

STRUCTURE OF HOCHSCHILD COHOMOLOGY OF PATH ALGEBRAS AND DIFFERENTIAL FORMULATION OF EULER'S POLYHEDRON FORMULA*

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Abstract. This article studies the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$ of derivations on the path algebra $\mathbf{k}\Gamma$ of a quiver Γ and the Lie algebra on the first Hochschild cohomology group $HH^1(\mathbf{k}\Gamma)$. We relate these Lie algebras to the algebraic and combinatorial properties of the path algebra. Characterizations of derivations on a path algebra are obtained, leading to a canonical basis of $\text{Der}(\mathbf{k}\Gamma)$ and its Lie algebra properties. Special derivations are associated to the vertices, arrows and faces of a quiver, and the concepts of a connection matrix and boundary matrix are introduced to study the relations among these derivations, giving rise to an interpretation of Euler's polyhedron formula in terms of derivations. By taking dimensions, this relation among spaces of derivations recovers Euler's polyhedron formula. This relation also leads to a combinatorial construction of a canonical basis of the Lie algebra $HH^1(\mathbf{k}\Gamma)$, together with a new semidirect sum decomposition of $HH^1(\mathbf{k}\Gamma)$.

Key words. quiver, path algebra, Hochschild cohomology, Lie algebra, differential algebra, graph, Euler's polyhedron formula, connection matrix.

AMS subject classifications. 16E40, 16G20, 05E15, 05C25, 12H05, 16W25, 16S32.

1. Introduction. This paper studies the structure of the Lie algebra of derivations on the path algebra $\mathbf{k}\Gamma$ of a quiver Γ and the Lie algebra of outer derivations on the path algebra, also known as the first Hochschild cohomology group $HH^1(\mathbf{k}\Gamma)$. This study has two motivations, one from Hochschild cohomology and one from differential algebra. We determine a canonical basis and their multiplication constants for these two Lie algebras, and relate it to the combinatorial properties of the quiver, such as Euler's Polyhedron Theorem.

The study of Hochschild cohomology of quiver related algebras started with the dimension formula for $HH^n(\mathbf{k}\Gamma)$ given by Happel in 1989 [11], who showed that for an acyclic quiver Γ and a field \mathbf{k} ,

$$HH^0(\mathbf{k}\Gamma) = \mathbf{k}, \quad \dim_{\mathbf{k}} HH^1(\mathbf{k}\Gamma) = 1 - |V| + \sum_{\alpha \in E} v(\alpha), \quad HH^i(\mathbf{k}\Gamma) = 0, \quad \forall i \geq 2$$

where $v(\alpha) = \dim_{\mathbf{k}} t(\alpha)\mathbf{k}\Gamma h(\alpha)$, while V and E are respectively the sets of vertices and arrows of Γ . Afterwards, there have been extensive studies on the dimensions of the Hochschild cohomology groups of quiver related algebras, such as truncated path algebras, monomial algebras, schurian algebras and 2-nilpotent algebras [1, 6, 11, 19, 20, 23, 24, 26, 29]. In [24], the first Hochschild cohomology group of a monomial algebra of radical square zero is given as the splitting extension of a semi-simple Lie algebra by the radical due to Levi-Malcev Theorem where the semisimple Lie algebra is related to the cardinality of parallel arrows and the radical to the Euler characteristic of the quiver. The quiver is supposed to be planar.

Further understanding of the Lie algebra $HH^1(\mathbf{k}\Gamma)$ would benefit from an explicit structure of this Lie algebra, such as a canonical basis and the corresponding multiplication constants. This is what we achieve in this paper. We find that the choice

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of the basis of $HH^1(\mathbf{k}\Gamma)$ is related to the combinatorics, such as Euler's formula, and the topology, such as the genus, of the quiver.

Our second motivation is differential algebra which has its origin in the algebraic study of differential equations [13, 22, 25] and is a natural yet profound extension of commutative algebra and the related algebraic geometry. After many years of developments, the theory has expanded into a vast area in mathematics [5, 13, 25]. Furthermore, differential algebra has found important applications in arithmetic geometry, logic and computational algebra, especially in the profound work of W. Wu on mechanical proof of geometric theorems [27, 28].

Most of the study on differential algebra has been for commutative algebras and fields. Recently, there have been interests to study differential algebra for noncommutative algebras. For instance, in connection with combinatorics, differential structures were found on heap ordered trees [9] and on decorated rooted trees [10].

This paper gives a differential study of the path algebra of a quiver, as a first step in the study of differential structures on Artinian algebras. According to the well-known Gabriel Theorem [2, 3], a basic algebra over an algebraically closed field is a quotient of the path algebra of its Ext-quiver modulo an admissible ideal. More generally, by [18], an Artinian algebra over a perfect field is isomorphic to a quotient of the generalized path algebra of its natural quiver. Thus if we can determine the differential structures on path algebras (resp. generalized path algebras), including their differential ideals, then by taking the quotients of these algebras modulo their differential ideals, we will be able to obtain the differential structure on a basic algebra (resp. an Artinian algebra). For more related references, see [15, 16, 17].

In Section 2 we characterize when a linear operator on a path algebra is a derivation. These characterizations of derivations allow us to obtain in Section 3 a canonical basis of the Lie algebra of derivations on a path algebra and obtain a structure theorem of $\text{Der}(\mathbf{k}\Gamma)$. This structure theorem is then applied to study Lie algebra properties of $\text{Der}(\mathbf{k}\Gamma)$. In Section 4, we focus on three types of derivations of combinatorial nature, namely derivations from the vertices, arrows and faces of the quiver respectively. Dimension formulas of the spaces spanned by these derivations are proved and the relations among them are determined. In Section 5 we give two applications of these dimension formulas. We first revisit Euler's Polyhedron Theorem from a differential viewpoint, and prove that the linear space of edge derivations is the direct sum of the linear spaces of vertex derivations and face derivations. Taking dimensions of the spaces in this direct sum decomposition gives the original formula of Euler. We next apply the combinatorial derivations to obtain a canonical basis for $HH^1(\mathbf{k}\Gamma)$. This basis allows us to perform computations in this Lie algebra and to factor it into a semidirect sum of an abelian Lie subalgebra and an Lie ideal. This given basis and the semidirect sum decomposition, i.e. the splitting extension, of the first Hochschild cohomology are in the context of combinatorics and topology. The splitting extension given here is not obtained from a semi-simple Lie algebra by the radical, hence is different from the one in [24] depending upon Levi-Malcev theorem for a finite-dimensional Lie algebra.

2. Derivations on path algebras. The main purpose of this section is to provide necessary and sufficient conditions for a linear operator on a path algebra to be a differential operator.

2.1. Derivations. We briefly recall concepts, notations and facts on differential algebras, Lie algebras and path algebras of quivers. Further details on these three subjects can be found in [5, 10, 13], in [12] and in [7, 2, 3, 15], respectively.

Let \mathbf{k} be a field and let A be a \mathbf{k} -algebra. Let $Lie(A) = (A, [,])$ denote the Lie algebra structure on A with the Lie bracket

$$[x, y] := xy - yx, \quad x, y \in A.$$

A **derivation** (or a **differential operator**) on A is a \mathbf{k} -linear map $D : A \rightarrow A$ such that

$$D(xy) = D(x)y + xD(y), \quad \forall x, y \in A.$$

Let $\text{Der}(A)$ denote the set of derivations on A . Then with the Lie bracket

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1, \quad D_1, D_2 \in \text{Der}(A),$$

$\text{Der}(A)$ is a Lie algebra, called the **Lie algebra of derivations** on A .

For $a \in A$, define the **inner derivation**

$$(1) \quad D_a : A \rightarrow A, \quad D_a(b) = (ad_a)(b) := ab - ba, \quad b \in A.$$

Then the map

$$(2) \quad \mathfrak{D} : Lie(A) \rightarrow \text{Der}(A), \quad \mathfrak{D}(a) = D_a, \quad a \in A,$$

from Eq. (1) is a Lie algebra homomorphism whose kernel is $C(A)$, the center of A and also the zeroth Hochschild cohomology group $HH^0(A)$.

The subset $\text{InDer}(A) := \text{im } \mathfrak{D} \subseteq \text{Der}(A)$ is a Lie ideal. The quotient Lie algebra

$$(3) \quad \text{OutDer}(A) := \text{Der}(A)/\text{InDer}(A)$$

is called the **Lie algebra of outer derivations**. As is well-known [21, §11.5], $\text{OutDer}(A)$ is also the **first Hochschild cohomology group** $HH^1(A)$.

We will study $\text{Der}(A)$ and $HH^1(A)$ when $A = \mathbf{k}\Gamma$ is the path algebra of a connected quiver Γ .

2.2. Path algebras. A **quiver** is a quadruple $\Gamma = (V, E, t, h)$ consisting of a set V of **vertices**, a set E of **arrows** and a pair of maps $h, t : E \rightarrow V$ such that for any arrow $a \in E$, $h(a)$ is called the **head** of a and $t(a)$ is called the **tail** of a . When there is no danger of confusion, we also denote $\Gamma = (V, E)$. A quiver is called **trivial** if $E = \emptyset$.

Let p be a **path** in Γ consisting of the ordered list $v_0, p_1, v_1, \dots, v_{\ell-1}, p_\ell, v_\ell$, with $v_i \in V, 0 \leq i \leq \ell$ and $p_j \in E, 1 \leq j \leq \ell$, such that $t(p_j) = v_{j-1}$ and $h(p_j) = v_j, 1 \leq j \leq \ell$. The integer $\ell \geq 0$ is called the **length** of the path p and is denoted by $\ell(p)$. The path p is expressed as

$$p = v_0 p_1 v_1 \cdots v_{\ell-1} p_\ell v_\ell,$$

called the **standard decomposition** of p , and the expression

$$p = p_1 \cdots p_\ell,$$

is called the **decomposition of p into arrows**. Both decompositions are unique.

Let \mathcal{P} denote the set of paths of Γ . For $p \in \mathcal{P}$ let $t(p)$ and $h(p)$ denote the **tail** and the **head** of p respectively.

Let

$$\mathbf{k}\Gamma = \bigoplus_{p \in \mathcal{P}} \mathbf{k}p,$$

denote the **path algebra** of Γ where the product is given by

$$p \cdot q := \delta_{h(p),t(q)}pq := \begin{cases} pq, & h(p) = t(q), \\ 0, & \text{otherwise.} \end{cases}$$

Here $\delta_{h(p),t(q)}$ is the delta function. To simplify notations, we often suppress the symbol \cdot and denote $p \cdot q = pq$, with the convention that $pq = 0$ when $h(p) \neq t(q)$.

We will use the following notations on quivers and their path algebras.

DEFINITION 2.1.

- (a) For two paths p and q , denote $p \parallel q$ and called p and q **parallel**, if $t(p) = t(q)$ and $h(p) = h(q)$.
- (b) For two paths p and q , if $q = pr$ (resp. $q = rp$) for some path r , then call p a **tail** (resp. **head**) of q and denote by $p \leq_t q$ (resp. $p \leq_h q$). Such an r is unique for given p, q .
- (c) A path p is called **acyclic** if $h(p) \neq t(p)$. The set of acyclic paths is denoted by \mathcal{P}_A .
- (d) A quiver Γ is called **acyclic** if Γ has no oriented cycles, that is, if $\mathcal{P} \setminus V = \mathcal{P}_A$.

In this paper, we always assume that a quiver Γ is finite, that is, its vertex set and arrow set are both finite. By the linearity of a derivation, we have:

LEMMA 2.2. *Let A be a \mathbf{k} -algebra with a linear basis X . Then a linear operator $D : A \rightarrow A$ is a derivation if and only if*

$$(4) \quad D(xy) = D(x)y + xD(y), \quad \forall x, y \in X.$$

In particular, a linear operator $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ is a derivation if and only if Eq. (4) holds for all $x, y \in \mathcal{P}$.

2.3. Necessary and sufficient conditions for a derivation. We now characterize a derivation on a path algebra $\mathbf{k}\Gamma$ in terms of the paths \mathcal{P} of Γ . These characterizations will be applied in the next section to determine all derivations on a path algebra.

Let $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ be a linear operator. Then for any $p \in \mathcal{P}$,

$$D(p) = \sum_{q \in \mathcal{P}} c_q^p q,$$

for unique $c_q^p \in \mathbf{k}$. We will use this notation for the rest of this paper. We also use the convention that, for the empty set \emptyset ,

$$\sum_{q \in \emptyset} c_q^p q = 0.$$

THEOREM 2.3. *Let Γ be a quiver. A linear operator $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ is a derivation if and only if D satisfies the following conditions.*

(a) For $v \in V$,

$$(5) \quad D(v) = \sum_{q \in \mathcal{P}_A, t(q)=v} c_q^v q + \sum_{q \in \mathcal{P}_A, h(q)=v} c_q^v q = \sum_{q \in \mathcal{P}_A, t(q)=v \text{ or } h(q)=v} c_q^v q.$$

(b) For $p \in \mathcal{P} \setminus V$,

$$(6) \quad D(p) = \sum_{q \in \mathcal{P}_A, h(q)=t(p)} c_q^{t(p)} qp + \sum_{q \parallel p} c_q^p q + \sum_{q \in \mathcal{P}_A, t(q)=h(p)} c_q^{h(p)} pq,$$

where the coefficients c_q^p are subject to the following conditions.

(i) For any path $q \in \mathcal{P}_A$,

$$(7) \quad c_q^{h(q)} + c_q^{t(q)} = 0.$$

(ii) For any path $p = p_1 p_2$ with $p_1, p_2 \in \mathcal{P} \setminus V$ and $q \parallel p$, we have

$$(8) \quad c_q^p = c_q^{p_1 p_2} = \begin{cases} c_{q_1}^{p_1} + c_{q_2}^{p_2}, & \text{if } p_2 \leq_h q \text{ with } q = q_1 p_2 \text{ and } p_1 \leq_t q \text{ with } q = p_1 q_2, \\ c_{q_1}^{p_1}, & \text{if } p_2 \leq_h q \text{ with } q = q_1 p_2 \text{ and } q_1 \not\leq_t p_1, \\ c_{q_2}^{p_2}, & \text{if } p_1 \leq_t q \text{ with } q = p_1 q_2 \text{ and } q_2 \not\leq_h p_2, \\ 0, & \text{if } p_2 \not\leq_h q \text{ and } p_1 \not\leq_t q. \end{cases}$$

Proof. (\implies) Let $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ be a linear operator. For a given $v \in V$, since $vv = v$, we have $D(v) = D(vv) = D(v)v + vD(v)$. Thus

$$(9) \quad \sum_{q \in \mathcal{P}} c_q^v q = \left(\sum_{q \in \mathcal{P}} c_q^v q \right) v + v \left(\sum_{q \in \mathcal{P}} c_q^v q \right) = \sum_{q \in \mathcal{P}, h(q)=v} c_q^v q + \sum_{q \in \mathcal{P}, t(q)=v} c_q^v q$$

since $qv = 0$ unless $h(q) = v$ and $vq = 0$ unless $t(q) = v$.

Similarly, from

$$D(v) = D(v^3) = D(v)v^2 + vD(v)v + v^2D(v) = D(v)v + vD(v)v + vD(v),$$

we have

$$\begin{aligned} \sum_{q \in \mathcal{P}} c_q^v q &= \left(\sum_{q \in \mathcal{P}} c_q^v q \right) v + v \left(\sum_{q \in \mathcal{P}} c_q^v q \right) v + v \left(\sum_{q \in \mathcal{P}} c_q^v q \right) \\ &= \sum_{q \in \mathcal{P}, h(q)=v} c_q^v q + \sum_{q \in \mathcal{P}, h(q)=v, t(q)=v} c_q^v q + \sum_{q \in \mathcal{P}, t(q)=v} c_q^v q. \end{aligned}$$

Comparing this with Eq. (9), we obtain $\sum_{q \in \mathcal{P}, h(q)=v, t(q)=v} c_q^v q = 0$. Then Eq. (5) follows from Eq. (9).

Also, for a given path $p \in \mathcal{P} \setminus V$, we have

$$\begin{aligned} D(p) &= D(t(p)ph(p)) \\ &= D(t(p))ph(p) + t(p)D(p)h(p) + t(p)pD(h(p)) \\ &= D(t(p))p + t(p)D(p)h(p) + pD(h(p)) \\ &= D(t(p))p + \sum_{q \parallel p} c_q^p q + pD(h(p)). \end{aligned}$$

By Eq. (5), we have

$$D(t(p))p = \left(\sum_{q \in \mathcal{P}_A, t(q)=t(p)} c_q^{t(p)} q + \sum_{q \in \mathcal{P}_A, h(q)=t(p)} c_q^{t(p)} q \right) p = \sum_{q \in \mathcal{P}_A, h(q)=t(p)} c_q^{t(p)} qp$$

since $qp = 0$ if $h(q) \neq t(p)$. Similarly, $pD(h(p)) = \sum_{q \in \mathcal{P}_A, t(q)=h(p)} c_q^{h(p)} pq$. This proves

Eq. (6).

Thus we only need to prove Eq. (7) and Eq. (8) in order to complete the proof of (\implies). For this purpose, we prove a lemma.

LEMMA 2.4.

- (a) Let $p_1, p_2 \in \mathcal{P}$. Suppose $p_1 p_2 = 0$ and Eq. (5) and Eq. (8) hold for p_1 and p_2 . Then $D(p_1 p_2) = D(p_1) p_2 + p_1 D(p_2)$ if and only if, for every $q \in \mathcal{P}$ with $t(q) = p_1$ and $h(q) = p_2$, Eq. (7) holds.
- (b) Let $p_1, p_2 \in \mathcal{P} \setminus V$. Suppose $p_1 p_2 \neq 0$ and Eq. (6) holds for p_1, p_2 and $p_1 p_2$. Then $D(p_1 p_2) = D(p_1) p_2 + p_1 D(p_2)$ if and only if, for every $q \parallel p$, Eq. (8) holds.

Proof. (a). Since $p_1 p_2 = 0$ and D is linear, we have $D(p_1 p_2) = 0$. To compute $D(p_1) p_2 + p_1 D(p_2)$, first consider the case when p_1, p_2 are in V .

$$\begin{aligned} & D(p_1) p_2 + p_1 D(p_2) \\ &= \left(\sum_{q \in \mathcal{P}_A, h(q)=p_1} c_q^{p_1} q + \sum_{q \in \mathcal{P}_A, t(q)=p_1} c_q^{p_1} q \right) p_2 \\ &+ p_1 \left(\sum_{q \in \mathcal{P}_A, h(q)=p_2} c_q^{p_2} q + \sum_{q \in \mathcal{P}_A, t(q)=p_2} c_q^{p_2} q \right) \quad (\text{by Eq. (6)}) \\ &= \sum_{q \in \mathcal{P}_A, t(q)=p_1, h(q)=p_2} c_q^{p_1} q + \sum_{q \in \mathcal{P}_A, t(q)=p_1, h(q)=p_2} c_q^{p_2} q \quad (\text{since } p_1 \neq p_2) \\ &= \sum_{q \in \mathcal{P}_A, t(q)=p_1, h(q)=p_2} (c_q^{p_1} + c_q^{p_2}) q. \end{aligned}$$

Thus $D(p_1) p_2 + p_1 D(p_2) = 0$ if and only if all the coefficients in the last sum are zero. That is, $c_q^{h(q)} + c_q^{t(q)} = 0$ for all $q \in \mathcal{P}$ with $t(q) = p_1, h(q) = p_2$. This proves (a).

Next consider the case when p_1 and p_2 are in $\mathcal{P} \setminus V$. By a similar computation¹, we have:

$$D(p_1) p_2 + p_1 D(p_2) = \sum_{t(q)=h(p_1), h(q)=t(p_2)} (c_q^{h(p_1)} + c_q^{t(p_2)}) p_1 q p_2.$$

For any two distinct $p_1, p_2 \in \mathcal{P}$ with $t(q) = h(p_1)$ and $h(q) = t(p_2)$, the corresponding $p_1 q p_2$ are non-zero and are distinct paths in the basis \mathcal{P} of $\mathbf{k}\Gamma$. Thus $D(p_1) p_2 + p_1 D(p_2) = 0$ if and only if all the coefficients in the last sum are zero. That is, $c_q^{h(p_1)} + c_q^{t(p_2)} = 0$ for all $q \in \mathcal{P}$ with $t(q) = h(p_1), h(q) = t(p_2)$. This proves (a) in this case.

¹For details of this and latter computations, see the online version of the paper arXiv:1010.1980v2.

The cases when one of p_1, p_2 is in V and the other one is in $\mathcal{P} \setminus V$ can be verified in the same way.

(b). By Eq. (6) we have

$$(10) \quad D(p_1 p_2) = \sum_{q \in \mathcal{P}_A, h(q)=t(p_1)} c_q^{t(p_1)} q p_1 p_2 + \sum_{q \parallel p_1 p_2} c_q^{p_1 p_2} q + \sum_{q \in \mathcal{A}, t(q)=h(p_2)} c_p^{h(p_2)} p_1 p_2 q$$

since $t(p_1 p_2) = t(p_1)$ and $h(p_1 p_2) = h(p_2)$. Similarly,

$$D(p_1) p_2 = \left(\sum_{q \in \mathcal{P}_A, h(q)=t(p_1)} c_q^{t(p_1)} q p_1 + \sum_{q_1 \parallel p_1} c_{q_1}^{p_1} q_1 + \sum_{q \in \mathcal{P}_A, t(q)=h(p_1)} c_q^{h(p_1)} p_1 q \right) p_2.$$

Since $h(q) \neq t(q) = h(p_1) = t(p_2)$, we have $q p_2 = 0$ for q in the last sum. Thus we obtain

$$(11) \quad D(p_1) p_2 = \sum_{q \in \mathcal{P}_A, h(q)=t(p_1)} c_q^{t(p_1)} q p_1 p_2 + \sum_{q_1 \parallel p_1} c_{q_1}^{p_1} q_1 p_2.$$

By the same argument, we have

$$(12) \quad p_1 D(p_2) = \sum_{q_2 \parallel p_2} c_{q_2}^{p_2} p_1 q_2 + \sum_{q \in \mathcal{P}_A, t(q)=h(p_2)} c_q^{h(p_2)} p_1 p_2 q.$$

Thus by equations (10), (11) and (12), we see that $D(p_1 p_2) = D(p_1) p_2 + p_1 D(p_2)$ if and only if

$$(13) \quad \sum_{q \parallel p_1 p_2} c_q^{p_1 p_2} q = \sum_{q_1 \parallel p_1} c_{q_1}^{p_1} q_1 p_2 + \sum_{q_2 \parallel p_2} c_{q_2}^{p_2} p_1 q_2.$$

In the sum on the left hand side, the paths $q \parallel p_1 p_2$ can be divided into the disjoint union of the following four subsets:

$$(14) \quad \mathcal{P}_1 := \{q \in \mathcal{P} \mid p \parallel p_1 p_2, p_1 \leq_t q \text{ and } p_2 \leq_h q\},$$

$$(15) \quad \mathcal{P}_2 := \{q \in \mathcal{P} \mid q \parallel p_1 p_2, p_1 \leq_t q \text{ and } p_2 \not\leq_h q\},$$

$$(16) \quad \mathcal{P}_3 := \{q \in \mathcal{P} \mid q \parallel p_1 p_2, p_1 \not\leq_t q \text{ and } p_2 \leq_h q\},$$

$$(17) \quad \mathcal{P}_4 := \{q \in \mathcal{P} \mid q \parallel p_1 p_2, p_1 \not\leq_t q \text{ and } p_2 \not\leq_h q\}.$$

Thus the left hand side of Eq. (13) becomes

$$(18) \quad \sum_{q \parallel p_1 p_2} c_q^{p_1 p_2} q = \sum_{q \in \mathcal{P}_1} c_q^{p_1 p_2} q + \sum_{q \in \mathcal{P}_2} c_q^{p_1 p_2} q + \sum_{q \in \mathcal{P}_3} c_q^{p_1 p_2} q + \sum_{q \in \mathcal{P}_4} c_q^{p_1 p_2} q.$$

By the definitions of \leq_t and \leq_h , for the two sums on the right hand side of Eq. (13), we have respectively

$$(19) \quad \{q_1 p_2 \mid q_1 \parallel p_1\} = \mathcal{P}_1 \cup \mathcal{P}_3$$

and

$$\{p_1 q_2 \mid q_2 \parallel p_2\} = \mathcal{P}_1 \cup \mathcal{P}_2.$$

Thus the right hand side of Eq. (13) becomes

$$(20) \quad \sum_{q \in \mathcal{P}_1 \text{ with } q=q_1 p_2 = p_1 q_2} (c_{q_1}^{p_1} + c_{q_2}^{p_2})q + \sum_{q \in \mathcal{P}_3 \text{ with } q=q_1 p_2} c_{q_1}^{p_1} q_1 p_2 + \sum_{q \in \mathcal{P}_2 \text{ with } q=p_1 q_2} c_{q_2}^{p_2} p_1 q_2.$$

Now comparing the coefficients on the two sides of Eq. (13) using Eq. (18) and Eq. (20), we obtain Eq. (8). \square

Now we return to the proof of Theorem 2.3. For any path $p \in \mathcal{P} \setminus V$ with $t(p) \neq h(p)$, we have $t(p)h(p) = 0$. So applying Lemma 2.4.(a) to $p_1 = t(p)$ and $p_2 = h(p)$ in Eq. (5), we obtain Eq. (7).

Finally let $p = p_1 p_2$ with $p_1, p_2 \in \mathcal{P} \setminus V$. Applying Lemma 2.4.(b) to $p = p_1 p_2$, we obtain Eq. (8).

(\Leftarrow) Suppose a linear operator $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ is given by Eq. (5) and (6) subject to the conditions Eq. (7) and (8). By Lemma 2.2, to show that D is a derivation we just need to show that Eq. (4) holds for $X = \mathcal{P}$. Thus we only need to verify Eq. (4) in the following four cases.

- (a) $x, y \in V$;
- (b) $x \in V, y \in \mathcal{P} \setminus V$;
- (c) $x \in \mathcal{P} \setminus V, y \in V$;
- (d) $x, y \in \mathcal{P} \setminus V$.

The checking of each case is routine and is omitted. \square

2.4. A variation of Theorem 2.3. For the convenience of later applications, we give another formulation of Theorem 2.3 on the condition of a derivation on a path algebra.

COROLLARY 2.5. *Let Γ be a quiver. A linear operator $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ is a derivation if and only if D is determined by its action on the basis \mathcal{P} as follows.*

- (a) Let $v \in V$. Then

$$(21) \quad D(v) = \sum_{q \in \mathcal{P}_A, t(q)=v} c_q^{t(q)} q - \sum_{q \in \mathcal{P}_A, h(q)=v} c_q^{t(q)} q.$$

- (b) Let $p \in \mathcal{P} \setminus V$. Then

$$(22) \quad D(p) = \sum_{q \in \mathcal{P}_A, t(q)=h(p)} c_q^{t(q)} p q + \sum_{q \parallel p} c_q^p q - \sum_{q \in \mathcal{P}_A, h(q)=t(p)} c_q^{t(q)} q p,$$

where the coefficients c_q^p are subject to the following condition: For any path $p = p_1 p_2$ with $p_1, p_2 \in \mathcal{P} \setminus V$ and $q \parallel p$, we have

$$c_q^p = c_q^{p_1 p_2} = \begin{cases} c_{q_1}^{p_1} + c_{q_2}^{p_2}, & \text{if } p_2 \leq_h q \text{ with } q = q_1 p_2 \text{ and } p_1 \leq_t q \text{ with } q = p_1 q_2, \\ c_{q_1}^{p_1}, & \text{if } p_2 \leq_h q \text{ with } q = q_1 p_2 \text{ and } q_1 \not\leq_t p_1, \\ c_{q_2}^{p_2}, & \text{if } p_1 \leq_t q \text{ with } q = p_1 q_2 \text{ and } q_2 \not\leq_h p_2, \\ 0, & \text{if } p_2 \not\leq_h q \text{ and } p_1 \not\leq_t q. \end{cases}$$

Proof. We only need to show that the condition Eq (7) imposed to Eq. (5) and Eq. (6) in Theorem 2.3 amount to Eq. (21) and Eq. (22). First, applying Eq. (7), that is $c_p^{h(p)} = -c_p^{t(p)}$ for $p \in \mathcal{P} \setminus V$, to Eq. (5) gives us Eq. (21).

Similarly apply Eq. (7) to Eq. (6). For the coefficients in the first sum, we have $c_q^{h(p)} = c_q^{t(q)}$ by the restriction of the sum. For the coefficients in the third sum of Eq. (6) we have $c_q^{t(p)} = c_p^{h(q)} = -c_q^{t(q)}$. Thus the first and third sums in Eq. (6) agree with the corresponding sums in Eq. (22). This is what we need. \square

3. The Structure of Lie algebra $\text{Der}(\mathbf{k}\Gamma)$ for a quiver Γ . In this section, we apply the characterizations (Theorem 2.3 and Corollary 2.5) of a derivation on a path algebra to study the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$. We first display a canonical basis for this Lie algebra and then use the basis to establish the multiplication structure of this Lie algebra. As applications, basic properties of this Lie algebra are studied.

3.1. The derivation $D_{r,s}$. Let $r \in E$ and $s \parallel r$. We construct a linear operator

$$D_{r,s} : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$$

by defining $D_{r,s}(p)$ for $p \in \mathcal{P}$ by induction on the length $\ell(p)$ of p .

When $\ell(p) = 0$, i.e., $p \in V$, we define

$$(23) \quad D_{r,s}(p) = 0.$$

Assume that $D_{r,s}(p)$ have been defined for $p \in \mathcal{P}$ with $\ell(p) = n \geq 0$. Consider $p \in \mathcal{P}$ with $\ell(p) = n + 1$. Then $p = p_1\tilde{p}$ with $p_1 \in E$ and $\tilde{p} \in \mathcal{P}$ with $\ell(\tilde{p}) = n$. We then define

$$(24) \quad D_{r,s}(p) = \begin{cases} s\tilde{p} + p_1D_{r,s}(\tilde{p}), & p_1 = r, \\ p_1D_{r,s}(\tilde{p}), & p_1 \neq r. \end{cases}$$

PROPOSITION 3.1. *For a quiver $\Gamma = (V, E)$, let $r \in E$ and $s \parallel r$. The linear operator $D_{r,s}$ recursively defined by Eqs. (23) and (24) have the following explicit formula. For any $p \in \mathcal{P}$ with the standard decomposition $p = v_0p_1v_1 \cdots p_kv_k$ with $v_0, \dots, v_k \in V, p_1, \dots, p_k \in E$, we have*

$$(25) \quad D_{r,s}(p) = \begin{cases} 0, & k = 0, \\ \sum_{i=1}^k v_0p_{i,1}v_1 \cdots p_{i,k}v_k, & k > 0, \end{cases}$$

where $p_{i,j}$ or, more precisely, $p_{i,j}^{(r,s)}$ is defined by

$$(26) \quad p_{i,j} := p_{i,j}^{(r,s)} := \begin{cases} s, & i = j, p_i = r, \\ 0, & i = j, p_i \neq r, \\ p_j, & i \neq j. \end{cases}$$

For example, for $p \in \mathcal{P}$ with standard decomposition $p = v_0p_1v_1p_2v_2p_3v_3p_4v_4$ where $p_1, p_3 = r$ and $p_2, p_4 \neq r$, we have

$$D_{r,s}(p) = v_0sv_1p_2v_2rv_3p_4v_4 + v_0rv_1p_2v_2sv_3p_4v_4.$$

Proof. Let D be defined by Eq. (25). We just need to show that $D_{r,s}(p) = D(p)$ for all $p \in \mathcal{P}$. We prove this by induction on $\ell(p)$. When $\ell(p) = 0$, then $D_{r,s}(p) = 0 = D(p)$ by the definitions of $D_{r,s}$ and D . Assume the equation holds for $\ell(p) = k$ for $k \geq 0$, and consider $p \in \mathcal{P}$ with $\ell(p) = k + 1$. Then $p = v_0p_1\tilde{p} = p_1\tilde{p}$ with $p_1 \in E$

and $\tilde{p} \in \mathcal{P}$ with $\ell(\tilde{p}) = k$. Let $\tilde{p} = v_1 p_2 v_2 \cdots p_{k+1} v_{k+1}$ be the standard decomposition of \tilde{p} . Then by the induction hypothesis we have

$$\begin{aligned} D_{r,s}(p) &= \begin{cases} s\tilde{p} + p_1 D_{r,s}(\tilde{p}), & p_1 = r \\ p_1 D_{r,s}(\tilde{p}), & p_1 \neq r \end{cases} \\ &= \begin{cases} v_0 s v_1 p_2 v_2 \cdots p_{k+1} v_{k+1} + v_0 p_1 \sum_{i=2}^{k+1} v_1 p_{i,2} v_2 \cdots p_{i,k+1} v_{k+1}, & p_1 = r \\ v_0 p_1 \sum_{i=2}^{k+1} v_1 p_{i,2} v_2 \cdots p_{i,k+1} v_{k+1}, & p_1 \neq r \end{cases} \\ &= \begin{cases} v_0 s v_1 p_2 v_2 \cdots p_{k+1} v_{k+1} + \sum_{i=2}^{k+1} v_0 p_1 v_1 p_{i,2} v_2 \cdots p_{i,k+1} v_{k+1}, & p_1 = r \\ \sum_{i=2}^{k+1} v_0 p_1 v_1 p_{i,2} v_2 \cdots p_{i,k+1} v_{k+1}, & p_1 \neq r \end{cases} \\ &= \sum_{i=1}^{k+1} v_0 p_{i,1} v_1 \cdots p_{i,k+1} v_{k+1}, \end{aligned}$$

where $p_{i,j}$ is defined by Eq. (26). Since this agrees with $D(p)$, the induction is completed. \square

THEOREM 3.2. *For $r \in E$ and $s \parallel r$, the linear operator $D_{r,s} : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ defined by Eq. (23) and Eq. (24) is a derivation.*

Proof. By Lemma 2.2, we only need to verify

$$(27) \quad D_{r,s}(pq) = D_{r,s}(p)q + pD_{r,s}(q), \quad \forall p, q \in \mathcal{P}.$$

We will prove this by induction on $\ell(p)$.

Let $\ell(p) = 0$. Then $pq = q$ if $p = t(q)$ and $pq = 0$ if $p \neq t(q)$. First consider the case when $D_{r,s}(q) = 0$. Then both sides of Eq. (27) are zero. So we are done. Next consider the case when $D_{r,s}(q) \neq 0$. Then we have $t(D_{r,s}(q)) = t(q)$ by the definition of $D_{r,s}(q)$. Thus if $t(q) \neq p$, then both sides of Eq. (27) are zero and we are done again. If $t(q) = p$, then both sides of Eq. (27) equal to $D_{r,s}(q)$, as needed.

Next assume that Eq. (27) has been proved for $p \in \mathcal{P}$ with $\ell(p) = n \geq 0$ and consider $p \in \mathcal{P}$ with $\ell(p) = n + 1$. Then we can write $p = p_1 \tilde{p}$ and obtain

$$\begin{aligned} D_{r,s}(pq) &= D_{r,s}(p_1 \tilde{p} q) \\ &= \begin{cases} s\tilde{p}q + p_1 D_{r,s}(\tilde{p}q), & p_1 = r, \\ p_1 D_{r,s}(\tilde{p}q), & p_1 \neq r. \end{cases} \quad (\text{by Eq. (24)}) \\ &= \begin{cases} s\tilde{p}q + p_1 (D_{r,s}(\tilde{p})q + \tilde{p}D_{r,s}(q)), & p_1 = r, \\ p_1 (D_{r,s}(\tilde{p})q + \tilde{p}D_{r,s}(q)), & p_1 \neq r. \end{cases} \quad (\text{by induction hypothesis}) \\ &= \begin{cases} D_{r,s}(p)q + pD_{r,s}(q), & p_1 = r, \\ D_{r,s}(p)q + pD_{r,s}(q), & p_1 \neq r. \end{cases} \quad (\text{by Eq. (24)}) \end{aligned}$$

This completes the induction. \square

As an immediate consequence of Theorem 2.3 and Theorem 3.2, we prove the following existence theorem of derivations on path algebras. Note that the zero map on any algebra is a derivation.

COROLLARY 3.3. *There is a nonzero derivation on the path algebra $\mathbf{k}\Gamma$ of a quiver Γ if and only if Γ is a non-trivial quiver, that is, Γ has at least one arrow. Equivalently, $\text{Der}(\mathbf{k}\Gamma)$ is a non-zero Lie algebra if and only if Γ is a non-trivial quiver.*

Proof. Suppose Γ contains only vertices. Let $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ be a derivation. Then by Eq. (5), we have $D(v) = \sum_{q \in \mathcal{P}_A, t(q)=v \text{ or } h(q)=v} c_q^v q$. Since $\mathcal{P}_A = \emptyset$ in this case, $D(v) = 0$ for all $v \in V$. Since V is a basis of $\mathbf{k}\Gamma$, D is the zero map.

Conversely, suppose Γ contains an arrow p_0 . Then we have the derivation D_{p_0, p_0} by Theorem 3.2. Since $D_{p_0, p_0}(p_0) = p_0$ is nonzero, we have obtained a non-zero derivation on $\mathbf{k}\Gamma$. \square

3.2. A canonical basis of $\text{Der}(\mathbf{k}\Gamma)$. We now display a canonical basis of the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$ for a quiver Γ . For a given $s \in \mathcal{P}$, we have the inner derivation

$$D_s : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma, \quad D_s(q) = sq - qs, \quad \forall q \in \mathcal{P}.$$

THEOREM 3.4. *Let Γ be a quiver. A basis of the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$ is given by the set*

$$\mathfrak{B} := \mathfrak{B}_1 \cup \mathfrak{B}_2$$

where

$$(28) \quad \mathfrak{B}_1 := \{D_s \mid s \in \mathcal{P}_A\} \quad \text{and} \quad \mathfrak{B}_2 := \{D_{r,s} \mid r \in E, s \parallel r\}.$$

Thus $\text{Der}(\mathbf{k}\Gamma) = \mathfrak{D}_1 \oplus \mathfrak{D}_2$ where \mathfrak{D}_i are the \mathbf{k} -linear space with bases \mathfrak{B}_i for $i = 1, 2$.

We will call \mathfrak{B} the **canonical basis** of $\text{Der}(\mathbf{k}\Gamma)$.

Proof. Since the operators in \mathfrak{B} are derivations by Theorem 3.2, we only need to verify that the operators in \mathfrak{B} are linearly independent and that \mathfrak{B} spans the whole space of derivations.

Step 1. \mathfrak{B} is linearly independent. Suppose there are $c_s, c_{r,s} \in \mathbf{k}$ such that

$$D := \sum_{s \in \mathcal{P}_A} c_s D_s + \sum_{r \in E, s \parallel r} c_{r,s} D_{r,s} = 0.$$

Then for any given $s_0 \in \mathcal{P}_A$, by the definitions of D_s and $D_{r,s}$ we have

$$\begin{aligned} 0 &= D(h(s_0)) \\ &= \sum_{s \in \mathcal{P}_A} c_s D_s(h(s_0)) \\ &= \sum_{s \in \mathcal{P}_A} c_s (s h(s_0) - h(s_0) s) \\ &= \sum_{s \in \mathcal{P}_A, h(s)=h(s_0)} c_s s - \sum_{s \in \mathcal{P}_A, t(s)=h(s_0)} c_s s. \end{aligned}$$

Since $h(s) \neq t(s)$ in the sums, the index sets of the two sums are disjoint. Thus both of the sums equal to zero and hence $c_s = 0$ for all $s \in \mathcal{P}_A$ with $h(s) = h(s_0)$. In particular, $c_{s_0} = 0$. Thus,

$$D = \sum_{r \in E, s \parallel r} c_{r,s} D_{r,s}.$$

Further, for any given $r_0 \in E$ and $s_0 \parallel r_0$, by the definitions of D_s and $D_{r,s}$ we have $0 = D(r_0) = \sum_{s_0 \neq s \in \mathcal{P}, s \parallel r_0} c_{r_0,s} s + c_{r_0,s_0} s_0$, and hence $c_{r_0,s_0} = 0$.

Thus we have proved that \mathfrak{B} is linearly independent.

Step 2. \mathfrak{B} is a spanning set of derivations on $\mathbf{k}\Gamma$. Let $D : \mathbf{k}\Gamma \rightarrow \mathbf{k}\Gamma$ be a given derivation. Then D is defined by Eq. (21) and Eq. (22) in Corollary 2.5. In particular, for $p \in E$, by Eq. (22) we have

$$(29) \quad D(p) = \sum_{q \in \mathcal{P}_A, t(q)=h(p)} c_q^{t(q)} pq + \sum_{q \parallel p} c_q^p q - \sum_{q \in \mathcal{P}_A, h(q)=t(p)} c_q^{t(q)} qp$$

for certain coefficients $c_q^{t(q)} \in \mathbf{k}$ where $q \in \mathcal{P}_A$ with $t(q) = h(p)$ and $c_q^p \in \mathbf{k}$ where $q \parallel p$.

We claim that D agrees with the operator \bar{D} defined by the linear combination

$$\bar{D} = - \sum_{s \in \mathcal{P}_A} c_s^{t(s)} D_s + \sum_{r \in E, s \in \mathcal{P}, h(s)=h(r), t(s)=t(r)} c_s^r D_{r,s},$$

obtained by the same coefficients appeared in Eq. (29). As a linear combination of derivations, \bar{D} is also a derivation. Any path in \mathcal{P} is either a vertex or a product of arrows. Thus by the product rule of derivations, to show the equality of D and \bar{D} , we only need to verify that $D(q) = \bar{D}(q)$ for each $q = v \in V$ and $q = p \in E$.

First let $q = v \in V$. Since $D_{r,s}(v) = 0$, we have

$$\bar{D}(v) = - \sum_{s \in \mathcal{P}_A} c_s^{t(s)} D_s(v) = - \sum_{s \in \mathcal{P}_A} c_s^{t(s)} (sv - vs) = - \sum_{s \in \mathcal{P}_A, h(s)=v} c_s^{t(s)} sv + \sum_{s \in \mathcal{P}_A, t(s)=v} c_s^{t(s)} vs.$$

Here the last equality follows since, in the first sum, sv is s if $h(s) = v$ and is zero otherwise, and in the second sum, vs is s if $t(s) = v$ and is zero otherwise. Thus $\bar{D}(v)$ agrees with $D(v)$ as defined in Eq. (21).

Next let $q = p \in E$. Then $D_{r,s}(p)$ is s if $r = p$ and is 0 otherwise. Thus we have

$$\begin{aligned} \bar{D}(p) &= - \sum_{s \in \mathcal{P}_A} c_s^{t(s)} D_s(p) + \sum_{r \in E, s \parallel r} c_s^r D_{r,s}(p) \\ &= - \sum_{s \in \mathcal{P}_A} c_s^{t(s)} (-ps + sp) + \sum_{s \parallel p} c_s^p s \\ &= \sum_{s \in \mathcal{P}_A, t(s)=h(p)} c_s^{t(s)} ps - \sum_{s \in \mathcal{P}_A, h(s)=t(p)} c_s^{t(s)} sp + \sum_{s \parallel p} c_s^p s. \end{aligned}$$

This agrees with $D(p)$ in Eq. (29). Thus $\bar{D} = D$, showing that \mathfrak{B} spans $\text{Der}(\mathbf{k}\Gamma)$.

The proof of Theorem 3.4 is completed. \square

3.3. A Structure theorem of $\text{Der}(\mathbf{k}\Gamma)$. Let $p \in E$ and let $\sum_{i=1}^k c_i q_i \in \mathbf{k}\Gamma$ where $c_i \in \mathbf{k}$ and $q_i \in \mathcal{P}$ with $q_i \parallel p$. We denote

$$(30) \quad D_{p, \sum_{i=1}^k c_i q_i} = \sum_{i=1, q_i \parallel p}^k c_i D_{p, q_i}.$$

THEOREM 3.5. (Basis Theorem of $\text{Der}(\mathbf{k}\Gamma)$) For derivations in $\text{Der}(\mathbf{k}\Gamma)$, the following relations hold.

$$(31) \quad [D_p, D_r] = D_{[p,r]}, \quad \text{for } p, r \in \mathcal{P},$$

$$(32) \quad [D_p, D_{r,s}] = -D_{D_{r,s}(p)}, \quad \text{for } r \in E, p, s \in \mathcal{P}, s \parallel r,$$

$$(33) \quad [D_{r,s}, D_{p,q}] = D_{p, D_{r,s}(q)} - D_{r, D_{p,q}(s)}, \quad \text{for } r, p \in E, s \parallel r, q \parallel p.$$

Proof. Eq. (31) follows from the fact that the map in Eq. (2) is a Lie algebra homomorphism.

Eq. (32) follows from a direct computation using the definitions of D_p and $D_{r,s}$.

Finally let $r, p \in E$ and $s \parallel r$ and $q \parallel p$. Since both sides of Eq. (33) are derivations, by the product rule of derivations, we only need to prove that, for $t \in E$, the following holds

$$(34) \quad [D_{r,s}, D_{p,q}](t) = D_{p, D_{r,s}(q)}(t) - D_{r, D_{p,q}(s)}(t).$$

Let such a t be given. If $t \neq r, p$, then both sides of Eq. (34) are zero. If $t = p$, then both sides of Eq. (34) equal to $D_{r,s}(q) - D_{p,q}(s)$ if $t = r$ and equal to $D_{r,s}(q)$ if $t \neq r$. If $t = r$, then both sides of Eq. (34) equal to $D_{r,s}(q) - D_{p,q}(s)$ if $t = p$ and equal to $-D_{p,q}(s)$ if $t \neq p$. This proves Eq. (33). \square

Note that we usually do not require D_p or D_r to be in \mathfrak{B}_1 . So $h(p) = t(p)$ or $h(r) = t(r)$ are allowed. In fact, even when D_p and D_r are in \mathfrak{B}_1 , $D_{[r,p]} = D_{rp} - D_{pr}$ might not be in the linear space \mathfrak{D}_1 spanned by \mathfrak{B}_1 . For example, if r is a path from a vertex v_1 to another vertex $v_2 \neq v_1$ and p is a path from v_2 to v_1 , then D_r and D_p are in \mathfrak{B}_1 . But pr and rp are both oriented cycles, so D_{pr} and D_{rp} are not in \mathfrak{B}_1 .

3.4. Ideals and nilpotency. We next apply Theorem 3.5 to study Lie algebra properties of $\text{Der}(\mathbf{k}\Gamma)$.

PROPOSITION 3.6. *For any non-trivial quiver Γ , the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$ is not nilpotent.*

Proof. By the well-known Engel theorem [12], a Lie algebra \mathfrak{g} is nilpotent if and only if for all its elements g , the adjoint derivation

$$\mathbf{ad} g : \mathfrak{g} \rightarrow \mathfrak{g}, \quad h \mapsto [g, h], \quad h \in \mathfrak{g},$$

is nilpotent. Let p be an arrow in Γ . Then we have $\mathbf{ad} D_{p,p}(D_p) = D_p$ and thus for any natural number n , $(\mathbf{ad} D_{p,p})^n(D_p) = D_p$. So $\mathbf{ad} D_{p,p}$ is not nilpotent. \square

THEOREM 3.7. *Let Γ be an acyclic quiver. Let $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}$ and $\mathfrak{D}_1 = \mathbf{k}\mathfrak{B}_1$, $\mathfrak{D}_2 = \mathbf{k}\mathfrak{B}_2$ be defined in Theorem 3.4. Then,*

- (a) \mathfrak{D}_1 (resp. \mathfrak{D}_2) is an ideal (resp. subalgebra) of the Lie algebra $\text{Der}(\mathbf{k}\Gamma)$;
- (b) **(Structure theorem of $\text{Der}(\mathbf{k}\Gamma)$)** $\text{Der}(\mathbf{k}\Gamma)$ is a semi-direct sum of the Lie ideal \mathfrak{D}_1 and the Lie subalgebra \mathfrak{D}_2 , that is,

$$\text{Der}(\mathbf{k}\Gamma) = \mathfrak{D}_1 \ltimes \mathfrak{D}_2;$$

- (c) *If Γ is also a finite quiver, then \mathfrak{D}_1 is a nilpotent Lie algebra.*

Proof. (a). Since Γ does not contain any oriented cycles, for any $p, r \in \mathcal{P}_A$, we have $rp \in \mathcal{P}_A$ unless $rp = 0$ and $pr \in \mathcal{P}_A$ unless $pr = 0$. Thus by Eq. (31), \mathfrak{D}_1 is closed under the Lie bracket. Further, for $p \in \mathcal{P}_A$ and $r \in E, s \parallel r$, by the acyclicity of r and the definition of $D_{r,s}$ in Eq. (24), $D_{r,s}(p) = \sum_i c_i q_i$ for $q_i \in \mathcal{P} \setminus V$. Since $\mathcal{P} \setminus V = \mathcal{P}_A$ by assumption, we see that $D_{D_{r,s}(p)} = \sum_i c_i D_{q_i}$ is in \mathfrak{D}_1 . Thus $[D_{r,s}, D_p]$ is in \mathfrak{D}_1 by Eq. (32). Since $\text{Der}(\mathbf{k}\Gamma) = \mathfrak{D}_1 \oplus \mathfrak{D}_2$ by Theorem 3.4, this proves that \mathfrak{D}_1 is an ideal of $\text{Der}(\mathbf{k}\Gamma)$.

By Eq. (33), \mathfrak{D}_2 is a Lie subalgebra of $\text{Der}(\mathbf{k}\Gamma)$.

(b). This follows from Item (a) and Theorem 3.4.

(c). We first note that the minimal length of $\mathcal{P}' := \mathcal{P} \setminus V$ is one. We then note that, for

$$\mathcal{P}^{(2)} := [\mathcal{P}', \mathcal{P}'] := \{[p, q] = pq - qp \mid p, q \in \mathcal{P}'\},$$

the minimal length is two unless $[\mathcal{P}', \mathcal{P}'] = 0$. Let $\mathfrak{g} := \mathbf{k}(\mathcal{P} \setminus V)$. Since $\mathcal{P}^{(2)} \subseteq \mathfrak{g}$, \mathfrak{g} is a Lie subalgebra of $\mathbf{k}\Gamma$. By an inductive argument, we see that, for the recursively defined $\mathcal{P}^{(n+1)} := [\mathcal{P}', \mathcal{P}^{(n)}]$, its minimal length is $n + 1$ unless $\mathcal{P}^{(n+1)} = 0$.

On the other hand, by our acyclicity and finiteness assumptions on Γ , the lengths of paths in Γ is bounded by $|V|$: suppose there is a path p of length $|V| + 1$ with its standard decomposition $p = v_0 p_1 v_1 \cdots v_{|V|} p_{|V|+1} v_{|V|+1}$. Then there are $0 \leq i < j \leq |V| + 1$ such that $v_i = v_j$. This shows that p contains an oriented cycle, contradicting the acyclicity assumption.

Combining the above two points, we see that $\mathcal{P}^{(n)} = 0$ for large enough n . Thus \mathfrak{g} is nilpotent.

Under the acyclic assumption, we have $\mathcal{P} \setminus V = \mathcal{P}_A$. So the Lie algebra homomorphism $\mathfrak{D} : \mathbf{k}\Gamma \rightarrow \text{Der}(\mathbf{k}\Gamma)$ from Eq. (2) sends the above Lie algebra \mathfrak{g} to $\mathfrak{D}_A = \mathbf{k}\mathfrak{B}_1$ surjectively. Thus \mathfrak{D}_A is nilpotent. \square

We note that when the restriction that Γ is acyclic is removed, the first statement of Theorem 3.7 is no longer true. This is because in Eq. (31): $[D_p, D_r] = D_{[p,r]}$, the right hand side might not be in \mathfrak{B}_1 even if D_p and D_r are. See the remark and example after Theorem 3.5.

3.5. Inner derivations and the canonical basis. We now express inner derivations in $\text{InDer} \mathbf{k}\Gamma$ in terms of the canonical basis.

PROPOSITION 3.8. *Let $q \in \mathcal{P}$ be such that $h(q) = t(q)$. Let $v_0 = h(q)$. We have*

$$(35) \quad D_q = \sum_{p \in E, t(p)=v_0} D_{p,qp} - \sum_{r \in E, h(r)=v_0} D_{r,rq}.$$

Proof. Note that both sides of the equation are derivations and $\mathbf{k}\Gamma$ is generated by $V \cup E$ as a \mathbf{k} -algebra. So by the product formula of derivations, we only need to verify that the two sides agree when acting on V and E . This follows from a routine computation. \square

4. Combinatorial derivations and their relations. In this section, we study combinatorial derivations on a path algebra $\mathbf{k}\Gamma$, namely derivations from the combinatorial objects of vertices, edges and faces of Γ . We define various relation matrices, study their ranks and obtain dimensional formulas of these derivations. These dimensional formula will be applied in the next section to give a strengthened form of Euler’s polyhedron formula and to determine the structure of the Lie algebra $HH^1(\mathbf{k}\Gamma)$ in the context of combinatorics and topology.

4.1. Combinatorial derivations and their relation matrices. We will consider a **quiver Γ of genus g** which is defined to be a quiver together with a fixed embedding of Γ into an oriented surface S of genus g such that g is the smallest. Such a quiver is called a **topological quiver**. Note that in this paper only oriented

surfaces are considered since we will use Eulers Polyhedron Theorem (Theorem 5.1) which is about a quiver embedded into an oriented surface.

A quiver Γ is called **connected** if the underlying set of Γ is connected. The set F of **faces** of Γ is the set of connected components of $S \setminus \Gamma$, or more precisely the complement of the underlying set of Γ in S .

When the genus g is zero, we can take the oriented surface S to be the Riemann sphere, through the stereographic projection from the Riemann sphere to \mathbb{R}^2 .

4.1.1. Combinatorial derivations from a quiver. We put together various combinatorially defined derivations on the path algebra of a quiver $\Gamma = (V, E)$. We recall the following notations.

- (a) For $v \in V$, we call D_v a **vertex derivation** and let \mathfrak{D}_V denote the linear space spanned by $\{D_v \mid v \in V\}$, called the **space of vertex derivations**.
- (b) For $p \in E$, we call $D_{p,p}$ an **edge derivation** and let \mathfrak{D}_E denote the linear space spanned by $\{D_{p,p} \mid p \in E\}$, called the **space of edge derivations**.

For a face $f \in F$, let \mathfrak{c}_f be the boundary of f . It is an unoriented cycle of Γ consisting of arrows that are not necessarily in one direction, called a **primitive cycle** of Γ . Thus the set $C := C_\Gamma$ of primitive cycles of Γ is in bijection with the set F of faces of Γ . For a primitive cycle $\mathfrak{c} = \mathfrak{c}_f$ of a face f , we will define a **face derivation** $D_{\mathfrak{c}} := D_{\mathfrak{c}_f}$ on $\mathbf{k}\Gamma$. First define

$$D_{\mathfrak{c}}(v) = 0, v \in V; \quad D_{\mathfrak{c}}(p) = \begin{cases} p, & p \text{ is clockwise on } \mathfrak{c}, \\ -p, & p \text{ is counterclockwise on } \mathfrak{c}, \quad p \in E. \\ 0, & p \text{ is not on } \mathfrak{c}. \end{cases}$$

Here being clockwise or counterclockwise is viewed from inside the face for the primitive cycle. We then expand $D_{\mathfrak{c}}$ to $\mathbf{k}\Gamma$ by the product rule, noting that $\mathbf{k}\Gamma$ is the algebra generated by $V \cup E$.

If a cycle is shared by two faces, such as in the quiver of an oriented loop, there will be two face derivations from the two faces. Also, if p is an edge in the interior of \mathfrak{c} , then $D_{\mathfrak{c}}(p) = 0$. The name face derivation is justified by the following alternative description of $D_{\mathfrak{c}}$.

LEMMA 4.1. *Suppose a primitive cycle $\mathfrak{c} \in C_\Gamma$ is comprised of an ordered list of arrows $p_1, \dots, p_s \in E$. Then*

$$(36) \quad D_{\mathfrak{c}} = \pm D_{p_1,p_1} \pm \dots \pm D_{p_s,p_s},$$

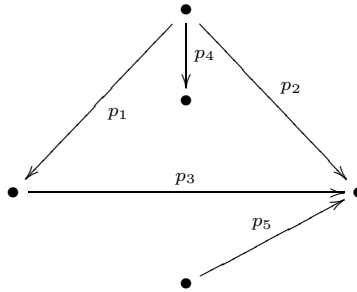
where a $\pm D_{p_i,p_i}$ is D_{p_i,p_i} if p_i is in clockwise direction when viewed from the interior of the face of \mathfrak{c} and is $-D_{p_i,p_i}$ otherwise. In particular, $D_{\mathfrak{c}}$ is a derivation.

Proof. We only need to check that the two operators agree on $V \cup E$. But this is clear from the definition of $D_{\mathfrak{c}}$. \square

Let \mathfrak{D}_F denote the linear span of $\{D_{\mathfrak{c}_f} \mid f \in F\}$, called the **space of face derivations**.

Since the concept of a face derivation will be important for the rest of the paper, we make the following remarks and illustrate their contents by the following quiver Γ .

(37)



The quiver Γ has two primitive cycles: the cycle \mathfrak{c}_1 of the finite face of Γ and the cycle \mathfrak{c}_0 of the infinite face of Γ .

REMARK 4.2.

- (a) In the case a quiver Γ is planar, for the boundary \mathfrak{c}_0 of the infinite face, an arrow on \mathfrak{c}_0 is in clockwise direction when viewed from the interior of the infinite face means that the arrow is in counterclockwise direction when viewed from the interior of the quiver. For the quiver in the diagram (37), the arrow p_2 is clockwise for the primitive cycle from the finite face, but is counterclockwise for the primitive cycle from the infinite face.
- (b) If an arrow p is on the boundary of two faces of Γ , then p will be in the clockwise direction on one boundary and in the counterclockwise direction on the other. Thus $D_{p,p}$ will have a plus sign in $D_{\mathfrak{c}}$ for the primitive cycle \mathfrak{c} of one boundary and will have a minus sign in $D_{\mathfrak{c}}$ for the other. For our example of Γ , D_{p_1,p_1} has a minus sign in \mathfrak{c}_1 and a plus sign in \mathfrak{c}_0 .
- (c) If an arrow p is not on the boundary of two faces of Γ , then both sides of the arrow are in the same face of Γ . In other words, p will appear twice in $D_{\mathfrak{c}}$ for the primitive cycle \mathfrak{c} of this face, once in the clockwise direction, once in the counterclockwise direct. As a result, $D_{p,p}$ will appear exactly twice in $D_{\mathfrak{c}}$, once with a positive sign and once with a negative sign. Consequently, there will be no contribution of $D_{p,p}$ in $D_{\mathfrak{c}}$. For our example of Γ , both sides of p_4 are in the finite face of Γ . The primitive cycle \mathfrak{c}_1 of this face gives

$$D_{\mathfrak{c}_1} = -D_{p_1,p_1} + D_{p_4,p_4} - D_{p_4,p_4} + D_{p_2,p_2} - D_{p_3,p_3} = -D_{p_1,p_1} + D_{p_2,p_2} - D_{p_3,p_3}.$$

So D_{p_4,p_4} does not contribute to $D_{\mathfrak{c}_1}$. Likewise, D_{p_5,p_5} does not contribute to $D_{\mathfrak{c}_0}$ from the infinite face.

- (d) The previous remark applies in particular when a quiver contains only a unique primitive cycle \mathfrak{c}_0 (in the case the quiver is planar, the unique primitive cycle is just the boundary of the infinite face, which is not proper). This is because for such a quiver, no arrow can appear on the boundary of two faces. Thus for such a quiver, we have $D_{\mathfrak{c}_0} = 0$.
- (e) By Remark 4.2.(b) and (c), each arrow $p \in E$ will appear in exactly one $D_{\mathfrak{c}}$ with a plus sign and in exactly another (or the same) $D_{\mathfrak{c}}$ with a minus sign. Consequently, we have

$$(38) \quad \sum_{i=1}^{\gamma_2} D_{\mathfrak{c}_i} = 0.$$

- (f) For an oriented cycle q of Γ , let \mathbf{c} be the corresponding primitive cycle. Then the $D_{\mathbf{c}}$ defined here is different with D_q given in Eq. (35).

4.1.2. Relation matrices. Denote $\gamma_0 = |V|$, $\gamma_1 = |E|$, $\gamma_2 = |F| = |C|$. They are all finite since Γ is finite. With these notations, we will use the following enumerations of sets.

$$(39) \quad V = \{v_i \mid 1 \leq i \leq \gamma_0\}, \quad E = \{p_k \mid 1 \leq k \leq \gamma_1\}, \quad C = \{\mathbf{c}_j \mid 1 \leq j \leq \gamma_2\}.$$

When $\gamma_2 = 1$, there are only a face and a primitive cycle in Γ . In this case, the underlying non-oriented graph of Γ is a tree.

By Eq. (35) and Eq. (36), we have the following system of linear relations.

$$(40) \quad D_{v_i} = \sum_{k=1}^{\gamma_1} c_{i,k} D_{p_k,p_k} = \sum_{p \in E, t(p)=v_i} D_{p,p} - \sum_{r \in E, h(r)=v_i} D_{r,r}, \quad 1 \leq i \leq \gamma_0,$$

$$(41) \quad D_{\mathbf{c}_j} = \sum_{k=1}^{\gamma_1} c_{\gamma_0+1+j,k} D_{p_k,p_k} = \sum_{k=1}^{\gamma_1} d_{j,k} D_{p_k,p_k}, \quad 1 \leq j \leq \gamma_2.$$

where all $d_{j,k} = \pm 1$, or $= 0$. Thus by Lemma 4.1, we find that \mathfrak{D}_F is a subspace of \mathfrak{D}_E .

DEFINITION 4.3.

- (a) Define the **(differential) vertex-arrow matrix** C_{va} of Γ to be the coefficient matrix of the linear system in Eq. (40).
- (b) Define the **(differential) cycle-arrow matrix** C_{ca} of Γ to be the coefficient matrix of the linear system in Eq. (41).
- (c) Define the **connection matrix** of Γ to be the coefficient matrix C_{Γ} of the combined linear system in Eq. (40) and Eq. (41), that is, the $(\gamma_0 + \gamma_2) \times \gamma_1$ -matrix $\begin{bmatrix} C_{va} \\ C_{ca} \end{bmatrix}$.
- (d) Define the **boundary matrix** of a quiver Γ to be the $\gamma_2 \times \gamma_2$ matrix $B_{\Gamma} = [e_{j,r}]_{0 \leq j,r \leq \gamma_2-1}$ in which $e_{j,j}$ is the number of arrows on \mathbf{c}_j that are also on \mathbf{c}_r for some $r \neq j$, and $-e_{j,r}$ for $r \neq j$ is the number of arrows on \mathbf{c}_j that are also on \mathbf{c}_r .

REMARK 4.4.

- (a) The matrix C_{va} encodes the relationship between the vertex derivations D_{v_i} , $1 \leq i \leq \gamma_0$ and the edge derivations D_{p_j,p_j} , $1 \leq j \leq \gamma_1$. Since the edge derivations D_{p_k,p_k} , $1 \leq k \leq \gamma_1$, are linearly independent the rank of the row space of C_{va} is $\dim \mathfrak{D}_V$.
- (b) The matrix C_{ca} encodes the relationship between the primitive cycle derivations $D_{\mathbf{c}_k}$, $0 \leq k \leq \gamma_2 - 1$ and the arrow derivations D_{p_j,p_j} , $1 \leq j \leq \gamma_1$. The rank of the row space of C_{ca} is just $\dim \mathfrak{D}_F$.
- (c) By the definition of $e_{j,j}$ and $e_{j,r}$ for $r \neq j$, the matrix B_{Γ} is independent of the direction of the quiver Γ and only depends on the underlying graph.

In order to study the rank of the boundary matrix, we recall preparatory concepts and results of matrices.

DEFINITION 4.5.

- (a) A square matrix M over a number field is called **irreducible** if M cannot be written as a block matrix $M = \begin{bmatrix} M_1 & O \\ M_3 & M_2 \end{bmatrix}$ where O is a zero matrix and M_1 and M_2 are both square matrices.
- (b) An $n \times n$ -matrix $M = [m_{i,j}]$ is called **weakly diagonally dominant** if $|m_{i,i}| \geq \sum_{j=1, j \neq i}^n |m_{i,j}|$ for $i = 1, \dots, n$, with strict inequality for at least one i .

THEOREM 4.6. ([14, Section 10.7]) *Let $M = [m_{ij}]$ be an $n \times n$ irreducible matrix over a number field. If M is weakly diagonally dominant, then M is invertible.*

Using of this fact, we prove the following

LEMMA 4.7. *Let $M = [m_{ij}]$ be an $n \times n$ irreducible matrix with entries in a number field. If, for each $i = 1, \dots, n$, $\sum_{j=1}^n m_{i,j} = 0$, $m_{i,i} > 0$ and $m_{i,j} \leq 0$ for $j \neq i$, then the rank of M is $n - 1$.*

Proof. Since $\sum_{j=1}^n m_{i,j} = 0$ for each $i = 1, \dots, n$, the sum of the n column vectors of M is zero. Thus the rank of M is less than or equal to $n - 1$.

On the other hand, consider the $(n - 1) \times (n - 1)$ -submatrix M_1 consisting of the first $n - 1$ rows and columns of M , that is $M_1 = [m_{i,j}]_{1 \leq i,j \leq n-1}$. Since $\sum_{j=1}^n m_{i,j} = 0$ for each $i = 1, \dots, n - 1$, we have

$$m_{i,i} = - \sum_{j=1, j \neq i}^n m_{i,j} = - \sum_{j=1, j \neq i}^{n-1} m_{i,j} - m_{i,n} \geq - \sum_{j=1, j \neq i}^{n-1} m_{i,j} = \sum_{j=1, j \neq i}^{n-1} |m_{i,j}|.$$

Since M is irreducible and symmetric, there is at least one $m_{i_0,n} \neq 0$ in the last column of M other than $m_{n,n}$. Otherwise we would have $M = \begin{bmatrix} M_1 & \vec{0} \\ M_0 & m_{n,n} \end{bmatrix}$ for the zero vector $\vec{0}$, contradicting the irreducibility condition on M . Then, since $m_{i,j} \leq 0$ for $i \neq j$ by assumption, we have $m_{i_0,n} < 0$. Therefore,

$$m_{i_0,i_0} = - \sum_{j=1, j \neq i_0}^{n-1} m_{i_0,j} - m_{i_0,n} > \sum_{j=1, j \neq i_0}^{n-1} |m_{i_0,j}|.$$

Thus, M_1 is a weakly diagonally dominant matrix. Then by Theorem 4.6, M_1 is invertible. Together with the observation made at the beginning of the proof, we conclude that $rk(M) = n - 1$. \square

4.1.3. Ranks of relation matrices.

THEOREM 4.8. *Let Γ be a connected finite quiver with no loops. Then the following statements hold:*

- (a) *The vertex-arrow matrix C_{va} has rank $\gamma_0 - 1$.*
- (b) *The cycle-arrow matrix C_{ca} has rank $\gamma_2 - 1$.*
- (c) *Suppose that the ground field \mathbf{k} has characteristic 0. Then the rank of the boundary matrix B_Γ is $\gamma_2 - 1$.*

Proof. (a) Since $e = \sum_{i=1}^{\gamma_0} v_i$ is the identity of $\mathbf{k}\Gamma$, we have

$$(42) \quad D_e = \sum_{i=1}^{\gamma_0} D_{v_i} = 0.$$

If $\gamma_0 = 1$, then $V = \{v_1\}$ and $D_{v_1} = 0$ by Eq. (42). Thus $C_{va} = 0$ and its rank is $0 = \gamma_0 - 1$.

If $\gamma_0 \geq 2$, then by Eq. (42), D_{v_1} is a linear combination of $D_{v_i}, 2 \leq i \leq \gamma_0$. Thus $\text{rk}(G) \leq \gamma_0 - 1$. So we just need to prove $\text{rk}(C_{va}) \geq \gamma_0 - 1$. We will show this by induction on $\gamma_0 \geq 2$.

First assume $\gamma_0 = 2$. Since Γ is connected, by Eq. (40) the rows of C_{va} are non-zero. Then $\text{rk}(C_{va}) \geq 1$, as needed.

Next assume that the statement holds when $\gamma_0 = n$ for $n \geq 2$ and consider a connected acyclic quiver Γ with $\gamma_0 = n + 1$. By first listing the arrows p_1, \dots, p_r of Γ that are connected to v_1 , we see that C_{va} is a block matrix of the form

$$C_{va} = \begin{bmatrix} \vec{e} & \vec{0} \\ B & \overline{G} \end{bmatrix},$$

where \vec{e} is a row vector of dimension r with entries ± 1 , $\vec{0}$ is a zero row vector of dimension $\gamma_1 - r$, B is a $(\gamma_0 \times r)$ -matrix and \overline{G} is in fact the vertex-arrow matrix $G_{\overline{\Gamma}}$ of the quiver $\overline{\Gamma}$ obtained by deleting the vertex v_1 and its attached arrows from Γ . Since $\overline{\Gamma}$ has n vertices, by the induction hypothesis, the rank of \overline{G} is at least $n - 1$. Thus there is a non-singular submatrix \overline{H} of \overline{G} of size $(n - 1) \times (n - 1)$. Adding back the first row and first column of G to this submatrix \overline{H} , we obtain a submatrix H of C_{va} of size $n \times n$. Since the added first row is $(\pm 1, 0, \dots, 0)$, the added first column is not a linear combination of the other columns in H . Thus H is non-singular and the rank of C_{va} is at least n . This completes the induction and hence the proof of Item (a).

(b) We prove by induction on γ_2 , which is also the number of rows of C_{ca} .

When $\gamma_2 = 1$, \mathbf{c}_0 is the unique primitive cycle. In this case, Γ is topologically homeomorphic to a point in the oriented surface S . By Remark 4.2(d), we have $D_{\mathbf{c}_1} = 0$. Thus, the unique row of C_{ca} is zero. So, $\text{rk}(C_{ca}) = 0 = \gamma_2 - 1$.

When $\gamma_2 = 2$, the rows of C_{ca} are non-zero by definition and the sum of the only two rows of C_{ca} is zero by Eq. (38). Hence, $\text{rk}(C_{ca}) = 1 = \gamma_2 - 1$.

Assume that the statement is verified when $\gamma_2 = n \geq 2$ and consider Γ with $\gamma_2 = n + 1$. By reordering the arrows p_1, \dots, p_{γ_1} of Γ if necessary, we can assume that the arrows of \mathbf{c}_1 are p_1, \dots, p_s where $s \geq 2$ since Γ has not loops. Thus the coefficients $c_{\gamma_0+1,1}, c_{\gamma_0+1,2}, \dots, c_{\gamma_0+1,\gamma_1}$ of \mathbf{c}_1 in Eq. (41) satisfy that $c_{\gamma_0+1,1}, \dots, c_{\gamma_0+1,s}$ are all ± 1 and $c_{\gamma_0+1,s+1} = \dots = c_{\gamma_0+1,\gamma_1} = 0$. Then, since the sum of the row vectors of C_{ca} is zero by Eq. (38), by reordering the primitive cycles $\mathbf{c}_1, \dots, \mathbf{c}_n$ if necessary, we can also assume that the first coefficient of \mathbf{c}_1 satisfies $c_{\gamma_0+2,1} = -c_{\gamma_0+1,1}$. Thus, there is r between 1 and s such that $c_{\gamma_0+2,i} = -c_{\gamma_0+1,i}$ for $i = 1, \dots, r$ and $c_{\gamma_0+2,j} = 0$ for $j = r + 1, \dots, s$. Then the first two rows of C_{ca} are of the form

$$\begin{bmatrix} \pm 1, \dots, \pm 1, & \pm 1, \dots, \pm 1, & 0, \dots, 0 \\ \mp 1, \dots, \mp 1, & \underbrace{0, \dots, 0}_{s-r \text{ terms}}, & \underbrace{*, \dots, *}_{\gamma_1-r \text{ terms}} \end{bmatrix}.$$

Here the signs in the second row for the first r terms are opposite to the signs in the corresponding terms in the first row. Thus, the matrix C_{ca} has the form

$$C_{ca} = \begin{bmatrix} \vec{a}_1 & \vec{b}_1 \\ -\vec{a}_1 & \vec{b}_2 \\ O & \overline{K} \end{bmatrix},$$

where \vec{a}_1 is a row vector of dimension r with entries ± 1 , O is a zero matrix of size $(n - 1) \times r$, \vec{b}_1, \vec{b}_2 are row vectors of dimension $\gamma_1 - r$ and \overline{K} is a matrix of size $(n - 1) \times (\gamma_1 - r)$. We can write

$$\vec{b}_1 = (c_{\gamma_0+1,r+1} \cdots c_{\gamma_0+1,s} \ 0 \cdots 0) = (\pm 1 \cdots \pm 1 \ 0 \cdots 0)$$

$$\vec{b}_2 = (0 \cdots 0 \ c_{\gamma_0+2,s+1} \cdots c_{\gamma_0+2,\gamma_1}) = (0 \cdots 0 \ * \cdots *).$$

Deleting the arrows p_1, \dots, p_r , we get a quiver $\overline{\Gamma}$ with n primitive cycles $\mathbf{c}'_1, \mathbf{c}_3, \dots, \mathbf{c}_n$ where \mathbf{c}'_1 is obtained via amalgamating \mathbf{c}_1 and \mathbf{c}_2 . Moreover, with

$$\vec{b} := \vec{b}_1 + \vec{b}_2 = (c_{\gamma_0+1,r+1} \cdots c_{\gamma_0+1,s} \ c_{\gamma_0+2,s+1} \cdots c_{\gamma_0+2,\gamma_1}),$$

we find that the cycle-arrow matrix $C_{ca, \overline{\Gamma}}$ of the quiver $\overline{\Gamma}$ is just $\begin{bmatrix} \vec{b} \\ \overline{K} \end{bmatrix}$.

By the induction hypothesis, $\text{rk}(C_{ca, \overline{\Gamma}}) = n - 1$. By Eq. (38), $-\vec{b}$ is the sum of all rows of \overline{K} . Hence, $\text{rk}(\overline{K}) = n - 1$, that is, \overline{K} has full row rank. Since $r \geq 1$, the second row $(-\vec{e}_1 \ \vec{b}_2)$ of C_{ca} is linearly independent from the last $n - 1$ rows of C_{ca} . So, $\text{rk}(C_{ca}) \geq (n - 1) + 1 = n$. By Eq. (38), the row vectors of C_{ca} are linearly dependent. Therefore, $\text{rk}(C_{ca}) = n = \gamma_2 - 1$, completing the induction.

(c) By definition, B_Γ is a $(\gamma_2 \times \gamma_2)$ -matrix.

In the special case of $\gamma_2 = 1$, there are no cycles on Γ except the boundary \mathbf{c}_1 for the unique face. So all the arrows are on \mathbf{c}_1 only. Thus $B_\Gamma = 0$ and hence its rank is 0.

For the case when $\gamma_2 > 1$, we apply Lemma 4.7. For this we just need to verify that B_Γ satisfies the conditions for M in the lemma as follows.

- By definition, B_Γ has entries in \mathbb{Z} and hence in a number field, say \mathbb{Q} .
- By Definition 4.3 on $e_{i,j}$, B_Γ is symmetric and $e_{i,j} \leq 0$ for $i \neq j$.
- B_Γ is irreducible since Γ is connected.
- Since $\gamma_0 > 0$, each cycle $\mathbf{c}_j, 1 \leq j \leq \gamma_2$ shares at least one arrow with the cycle of a neighboring face. Thus $e_{j,j} > 0$.
- By the definition of $e_{j,r}, 1 \leq j, r \leq \gamma_2$, the sum of the entries of each row of B_Γ is zero. Thus the sum of the columns of B_Γ is zero. Since B_Γ is symmetric, the sum of the rows of B_Γ is also zero.

Thus Item (c) is proved. \square

4.2. Dimensions of combinatorial derivations. From Theorem 4.8, we have the following dimensional formulas of combinatorial derivations.

THEOREM 4.9. *Let Γ be a connected finite acyclic quiver. Then,*

- (a) $\dim \mathfrak{D}_V = |V| - 1$;
- (b) $\dim \mathfrak{D}_F = |F| - 1$;
- (c) \mathfrak{D}_V and \mathfrak{D}_F are linearly disjoint subspaces of \mathfrak{D}_E .

The following is a direct consequence of Theorem 4.9. More applications of the theorem will be given in Section 5.

COROLLARY 4.10. *The dimensions of the spaces of derivations $\mathfrak{D}_V, \mathfrak{D}_E$ and \mathfrak{D}_F of a connected acyclic quiver Γ only depend on the underlying graph of Γ and not depend on the choice of orientations of the edges.*

Proof. Item (a) follow directly from of Theorem 4.8.(a) by Eq. (40). Similarly, Item (b) follows directly from Theorem 4.8.(b) by Eq. (41). So we just need to prove Item (c).

Let D be in $\mathfrak{D}_V \cap \mathfrak{D}_F$. Then there are constants $a_i, 1 \leq i \leq \gamma_0$, and $b_j, 1 \leq j \leq \gamma_2$, in \mathbf{k} such that

$$(43) \quad \sum_{i=1}^{\gamma_0} a_i D_{v_i} = D = \sum_{j=1}^{\gamma_2} b_j D_{c_j}.$$

To prove Proposition 4.9.(c), we just need to show $D = 0$.

We first rewrite the left sum in Eq. (43). By Eq. (40), we obtain

$$\sum_{i=1}^{\gamma_0} a_i D_{v_i} = \sum_{i=1}^{\gamma_0} a_i \sum_{k=1}^{\gamma_1} c_{i,k} D_{p_k, p_k} = \sum_{k=1}^{\gamma_1} \left(\sum_{i=1}^{\gamma_0} c_{i,k} a_i \right) D_{p_k, p_k}.$$

Since by definition, $c_{i,k} = \begin{cases} 1, & t(p_k) = v_i, \\ -1, & h(p_k) = v_i, \\ 0, & \text{otherwise,} \end{cases}$ we obtain

$$(44) \quad \sum_{i=1}^{\gamma_0} a_i D_{v_i} = \sum_{k=1}^{\gamma_1} (a_{t(p_k)} - a_{h(p_k)}) D_{p_k, p_k}.$$

On the other hand, by Eq. (41), the right sum of Eq. (43) can be written as

$$(45) \quad \sum_{j=1}^{\gamma_2} b_j D_{c_j} = \sum_{j=1}^{\gamma_2} b_j \left(\sum_{k=1}^{\gamma_1} d_{j,k} D_{p_k, p_k} \right) = \sum_{k=1}^{\gamma_1} \left(\sum_{j=1}^{\gamma_2} b_j d_{j,k} \right) D_{p_k, p_k}.$$

Here by Eq. (36), $d_{j,k} = \begin{cases} 1, & \text{if } p_k \text{ is on } c_j \text{ in clockwise direction,} \\ -1, & \text{if } p_k \text{ is on } c_j \text{ in counter clockwise direction,} \\ 0, & \text{if } p_k \text{ is not on } c_j. \end{cases}$

By Remark 4.2.(e), any given arrow p_k appears twice on the boundaries of the cycles $c_j, 1 \leq j \leq \gamma_2$, once in clockwise direction and once in counter clockwise direction. Let $1 \leq x(p_k) \leq \gamma_2$ (resp. $1 \leq y(p_k) \leq \gamma_2$) denote the label of the cycle containing p_k in the clockwise (resp. counter clockwise) direction. It is possible that $x(p_k) = y(p_k)$. Then we have $d_{x(p_k),k} = 1$ and $d_{y(p_k),k} = -1$. Then Eq. (45) becomes

$$(46) \quad \sum_{j=1}^{\gamma_2} b_j D_{c_j} = \sum_{k=1}^{\gamma_1} (b_{x(p_k)} - b_{y(p_k)}) D_{p_k, p_k}.$$

Thus by Eq. (44) and (46), we find that Eq. (43) is equivalent to

$$(47) \quad D = \sum_{k=1}^{\gamma_1} (a_{t(p_k)} - a_{h(p_k)}) D_{p_k, p_k} = \sum_{k=1}^{\gamma_1} (b_{x(p_k)} - b_{y(p_k)}) D_{p_k, p_k}.$$

Since the set $\{D_{p_k, p_k} \mid 1 \leq k \leq \gamma_1\}$ of derivations is linearly independent, we thus obtain the following system of linear equations:

$$(48) \quad a_{t(p_k)} - a_{h(p_k)} = b_{x(p_k)} - b_{y(p_k)}, \quad 1 \leq k \leq \gamma_1.$$

LEMMA 4.11. *For $1 \leq k \leq \gamma_1$, a change of the direction of p_k leads to a change of signs on both sides of the k -th equation in Eq. (48), yielding an equivalent equation.*

Proof. Let Γ' be the quiver with p_k replaced by q_k in the opposite direction. Then $t(p_k) = h(q_k)$ and $h(p_k) = t(q_k)$. Then the left hand side of Eq. (48) becomes

$$a_{h(q_k)} - a_{t(q_k)} = -(a_{t(q_k)} - a_{h(q_k)}).$$

Likewise, by the definition of $x(p_k)$ and $y(p_k)$ we have $x(q_k) = y(p_k)$ and $y(q_k) = x(p_k)$. This reverses the sign on the right hand side of Eq. (48). \square

LEMMA 4.12. *Any set of solutions $b_j, 1 \leq j \leq \gamma_2$, in the system Eq. (48) satisfies the following system of linear equations:*

$$(49) \quad e_{j,j}b_j + \sum_{1 \leq r \leq \gamma_2, r \neq j} e_{j,r}b_r = 0, \quad 1 \leq j \leq \gamma_2,$$

where $e_{i,j}$ is defined in Definition 4.3.

Proof. Fix a $1 \leq j \leq \gamma_2$. Let p_{k_1}, \dots, p_{k_s} be the arrows on \mathfrak{c}_j that are also on \mathfrak{c}_r for some $r \neq j$, that is, these are the arrows on \mathfrak{c}_j that are not in the interior of \mathfrak{c}_j . Because of Lemma 4.11, we can change the directions of some of the arrows p_{k_1}, \dots, p_{k_s} so that all the arrows in \mathfrak{c}_j go clockwise, without changing the solution set of the system in Eq. (48). After this is done, when we add the k_ℓ -th equations

$$a_{t(p_{k_\ell})} - a_{h(p_{k_\ell})} = b_{x(p_{k_\ell})} - b_{y(p_{k_\ell})}, \quad 1 \leq \ell \leq s,$$

to the system (48), the left hand sides add up to zero. For the right hand side, we have $x(p_{k_\ell}) = j, 1 \leq \ell \leq s$, and $y(p_{k_\ell})$ is the label of the other cycle that p_{k_ℓ} is on. Note that $s = e_{j,j}$ by definition. Thus the right hand sides add up to

$$e_{j,j}b_j + \sum_{1 \leq r \leq \gamma_2, r \neq j} e_{j,r}b_r,$$

for $e_{j,j}$ and $e_{j,r}$ as defined in the lemma. This finishes the proof. \square

We can now complete the proof of Theorem 4.9.(c) as follows. Let D be in $\mathfrak{D}_V \cap \mathfrak{D}_F$. Then there are $a_i, 1 \leq i \leq \gamma_0$ and $b_j, 1 \leq j \leq \gamma_2$ such that they satisfy Eq. (43). Then they satisfy the linear system in Eq. (48). Then by Lemma 4.12, b_j satisfies the system of linear equations in Eq. (49). Since the coefficient matrix of this system is the boundary matrix B_Γ by Lemma 4.12 and hence has rank $\gamma_2 - 1$ by Theorem 4.8.(c), the system in Eq. (49) has unique nonzero solutions b_i up a constant. But the choice of $b_{x(p_k)} = b_{y(p_k)}, 1 \leq k \leq \gamma_1$ together with $a_{t(p_k)} = a_{h(p_k)}, 1 \leq i \leq \gamma_0$ is already a nonzero solution of Eq. (48) and hence of Eq. (49). Thus this gives the unique solution of Eq. (48). Hence by Eq. (47), we have $D = 0$. Therefore $\mathfrak{D}_V \cap \mathfrak{D}_F = 0$, showing that \mathfrak{D}_V and \mathfrak{D}_F are linearly disjoint.

Now the proof of Theorem 4.9(c) is completed. \square

5. Euler’s formula and Hochschild cohomology. In this section we give two applications of the dimensional formulas of combinatorial derivations in Theorem 4.9. We first present a differential enrichment of Euler’s Polyhedron Theorem (Theorem 5.1) by showing that the numerical relation in Euler’s formula among the geometric objects of vertices, edges and faces of the underlying graph of a quiver comes from an algebraic relation among the spaces of derivations associated to these geometric objects.

5.1. Euler's polyhedron formula from a differential point of view. We recall the following classical result from [4] and [8].

THEOREM 5.1. (Euler's Polyhedron Theorem) *For any connected quiver Γ of genus g , we have*

$$(50) \quad |V| - |E| + |F| = 2 - 2g.$$

Recall that the genus of a quiver Γ is defined to be the smallest genus of an oriented surface in which the quiver can be embedded. This genus is used in Euler's Polyhedron Theorem (Theorem 5.1) by the theory of topological graphs [8]. Our results in this paper, e.g. Theorem 5.2 and Proposition 5.4, are based on this Euler's theorem and will use the same genus.

We show that there is a strengthening of Euler's numerical formula in the context of derivations.

THEOREM 5.2. (Differential Formulation of Euler's Polyhedron Theorem) *For a connected finite acyclic quiver Γ of genus g , the spaces $\mathfrak{D}_V, \mathfrak{D}_E, \mathfrak{D}_F$ of vertex derivations, arrow derivations and face derivations satisfy the following relation.*

$$(51) \quad \dim_{\mathbf{k}} \mathfrak{D}_E / (\mathfrak{D}_V \oplus \mathfrak{D}_F) = 2g,$$

where the direct sum is the interior sum of subspaces.

In particular, in the case the genus $g = 0$,

$$(52) \quad \mathfrak{D}_E = \mathfrak{D}_V \oplus \mathfrak{D}_F.$$

Proof. By Theorem 4.9, we have

$$\dim(\mathfrak{D}_V \oplus \mathfrak{D}_F) = \dim \mathfrak{D}_V + \dim \mathfrak{D}_F = |V| - 1 + |F| - 1.$$

But, $\dim \mathfrak{D}_E = |E|$. By Euler's formula in Theorem 5.1,

$$(53) \quad \dim \mathfrak{D}_E - \dim(\mathfrak{D}_V \oplus \mathfrak{D}_F) = |E| - (|V| - 1 + |F| - 1) = 2g.$$

This gives Eq. (51) since $\mathfrak{D}_V \oplus \mathfrak{D}_F$ is a subspace of \mathfrak{D}_E . \square

REMARK 5.3. As we see above, Euler's theorem is used in the proof of Theorem 5.2. Conversely, the equation $\dim_{\mathbf{k}} \mathfrak{D}_E / (\mathfrak{D}_V \oplus \mathfrak{D}_F) = 2g$ gives Euler's formula by Eq. (53). Thus, as its name suggests, Theorem 5.2 gives a strengthened form of Euler's theorem from the view point of derivations. It would be interesting to find a proof of Theorem 5.2 without using Euler's theorem.

5.2. The Structure of Lie algebra $HH^1(\mathbf{k}\Gamma)$. We next apply Theorem 4.9 to study $HH^1(\mathbf{k}\Gamma)$. We first give a dimensional formula of $HH^1(\mathbf{k}\Gamma)$. Then we obtain a canonical basis of $HH^1(\mathbf{k}\Gamma)$.

5.2.1. A dimensional formula of $HH^1(\mathbf{k}\Gamma)$. Denote

$$\mathcal{P}_L := \{s \in \mathcal{P} \mid h(s) = t(s)\}, \quad \mathfrak{B}_L := \{D_s \mid s \in \mathcal{P}_L\}, \quad \mathfrak{D}_L := \mathbf{k}\mathfrak{B}_L = \mathfrak{D}(\mathbf{k}\mathcal{P}_L).$$

Then on one hand we have the disjoint union $\mathcal{P} = \mathcal{P}_A \sqcup \mathcal{P}_L$, and hence $\mathbf{k}\Gamma = \mathbf{k}\mathcal{P}_A \oplus \mathbf{k}\mathcal{P}_L$. Since $\ker \mathfrak{D} = \mathbf{k} \subseteq \mathbf{k}\mathcal{P}_L$ for the linear map $\mathfrak{D} : \mathbf{k}\Gamma \rightarrow \text{Der}(\mathbf{k}\Gamma)$ in Eq. (2), we have

$$(54) \quad \text{InDer}(\mathbf{k}\Gamma) = \mathfrak{D}(\mathbf{k}\Gamma) \cong \mathfrak{D}_1 \oplus \mathfrak{D}_L.$$

On the other hand, since $\mathfrak{D}_1 \subseteq \text{InDer}(\mathbf{k}\Gamma)$ and $\text{Der}(\mathbf{k}\Gamma) = \mathfrak{D}_1 \oplus \mathfrak{D}_2$ from Theorem 3.4, we have

$$(55) \quad \text{InDer}(\mathbf{k}\Gamma) = \text{InDer}(\mathbf{k}\Gamma) \cap (\mathfrak{D}_1 \oplus \mathfrak{D}_2) = \mathfrak{D}_1 \oplus (\text{InDer}(\mathbf{k}\Gamma) \cap \mathfrak{D}_2).$$

By Proposition 3.8, we have $\mathfrak{D}_L \subseteq \text{InDer}(\mathbf{k}\Gamma) \cap \mathfrak{D}_2$. Thus from Eq. (54) and Eq. (55) we obtain

$$\text{InDer}(\mathbf{k}\Gamma) \cap \mathfrak{D}_2 = \mathfrak{D}_L.$$

Therefore we have

$$(56) \quad \begin{aligned} HH^1(\mathbf{k}\Gamma) &= \text{Der}(\mathbf{k}\Gamma)/\text{InDer}(\mathbf{k}\Gamma) \\ &= (\text{InDer}(\mathbf{k}\Gamma) + \mathfrak{D}_2)/\text{InDer}(\mathbf{k}\Gamma) \\ &\cong \mathfrak{D}_2/(\text{InDer}(\mathbf{k}\Gamma) \cap \mathfrak{D}_2) \\ &= \mathfrak{D}_2/\mathfrak{D}_L, \end{aligned}$$

giving us the following commutative diagram of exact sequences of Lie algebras.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{InDer}(\mathbf{k}\Gamma) & \longrightarrow & \text{Der}(\mathbf{k}\Gamma) & \longrightarrow & HH^1(\mathbf{k}\Gamma) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{D}_L & \longrightarrow & \mathfrak{D}_2 & \longrightarrow & \mathfrak{D}_2/\mathfrak{D}_L \longrightarrow 0. \end{array}$$

An **almost oriented cycle** in a quiver Γ is defined to be a pair (p, r) where $p \in E$ and $r \in \mathcal{P}$ with $r \neq p$ and $r \parallel p$. It is so namely since p and r form an oriented cycle by reversing the arrow p . Let Γ_{AL} be the set of almost oriented cycles of Γ . Denote

$$(57) \quad \mathfrak{B}_E := \{D_{p,p} \mid p \in E\}, \quad \mathfrak{B}_{AL} := \{D_{p,r} \mid (p,r) \in \Gamma_{AL}\}, \quad \mathfrak{D}_{AL} = \mathbf{k}\mathfrak{B}_{AL}.$$

Then by the definition of \mathfrak{B}_2 in Eq. (28), we have the disjoint union

$$(58) \quad \mathfrak{B}_2 = \mathfrak{B}_E \sqcup \mathfrak{B}_{AL} \quad \text{and} \quad \mathfrak{D}_2 = \mathfrak{D}_E \oplus \mathfrak{D}_{AL}.$$

From this and Eq.(56), we obtain

$$(59) \quad HH^1(\mathbf{k}\Gamma) \cong \mathfrak{D}_2/\mathfrak{D}_V \cong (\mathfrak{D}_E/\mathfrak{D}_V) \oplus \mathfrak{D}_{AL}.$$

PROPOSITION 5.4. *Let Γ be a connected acyclic quiver. Then*

$$\dim_{\mathbf{k}} HH^1(\mathbf{k}\Gamma) = |F| + |\Gamma_{AL}| - 1 + 2g.$$

In the case $g = 0$, $HH^1(\mathbf{k}\Gamma) \neq 0$ if and only if Γ contains an unoriented cycle.

Proof. By Eq. (59), Theorem 4.8 and Euler's Theorem 5.1, we have

$$\dim HH^1(\mathbf{k}\Gamma) = \dim \mathfrak{D}_E - \dim \mathfrak{D}_V + \dim \mathfrak{D}_{AL} = |E| - |V| + 1 + |\Gamma_{AL}| = |F| + |\Gamma_{AL}| + 2g - 1.$$

Then the last statement follows easily. \square

Compare Proposition 5.4 with Happel's formula [11], $\dim_{\mathbf{k}} HH^1(\mathbf{k}\Gamma) = 1 - |V| + \sum_{\alpha \in E} v(\alpha)$, where $v(\alpha) = \dim_{\mathbf{k}} t(\alpha)\mathbf{k}\Gamma h(\alpha)$. The two formulas can be easily derived from each other. Our formula makes it easy to guess a canonical basis of $HH^1(\mathbf{k}\Gamma)$. Indeed, verifying that this guess actually works is the motivation behind the introduction of the combinatorial derivations in Section 4.

5.2.2. A canonical basis of $HH^1(\mathbf{k}\Gamma)$. Proposition 5.4 suggests that a canonical basis of $HH^1(\mathbf{k}\Gamma)$ can be obtained from derivations defined from the faces (through their unoriented cycles) and Γ_{AL} . We show that this is indeed the case.

Denote

$$(60) \quad \mathfrak{B}_F^- = \{D_{c_i} \mid 2 \leq i \leq \gamma_2\},$$

then by Theorem 4.9.(b) and Eq. (38), we have $\mathfrak{D}_F = \mathbf{k}\mathfrak{B}_F^-$. Our main result on outer derivations is the following

THEOREM 5.5. (Basis Theorem of the First Hochschild Cohomology) *Let Γ be a connected finite acyclic quiver over a ground field \mathbf{k} of characteristic 0. Then the disjoint union*

$$\mathfrak{B}_{AL} \sqcup \mathfrak{B}_F^- \sqcup \mathfrak{B}^*$$

forms a basis of $HH^1(\mathbf{k}\Gamma)$, where \mathfrak{B}^ is any basis of $\mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F)$.*

Proof. We have $\mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F) \cong (\mathfrak{D}_E/\mathfrak{D}_V)/((\mathfrak{D}_V \oplus \mathfrak{D}_F)/\mathfrak{D}_V)$. It follows that

$$\begin{aligned} \mathfrak{D}_E/\mathfrak{D}_V &\cong \mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F) \oplus (\mathfrak{D}_V \oplus \mathfrak{D}_F)/\mathfrak{D}_V \\ &\cong \mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F) \oplus \mathfrak{D}_F. \end{aligned}$$

Thus by Eq. (59), $HH^1(\mathbf{k}\Gamma) \cong (\mathfrak{D}_E/\mathfrak{D}_V) \oplus \mathfrak{D}_{AL} \cong \mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F) \oplus \mathfrak{D}_F \oplus \mathfrak{D}_{AL}$.

\mathfrak{D}_F has a basis \mathfrak{B}_F^- by Theorem 4.9.(b) and \mathfrak{D}_{AL} has a basis \mathfrak{B}_{AL} . Since they are disjoint by Eq. (58), we obtain Theorem 5.5 for any basis \mathfrak{B}^* of $\mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F)$. \square

We have not found a canonical way to determine \mathfrak{B}^* except when the genus is zero, see the following Corollary 5.6.

We require that \mathbf{k} is of characteristic 0 in Theorem 5.5 because its proof depends on Theorem 4.6. When the genus g is 0, Γ is a planar quiver. In this case, by Euler's theorem, $\dim \mathfrak{D}_E/(\mathfrak{D}_V \oplus \mathfrak{D}_F) = 0$. Also, we can take c_1 to be the primitive cycle for the unique unbounded face. Thus we can make the choice of \mathfrak{B}_F^- as well as \mathfrak{B}_{AL} completely canonical. Hence, we have

COROLLARY 5.6. (Basis Theorem for Planar Quiver) *Let Γ be a connected finite planar acyclic quiver and let the ground field \mathbf{k} be of characteristic 0. Then the disjoint union*

$$\mathfrak{B}_{AL} \cup \{D_c \mid c \text{ is a bounded primitive cycle}\}$$

forms a basis of $HH^1(\mathbf{k}\Gamma)$.

5.2.3. Lie algebra structure of $HH^1(\mathbf{k}\Gamma)$. From the basis $\mathfrak{B}_{AL} \cup \mathfrak{B}_F^-$ in Corollary 5.6, the structural constants of the Lie algebra $HH^1(\mathbf{k}\Gamma)$ can be computed explicitly by Eq. (33) since we have $\mathfrak{B}_{AL} \subseteq \mathfrak{B}_2$ and $\mathfrak{B}_F \subseteq \mathbf{k}\mathfrak{B}_E = \mathfrak{D}_E \subseteq \mathfrak{D}_2 = \mathbf{k}\mathfrak{B}_2$. Concretely, for $D_c, D_q \in \mathfrak{B}_F$, let

$$\begin{aligned} D_c &= a_1 D_{p_1, p_1} + a_2 D_{p_2, p_2} + \cdots + a_{\gamma_1} D_{p_{\gamma_1}, p_{\gamma_1}}, \\ D_q &= b_1 D_{p_1, p_1} + b_2 D_{p_2, p_2} + \cdots + b_{\gamma_1} D_{p_{\gamma_1}, p_{\gamma_1}} \end{aligned}$$

for the arrow set $E = \{p_1, p_2, \dots, p_{\gamma_1}\}$. Then by Eq. (33) we have

$$\begin{aligned} [D_c, D_q] &= \sum_{i,j=1}^{\gamma_1} a_i b_j [D_{p_i, p_i}, D_{p_j, p_j}] \\ &= \sum_{i,j=1}^{\gamma_1} a_i b_j (D_{p_j, D_{p_i, p_i}(p_j)} - D_{p_i, D_{p_j, p_j}(p_i)}) \\ &= \sum_{i=1}^{\gamma_1} a_i b_i (D_{p_i, p_i} - D_{p_i, p_i}) \\ &= 0. \end{aligned}$$

This means that $\mathfrak{D}_F = \mathbf{k}\mathfrak{B}_F$ is an abelian Lie sub-algebra of the Lie algebra $HH^1(\mathbf{k}\Gamma)$.

For $D_{r,s} \in \mathfrak{B}_{AL}$, by Eq. (33) we have

$$\begin{aligned} [D_{r,s}, D_c] &= a_1 [D_{r,s}, D_{p_1, p_1}] + \cdots + a_{\gamma_1} [D_{r,s}, D_{p_{\gamma_1}, p_{\gamma_1}}] \\ &= a_1 (D_{p_1, D_{r,s}(p_1)} - D_{r, D_{p_1, p_1}(s)}) + \cdots + a_{\gamma_1} (D_{p_{\gamma_1}, D_{r,s}(p_{\gamma_1})} - D_{r, D_{p_{\gamma_1}, p_{\gamma_1}}(s)}) \\ &= (a_1 D_{p_1, D_{r,s}(p_1)} + \cdots + a_{\gamma_1} D_{p_{\gamma_1}, D_{r,s}(p_{\gamma_1})}) \\ &\quad - (a_1 D_{r, D_{p_1, p_1}(s)} + \cdots + a_{\gamma_1} D_{r, D_{p_{\gamma_1}, p_{\gamma_1}}(s)}). \end{aligned}$$

Write $r = p_{i_0}$ for some $1 \leq i_0 \leq \gamma_1$. Then $a_1 D_{p_1, D_{r,s}(p_1)} + \cdots + a_{\gamma_1} D_{p_{\gamma_1}, D_{r,s}(p_{\gamma_1})} = a_{i_0} D_{r,s}$. Also, write $s = p_{i_1} \cdots p_{i_t}$ for arrows p_{i_1}, \dots, p_{i_t} . Thus, $a_1 D_{r, D_{p_1, p_1}(s)} + \cdots + a_{\gamma_1} D_{r, D_{p_{\gamma_1}, p_{\gamma_1}}(s)} = (a_{i_1} + \cdots + a_{i_t}) D_{r,s}$. In summary,

$$[D_{r,s}, D_c] = (a_{i_0} - a_{i_1} - \cdots - a_{i_t}) D_{r,s},$$

which means that $D_{r,s}$ is the eigenvector under the adjoint action of D_c with eigenvalue $-a_{i_0} + a_{i_1} + \cdots + a_{i_t}$. It follows that $\mathfrak{D}_{AL} = \mathbf{k}\mathfrak{B}_{AL}$ is a Lie ideal of $HH^1(\mathbf{k}\Gamma)$. We have proved the following result.

THEOREM 5.7. (Structure Theorem of the First Hochschild Cohomology)
Let Γ be a connected planar finite acyclic quiver and let the ground field \mathbf{k} be of characteristic 0. Then the Lie algebra $HH^1(\mathbf{k}\Gamma)$ is the semi-direct sum of the Lie ideal \mathfrak{D}_{AL} and the abelian Lie subalgebra \mathfrak{D}_F :

$$HH^1(\mathbf{k}\Gamma) = \mathfrak{D}_{AL} \rtimes_{\varphi} \mathfrak{D}_F.$$

Here the action φ of \mathfrak{D}_F on \mathfrak{D}_{AL} is given as follows. For each given D_c with $\mathbf{c} = a_1 D_{p_1, p_1} + a_2 D_{p_2, p_2} + \cdots + a_{\gamma_1} D_{p_{\gamma_1}, p_{\gamma_1}}$ and $D_{r,s} \in \mathfrak{D}_{AL}$ with $r = p_{i_0}$ and $s = p_{i_1} \cdots p_{i_t}$, $D_{r,s}$ is the eigenvector under the adjoint action of D_c with eigenvalue $-a_{i_0} + a_{i_1} + \cdots + a_{i_t}$.

As noted in the introduction, the basis and semidirect sum decomposition of the first Hochschild cohomology in Theorem 5.5, Corollary 5.6 are Theorem 5.7, which are different from the one in [24].

From this proposition, we conclude that when two quivers are defined on the same unoriented graph, their outer differential Lie algebras are not isomorphic in general unless they have the same set of almost oriented cycles.

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