$
\begin{aligned}
0 \geq\ & t_0\left[4(1-bk_1)\sum_{\alpha,\beta=1}^{n}|f_{a\beta}|^2\right] \\
&+t_0\left[n-2b\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right)-8bk_1n^2\right. \\
&\left.-\left(2b^2k_1n+\frac{2b}{na^2}\right)F(p(t_0),t_0,b,\ell)\right]f_0^2 \\
&+\left[\frac{-2(1+a)}{na^2}F-2k_0-2k_1n(b+1)-\frac{2}{b}\right. \\
&\left.-2\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right)\right]|\nabla_b f|^2 .
\end{aligned}
\tag{3.16}
$$

Next, by using (3.15) and noting that $(1+a)<0$

$$
\begin{aligned}
& n-2b\lambda\left[\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right]-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F(p(t_0),t_0,b,\ell) \\
&= n-2b\lambda\left[\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right]-8bk_1n^2 \\
&\quad-\left(2b^2k_1n+\frac{2b}{na^2}\right)\left[\frac{na^2}{-(1+a)}\right]\left\{k_0+k_1n(b+1)+\frac{1}{b}\right. \\
&\quad\left.+\lambda\left[\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right]\right\} \\
&= n-2b\lambda\left[\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right]-8bk_1n^2+\frac{2b}{1+a}\left(a^2bk_1n^2+1\right)[k_0+k_1n(b+1)] \\
&\quad+\frac{2}{1+a}\left(a^2bk_1n^2+1\right)+\frac{2b\lambda}{1+a}\left(a^2bk_1n^2+1\right)\left[\frac{\ell}{\ell-1}+\frac{1}{n(\ell-1)}+\frac{1}{na(\ell-1)}\right] \\
&= n+\frac{2}{1+a}+\frac{2b}{1+a}\left\{a^2k_1n^2+\left(a^2bk_1n^2+1\right)[k_0+k_1n(b+1)]\right\}-8bk_1n^2 \\
&\quad+\frac{2b\lambda\ell}{\ell-1}\left[-1-\frac{1}{na\ell}+\frac{1}{1+a}\left(a^2bk_1n^2+1\right)\left(1+\frac{1}{n\ell}+\frac{1}{na\ell}\right)\right].
\end{aligned}
$$

Now, choosing $(1+a)=-\frac{3}{n}$, one obtains

$$
\begin{aligned}
& n-2b\lambda\left(\frac{\ell}{\ell-1}+\frac{1}{na(\ell-1)}\right)-8bk_1n^2-\left(2b^2k_1n+\frac{2b}{na^2}\right)F(p(t_0),t_0,b,\ell) \\
&\geq \frac{n}{3}-\frac{2bn}{3}\left\{(n+3)^2k_1+\left[(n+3)^2bk_1+1\right][k_0+k_1n(b+1)]\right\}-8bk_1n^2 \\
&\quad+\frac{2b\ell}{\ell-1}\lambda\left\{-1-\frac{(n+1)}{3}\left[(n+3)^2bk_1+1\right]\right\}.
\end{aligned}
$$

By choosing $b < \frac{1}{k_1+k_0}$, then we have

$$\begin{aligned}
& n - 2b\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{na(\ell-1)}\right) - 8bk_1n^2 - \left(2b^2k_1n + \frac{2b}{na^2}\right)F\left(p\left(t_0\right), t_0, b, \ell\right) \\
\geq\ & \frac{n}{3} - \frac{2bn}{3}\left\{(n+3)^2 k_1 + \left[(n+3)^2 + 1\right]\left[k_0 + \frac{nk_1\left(k_1+k_0+1\right)}{k_1+k_0}\right] + 12k_1n\right\} \\
& -\frac{2b\ell}{\ell-1}\lambda\left\{1 + \frac{(n+1)}{3}\left[(n+3)^2+1\right]\right\} \\
\geq\ & \frac{n}{3} - \frac{n}{3}b\left\{\{2(n+1)(n+3)^2 + 26n + 2\left[(n+3)^2+1\right]\frac{1}{k_1+k_0}\}k_1\right. \\
& \left. +2\left[(n+3)^2+1\right]k_0 + \left(\frac{2(n+1)\left[(n+3)^2+1\right]+6}{n}\right)\frac{\ell}{(\ell-1)}\lambda.\right\}.
\end{aligned}$$

Define

$$\begin{aligned}
H &= H(k_0, k_1, \ell, \lambda) \\
&= \{2(n+1)(n+3)^2 + 26n + 2\left[(n+3)^2+1\right]\frac{1}{k_1+k_0}\}k_1 \\
&\quad +2\left[(n+3)^2+1\right]k_0 + \left(\frac{2(n+1)\left[(n+3)^2+1\right]+6}{n}\right)\frac{\ell}{(\ell-1)}\lambda.
\end{aligned}$$

Thus for any b such that $bH(k_0, k_1, l, \lambda) < 1$ (Note this condition also implies $b < \frac{1}{k_1+k_0}$), we have

$$n - 2b\lambda\left(\frac{\ell}{\ell-1} + \frac{1}{na(\ell-1)}\right) - 8bk_1n^2 - \left(2b^2k_1n + \frac{2b}{na^2}\right)F\left(p\left(t_0\right), t_0, b, \ell\right) > 0.$$

This gives a contradiction to (3.17).

Hence, for $(1+a) = -\frac{3}{n}$

$$|\nabla_b f|^2 + bf_0^2 < \frac{(n+3)^2}{3}\left\{k_0 + 2k_1n + \frac{1}{b} + \lambda\left[\frac{\ell}{\ell-1} + \frac{3}{n(n+3)(\ell-1)}\right]\right\}.$$

Let $b \to \frac{1^-}{H}$, and note $\ell > 1$, we have

$$\begin{aligned}
& |\nabla_b f|^2 + \frac{1}{H}f_0^2 \\
\leq\ & \frac{(n+3)^2}{3}\left\{k_0 + 26k_1n + 2(n+3)^2k_1 + 2\left[(n+3)^2+1\right]\left[k_0 + \frac{nk_1\left(k_1+k_0+1\right)}{k_1+k_0}\right]\right. \\
& \left. +\frac{\ell}{(\ell-1)}\lambda\left\{\frac{6}{n} + \frac{2(n+1)}{n}\left[(n+3)^2+1\right] + 1 + \frac{3}{n(n+3)}\right\}\right\} \\
=\ & \frac{(n+3)^2}{3}\left\{k_0 + 26k_1n + 2(n+3)^2k_1 + 2\left[(n+3)^2+1\right]\left[k_0 + \frac{nk_1\left(k_1+k_0+1\right)}{k_1+k_0}\right]\right\} \\
& +\frac{\ell}{(\ell-1)}\lambda\left\{\frac{(n+3)^2}{3}\left[\frac{6}{n} + \frac{2(n+1)}{n}\left((n+3)^2+1\right) + 1 + \frac{3}{n(n+3)}\right]\right\}.
\end{aligned}$$

Hence

$$|\nabla_b f|^2 + \frac{1}{H} f_0^2 \leq Q + \frac{\ell}{(\ell-1)} \lambda G_n$$

where H, Q, and G_n are constants defined in (1.5).

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. As before, we assume $\min_M u(x) = -1$ and $\max_M u(x) \leq 1$. Let $\gamma : [0,1] \to M$ be a minimal horizontal geodesic joinging the points between $\max u$ and $\min u$.

Then

$$\int_0^1 \frac{d}{dt} \ln [u(\gamma(t)) + \ell]\, dt = \ln \max (u+\ell) - \ln \min (u+\ell) \tag{3.17}$$
$$= \ln \max (u+\ell) - \ln (\ell - 1)$$
$$\geq \ln \left(\frac{\ell}{\ell-1} \right).$$

On the other hand, by Theorem 1.1 one obtains

$$\int_0^1 \frac{d}{dt} \ln [u(\gamma(t)) + \ell]\, dt \tag{3.18}$$
$$\leq \int_0^1 |\nabla_b \ln (u+\ell)| \, |\gamma'(t)|\, dt$$
$$\leq d \left\{ \left[Q + \lambda \frac{\ell}{\ell-1} G_n \right] \right\}^{\frac{1}{2}}$$

where $d = \operatorname{diam}(M)$.

From Theorem 1.1, (3.17) and (3.18), we have

$$\left[\lambda \frac{\ell}{\ell-1} G_n \right] \geq \frac{1}{d^2} \left[\ln \left(\frac{\ell}{\ell-1} \right) \right]^2 - Q.$$

Let $t = \frac{\ell-1}{\ell}$. This implies that for any $0 < t < 1$,

$$\lambda G_n \geq \left[\frac{1}{d^2} (\ln t)^2 - Q \right] t.$$

Now, we define a real function

$$g : \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1) \to \mathbb{R}$$

by

$$g(A,B,t) = \left(A (\ln t)^2 - B \right) t.$$

This function has maximum point at $t_0 = \exp\left(-1 - \sqrt{1 + 2\frac{B}{A}}\right)$ and

$$g(t_0) = 2A \left[1 + \sqrt{1 + 2\frac{B}{A}} \right] \exp \left(-1 - \sqrt{1 + 2\frac{B}{A}} \right).$$

Taking $A = \frac{1}{d^2}$ and $B = Q(k_0, k_1, n)$, we get

$$\lambda \geq \frac{2}{d^2 G_n}\left[1+\sqrt{1+2Qd^2}\right]\exp\left(-1-\sqrt{1+2Qd^2}\right).$$

This completes the proof of the Theorem 1.2. □

Appendix A. We give a brief introduction to pseudohermitian geometry (see [L1], [L2] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{A.1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Actually we can always choose Z_α such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$; hence, throughout this note, we assume $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$.

The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle.$$

We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle \overline{Z}, \overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle u, v\rangle = \int_M u\overline{v}\ d\mu,$$

for functions u and v.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \theta_{\bar{\alpha}}{}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \theta_\alpha{}^\beta + \theta \wedge \tau^\beta, \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \theta_\alpha{}^\beta + \theta_{\bar{\beta}}{}^{\bar{\alpha}}. \end{aligned} \tag{A.2}$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha &= \overline{\Pi_{\bar{\beta}}{}^{\bar{\alpha}}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha, \\ \Pi_0{}^\alpha &= \Pi_\alpha{}^0 = \Pi_0{}^{\bar{\beta}} = \Pi_{\bar{\beta}}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that ${\Pi_\beta}^\alpha$ can be written

$$\text{(A.3)} \qquad {\Pi_\beta}^\alpha = {R_\beta}^\alpha{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + {W_\beta}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u,\ u_{\alpha\bar\beta} = Z_{\bar\beta} Z_\alpha u - {\omega_\alpha}^\gamma (Z_{\bar\beta}) Z_\gamma u$.

For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle_{L_\theta} = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_\alpha u_{\bar\alpha} Z_\alpha + u_\alpha Z_{\bar\alpha}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$$

by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta}u_{\overline{\alpha\beta}} + u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr\left((\nabla^H)^2 u\right) = \textstyle\sum_\alpha (u_{\alpha\bar\alpha} + u_{\bar\alpha\alpha}).$$

Next we recall the following definition.

Definition A.1. A piecewise smooth curve $\gamma : [0,1] \longrightarrow M$ is said to be the horizontal if $\gamma'(t) \in \xi$ whenever $\gamma'(t)$ exists. The length of γ is then defined by

$$l(\gamma) := \int_0^1 dt \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_{L_\theta}}.$$

The Carnot-Carathéodory distance between two points $p, q \in M$ is

$$d(p,q) := \inf\{l(\gamma) : \gamma \in C_{p,q}\},$$

where $C_{p,q}$ denote the set of all horizontal curves joining p and q. By Chow connectivity theorem [Cho], there always exists a horizontal curve joining p and q, so the distance is finite. The CR diameter d is defined by

$$\text{(A.4)} \qquad d := \sup\{d(p,q) : p, q \in M\}.$$

Finally, we state the following commutation relations ([L1]). Let φ be a scalar function and $\sigma = \sigma_\alpha \theta^\alpha$ be a $(1,0)$ form, then we have

$$\text{(A.5)} \qquad \begin{array}{rcl} \varphi_{\alpha\beta} &=& \varphi_{\beta\alpha}, \\ \varphi_{\alpha\bar\beta} - \varphi_{\bar\beta\alpha} &=& ih_{\alpha\overline{\beta}}\varphi_0, \\ \varphi_{0\alpha} - \varphi_{\alpha 0} &=& A_{\alpha\beta}\varphi_{\bar\beta}, \\ \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &=& \sigma_{\alpha,\bar\gamma}A_{\gamma\beta} - \sigma_\gamma A_{\alpha\beta,\bar\gamma}, \\ \sigma_{\alpha,0\bar\beta} - \sigma_{\alpha,\bar\beta 0} &=& \sigma_{\alpha,\gamma}A_{\bar\gamma\bar\beta} - \sigma_\gamma A_{\bar\gamma\bar\beta,\alpha}, \end{array}$$

and

$$
\begin{aligned}
\sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma}\sigma_{\beta} - iA_{\alpha\beta}\sigma_{\gamma}, \\
\sigma_{\alpha,\bar{\beta}\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\overline{\beta}}A_{\bar{\gamma}\bar{\rho}}\sigma_{\rho} - ih_{\alpha\overline{\gamma}}A_{\bar{\beta}\bar{\rho}}\sigma_{\rho}, \\
\sigma_{\alpha,\beta\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\overline{\gamma}}\sigma_{\alpha,0} + R_{\alpha\bar{\rho}\beta\bar{\gamma}}\sigma_{\rho}.
\end{aligned} \tag{A.6}
$$

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