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A MATHEMATICAL THEORY OF QUANTUM SHEAF COHOMOLOGY*

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Abstract. The purpose of this paper is to present a mathematical theory of the half-twisted $(0,2)$ gauged linear sigma model and its correlation functions that agrees with and extends results from physics. The theory is associated to a smooth projective toric variety X and a deformation $\mathcal{E}$ of its tangent bundle T_X. It gives a quantum deformation of the cohomology ring of the exterior algebra of $\mathcal{E}^*$. We prove that in the general case, the correlation functions are independent of 'nonlinear' deformations. We derive quantum sheaf cohomology relations that correctly specialize to the ordinary quantum cohomology relations described by Batyrev in the special case $\mathcal{E} = T_X$.

Key words. Quantum cohomology, quantum shear cohomology, toric varieties, primitive collection, gauged linear sigma model.

AMS subject classifications. Primary 32L10, 81T20; Secondary 14N35.

1. Introduction.

The gauged linear sigma model (GLSM) was introduced in [Wit93] as a quantum field theory that is closely related to the nonlinear sigma model (NLSM), but easier to analyze for both (2,2) and (0,2) versions. Quantum cohomology relations for the (2,2) GLSM were described in [Bat93] and elaborated on by [MP95]. In this paper, we present a mathematical theory for the (0,2) GLSM and derive analogous results. To put this work in context, we give some background and motivation from physics before focusing on the mathematical formulation. More details from the physics perspective are given in the paper [DGKS11] by the authors.

The (0,2) NLSM is a physical theory associated to a Calabi-Yau threefold X and a vector bundle $\mathcal{E}$ on X satisfying $c_1(\mathcal{E}) = 0$ and the Green-Schwarz anomaly cancellation condition $c_2(\mathcal{E}) = c_2(T_X)$. The half-twisted (0,2) NLSM (sometimes called the $A/2$ model) is a closely-related but simpler theory that can be constructed in a much more general situation.

DEFINITION 1.1. *A holomorphic vector bundle $\mathcal{E} \to X$ on a compact Kähler manifold X is* omalous *if it satisfies the equalities*
(i) $c_1(\mathcal{E}) = c_1(T_X)$,
(ii) $c_2(\mathcal{E}) = c_2(T_X)$
in the cohomology of X.

These conditions extend the usual Green-Schwarz anomaly cancellation condition of heterotic string theory. (The alternate spelling 'homalous' may be more correct linguistically, but it is unpronounceable.)

Given an omalous bundle $\mathcal{E}$ on X, the half-twisted (0,2) NLSM can be defined as a physical theory associated to maps from a genus-zero Riemann surface Σ to X, with fermions associated to $\mathcal{E}$. This quantum field theory possesses a "quasi-topological subsector"; a subalgebra of vertex operators conjectured to be independent of the

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complex structure on X and referred to as the *quantum sheaf cohomology* of $\mathcal{E} \to X$ [ABS04, ADE06]. The operators in the quasi-topological sector are in one-to-one correspondence with the sheaf cohomology $\oplus_{p,q} H^q(X, \bigwedge^p \mathcal{E}^\vee)$.

Ignoring quantum corrections, the product of operators corresponds to the cup product of corresponding classes

$$H^q(X, \textstyle\bigwedge^p \mathcal{E}^\vee) \otimes H^{q'}(X, \textstyle\bigwedge^{p'} \mathcal{E}^\vee) \xrightarrow{\cup} H^{q+q'}(X, \textstyle\bigwedge^{p+p'} \mathcal{E}^\vee),$$

where the cup product $\cup$ refers to the usual cup product on cohomology followed by the product in the exterior algebra of $\mathcal{E}^\vee$. In the full quantum theory, instanton corrections modify the product, but for now we discuss the classical algebra, which we call the *polymology* of $\mathcal{E}$.

Definition 1.2. *The* polymology *of a vector bundle* $\mathcal{E}$ *is the associative algebra*

$$H^*_{\mathcal{E}}(X) := \bigoplus_{p,q} H^q(X, \textstyle\bigwedge^p \mathcal{E}^\vee) \tag{1}$$

equipped with the cup product.

If $\mathcal{E} = T_X$, i.e. if the (0,2) theory is actually a (2,2) theory, then the polymology is canonically isomorphic to the ordinary cohomology of X by Hodge theory.

The polymology can be defined for any vector bundle, but if $\mathcal{E}$ is omalous, a choice of isomorphism $\det \mathcal{E}^\vee \simeq \omega_X$ induces an isomorphism $\psi : H^n(X, \det \mathcal{E}^\vee) \simeq H^n(X, \omega_X)$, where n is the dimension of X. This isomorphism in turn induces a pairing

$$(\alpha, \beta) = \int_X \psi(\alpha \cup \beta) \tag{2}$$

satisfying $(\alpha \cup \beta, \gamma) = (\alpha, \beta \cup \gamma)$, which is perfect by Serre duality:

$$\begin{aligned} H^p(X, \textstyle\bigwedge^q \mathcal{E}^\vee)^\vee &\simeq H^{n-p}(X, \textstyle\bigwedge^q \mathcal{E} \otimes \omega_X) \\ &\simeq H^{n-p}(X, \textstyle\bigwedge^q \mathcal{E} \otimes \bigwedge^n \mathcal{E}^\vee) \\ &\simeq H^{n-p}(X, \textstyle\bigwedge^{n-q} \mathcal{E}^\vee). \end{aligned} \tag{3}$$

It follows easily that the polymology of an omalous vector bundle admits the structure of a bigraded Frobenius algebra.

The classical correlation functions can be identified with the pairing (2). Note that as $H^n(X, \bigwedge^n \mathcal{E}^\vee)$ is isomorphic to the complex numbers, any such isomorphism defines the trace in the Frobenius algebra structure. However, this isomorphism is not canonical. We deal with this normalization issue by simply defining the classical correlation functions to live in the one-dimensional vector space $H^n(X, \bigwedge^n \mathcal{E}^\vee)$.

Before discussing the *quantum sheaf cohomology* of $(X, \mathcal{E})$, a brief discussion is in order about two relevant quantum field theories: the (2,2) NLSM and the (2,2) GLSM. We list a few salient features of these theories here, which allows us to discuss their analogues for the (0,2) GLSM and the (0,2) NLSM. For the convenience of the reader, a fuller review of these (2,2) theories follows in Section 2.

Given a smooth projective variety X, the quantum corrections to any of these quantum field theories can be computed perturbatively using a compactification of the space of holomorphic maps $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = \beta$, for each $\beta \in H_2(X, \mathbb{Z})$, and performing an integration over this compactification. The (2,2) NLSM is well-understood mathematically as ordinary quantum cohomology, with the appropriate compactification being the moduli space $\overline{M}_{0,3}(X,\beta)$ of genus 0 stable maps of class β. If X is a toric variety, there is the linear sigma model moduli space X_β, which is a toric compactification used for the (2,2) GLSM [Wit93, MP95], leading to a quantum cohomology ring whose structure was described by Batyrev in [Bat93]. Either of these quantum cohomology rings are deformations of the ordinary cohomology ring $H^*(X)$. A comparison between Batyrev's quantum cohomology ring and the ordinary quantum cohomology ring follows from [Giv98] and is described in [CK00]. These two cohomology rings become identified after a change in variables (the *mirror map*). The Batyrev quantum cohomology ring is identical to the usual quantum cohomology ring if X is Fano.

The quantum sheaf cohomology described in this paper arises from the half-twisted (0,2) GLSM, and extends Batyrev's quantum cohomology ring. We will also explain how the identical moduli space X_β used in the (2,2) GLSM moduli space can also be used to describe the (0,2) GLSM, independent of $\mathcal{E}$.

Quantum sheaf cohomology is a quantum deformation of the Frobenius algebra structure on the polymology of $(X,\mathcal{E})$. This is analogous to either of the two versions of the quantum cohomology of X mentioned above.

Physics tells us from the half-twisted (0,2) NLSM that a quantum sheaf cohomology ring is associated to $(X,\mathcal{E})$ for any omalous vector bundle $\mathcal{E}$ on X. Unfortunately, a mathematically-precise version of such a theory does not yet exist. However, physical arguments providing an approach to such a mathematical version are given in [KS06]. Furthermore, one can speculate that the relevant compactification of the space of maps will be $\overline{M}_{0,3}(X,\beta)$ as in the (2,2) NLSM, independent of the choice of bundle $\mathcal{E}$. One can also speculate that the GLSM and NLSM versions of quantum sheaf cohomology will be identified by a change of variables analogous to the mirror map.

We now describe the ingredients of the half-twisted (0,2) GLSM. Although such theories are more general, we will only describe them in the situation we consider. Let X be a smooth projective toric variety, and let $\mathcal{E}$ be a deformation of the tangent bundle T_X arising from a deformation of the toric Euler sequence (to be described in (9) below). The half-twisted (0,2) GLSM is associated to $(X,\mathcal{E})$, and has a *quasi-topological sector* whose operators are generated by the symmetric algebra on $H^2(X,\mathbb{C})$ and become isomorphic to $H^*_{\mathcal{E}}(X)$ in the classical limit. In physics, the quasi-topological sector arises as the set of operators lying in the kernel of the scalar supercharge whose holomorphic conformal weight vanishes.

One of our main results is the calculation of the classical polymology of $(X,\mathcal{E})$ in Theorem 4.16. The algebra $H^*_{\mathcal{E}}(X)$ is naturally a quotient of this symmetric algebra, relating the GLSM to the NLSM. To each primitive collection in the toric variety X is associated a generator of the Stanley-Reisner ideal $\mathrm{SR}(X,\mathcal{E})$. The classical polymology is the quotient of this symmetric algebra by $\mathrm{SR}(X,\mathcal{E})$.

Since X_β is itself toric, the same result can be used to compute quantum corrections in all instanton sectors. We introduce a direct system of polymologies in Section 5.3 which allows us to sum over the instanton sectors and rigorously define the quantum sheaf cohomology ring by abstracting the physical notion of an operator. For each primitive collection, a quantum deformation of the corresponding Stanley-Reisner generators can be written down (63), as proposed in the physics literature. Our final main result is:

THEOREM 5.10. *The quantum sheaf cohomology relations (63) hold for all primitive collections K.*

Here is an outline of the rest of the paper.

We begin in Section 2 with a review of several variants of quantum cohomology and their relations, emphasizing the comparison of the GLSM and NLSM versions. This section is meant as motivation for our work on quantum sheaf cohomology: quantum sheaf cohomology should be the $(0,2)$ theory generalizing quantum cohomology which is its $(2,2)$ special case. In other words, quantum sheaf cohomology is the generalization of quantum cohomology where the tangent bundle is replaced by an omalous bundle. Much of the technical work done in this paper actually concerns ordinary (non-quantum) sheaf cohomology. When quantum sheaf cohomology is defined in Section 5.5 (Definition 5.9), it is obtained by putting together an infinite series of these ordinary sheaf cohomologies - much the way that quantum cohomology is assembled from copies of the ordinary cohomology. It is expected to come in the same variants as the $(2,2)$ quantum cohomology, i.e. both the NLSM and the GLSM. In this paper we focus on the GLSM variant. None of the material in Section 2 will be used in the rest of this work. The material in Section 2 is excerpted from [DGKS12] with minor edits, and is used here with the kind permission of the American Mathematical Society.

We get down to business in Section 3 by recalling standard concepts and notation from toric geometry. Then we recall the toric Euler sequence of a smooth toric variety X, which gives a presentation of its tangent bundle. Deformations of this sequence are presentations of deformations $\mathcal{E}$ of the tangent bundle, which complete the input data needed to define the $(0,2)$ GLSM. We then introduce a generalized Koszul complex that plays a fundamental role in our analysis. We conclude the section with the computation of the sheaf cohomology of certain T-invariant divisors, which enable us to chase through exact cohomology sequences associated with the generalized Koszul complex.

In Section 4 we compute the polymology of $(X,\mathcal{E})$. Let $W = H^2(X,\mathbb{C})$ and let $K \subset \Sigma(1)$ be a primitive collection, where $\Sigma(1)$ as usual denotes the edges of the fan for X. By a diagram chase, we associate to each K an element $Q_K \in \mathrm{Sym}^k W$, where $k = |K|$. We then define the Stanley-Reisner ideal $\mathrm{SR}(X,\mathcal{E})$ of $\mathcal{E}$ to be the ideal in $\mathrm{Sym}^* W$ generated by the Q_K. The main result of Section 4 is that the polymology of $\mathcal{E}$ is isomorphic to the quotient of $\mathrm{Sym}^* W$ by the Stanley-Reisner ideal of $\mathcal{E}$ (Theorem 4.16).

In Section 5 we describe the GLSM moduli space X_β associated to an effective class $\beta \in H_2(X,\mathbb{Z})$ and an induced vector bundle $\mathcal{E}_\beta$ on X_β. The correlation functions are defined in terms of the polymology of $(X_\beta,\mathcal{E}_\beta)$. We show that the polymology of $(X_\beta,\mathcal{E}_\beta)$ is a quotient of $\mathrm{Sym}^* W$ by an ideal generated by products of powers of factors of the Q_K given by (54). We then introduce a direct system of polymologies that will allow us to compare correlation functions for different β, and then define the correlation functions after introducing *four-fermi terms* in (59) that play a role for the $(0,2)$ GLSM similar to that of the virtual fundamental class of Gromov-Witten theory. Finally, we define the quantum sheaf cohomology ring abstractly and then compute the quantum cohomology relations in Theorem 5.10. In [MM09], predictions were made for the image of the relations in a localization of the ring (following a standard procedure in the physics literature). It is straightforward to verify that our relations descend to the relations of [MM09] in that localization. See [DGKS11] for details.

2. Quantum cohomology. The idea behind quantum cohomology is that the ordinary multiplication of cohomology classes on the variety X can be perturbed into a power series whose coefficients encode Gromov-Witten invariants, or intersections in moduli spaces of maps to X.

Fix a class $\beta \in H_2(X, \mathbb{Z})$, and let $M_\beta := M_{g,k}(X, \beta)$ be the moduli space of maps to X from a curve of genus g with k marked points. The complex structure of the curve, as well as the location of the marked points, are free to vary. We will usually restrict attention to $g = 0$.

There are several natural maps: the evaluation map $e : M_\beta \times \mathbb{P}^1 \to X$, the projection $\pi : M_\beta \times \mathbb{P}^1 \to M_\beta$, and k sections $s_i : M_\beta \to M_\beta \times \mathbb{P}^1$ for $i = 1, \ldots, k$ corresponding to the marked points.

The correlation function $\langle a_1, ..., a_k \rangle_\beta$ of cohomology classes $a_1, ..., a_k$ on X is roughly speaking the degree of the ordinary product on M_β of the induced classes $s_i^* e^* a_i$. Making sense of this requires several technicalities. In particular, we need to replace M_β by a good compactification X_β, and we need a well behaved notion of a virtual fundamental class.

For the formal definition, introduce the Novikov ring of X, which is the ring of formal power series $\Lambda := \mathbb{Z}[[H_2^+(X)]]$ on the semigroup $H_2^+(X)$ of effective classes in H_2. Elements of Λ are written

$$\lambda = \Sigma_{\beta \in H_2^+(X)} \lambda_\beta q^\beta, \tag{4}$$

where the q^β are formal symbols satisfying $q^{\beta_1+\beta_2} = q^{\beta_1} q^{\beta_2}$. Other variants of Λ are possible, and are frequently encountered. In this work we will not need power series, so we may as well replace Λ by the semigroup ring $\Lambda := \mathbb{Z}[H_2^+(X)]$.

The quantum cohomology ring is a Λ-algebra $QH^*(X)$. The additive structure, i.e. the underlying Λ-module, is $H^*(X) \otimes \Lambda$. By Poincaré duality, the quantum product $a_1 * a_2$ of elements $a_1, a_2 \in H^*(X)$ can be defined by specifying its Poincaré pairing with an arbitrary $a_3 \in H^*(X)$:

$$\langle a_1 * a_2, a_3 \rangle = \sum_\beta \langle a_1, a_2, a_3 \rangle_\beta q^\beta.$$

The multiplication is then extended to all of $QH^*(X)$ by Λ-linearity. If we set the quantum variables q to 0, i.e. take the constant coefficients in (4), we recover the classical multiplication on X.

The specific theory of quantum cohomology depends on the choice of a compactification. Several compactifications X_β of M_β are known. We will focus on two of these: the stable map, or NLSM (non-linear sigma model) compactification $\overline{M}_{0,3}(X, \beta)$, and the toric, or GLSM (=gauged linear sigma model) compactification, which we will denote by $\mathcal{M}_\beta$.

The stable map compactification is due to Kontsevich, and is defined for any smooth projective variety X, including Calabi-Yaus and Fanos. A point of the Kontsevich compactification explicitly depends on the location of the marked points. It is essentially geometric: the objects being parametrized are honest maps to X from various degenerations of the original curve Σ. This implies the existence of a universal curve over it, and makes possible the definition of Gromov-Witten invariants, or intersection numbers on all of X_β. We will denote the resulting quantum cohomology by $QH^*_{NLSM}(X)$.

On the other hand we have the toric, or GLSM compactification. This is defined for toric varieties X, and we will see in Section 5.1 below that it is a toric variety itself. The moduli of the marked points do not enter into the definition of the GLSM compactification. It can be described by a global quotient construction, or constructed from a fan which is obtained from the fan for X by replicating each edge an appropriate number of times. The data it parametrizes is algebraic rather than geometric, so there is no universal object over it, making the definition of correlation functions harder. (Many of the technical difficulties addressed in the present work are due to the absence of a universal curve.) Still, we get a quantum cohomology ring, denoted $QH^*_{GLSM}(X)$.

In most of this article we focus exclusively on the GLSM. Here we make a few comments on the NLSM. It is obtained by using Kontsevich's stable map compactification $\overline{M}_\beta := \overline{M}_{g,k}(X,\beta)$ of $M_\beta := M_{g,k}(X,\beta)$. In this case, the quantum product is called the small quantum cohomology ring. The big quantum cohomology ring is defined using the complete set of all Gromov-Witten invariants of X, which is equivalent to the Frobenius manifold structure of $H^*(X)$. The small quantum cohomology on the other hand, only involves the three-point functions. Physically, the big quantum cohomology ring includes gravity, while the small quantum cohomology ring does not. In general, gravity refers to variation of the metric. The relevant parameter for us is the complex structure of the source curve, which is determined by the metric. Since we restrict to genus zero, this amounts simply to the location of the k punctures. The big quantum cohomology ring encodes all Gromov-Witten invariants, *i.e.* intersection numbers on the Kontsevich compactification $\overline{M}_\beta$ of M_β, while the small quantum cohomology ring encodes only Gromov invariants, i.e. intersections on a fiber of $p : \overline{M}_\beta \to \overline{M}_{g,k}$. When $g = 0, k = 3$, the moduli space $\overline{M}_{g,k}$ is just a point, so the fiber of p is all of $\overline{M}_\beta$ and there is no difference between the two notions. However, as soon as k becomes bigger than 3, $\overline{M}_{0,k}$ becomes positive dimensional and the two notions diverge. For example, consider $X = \mathbb{P}^2$. Its ordinary cohomology is $H^*(X) = \mathbb{Z}[H]/(H^3)$, where H is the class of a line and H^2 the class of a point. The fact that there is a unique line through two distinct points says that $\langle H^2, H^2\rangle_1^{gravity} = 1$. Similarly, the fact that there is a unique conic through five general points says that $\langle H^2, H^2, H^2, H^2, H^2\rangle_2^{gravity} = 1$. If we fix only 4 points, we get a pencil of conics. As the conic varies in this pencil, the cross ratio of the 4 marked points on it varies linearly, so there is a unique conic in the pencil on which the cross ratio takes a specified value. This translates into $\langle H^2, H^2, H^2, H^2\rangle_2^{nogravity} = 1$.

The GLSM is a variant of the small quantum cohomology. The two objects $QH^*_{NLSM}(X)$ and $QH^*_{GLSM}(X)$ are distinct rings, but are related by the following results of Givental [Giv98]:

- There is a "mirror map", a (typically nonlinear) change in the coordinate variables q^β, which identifies the ring $QH^*_{NLSM}(X)$ with $QH^*_{GLSM}(X)$.
- The mirror map is completely determined by the (gravitational) Gromov-Witten theory of X.
- When X is Fano, the mirror map is the identity, so the two theories are completely equivalent.

The triviality of the mirror map in the Fano case is explained in [CK00].

3. The toric setting. We start by fixing some notation associated to toric varieties. A good general reference for toric varieties is [CLS11].

Let $X = X_\Sigma$ be a smooth projective toric variety of dimension n with fan Σ whose support lies in $N_\mathbb{R} \simeq \mathbb{R}^n$, where N is the lattice of one-parameter subgroups of the dense torus of X. We will denote by M the lattice of characters dual to N, by $\Sigma(1)$ the set of one-dimensional cones of the fan, and we will write $\bigoplus_\rho$ and $\sum_\rho$ in place of $\bigoplus_{\rho\in\Sigma(1)}$ and $\sum_{\rho\in\Sigma(1)}$, respectively. To each $\rho \in \Sigma(1)$ is associated a torus-invariant

Weil divisor denoted D_ρ, a unique minimal generator v_ρ of the semigroup $N \cap \rho$, and a canonical section $x_\rho \in H^0(X, \mathcal{O}_X(D_\rho))$. These canonical sections freely generate the homogeneous coordinate ring of X:

$$S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]. \tag{5}$$

The homogenous coordinate ring S has a natural grading by $\mathrm{Pic}(X)$, which assigns to x_ρ the degree $[D_\rho] \in \mathrm{Pic}(X)$.

For each T-invariant Weil divisor $D = \sum_\rho a_\rho D_\rho$ we have a natural isomorphism

$$S_{[D]} \simeq H^0(X, \mathcal{O}_X(D)), \tag{6}$$

where as usual, $S_{[D]}$ denotes the graded piece of S of degree $[D]$. To describe this isomorphism, we associate to D the polytope

$$\Delta_D = \{m \in M_\mathbb{R} \mid \langle m, v_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

Then the isomorphism (6) is conveniently described by the identification of basis elements

$$\prod_\rho x_\rho^{\langle m, v_\rho \rangle + a_\rho} \leftrightarrow \chi^m, \tag{7}$$

where m ranges over $\Delta_D \cap M$ and χ^m is the character associated to $m \in M$, thought of as a meromorphic function with at worst poles on D. Note in particular that for each ρ the trivial character χ^0 (i.e. the constant function 1) is the section of $\mathcal{O}_X(D_\rho)$ associated to x_ρ via the isomorphism (6) for $D = D_\rho$.

Since X is smooth and toric, the class group of Weil divisors, the Picard group, and the integral cohomology are all isomorphic. We associate to each $m \in M$ the element $e_m \in \mathbb{Z}^{\Sigma(1)}$ defined by $e_m(\rho) = \langle m, v_\rho \rangle$. Then $\mathrm{Pic}(X)$ admits a presentation as

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X) \to 0, \tag{8}$$

where the first non-trivial morphism is $m \mapsto e_m$ and the basis element of $\mathbb{Z}^{\Sigma(1)}$ dual to ρ maps to $[D_\rho]$.

Let $W = H^2(X, \mathbb{C})$. Then the tangent bundle of X fits into a short exact sequence known as the toric Euler sequence:

$$0 \longrightarrow \mathcal{O}_X \otimes_\mathbb{C} W^\vee \xrightarrow{E_0} \bigoplus_\rho \mathcal{O}_X(D_\rho) \longrightarrow T_X \longrightarrow 0. \tag{9}$$

Thinking of E_0 as an element of $\bigoplus_\rho S_{[D_\rho]} \otimes W$, the ρ^{th} component of E_0 is $x_\rho \otimes [D_\rho]$.

Recall that a collection of edges $K \subset \Sigma(1)$ is a *primitive collection* if K does not span any cone in Σ, but every proper subcollection of K does. Equivalently, the intersection of the divisors D_ρ with $\rho \in K$ is empty, but the intersection of any proper subset of these divisors is nonempty. Following the presentation of [CK00, §8.1.2], we define two ideals in the homogeneous coordinate ring S:

$$\begin{aligned} P(X) &= \left(\sum_\rho \langle m, v_\rho \rangle x_\rho \;\middle|\; m \in M \right) \\ SR(X) &= \left(\prod_{\rho \in K} x_\rho \;\middle|\; K \text{ a primitive collection of } \Sigma \right). \end{aligned} \tag{10}$$

The former is the ideal of linear equivalences, so that

$$\mathrm{Sym}(W) \simeq S/P(X). \tag{11}$$

The second ideal in (10) is known as the *Stanley-Reisner* ideal of X. It is well known [Ful93, Oda88] that there is an isomorphism of $\mathbb{Z}$-graded algebras

$$S/\left(P(X) + SR(X)\right) \simeq H^*(X, \mathbb{C}) \tag{12}$$

induced by sending a generator x_ρ of S to $[D_\rho]$.

For later use, we recall the description of toric varieties as quotients. There is a natural action of $G = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}^*)$ on $\mathbb{C}^{\Sigma(1)}$ following from the inclusion $G \subset \mathrm{Hom}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*)$ derived from (8). For each $\sigma \in \Sigma$ define

$$x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \tag{13}$$

and the *irrelevant ideal*

$$B(\Sigma) = \left(x^{\hat{\sigma}} \mid \sigma \in \Sigma\right). \tag{14}$$

Thinking of the x_ρ as coordinate functions on $\mathbb{C}^{\Sigma(1)}$, we define the subset $Z(\Sigma) \subset \mathbb{C}^{\Sigma(1)}$ as the vanishing locus of the irrelevant ideal. Then $Z(\Sigma)$ is G-invariant and

$$X = \left(\mathbb{C}^{\Sigma(1)} - Z(\Sigma)\right)/G. \tag{15}$$

For later use, we note that it is well known that $Z(\Sigma)$ can be described in terms of primitive collections. If $K = \{\rho_1, \ldots, \rho_k\}$ is a primitive collection, let $L_K \subset \mathbb{C}^{\Sigma(1)}$ be the linear subspace defined by $x_{\rho_1} = \ldots = x_{\rho_k} = 0$. Then

$$Z(\Sigma) = \cup_K L_K, \tag{16}$$

where the union is taken over all primitive collections K. The fan Σ can also be recovered from the set of primitive collections as the set of cones spanned by collections of edges that do not contain any primitive collection. See [CLS11] for example.

It will also be useful to note that $\mathrm{Pic}(X)$ can be recovered from G as

$$\mathrm{Pic}(X) \simeq \mathrm{Hom}(G, \mathbb{C}^*), \tag{17}$$

by duality for finitely-generated abelian groups.

3.1. Toric deformations of the tangent bundle. To define a half-twisted (0,2) GLSM, we need a presentation of a vector bundle $\mathcal{E}$ obtained from (9) by simply changing the map E_0:

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} W^\vee \xrightarrow{E} \bigoplus\nolimits_\rho \mathcal{O}_X(D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0. \tag{18}$$

Both the bundle and the presentation are required to define the GLSM. We will sometimes abuse terminology slightly by referring to the bundle $\mathcal{E}$ as a *toric deformation of the tangent bundle*, but we always have a fixed presentation (18) in mind. Specifying a map E is not sufficient; it is required that the cokernel $\mathcal{E}$ of E is locally free, or equivalently that

$$E^t : \oplus_\rho \mathcal{O}(-D_\rho) \to W \otimes \mathcal{O}_X \tag{19}$$

is surjective.

As with E_0, the map E can be viewed as a section of $\oplus_\rho H^0(X, \mathcal{O}(D_\rho)) \otimes W = \oplus_\rho S_{[D_\rho]} \otimes W$.

The components E_ρ of E can be thought of as W-valued sections of $\mathcal{O}(D_\rho)$. We will sometimes express these sections as

$$E_\rho = \sum_{m \in \Delta_{D_\rho} \cap M} a_{\rho m} \chi^m,$$

where $a_{\rho m} \in W$, or as

$$E_\rho = \sum_{m \in \Delta_{D_\rho} \cap M} a_{\rho m} x_\rho \prod_{\rho'} x_{\rho'}^{\langle m, v'_\rho \rangle}$$

using the identification (7).

We illustrate with $X = \mathbb{P}^1 \times \mathbb{P}^1$, which we describe by its standard 2-dimensional fan with edges $\rho_1, \ldots, \rho_4$ generated by $v_1 = (1,0)$, $v_2 = (-1,0)$, $v_3 = (0,1)$, and $v_4 = (0,-1)$ respectively. Denoting the associated coordinates by x_1, x_2, x_3, x_4, it is seen that the coordinates (x_1, x_2) and (x_3, x_4) are naturally identified with the usual homogeneous coordinates on the respective $\mathbb{P}^1$ factors. The sheaves $\mathcal{O}(D_i)$ are identified with the sheaf $\mathcal{O}(1,0)$ on X for $i = 1,2$ and with $\mathcal{O}(0,1)$ for $i = 3,4$. The short exact sequence (18) defining $\mathcal{E}$ becomes

$$0 \to \mathcal{O}_X \otimes_{\mathbb{C}} W^\vee \xrightarrow{E} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \to \mathcal{E} \to 0.$$

The map E is identified with a collection of W-valued sections $s_1, s_2 \in H^0(X, \mathcal{O}(1,0))$ and W-valued sections $s_3, s_4 \in H^0(X, \mathcal{O}(0,1))$. We can write

$$s_i = \begin{cases} \sum_{j=1}^2 a_{ij} x_j & i = 1,2 \\ \sum_{j=3}^4 a_{ij} x_j & i = 3,4 \end{cases} \tag{20}$$

with $a_{ij} \in W = H^2(X, \mathbb{C}) \simeq \mathbb{C}^2$.

In the general situation, the terms $a_{\rho m} \chi^m$ with $x_\rho \prod_{\rho'} x_{\rho'}^{\langle m, v'_\rho \rangle}$ a linear monomial in the homogeneous coordinates will be called the *linear terms*; the other terms will be called the *nonlinear terms*. A toric deformation of the tangent bundle containing only linear terms will be called a *linear deformation*. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we see from (20) that all deformations are linear, but this is not the case in general.

Linear deformations play a significant role in physicists' analyses of quantum sheaf cohomology [KS06, GK10, MM08, MM09] in localized rings, and we will see that they capture the essence of the polymology associated to a toric deformation of the tangent bundle.

Throughout, we will make extensive use of a generalized Koszul complex associated to deformations of the toric Euler sequence. In order to simplify notation, put $Z = \oplus_\rho \mathcal{O}(-D_\rho)$ and then for $0 \leq k$ and $0 \leq j \leq k$ define

$$\mathfrak{Z}_j^{(k)} := \bigwedge^j Z \otimes \mathrm{Sym}^{k-j} W.$$

Let $\mathcal{E}$ be a deformation of the tangent bundle in the preceding sense. The dual of the exact sequence (18) induces an injection $\bigwedge^k \mathcal{E}^\vee \to \bigwedge^k Z$ and maps $\alpha_j : \mathfrak{Z}_j^{(k)} \to \mathfrak{Z}_{j-1}^{(k)}$ defined as

$$\alpha_j : (z_1 \wedge \cdots \wedge z_j) \otimes s \mapsto \sum_{\ell=1}^{j} (-1)^{\ell-1} (z_1 \wedge \cdots \wedge \hat{z}_\ell \wedge \cdots \wedge z_j) \otimes \big[E^\vee(z_\ell) \odot s\big], \tag{21}$$

where E is the injection in (18) and $\odot$ is multiplication in Sym^*W. These maps may be arranged into an exact sequence

$$0 \longrightarrow \bigwedge^k \mathcal{E}^\vee \longrightarrow \mathfrak{Z}_k^{(k)} \longrightarrow \mathfrak{Z}_{k-1}^{(k)} \longrightarrow \cdots \longrightarrow \mathfrak{Z}_1^{(k)} \longrightarrow \mathfrak{Z}_0^{(k)} \longrightarrow 0. \tag{22}$$

Exactness follows since the maps in (22) are natural and the analogous complex formed from a short exact sequence of vector spaces is easily seen to be exact.

3.2. A vanishing result for certain toric line bundles. We will make extensive use of the line bundles $\mathcal{O}_X(D)$ associated to T-invariant divisors D on X, and in particular Theorem 3.1 below, which appears in [Dem70] and is reproduced here for convenience.

Consider a Weil divisor $D = \sum_\rho a_\rho D_\rho$ and define

$$\Sigma_{D,m} = \{\rho \in \Sigma(1) \mid \langle m, v_\rho \rangle < -a_\rho\}.$$

Then let $V_{D,m}$[1] be the union of polytopes in $N_\mathbb{R}$

$$V_{D,m} = \bigcup_{\substack{\sigma\in\Sigma \\ \sigma(1)\subset\Sigma_{D,m}}} \mathrm{Conv}(v_\rho \mid \rho \in \sigma).$$

Since $\mathcal{O}_X(-D)$ is a torus-equivariant bundle, $H^j(X, \mathcal{O}_X(-D))$ decomposes as a direct sum of weight spaces $H^j(X, \mathcal{O}_X(-D))_m$ with $m \in M$.

THEOREM 3.1 (Proposition 6 of [Dem70]). *Let $D = \sum_\rho a_\rho D_\rho$ be a T-invariant Weil divisor on X. For $m \in M$ and $p \geq 0$,*

$$H^p(X, \mathcal{O}_X(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C}).$$

Here $\widetilde{H}$ denotes the reduced cohomology. Consider a subset $K \subset \Sigma(1)$ and set $D_K = \sum_{\rho\in K} D_\rho$.

LEMMA 3.2. *For all j and all $K \subset \Sigma(1)$, $H^j(X, \mathcal{O}_X(-D_K)) = H^j(X, \mathcal{O}_X(-D_K))_0$. That is, the cohomology of $\mathcal{O}_X(-D_K)$ is purely of weight 0.*

Proof. By Theorem 3.1, $H^j(X, \mathcal{O}_X(-D_K))_m$ is the reduced cohomology of the topological space $V_{-D_K,m}$ obtained as follows: Σ determines a simplicial complex whose faces are $\mathrm{Conv}(v_\rho | \rho \in \sigma(1))$ for $\sigma \in \Sigma$. $V_{-D_K,m}$ is the subcomplex corresponding to those σ such that $\sigma(1)$ is contained in:

$$\Sigma_{-D_K,m} := \{\rho \in K | \langle m, v_\rho \rangle \leq 0\} \cup \{\rho \notin K | \langle m, v_\rho \rangle < 0\}.$$

Here we use the fact that the coefficients of the D_ρ in $-D_K$ are either 0 or -1. The set $\Sigma_{-D_K,m}$ is invariant under rescaling m by a positive integer, and therefore so are $V_{D,m}$ and $H^j(X, \mathcal{O}_X(-D_K))_m$. If the latter were non-vanishing for some non-zero m, $H^j(X, \mathcal{O}_X(-D_K))$ would not be finite dimensional, contradicting the projectivity of X. ☐

PROPOSITION 3.3. *Let Σ be a simplicial fan and $K \subset \Sigma(1)$. Setting $k = |K|$ and $D_K = \sum_{\rho\in K} D_\rho$ as before, we have that*

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[1] We are following the notation of [CLS11], adapting it slightly; our $V_{D,m}$ matches their $V_{D,m}^{\mathrm{simp}}$.

i) For all $\ell \geq k$, $H^\ell(X, \mathcal{O}_X(-D_K)) = 0$

ii) If $\cap_{\rho \in K} D_\rho \neq \emptyset$*, then for all* $\ell \in \mathbb{Z}$, $H^\ell(X, \mathcal{O}_X(-D_K)) = 0$.

iii) If K *is a primitive collection,*

$$H^\ell(X, \mathcal{O}_X(-D_K)) \simeq \begin{cases} \mathbb{C} & \ell = k-1 \\ 0 & \textit{otherwise} \end{cases}$$

iv) If K *is not a primitive collection, then* $H^{k-1}(X, \mathcal{O}_X(-D_K)) = 0$.

Proof. We use Theorem 3.1 and the notation therein throughout. By Lemma 3.2, we need only consider the torus-invariant part of the cohomology. The relevant set of one-cones is $\Sigma_{-D_K,0} = K$.

i) $V_{-D_K,0}$ is contained in the convex hull of k points and so can never contain a non-contractible $k-1$ cycle. Similarly, it does not contain any ℓ cycles with $\ell > k-1$.

ii) If the intersection is nonempty, $\mathrm{cone}\{v_\rho \mid \rho \in K\} \in \Sigma$ and the v_ρ are linearly independent since Σ is simplicial: thus $V_{-D_K,0}$ is a $k-1$ simplex.

iii) Consider a primitive collection K. Since K is primitive, every proper subset of K spans a cone in Σ, so the simplicial complex takes the form

$$V_{-D_K,0} = \bigcup_{\rho' \in K} \mathrm{Conv}(v_\rho \mid \rho \in K, \rho \neq \rho').$$

This set is precisely the boundary of the $(k-1)$-simplex $\mathrm{Conv}(v_\rho \mid \rho \in K)$, so that $V_{-D_K,0}$ is homeomorphic to the $(k-2)$ sphere and the last claim follows immediately.

iv) If K is not a primitive collection, we need only consider the situation where $\cap_{\rho \in K} D_\rho = \emptyset$. Then by comparison to the analysis in iii) above, we see that either $V_{-D_K,0}$ has dimension strictly less than $k-2$, or it has dimension $k-2$ and is homeomorphic to a proper subcomplex of the above simplicial triangulation of S^{k-2}. Either way we conclude that $\widetilde{H}^{k-2}(V_{-D_K,0}) = 0$ and we are done.

□

Remark 3.4. *An immediate consequence of ii) is that for all* $\ell \in \mathbb{Z}$ *and* $\rho \in \Sigma(1)$, $H^\ell(X, \mathcal{O}_X(-D_\rho)) = 0$.

In the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$, the primitive collections are given by $\{\rho_1, \rho_2\}$ and $\{\rho_3, \rho_4\}$ in the notation introduced in Section 3.1. The nonvanishing cohomology groups associated to these primitive collections by Proposition 3.3 are $H^1(X, \mathcal{O}(-2,0))$ and $H^1(X, \mathcal{O}(0,-2))$, respectively.

4. $H^*_{\mathcal{E}}(X)$ for toric deformations of the tangent bundle.

In this section, we study the algebra $H^*_{\mathcal{E}}(X)$, showing that as a bigraded vector space it is isomorphic to $H^*(X, \mathbb{C})$. Multiplicatively it is generated under the cup product by elements of $H^1(X, \mathcal{E}^\vee)$. We show that the relations among the generators may be given explicitly by defining an ideal analogous to $SR(X)$. Some of the results in this section are not used elsewhere in this paper, but we include them since they could be useful in applications to the NLSM.

4.1. Graded components.

We begin our study of $H^*_{\mathcal{E}}(X)$ by elucidating its vector space structure. In particular, we show that it is diagonal, in the sense that its graded components $H^q(X, \bigwedge^p \mathcal{E}^\vee)$ vanish unless $p = q$.

Proposition 4.1. *Let $\mathcal{E}$ be a locally-free toric Euler sequence deformation. Then for any p and $q \neq p$,*

$$H^q(X, \textstyle\bigwedge^p \mathcal{E}^\vee) = 0.$$

Proof. For $q = 0$, the proposition holds trivially, since $\bigwedge^p \mathcal{E}^\vee \subset \bigwedge^p Z$ and $H^0(X, \bigwedge^p Z) = 0$. Thus, assume that $q > 0$.

Consider the exact sequence in (22). For each $1 \leq j \leq p-1$ define

$$S_j^{(p)} = \ker\left(\mathfrak{Z}_j^{(p)} \to \mathfrak{Z}_{j-1}^{(p)}\right), \tag{23}$$

and set $S_0^{(p)} := \mathrm{Sym}^p W \otimes \mathcal{O}_X$ and $S_p^{(p)} := \bigwedge^p \mathcal{E}^\vee$. Then, for each $0 < \ell \leq p$, we have a short exact sequence

$$0 \longrightarrow S_\ell^{(p)} \longrightarrow \mathfrak{Z}_\ell^{(p)} \longrightarrow S_{\ell-1}^{(p)} \longrightarrow 0. \tag{24}$$

We first show that for $q > p$, $H^q(X, \bigwedge^p \mathcal{E}^\vee) = 0$. Consider the long exact sequence in cohomology induced by (24):

$$\cdots \to H^{q-1}(X, \mathfrak{Z}_\ell^{(p)}) \to H^{q-1}(X, S_{\ell-1}^{(p)}) \to H^q(X, S_\ell^{(p)}) \to H^q(X, \mathfrak{Z}_\ell^{(p)}) \to \cdots .$$

By Proposition 3.3, we have that for $q \geq \ell$, $H^q(X, \mathfrak{Z}_\ell^{(p)}) = 0$, so that for $q > \ell$ we have

$$H^{q-1}(X, S_{\ell-1}^{(p)}) \simeq H^q(X, S_\ell^{(p)}).$$

By varying ℓ, we obtain a chain of isomorphisms

$$H^{q-p}(X, S_0^{(p)}) \xrightarrow{\sim} H^{q-p+1}(X, S_1^{(p)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H^q(X, S_p^{(p)}),$$

which shows $H^q(X, \bigwedge^p \mathcal{E}^\vee) \simeq \mathrm{Sym}^p W \otimes H^{q-p}(X, \mathcal{O}_X) = 0$.

Now, assume that $q < p$. By Serre duality $H^q(X, \bigwedge^p \mathcal{E}^\vee)$ is dual to $H^{n-q}(X, \bigwedge^{n-p} \mathcal{E}^\vee)$, and $n - q > n - p$, so the latter vanishes by the above considerations. □

Corollary 4.2. *Let $\mathcal{E}$ be a toric deformation of the tangent bundle. Then there is a canonical isomorphism*

$$H^*_{\mathcal{E}}(X) \simeq \bigoplus_k H^k(X, \textstyle\bigwedge^k \mathcal{E}^\vee). \tag{25}$$

Remark 4.3. If we examine the particular case of $k = 1$, $\ell = 1$, we find that

$$H^1(X, \mathcal{E}^\vee) \simeq H^0(X, \mathcal{O}_X \otimes W) \simeq W \tag{26}$$

by dualizing (18) to obtain

$$0 \to \mathcal{E}^\vee \to Z \to W \otimes \mathcal{O} \to 0 \tag{27}$$

and using $H^0(X, Z) = H^1(X, Z) = 0$.

4.2. Generators. Now that we have a better idea of the linear structure of the algebra, we would like to identify a set of minimal generators. Corollary 4.2 and the fact that the cohomology of smooth projective toric varieties are generated by elements of $H^1(X,\Omega^1_X)$ lead us to suspect that elements of $H^1(X,\mathcal{E}^\vee)\simeq W$ generate $H^*_{\mathcal{E}}(X)$.

As multiplication is defined using the cup product and the algebra is diagonal, it is in fact commutative. For $1\le k\le n$, the cup product of k elements of $H^1(X,\mathcal{E}^\vee)$ is a linear map $\mathrm{Sym}^k H^1(X,\mathcal{E}^\vee)\to H^k(X,\bigwedge^k\mathcal{E}^\vee)$ that we will rewrite as

$$m_k : \mathrm{Sym}^k W \to H^k(X,{\textstyle\bigwedge^k}\mathcal{E}^\vee). \tag{28}$$

Lemma 4.4. *The map m_k can be identified with the extension class in* $\mathrm{Ext}^k(\mathrm{Sym}^k W\otimes\mathcal{O}_X,\bigwedge^k\mathcal{E}^\vee)\simeq \mathit{Hom}(\mathrm{Sym}^k W, H^k(X,\bigwedge^k\mathcal{E}^\vee))$ *associated to the generalized Koszul complex* (22).

Proof. First, m_1 is identified with the extension class $e\in\mathrm{Ext}^1(W\otimes\mathcal{O},\mathcal{E}^\vee)$ of (27), an exact sequence that can be reinterpreted as a quasi-isomorphism of $\mathcal{E}^\vee$ with the complex

$$C^\bullet : \qquad 0\to Z\to W\otimes\mathcal{O}\to 0$$

with Z in degree 0. Thus, e can be represented by the natural morphism of complexes $f: W\otimes\mathcal{O}\to C^\bullet[1]$ whose only nontrivial component is the identity map on $W\otimes\mathcal{O}$. We can then tensor f with itself k times to get a morphism

$$f^{\otimes k} : W^{\otimes k}\otimes\mathcal{O}\to (C^\bullet)^{\otimes k}[k].$$

However, $(C^\bullet)^{\otimes k}$ is quasiisomorphic to $(\mathcal{E}^\vee)^{\otimes k}$, a statement that we can rewrite as an exact sequence

$$0\to(\mathcal{E}^\vee)^{\otimes k}\to(C^\bullet)^{\otimes k}\to 0. \tag{29}$$

Noting that $W^{\otimes k}\otimes\mathcal{O}$ is the degree k component of $(C^\bullet)^{\otimes k}$, the above discussion identifies the extension class of (29) with

$$(m_1)^{\otimes k} : W^{\otimes k}\to H^k(X,{\textstyle\bigwedge^k}\mathcal{E}^\vee).$$

Finally, we define a natural action of the symmetric group S_k on the complex (29) by permuting the factors and inserting signs. Explicitly, the local sections of the degree j term of $(C^\bullet)^{\otimes k}$ consist of expressions

$$\sum s_{i_1}\otimes\ldots\otimes s_{i_k}, \tag{30}$$

where the s_i range over a basis of local sections of $Z\oplus(W\otimes\mathcal{O})$, and where each term in (30) has exactly j of its factors s_{i_α} belonging to $W\otimes\mathcal{O}$ and $k-j$ of its factors belonging to Z. Let $\sigma\in S_k$, and let $\epsilon_\sigma=\pm1$ be defined by the usual sign rule for the inherited $\mathbb{Z}_2$-grading on $(C^\bullet)^{\otimes k}[k]$, that a sign is picked up for each reordering of the factors belonging to Z, so that the action of the symmetric group is given by

$$\sigma\cdot(s_{i_1}\otimes\ldots\otimes s_{i_k})=\epsilon_\sigma s_{i_{\sigma(1)}}\otimes\ldots\otimes s_{i_{\sigma(k)}}. \tag{31}$$

Then (22) is identified with the S_k-invariant subcomplex of (29), and the extension class is m_k, the S_k-invariant part of m_1^k, as claimed. □

Since (22) can be reconstructed by splicing together the short exact sequences (24), the extension class m_k is the Yoneda product of the extension classes of the sequences (24). In more explicit terms, there are coboundary maps

$$\delta_j : H^j(X, S_j^{(k)}) \longrightarrow H^{j+1}(X, S_{j+1}^{(k)}), \tag{32}$$

which may be composed to a linear map

$$\delta_{k-1} \circ \delta_{k-2} \cdots \circ \delta_0 : H^0(X, \mathrm{Sym}^k W \otimes \mathcal{O}_X) \to H^k(X, \textstyle\bigwedge^k \mathcal{E}^\vee). \tag{33}$$

By first identifying $H^0(X, \mathrm{Sym}^k W \otimes \mathcal{O}_X)$ with $\mathrm{Sym}^k W$ and then applying the above composition of maps, we obtain precisely the linear map in (28). That is,

$$m_k = \delta_{k-1} \circ \delta_{k-2} \circ \cdots \circ \delta_0. \tag{34}$$

Proposition 4.5. *For all $1 \le k \le n$, m_k is surjective.*

Proof. Fix such a k; for all ℓ, $H^\ell(X, \mathfrak{Z}_\ell^{(k)}) = 0$ by Proposition 3.3. In the long exact sequence in cohomology induced by the exact sequence (24), we obtain the following subsequences for each $0 < \ell \le k$:

$$\cdots \longrightarrow H^{\ell-1}(X, \mathfrak{Z}_\ell^{(k)}) \longrightarrow H^{\ell-1}(X, S_{\ell-1}^{(k)}) \longrightarrow H^\ell(X, S_\ell^{(k)}) \longrightarrow 0 \tag{35}$$

Thus, both the coboundary maps (32) and their composition (33) are surjective. ☐

Remark 4.6. In fact, most of the coboundary maps are isomorphisms; $H^{\ell-1}(X, \mathfrak{Z}_\ell^{(k)})$ is non-vanishing iff there is a primitive collection of order ℓ in Σ, by Proposition 3.3. This observation allows us to characterize the kernel of the multiplication map: to find the relations amongst the generators.

4.3. Relations.

For each primitive collection K we will exhibit an explicit element $Q_K \in \ker m_k \subset \mathrm{Sym}^k W$, where $k = |K|$ as before. We will see later that $H^*_{\mathcal{E}}(X)$ is the quotient of $\mathrm{Sym} W$ by the ideal generated by the Q_K, with K varying over all primitive collections. For each K, we set

$$Z_K = \oplus_{\rho \in K} \mathcal{O}(-D_\rho),$$

and for each $\ell \le k$ we set

$$\mathfrak{Z}_\ell^K = \textstyle\bigwedge^\ell Z_K \otimes \mathrm{Sym}^{k-\ell} W.$$

Then the $\mathfrak{Z}_\ell^K$ are the terms of a subcomplex of (22). Note that

$$\mathfrak{Z}_k^K \simeq \mathcal{O}\left(-D_K\right),$$

and so $H^{k-1}(X, \mathfrak{Z}_k^K)$ is isomorphic to $\mathbb{C}$ by Proposition 3.3.

The following Lemma gives a procedure for computing Q_K. We will identify the cohomology of coherent sheaves $\mathcal{F}$ on X with Čech cohomology of the standard affine open cover $\{U_\sigma\}$ of X associated with the maximal cones σ of Σ, with values in $\mathcal{F}$. Using this open cover, we will denote p-cochains with values in $\mathcal{F}$ by $C^p(\mathcal{F})$ and p-cocycles by $Z^p(\mathcal{F}) = \ker \delta : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$.

Lemma 4.7.

i) *Given* $z_k \in Z^{k-1}(\mathfrak{Z}_k^K)$, *then one can simultaneously choose* $z_\ell \in C^{\ell-1}(\mathfrak{Z}_\ell^K)$ *so that* $\delta(z_\ell) = \alpha_{\ell+1}(z_{\ell+1})$ *for each* $1 \leq \ell \leq k-1$, *where the* α_i *are given by (21). Furthermore, the cochains* z_ℓ *can be chosen so that each of their components is a homogeneous polynomial in the bundle moduli* $\{a_{\rho m}\}$, *of degree* $k-\ell$.

ii) *Given choices of the* z_ℓ *as above, then* $\alpha_1(z_1) \in Z^0(\mathrm{Sym}^k W \otimes \mathcal{O}) \simeq \mathrm{Sym}^k W$ *depends only on the cohomology class* $[z_k] \in H^{k-1}(\mathfrak{Z}_k^K)$ *of* z_k *and not on the choice of representative* z_k *of that cohomology class or any of the choices made for* z_ℓ *above.*

iii) $\alpha_1(z_1) \in \ker(m_k)$.

Proof. For *i*), it is enough to show that the z_ℓ can be successively chosen in descending order to satisfy the required property. To start the induction, note that z_k is independent of the bundle moduli, so the required homogeneity is trivially satisfied. If z_ℓ has been chosen with $\ell > 1$, then

$$\delta(\alpha_\ell(z_\ell)) = \alpha_\ell(\delta(z_\ell)) = \alpha_\ell(\alpha_{\ell+1}(z_\ell)) = 0,$$

so that $\alpha_\ell(z_\ell) \in Z^{\ell-1}(\mathfrak{Z}_{\ell-1}^K)$ is a cocycle. Furthermore, since the nonzero components of α_ℓ are linear in the bundle moduli, it follows that the components of $\alpha_\ell(z_\ell)$ are homogeneous of degree $k-\ell+1$ in the bundle moduli. Since $H^{\ell-1}(\mathfrak{Z}_{\ell-1}^K) = 0$ by Proposition 3.3, it follows that $\alpha_\ell(z_\ell)$ is a coboundary so can be written as $\delta(z_{\ell-1})$ for some $z_{\ell-1} \in C^{\ell-2}(\mathfrak{Z}_{\ell-1}^K)$. Finally, $z_{\ell-1}$ can easily be chosen to be homogeneous of degree $k-(\ell-1)$ in the bundle moduli. We can for example take any $\mathbb{C}$-basis $\{u_\alpha\}$ for $Z^{\ell-1}(\mathfrak{Z}_{\ell-1}^K)$ and write $u_\alpha = \delta(v_\alpha)$ for some $v_\alpha \in C^{\ell-2}(\mathfrak{Z}_{\ell-1}^K)$. We can then write $\alpha_\ell(z_\ell) = \sum_\alpha f_\alpha u_\alpha$ for some f_α which are homogeneous functions of degree $k-(\ell-1)$ in the bundle moduli, and then take $z_{\ell-1} = \sum f_\alpha v_\alpha$.

For the first part of *ii*), note that $\alpha_1(z_1) \in Z^0(\mathrm{Sym}^k W \otimes \mathcal{O})$ is clear by $\delta(\alpha_1(z_1)) = \alpha_1(\delta(z_1)) = \alpha_1(\alpha_2(z_2)) = 0$.

Next, we prepare for an induction on ℓ by claiming that if $z_{\ell+1}, \ldots, z_k$ are fixed and choices are only allowed to be made in $z_1, \ldots, z_\ell$, then $\alpha_1(z_1)$ is independent of the allowed choices. The assertion of *ii*) is simply this claim for $\ell = k$.

Fix $\ell < k$ and suppose that z_r has been chosen for all $r > \ell$ and we want to choose z_ℓ so that $\delta(z_\ell) = \alpha_\ell(z_{\ell+1})$. Once any z_ℓ is chosen, then any other choice differs from z_ℓ by a cocycle in $Z^{\ell-1}(\mathfrak{Z}_\ell^K)$, which is necessarily a coboundary δy_ℓ for some $(\ell-2)$-cochain $y_\ell \in C^{\ell-2}(\mathfrak{Z}_\ell^K)$, since $H^{\ell-1}(\mathfrak{Z}_\ell^K) = 0$. If $\ell = k$, a separate argument is required, but the conclusion is the same: the only other choice for z_k is to modify it by the addition of a coboundary δy_k.

We start the induction with $\ell = 1$. Since $Z^0(\mathfrak{Z}_1^K) = H^0(\mathfrak{Z}_1^K) = 0$, there are no nontrivial cocyles, so that z_1 is unique and the independence is trivially true for $\ell = 1$. Now suppose that the claim is true for some $\ell < k$ and we show that it is true for $\ell+1$. As noted above, we can only change the choice of $z_{\ell+1}$ to $z_{\ell+1} + \delta(y_{\ell+1})$ and then we see what that does to the rest of the computation. Then $\alpha_{\ell+1}(z_{\ell+1})$ is replaced by

$$\alpha_{\ell+1}(z_{\ell+1} + \delta(y_{\ell+1})) = \alpha_{\ell+1}(z_{\ell+1}) + \delta(\alpha_{\ell+1}(y_{\ell+1})) = \delta(z_\ell) + \delta(\alpha_{\ell+1}(y_{\ell+1})).$$

Thus, we can continue the computation by replacing z_ℓ by $z_\ell + \alpha_{\ell+1}(y_{\ell+1})$. Other choices for modifying z_ℓ are possible, but by the inductive hypothesis, other choices won't affect $\alpha_1(z_1)$. At the next step, $\alpha_\ell(z_\ell)$ gets replaced by

$$\alpha_\ell(z_\ell + \alpha_{\ell+1}(y_{\ell+1})) = \alpha_\ell(z_\ell) + \alpha_\ell(\alpha_{\ell+1}(y_{\ell+1})) = \alpha_\ell(z_\ell),$$

and no change in $z_{\ell-1}$ is required. The inductive hypothesis takes care of the rest.

For *iii*), let $i_k : H^{k-1}(\mathfrak{Z}_k^{(k)}) \to H^{k-1}(S_{k-1}^{(k)})$ be the map in (35). Then we have constructed $\alpha_1(z_1)$ so that

$$(\delta_{k-2} \circ \cdots \circ \delta_0)\,(\alpha_1(z_1)) = i_k\,([z_k])\,.$$

By the exactness of (35), the image of i_k is the kernel of δ_{k-1}. So

$$m_k\,(\alpha_1(z_1)) = (\delta_{k-1} \circ \cdots \circ \delta_0)\,(\alpha_1(z_1)) = \delta_{k-1}\,(i_k\,([z_k])) = 0$$

as claimed. □

Note that Lemma 4.7 gives rise to a well-defined map

$$\ell_K : H^{k-1}\big(\mathcal{O}(-D_K)\big) \to \mathrm{Sym}^k W, \qquad \ell_k([z_k]) = \alpha_1(z_1). \tag{36}$$

We now examine the form of the image of ℓ_K. Let $z \in H^{k-1}(\mathfrak{Z}_k^K)$, and write $K = \{\rho_1, \ldots, \rho_k\}$.
Claim: $\ell_K(z)$ has the form

$$\sum_{m_1+m_2+\cdots+m_k=0} a_{\rho_1 m_1} \cdots a_{\rho_k m_k} \tag{37}$$

where each $m_i \in \Delta_{D_{\rho_i}} \cap M$.

Without the qualifier $m_1 + \ldots + m_k = 0$ in the sum, the claim follows immediately from part i) of Lemma 4.7. We put $z_k = z$ and apply the lemma, obtaining $z_1 \in C^0(\mathfrak{Z}_1^K)$, each of whose components is homogeneous of degree $k-1$ in the bundle moduli. Then $\ell_K(z) = \alpha_1(z_1)$ is homogeneous of degree k, i.e. has the required form. The restriction to $m_1 + \ldots + m_k = 0$ occurs because $a_{\rho_1 m_1} \cdots a_{\rho_k m_k}$ only arises in combination with a factor of $\chi^{m_1+\ldots+m_k}$, and $\ell_K(z) \in H^0(\mathrm{Sym}^k W \otimes \mathcal{O})$, which has pure weight 0. This proves the claim.

LEMMA 4.8. *Any primitive collection containing an edge ρ necessarily contains the edge ρ' if $D_{\rho'}$ is linearly equivalent to D_ρ.*

Proof. To see this, suppose $K = \{\rho_1, \ldots, \rho_k\}$ is a primitive collection. Following [Bat93], we have that $v_{\rho_1} + \ldots + v_{\rho_k}$ lies in the relative interior of a unique cone $\sigma \in \Sigma$, whose set of edges are disjoint from K. Letting the edges of σ be generated by primitive vectors $w_1, \ldots, w_l$ we then have an identity

$$v_{\rho_1} + \ldots + v_{\rho_k} = \sum_{i=1}^{l} c_i w_i \tag{38}$$

with all $c_i > 0$.

Suppose there is an edge ρ' such that $D_{\rho'}$ is linearly equivalent to one of the D_{ρ_i}. Then there exists an $m \in M$ such that

$$\langle m, v_{\rho'} \rangle = 1, \quad \langle m, v_{\rho_i} \rangle = -1, \quad \langle m, v' \rangle = 0$$

where v' is any edge generator distinct from v_{ρ_i} or $v_{\rho'}$. Applying $\langle m, \cdot \rangle$ to (38) we obtain

$$\sum_{j=1}^{k} \langle m, v_{\rho_j} \rangle = \sum_{i=1}^{l} c_i \langle m, w_i \rangle. \tag{39}$$

The only negative term in this equation is $\langle m, v_{\rho_i}\rangle$ on the left hand side. Therefore the left hand side must also contain a strictly positive term. This can only happen if $v_{\rho'}$ is one of the v_{ρ_j}. □

For later use, we note that the same argument shows that the $\{w_i\}$ are closed under the relation of linear equivalence of the corresponding divisors, and in fact linearly equivalent terms have to arise with identical coefficients c_i in (38) as the only way to get the identity $0 = c_i - c_j$.

We let $[K]$ denote the set of equivalence classes of edges in K induced by linear equivalence of the corresponding divisors. Similarly, we let $[K^-]$ denote the set of equivalence classes of the edges spanned by the w_i induced by linear equivalence of the corresponding divisors.

We now discuss the linear terms of E in more detail. For each ρ, write

$$E_\rho = \sum_{m\in\Delta_{D_\rho}\cap M} a_{\rho m}\, x_\rho \prod_{\rho'} x_{\rho'}^{\langle m, v'_\rho\rangle}.$$

The linear monomials occurring in E_ρ come in two forms: x_ρ, and $x_{\rho'}$ for certain $\rho' \neq \rho$. In the first case, the monomial x_ρ corresponds to $m = 0 \in \Delta_{D_\rho}$. In the second case, $x_{\rho'}$ for $\rho' \neq \rho$ can be a term in E_ρ if for some $m \in M$ we have

$$\langle m, v_{\rho'}\rangle = 1,\quad \langle m, v_\rho\rangle = -1,\ \text{and}\ \langle m, v_{\rho''}\rangle = 0$$

for all $\rho'' \neq \rho, \rho'$.

Note the linear monomials $x_{\rho'}$ correspond to linear equivalences $D_{\rho'} \sim D_\rho$. The associated m is the one corresponding to the character whose divisor satisfies

$$(\chi^m) = D_{\rho'} - D_\rho.$$

Now partition the divisors D_ρ into linear equivalence classes, inducing a corresponding equivalence relation on $\Sigma(1)$: the linear part of E_ρ becomes

$$E_\rho^{\mathrm{lin}} = \sum_{\rho'\in[\rho]} a_{\rho\rho'} x^{\rho'}, \tag{40}$$

where $[\rho]$ is the equivalence class of ρ under the equivalence relation just described.

For each such equivalence class $c = [\rho]$, we define a $|c| \times |c|$ matrix A_c with entries in W, where $|c|$ is the size of the equivalence class, i.e. the number of divisors $D_{\rho'}$ linearly equivalent to D_ρ (including D_ρ itself). The rows and columns of A_c can be naturally identified with the edges comprising the equivalence class. The (ρ, ρ') entry of A_c is the coefficient $a_{\rho\rho'}$ in (40) above.

Definition 4.9. *The matrix A_c is the* linear coefficient matrix *associated to the divisor class corresponding to c.*

Remark 4.10. Note that this matrix is precisely the one denoted $A^a_{(\alpha)}$ in equation 2.5 of [MM08].

Remark 4.11. For $\mathcal{E} = TX$ with its standard Euler sequence, A_c is diagonal with diagonal terms $[D_\rho]$.

Lemma 4.12. *The image of ℓ_K only depends on the linear terms of E.*

Proof. We show that each term of (37) depends only on the linear terms. Since each m_j is in $\Delta_{D_{\rho_j}}$, we know that $\langle m_j, v_{\rho_j}\rangle \geq -1$ while $\langle m_j, v_\rho\rangle \geq 0$ for $\rho \neq \rho_j$. Then the vanishing of $m_1 + \ldots + m_k$ implies that $\sum_{i=1}^k \langle m_i, v_{\rho_\ell}\rangle = 0$ for all ℓ, so we are left with two possibilities for each j:

1. $m_j = 0$
2. $m_j \neq 0$, and there exists a unique i with $i \neq j$ such that

$$\langle m_j, v_{\rho_\ell}\rangle = \begin{cases} 1 & \ell = i \\ -1 & \ell = j \\ 0 & \text{otherwise.} \end{cases}$$

As we have seen above, these m_i each are associated with linear terms. □

We can at last explicitly describe the image of ℓ_K.

LEMMA 4.13. *Let K be a primitive collection and $[K] = \{[D_\rho] | \rho \in K\}$ as before. Then the image of ℓ_K is the 1-dimensional space spanned by*

$$Q_K := \prod_{c \in [K]} \det A_c,$$

where the product is over the linear equivalence classes contained in K.

Proof. Our application of Theorem 3.1 to compute

$$H^{k-1}(X, \mathcal{O}(-D_K)) \simeq \widetilde{H}^{k-2}(S^{k-2}) \simeq \mathbb{C}$$

gives us more than the computation of the dimension of the cohomology group. If we choose an orientation of S^{k-2} (determined by an ordering of the edges in K), the fundamental class of S^{k-2} gives an almost canonical generator $\gamma \in H^{k-1}(X, \mathcal{O}(-D_K))$ depending only on this orientation.

If we now choose any two edges ρ and ρ' in the same equivalence class and interchange them in the ordering of K, this changes the orientation of S^{k-2}, and hence the sign of γ, while switching E_ρ and $E_{\rho'}$ and leaving everything else about the input data unchanged. Then $\ell_K(\gamma)$ is changed to $-\ell_K(\gamma)$. Thus, given an ordered primitive collection, $\ell_K(\gamma) \in \mathrm{Sym}^k W$ is a degree k polynomial function of the $a_{\rho m}$, completely antisymmetric within each equivalence class c of edges corresponding to linear equivalence of divisors. We conclude that up to multiple it is necessarily the product of determinants associated with blocks of linearly equivalent D_i. □

Since the expression $\det A_c$ will arise frequently in our computations, we write

$$Q_c := \det A_c.$$

In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, there are two linear equivalence classes of toric divisors: $\{D_1, D_2\}$ and $\{D_3, D_4\}$. The associated matrices and determinants are then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; A' = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}, \qquad Q = \det A, \; Q' = \det A' \tag{41}$$

in the notation introduced in Section 3.1.

For a perfect analogy with (12), we would have to express the polymology as a quotient of the homogeneous coordinate ring. Instead, we content ourselves with a description of the polymology as a quotient of Sym^*W. We define the polymological analogue of the Stanley-Reisner ideal in Sym^*W.

Definition 4.14. *The* Stanley-Reisner ideal of $\mathcal{E}$ *is*

$$SR(X,\mathcal{E}) = (Q_K \mid K \text{ a primitive collection of } \Sigma) \subset Sym^*W. \tag{42}$$

Remark 4.15. If $\mathcal{E} = TX$, then $Q_K = \prod_{\rho\in K}[D_\rho]$, and $SR(X, T_X)$ is the image of the usual Stanley-Reisner ideal $SR(X)$ in Sym^*W under the quotient (11).

We now have the main theorem of this section.

Theorem 4.16. *Let X be a smooth projective toric variety, $W = Pic(X)\otimes_{\mathbb{Z}}\mathbb{C}$, and $\mathcal{E}\to X$ a toric deformation of the tangent bundle of X. Then the polymology of $\mathcal{E}$ is isomorphic as a graded algebra to*

$$H^*_{\mathcal{E}}(X) \simeq Sym^*W/SR(X,\mathcal{E}) \tag{43}$$

Proof. The map $m_k : \mathrm{Sym}^*W \to H^*_{\mathcal{E}}(X)$ is surjective by Proposition 4.5. We only have to note that its kernel is $SR(X,\mathcal{E})$, by (35), Remark 4.6, and Lemma 4.13. □

5. Quantum sheaf cohomology. In the following, we denote the Mori cone of X by $\mathrm{NE}(X) \subset H_2(X,\mathbb{R})$, and its integral points by $\mathrm{NE}(X)_{\mathbb{Z}} = \mathrm{NE}(X)\cap H_2(X,\mathbb{Z})$. Since for a smooth toric variety the Mori cone is generated by the curves associated to the cones in $\Sigma(n-1)$ (see e.g. [CLS11]), $\mathrm{NE}(X)$ is polyhedral.

5.1. Moduli space. For each $\beta\in\mathrm{NE}(X)_{\mathbb{Z}}$, we describe the GLSM moduli space variety X_β after [Wit93, MP95]. We will think of X_β as a compactification of the space of holomorphic maps $\mathbb{P}^1\to X$ of class β, although in the GLSM the β arise instead as a natural index for the topological type of the gauge bundle on $\mathbb{P}^1$.

Fix a $\beta\in\mathrm{NE}(X)_{\mathbb{Z}}$, and let $d^\beta_\rho = D_\rho\cdot\beta$. An actual map $f:\mathbb{P}^1\to X$ can be described in homogeneous coordinates as

$$x_\rho = f_\rho, \qquad f_\rho \in H^0(\mathbb{P}^1,\mathcal{O}(d^\beta_\rho)).^2 \tag{44}$$

Accordingly, put

$$\mathbb{C}_\beta = \bigoplus_\rho H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(d^\beta_\rho)).$$

There is a natural action of $G = \mathrm{Hom}(\mathrm{Pic}(X),\mathbb{C}^*)$ on $\mathbb{C}_\beta$, where G acts on f_ρ as multiplication by $g([D_\rho])$. Thinking of a point in $\mathbb{C}_\beta$ as a map $\mathbb{C}^2\to\mathbb{C}^{\Sigma(1)}$, we define a set $Z_\beta\subset\mathbb{C}_\beta$ to be those tuples of sections defining a map whose image is contained in $Z(\Sigma)\subset\mathbb{C}^{\Sigma(1)}$. One can easily check that

$$X_\beta = (\mathbb{C}_\beta - Z_\beta)/G \tag{45}$$

is a toric variety.[3]

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[2] In the GLSM, the f_ρ are identified with the zero modes of certain charged chiral fields in the theory.

[3] In the GLSM, the constraint leading to the avoidance of Z_β arises from FI terms.

We can alternatively describe X_β as the toric variety X_{Σ_β} associated to a fan Σ_β. Let (t_0, t_1) be homogeneous coordinates on $\mathbb{P}^1$ and for each ρ with $d^\beta_\rho \geq 0$ write

$$f_\rho = \sum_{i=0}^{d^\beta_\rho} f_{\rho_i} t_0^i t_1^{d^\beta_\rho - i}. \tag{46}$$

Let $\mathbb{C}^*_\beta \subset \mathbb{C}_\beta$ be the subset on which all f_{ρ_i} are nonzero. Then $T_\beta = \mathbb{C}^*_\beta / G \subset X_\beta$ is a dense torus acting on X_β, giving it the structure of a toric variety. Defining the lattice of 1-parameter subgroups of T_β as N_β, we have a fan Σ_β in $N_\beta \otimes \mathbb{R}$ whose edges $\Sigma_\beta(1)$ naturally correspond to T_β-invariant divisors D_{ρ_i} defined by $f_{\rho_i} = 0$.

We would like to identify the edges in $\Sigma_\beta(1)$ as ρ_i with $\rho \in \Sigma(1)$ and $i = 0, \ldots, d^\beta_\rho$. Then the f_{ρ_i} would be naturally identified with the homogeneous coordinates of X_β.

There is however a subtlety in that not all of the divisors D_{ρ_i} defined by $f_{\rho_i} = 0$ correspond to edges in the fan Σ_β, since it can happen that D_{ρ_i} can be empty. Before explaining how these arise in general, an example will clarify the phenomenon.

We consider the Hirzebruch surface F_n described as a 2-dimensional toric variety whose edges $\rho_1 \ldots, \rho_4$ are respectively spanned by the vectors

$$v_1 = (1, 0),\ v_2 = (-1, n),\ v_3 = (0, 1),\ v_4 = (0, -1).$$

Related aspects of this example are also discussed in [DGKS11].

Let $\beta = D_{\rho_3}$, the class of the $-n$ curve. Then the $d^\beta_{\rho_i}$ are respectively $1, 1, -n, 0$ for $i = 1, \ldots, 4$, and so

$$\mathbb{C}_\beta = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))^2 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}).$$

Note that $f_{\rho_3} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-n))$ is identically zero, so that Z_β contains the hyperplane $f_{(\rho_4)_0} = 0$ owing to the primitive collection $K = \{\rho_3, \rho_4\}$ in the fan Σ of F_n. It follows that $D_{(\rho_4)_0}$ is empty, hence $(\rho_4)_0$ is not in the fan Σ_β. In fact, Σ_β is readily identified with the standard toric fan for $\mathbb{P}^3$, with the four edges $(\rho_i)_j$ for $i = 1, 2$ and $j = 0, 1$. We have $\Sigma_\beta(1) = \{(\rho_i)_j \mid\ i = 1, 2,\ \text{and } j = 0, 1\}$.

We now explain when D_{ρ_i} can be empty. If $d^\beta_\rho < 0$, then there are no f_{ρ_i}. If $d^\beta_\rho > 0$, then f_{ρ_i} can be zero while f_ρ from (46) is not identically zero, so D_{ρ_i} is certainly nonempty. It follows that D_{ρ_i} is empty precisely when $d^\beta_\rho = 0$ and for some primitive collection K of Σ containing ρ we have $d^\beta_{\rho'} < 0$ for all $\rho' \neq \rho$ in K.

We will consider these ρ_0 as *degenerate edges* of Σ_β.

To formalize these considerations and facilitate a uniform treatment, we introduce the notation

$$\widehat{\Sigma}_\beta(1) = \left\{\rho_i \mid \rho \in \Sigma(1),\ 0 \leq i \leq d^\beta_\rho\right\}. \tag{47}$$

In (47), $\widehat{\Sigma}_\beta(1)$ is a set of formal symbols. We can and will identify those ρ_i whose associated divisor D_{ρ_i} is nonempty with edges ρ_i of the fan $\Sigma_\beta(1)$. No confusion should result from this slight abuse of notation. Thus, $\widehat{\Sigma}_\beta(1)$ is an enhancement of the set of edges of the fan Σ_β by the degenerate edges.

In the case of F_n, we have $\widehat{\Sigma}_\beta(1) = \Sigma_\beta(1) \cup \{(\rho_4)_0\}$.

With this understanding, we can now alternatively specify the fan Σ_β by specifying the primitive collections within the set $\widehat{\Sigma}_\beta(1)$ of enhanced edges.

Recalling that $Z(\Sigma)$ is the union of the linear subspaces L_K, we infer that Z_β is the union over all K of the subspaces of sections defining maps whose images are contained in $L_K \subset \mathbb{C}^{\Sigma(1)}$. If $K = \{\rho_1, \ldots, \rho_k\}$, then this subspace is defined by imposing $f_{\rho_{j\,i}} = 0$ for $1 \leq j \leq k$.

Accordingly, for each primitive collection $K \subset \Sigma(1)$, define $K_\beta \subset \widehat{\Sigma}_\beta(1)$ as the set of all edges corresponding to those in K:

$$K_\beta = \left\{ \left\{ \rho_i \in \widehat{\Sigma}_\beta(1) \right\} \mid \rho \in K \right\}.$$

It is straightforward to see that the K_β are the primitive collections for X_β, with a natural extension of the notion of primitive collection to $\widehat{\Sigma}_\beta(1)$. This is essentially because we can recognize Z_β as the union of the linear subspaces L_{K_β}. Furthermore, the fan Σ_β consists of the cones spanned by all collections of edges in $\Sigma_\beta(1)$ that do not contain any primitive collection K_β. These general constructions of toric geometry identify Z_β with $Z(\Sigma_\beta)$. Note that if ρ_0 is a degenerate edge, then by this definition, the singleton set $\{\rho_0\}$ is a primitive collection. But this is exactly what we want, since ρ_0 is not part of the actual fan.

For later use, we introduce the following numerical function.

Definition 5.1. *Define the function $h^0 : \mathbb{Z} \to \mathbb{Z}$ by*

$$h^0(x) = \begin{cases} x+1 & x \geq -1 \\ 0 & x \leq -1, \end{cases}$$

or more concisely, $h^0(x) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(x)) = \max(0, x+1)$.

Note that for each edge $\rho \in \Sigma(1)$ there are $h^0(d^\beta_\rho)$ edges in $\Sigma_\beta(1)$.

Proposition 5.2. *For all $\beta \in NE(X)_\mathbb{Z}$, X_β is a smooth projective toric variety.*

Proof. See [CFK10, Example 7.2.3] for a proof that X_β is projective, and we have already explained that it is toric. For smoothness, it is elementary to show that a simplicial toric variety X is smooth if and only if, upon writing $X = (\mathbb{C}^{\Sigma(1)} - Z(\Sigma))/G$, the action of G on $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ is free. Thus, the smoothness of X implies that G acts freely on $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$.

This easily implies that the action of G on $\mathbb{C}^{\Sigma_\beta(1)} - Z(\Sigma_\beta)$ is also free as follows. Let $f \in \mathbb{C}^{\Sigma_\beta(1)} - Z(\Sigma_\beta)$ with f expressed in terms of the f_{ρ_i} as in (46). Then for any primitive collection K and edge $\rho \in K$, we can't have $f_{\rho_i} = 0$ for all i by the definition of K_β. It follows that the $f_{\rho_i}(t_0, t_1)$ can't all vanish for generic $(t_0, t_1) \in \mathbb{C}^2$, hence there exists $(t_0, t_1) \in \mathbb{C}^2$ with $f(t_0, t_1) \notin Z$, where $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)}$ has coordinates $f_\rho(t_0, t_1)$. If in addition such an f is a fixed point, then $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ would also be a fixed point, contradicting the smoothness of X. Therefore X_β is smooth as well. ◻

Remark 5.3. $X_0 = X$; the moduli space of constant maps is X itself.

5.2. Induced sheaf.

Since X_β is a smooth projective toric variety, we have its Euler exact sequence

$$0 \to \mathcal{O}_{X_\beta} \otimes_\mathbb{Z} \mathrm{Pic}(X_\beta)^\vee \xrightarrow{F_0} \bigoplus_{\rho_i \in \Sigma_\beta(1)} \mathcal{O}_{X_\beta}(D_{\rho_i}) \to T_{X_\beta} \to 0, \tag{48}$$

where the ρ_i^{th} component of the morphism F_0 is $f_{\rho_i} \otimes [D_{\rho_i}]$.

Recalling that D_{ρ_0} is empty for a degenerate edge ρ_0, by adding trivial line bundles to each of the first two nonzero bundles in (48) for each degenerate edge, we obtain a modification of the Euler sequence of X_β

$$0 \to \mathcal{O}_{X_\beta} \otimes W^\vee \xrightarrow{\hat{F}_0} \bigoplus_{\rho_i \in \widehat{\Sigma}_\beta(1)} \mathcal{O}_{X_\beta}(D_{\rho_i}) \to T_{X_\beta} \to 0, \tag{49}$$

where the ρ_i^{th} component of the morphism $\hat{F}_0$ is $f_{\rho_i} \otimes [D_\rho]$.

We add a few words of clarification on the relationship between (48) and (49), even though it is not essential for the sequel.

Let

$$\widehat{\mathbb{C}}_\beta = \widehat{\bigoplus}_\rho H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_\rho^\beta)),$$

where the hat over the direct sum means that we are omitting the components of $\mathbb{C}_\beta$ associated with degenerate edges. Let $G' \subset G$ be the subgroup which acts as the identity on the linear subspace of $\mathbb{C}_\beta$ defined by the vanishing of the f_{ρ_i} with $\rho_i \in \Sigma_\beta(1)$ (rather than $\widehat{\Sigma}_\beta(1)$). Then we have

$$X_\beta = \left(\widehat{\mathbb{C}}_\beta - \left(Z_\beta \cap \widehat{\mathbb{C}}_\beta\right)\right)/G', \tag{50}$$

where $\widehat{\mathbb{C}}_\beta$ is viewed as a subspace of $\mathbb{C}_\beta$ in the natural way. In fact, (50) is just the usual description of X_β as a quotient constructed from the fan $\Sigma_\beta(1)$.

By (17), $\mathrm{Pic}(X)$ is canonically isomorphic to $\mathrm{Hom}(G, \mathbb{C}^*)$ and $\mathrm{Pic}(X_\beta)$ is canonically isomorphic to $\mathrm{Hom}(G', \mathbb{C}^*)$. The inclusion $G' \hookrightarrow G$ therefore induces a mapping $\mathrm{Pic}(X) \to \mathrm{Pic}(X_\beta)$ which is needed in justifying the claimed relationship between (48) and (49). It is the modified version (49) of the toric euler sequence that gets deformed by a deformation $\mathcal{E}$ of TX. With slight abuse of notation, we will refer to deformations of the map $\hat{F}_0$ as giving rise to toric deformations of the tangent bundle of X_β. No confusion should result.

We now define the induced sheaf $\mathcal{E}_\beta$ precisely as dictated by the GLSM. On X, we defined a bundle $\mathcal{E}$ in terms of sections E_ρ of $\mathcal{O}_X(D_\rho) \otimes W$, for each $\rho \in \Sigma(1)$. We shall now associate to these sections corresponding sections of $\mathcal{O}_{X_\beta}(D_{\rho_i}) \otimes W$, leading to a toric deformation $\mathcal{E}_\beta$ of the tangent bundle of X_β in the modified sense just explained.

We express each E_ρ in terms of homogeneous coordinates; for emphasis we write this as $E_\rho = E_\rho(x)$. Over X_β we then make the substitutions $x_\rho = f_\rho(t_0, t_1)$ for each $\rho \in \Sigma(1)$ to obtain expressions that we abbreviate as $E_\rho(f(t))$. We now collect powers of t_0 and t_1, writing the result as

$$E_\rho(f(t)) = \sum_{i=0}^{d_\rho^\beta} E_{\rho_i}(f) t_0^i t_1^{d_\rho^\beta - i}. \tag{51}$$

The W-valued expressions $E_{\rho_i}(f)$ are then interpreted as the components of a toric deformation $\mathcal{E}_\beta$ of the tangent bundle of X_β in the modified sense, defined by the exact sequence

$$0 \to \mathcal{O}_{X_\beta} \otimes W^\vee \xrightarrow{\hat{F}} \bigoplus_{\rho_i \in \widehat{\Sigma}_\beta(1)} \mathcal{O}_{X_\beta}(D_{\rho_i}) \to \mathcal{E}_\beta \to 0, \tag{52}$$

where the components of $\hat{F}$ are as described above.

Proposition 5.4. *$\mathcal{E}_\beta$ is locally-free if $\mathcal{E}$ is.*

Proof. The local freeness of $\mathcal{E}$ is equivalent to the assertion that the $E_\rho(x)$ span W for all $x \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$—this is just the surjectivity of (19).

Now let $f \in C_\beta - Z_\beta$. Taking a generic $t = (t_0, t_1) \in \mathbb{C}^2$ as in the proof of Proposition 5.2, it follows that $f(t_0, t_1) \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)$, hence the $E_\rho(f(t))$ span W. But (51) says that $E_\rho(f(t))$ is in the span of the $E_{\rho_i}(f)$. It follows immediately that the $E_{\rho_i}(f)$ span W as well, hence E_β is locally free. □

We now turn to the computation of the polymology of $(X_\beta, \mathcal{E}_\beta)$. Since $\mathcal{E}_\beta$ is a toric deformation of the tangent bundle of X_β in the modified sense, its polymology algebra can be computed by the same method as in Section 4, resulting in a description of its polymology as a quotient of the symmetric algebra of W. The only change that is needed is to consider degenerate edges. Recall that for a degenerate edge ρ_0, D_{ρ_0} is empty. So we have to supplement Proposition 3.3 with

$$H^\ell(X_\beta, \mathcal{O}(-D_{\rho_0})) = \begin{cases} \mathbb{C} & \ell = 0 \\ 0 & \ell > 0 \end{cases}$$

for degenerate edges ρ_0. The reasoning in Section 4 produces an element $Q_{[\rho_0]} \in W$ associated to the primitive collection $\{\rho_0\}$ which is in the kernel of the map $W \to H^1(X_\beta, \mathcal{E}_\beta^\vee)$ arising from the long exact sequence associated to the dual of (52). We let $\widehat{SR}(X_\beta, \mathcal{E}_\beta)$ be the ideal in $\mathrm{Sym}^* W$ generated by $SR(X_\beta, \mathcal{E}_\beta)$ and the $Q_{[\rho_0]}$ just described.

To compute $\widehat{SR}(X_\beta, \mathcal{E}_\beta)$, we just need to compute the Q_{K_β}.

Fix a $\rho \in \Sigma(1)$ and write

$$E_\rho = \sum_{\rho'} a_{\rho\rho'} x_{\rho'} + \cdots ,$$

where the sum is over edges ρ' with $D_{\rho'}$ linearly equivalent to D_ρ and the $\cdots$ represent the omitted nonlinear terms. Said differently, A is the linear coefficient matrix A_c associated to the linear equivalence class c containing ρ. Then

$$\begin{aligned} E_\rho(f) &= \textstyle\sum_{\rho'} a_{\rho\rho'} f_{\rho'} + \cdots \\ &= \textstyle\sum_{\rho', i} a_{\rho\rho'} f_{\rho'_i} t_0^i t_1^{d_\beta - i} + \cdots , \end{aligned}$$

so that

$$E_{\rho_i} = \sum_{\rho'} a_{\rho\rho'} f_{\rho'_i} + \cdots . \tag{53}$$

Denoting the analogue of A_c for $\mathcal{E}_\beta$ by $(A_\beta)_c$, equation (53) says that $(A_\beta)_c$ has a block diagonal form consisting of $h^0(d_\beta)$ copies of A_c. Thus

$$\det{(A_\beta)}_c = Q_c^{h^0(d_\beta)}.$$

It follows immediately that

$$Q_{K_\beta} = \prod_{c\in[K]} Q_c^{h^0(D_c\cdot\beta)} \tag{54}$$

and

$$H^*_{\mathcal{E}_\beta} = \mathrm{Sym}^* W/\widehat{SR}(X_\beta, \mathcal{E}_\beta),$$

where

$$\widehat{SR}(X_\beta, \mathcal{E}_\beta) = \big(Q_{K_\beta} \mid K \text{ a primitive collection of } \Sigma\big) \subset \mathrm{Sym}^* W.$$

Note that if K is a primitive collection of Σ containing an edge ρ of Σ with ρ_0 a degenerate edge of Σ_β, then from (54) we get $Q_{K_\beta} = Q_{[\rho]}$, where $[\rho] = \{\rho\}$ is the linear equivalence class of ρ. But this is precisely the linear part of $\widehat{SR}(X_\beta, \mathcal{E}_\beta)$ by our earlier discussion.

As with the classical case, we will define the correlation functions in the sector labelled by β as elements of the one-dimensional vector space $H^{n_\beta}(X_\beta, \bigwedge^{n_\beta}\mathcal{E}_\beta)$, where n_β is the dimension of X_β. The precise definition will be spelled out later, but first we have to grapple with the normalization issue.

The correlation functions will be obtained as usual by adding the contributions over the sectors β. However, since these contributions live in *different* one-dimensional vector spaces, we will describe a distingushed space H^*, together with a collection of isomorphisms

$$H^{n_\beta}_{\mathcal{E}_\beta} \simeq H^*$$

for each β, in which the contributions will be summed. This will be accomplished by assembling the $H^{n_\beta}_{\mathcal{E}_\beta}$ into a direct system in the next section.

5.3. Direct system of polymologies. For every $\beta \in H_2(X,\mathbb{Z})$, we have constructed an induced deformation $\mathcal{E}_\beta$ of the toric Euler sequence on X_β. The algebra $H^*_{\mathcal{E}_\beta}(X_\beta)$ is generated by elements of $H^1(X_\beta, \mathcal{E}_\beta^\vee) \simeq W$.

We now construct a direct system from these polymologies and show that the one-dimensional spaces $H^{n_\beta}(X_\beta, \bigwedge^{n_\beta}\mathcal{E}_\beta^\vee)$ are preserved by the maps of the direct system, hence by restriction also give a direct system. We will show that the direct limit is a one-dimensional vector space, in which the correlation functions take their values.

For each c corresponding to a linear equivalence class of divisors D_ρ, we put $d_c^\beta = d_\rho^\beta$, for any ρ in the equivalence class.

Definition 5.5. *For classes $\beta, \beta' \in H_2(X,\mathbb{Z})$, we say that β'* dominates *β if $\beta' - \beta$ is effective and $h^0(d_c^{\beta'}) \geq h^0(d_c^\beta)$ for all linear equivalence classes c of the irreducible toric divisors D_ρ.*

If β' dominates β, we define the expression

$$R_{\beta'\beta} = \prod_c Q_c^{h^0(d_c^{\beta'})-h^0(d_c^\beta)} \in \mathrm{Sym}^* W. \tag{55}$$

Lemma 5.6. *Suppose that β' dominates β. Then*

$$R_{\beta'\beta}\left(SR(X_\beta, \mathcal{E}_\beta)\right) \subset SR(X_{\beta'}, \mathcal{E}_{\beta'}).$$

Proof. We will show that for each primitive collection K, $R_{\beta'\beta}Q_{K_\beta}$ is a multiple of $Q_{K_{\beta'}}$. This will suffice to prove the lemma, by the definitions of $SR(X_\beta, \mathcal{E}_\beta)$ and $SR(X_{\beta'}, \mathcal{E}_{\beta'})$.

For this, it suffices to compare the powers of Q_c occurring in $R_{\beta'\beta}Q_{K_\beta}$ and $Q_{K_{\beta'}}$, for each c. If $c \in [K]$, then the exponent of Q_c in $R_{\beta'\beta}Q_{K_\beta}$ is

$$h^0(d_c^{\beta'}) - h^0(d_c^\beta) + h^0(d_c^\beta) = h^0(d_c^{\beta'}),$$

which is the exponent of Q_c in $Q_{K_{\beta'}}$.

If $c \notin [K]$, then the exponent of Q_c in $R_{\beta'\beta}Q_{K_\beta}$ is $h^0(d_c^{\beta'}) - h^0(d_c^\beta)$, the exponent of Q_c in $Q_{K_{\beta'}}$ is 0, and the required inequality holds by the dominance assumption. $\square$

Whenever β' dominates β, multiplication by $R_{\beta'\beta}$ induces a well-defined map

$$f_{\beta'\beta} : H^*_{\mathcal{E}_\beta}(X_\beta) \to H^*_{\mathcal{E}_{\beta'}}(X_{\beta'})$$

by Lemma 5.6. It is straightforward to verify that the $\{f_{\beta'\beta}\}$ form a direct system. We have to show that the maps are compatible and that any β_1 and β_2 are dominated by some β. Compatibility is obvious, and given any β_1 and β_2, choose a β_3 effective that has positive intersection with each D_c[4] and set $\beta = \beta_1 + \beta_2 + n\beta_3$ for some $n \gg 0$. Then β dominates both β_1 and β_2.

For simplicity of notation, let $H^{n_\beta}_{\mathcal{E}_\beta}$ be the degree n_β part of $\mathrm{Sym}^* W/(SR(X_\beta), \mathcal{E}_\beta)$ (which is canonically isomorphic to $H^{n_\beta}(X_\beta, \bigwedge^{n_\beta} \mathcal{E}^\vee_\beta)$).

Lemma 5.7.

i) *If β' dominates β, then $f_{\beta'\beta}(H^{n_\beta}_{\mathcal{E}_\beta}) \subset H^{n_{\beta'}}_{\mathcal{E}_{\beta'}}$. Thus the maps*

$$g_{\beta'\beta} := f_{\beta'\beta}|_{H^{n_\beta}_{\mathcal{E}_\beta}} : H^{n_\beta} \to H^{n_{\beta'}}_{\mathcal{E}_{\beta'}}$$

also form a direct system.

ii) *If X_β and $X_{\beta'}$ are nonempty, then $g_{\beta'\beta}$ is an isomorphism for deformations $\mathcal{E}$ sufficiently close to TX in the moduli space of toric deformations of TX.*

It follows immediately from Lemma 5.7 that the direct limit

$$H^* := \varinjlim_{g} H^{n_\beta}_{\mathcal{E}_\beta}$$

is a 1-dimensional vector space, and the induced maps $i_\beta : H^*_\beta \to H^*$ are isomorphisms. The correlation functions will all take values in H^*.

Proof of Lemma 5.7. The emptiness of X_β is equivalent to $D_c \cdot \beta < 0$ for all c that are part of some fixed primitive collection K, by the definition of the primitive collections for X_β.

For i), we just have to show that the cohomology degrees are compatible with $g_{\beta'\beta}$.

Noting that Q_c has degree $|c|$, the number of divisors D_ρ in the corresponding linear equivalence class, we see that

$$\begin{aligned} \deg R_{\beta'\beta} &= \textstyle\sum_c |c| \left(h^0(d_c^{\beta'}) - h^0(d_c^\beta) \right) \\ &= \textstyle\sum_\rho \left(h^0(d_\rho^{\beta'}) - h^0(d_\rho^\beta) \right). \end{aligned}$$

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[4] An intersection of ample divisors suffices.

Thus we must show

$$n_{\beta'} = n_\beta + \sum_\rho \left(h^0(d_\rho^{\beta'}) - h^0(d_\rho^\beta) \right). \tag{56}$$

Before computing $n_\beta = \dim X_\beta$ we note that for a general toric variety X we have

$$\dim X = |\Sigma(1)| - h^2(X) = \left(\sum_\rho 1 \right) - h^2(X),$$

as follows from the quotient description (15) and $\dim G = \operatorname{rank}(Pic(X)) = h^2(X)$. Applying the same calculation to X_β, we count the edges in $\Sigma_\beta(1)$ and recall that $h^2(X_\beta) = h^2(X)$ to conclude

$$n_\beta = \dim X_\beta = \left(\sum_\rho h^0(d_\rho^\beta) \right) - h^2(X), \tag{57}$$

and (56) follows immediately.

For ii), we just have to show that the map $g_{\beta'\beta}$ of one-dimensional vector spaces is nonzero.

Claim. If $R_{\beta'\beta} \notin SR(X_{\beta'}, \mathcal{E}_{\beta'})$, then $g_{\beta'\beta}$ is an isomorphism.

To justify the claim, the hypotheses can be restated as saying that $[R_{\beta'\beta}]$ is nonzero in $H^*_{\mathcal{E}_{\beta'}}$. Since $X_{\beta'}$ is smooth and $\mathcal{E}_{\beta'}$ is locally free, $H^*_{\mathcal{E}_{\beta'}}$ satisfies Poincaré duality. Thus, there exists an element $p \in \operatorname{Sym}^* W$ such that $[pR_{\beta'\beta}]$ is a nonzero element of $H^{n_{\beta'}}_{\mathcal{E}_{\beta'}}$. Thus $g_{\beta'\beta}(p) \neq 0$, justifying the claim.

It therefore suffices to show that $R_{\beta'\beta} \notin SR(X_{\beta'}, \mathcal{E}_{\beta'})$. We do this by first verifying it for $\mathcal{E} = TX$. Once we show that, we have proven the second part of the lemma in a neighborhood of TX by the closedness of the ideal membership condition.

We now assume that $\mathcal{E} = T_X$ and identify the polymology of $\mathcal{E}$ with the cohomology of X. We will show that $g_{\beta'\beta}$ applied to the cohomology class of a point of X_β is the cohomology class of a point of $X_{\beta'}$.[5] Since the class of a point is nontrivial, it follows that $g_{\beta'\beta}$ is nonzero. In this case, it is easy to see that $Q_c = [\prod_{\rho \in c} x_\rho]$ by looking at the components of the toric Euler sequence (9).

Recall that $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ is the homogeneous coordinate ring of X. We have the ring homorphism $S \to \operatorname{Sym}^* W$ defined by taking x_ρ to the cohomology class of D_ρ. Starting with a polynomial p, we can take its image in $\operatorname{Sym}^* W$ and then take a further quotient by $\mathrm{SR}(X)$ to get a cohomology class $[p] \in H^*(X)$. We also let S_β be the homogeneous coordinate ring of X_β, with a similar map to $\operatorname{Sym}^* W$ and we use the notation $[p]_\beta$ for the image of a polynomial in $H^*(X_\beta)$.

If β' dominates β, then S_β can be regarded as a subring of $S_{\beta'}$ in a natural way as follows. Recall that S_β is generated by the variables $x_{\rho_1}, \ldots, x_{\rho_{d_\rho^\beta}}$ while $S_{\beta'}$ is generated by the variables $x_{\rho_1}, \ldots, x_{\rho_{d_\rho^{\beta'}}}$. The dominance assumption implies that $d_\rho^{\beta'} \geq d_\rho^\beta$ for each ρ, so the shared nomenclature of the variables provides a natural embedding of S_β in $S_{\beta'}$. Furthermore $R_{\beta'\beta}$ is simply the image in $\operatorname{Sym} W$ of the

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[5] This matching of point classes appeared in the context of ordinary cohomology in [MP95], and was part of our motivation for the definition of $R_{\beta'\beta}$.

product

$$m = \prod_{\rho} \prod_{i=d_\rho^\beta+1}^{d_\rho^{\beta'}} x_{\rho_i}$$

corresponding to the additional edges added in going from X_β to $X_{\beta'}$.

When we multiply $R_{\beta'\beta}$ as represented by m by the class of a point represented by a product of variables in S_β corresponding to a maximal cone σ of X_β, it is clear that multiplication by m corresponds to simply appending the additional edges $\rho_{d_\rho^\beta+1}, \ldots, \rho_{d_\rho^{\beta'}}$ needed to complete σ to a maximal cone of $X_{\beta'}$. Since the resulting monomial represents the class of a point, a nonzero cohomology class, the resulting monomial in $S_{\beta'}$ cannot be in the Stanley-Reisner ideal.

Part of the above argument appeared in [MP95]. $\square$

5.4. Correlation functions. In this section, we will define the correlation functions. In the half-twisted GLSM, correlation functions in sector β can be nonzero only if the operators have degree $c_1(X)\cdot\beta+\dim(X)$. Since $c_1(X) = \sum_\rho D_\rho$, it follows that

$$c_1(X) \cdot \beta = \sum_\rho d_\rho^\beta,$$

hence

$$c_1(X) \cdot \beta + \dim(X) = \sum_\rho \left(d_\rho^\beta + 1\right) - h^2(X). \tag{58}$$

The formula (58) plays the role of the virtual or expected dimension of Gromov-Witten theory.

Note that (57) and (58) differ only in that $h^0(d_\rho^\beta)$ in (57) is replaced by $d_\rho^\beta + 1$ in (58). These are in fact equal, unless $d_\rho^\beta \leq -2$ for some ρ. Such a situation is the analogue of excess dimension in Gromov-Witten theory. In our situation, we have both the excess dimension of X_β and the excess rank of $\mathcal{E}_\beta$.

In this case, to compensate, we need something to play the role of the obstruction classes of Gromov-Witten theory. These were called four-fermi terms in [KS06] because of how they arose in the path integral.

Define $h^1(x) = h^1(\mathcal{O}_{\mathbb{P}^1}(x))$ in analogy with Definition 5.1. Then our formula for the four-fermi terms is

$$F_\beta = \prod_c Q_c^{h^1(d_c^\beta)}, \tag{59}$$

where the product is over all linear equivalence classes c of the divisors D_ρ.

At last, we can define the correlation functions in sector β. Let

$$p \in \mathrm{Sym}^{c_1(X)\cdot\beta+\dim X} W.$$

A simple computation of degree shows that $pF_\beta \in H^{n_\beta}_{\mathcal{E}_\beta}$: the degree of pF_β is

$$\begin{aligned}\deg(pF_\beta) &= c_1(X)\cdot\beta + \dim X + \sum_c |c| h^1(d_c^\beta)\\ &= \sum_\rho \left(d_\rho^\beta + 1\right) - h^2(X) + \sum_\rho h^1(d_\rho^\beta)\\ &= \sum_\rho h^0(d_\rho^\beta) - h^2(X)\\ &= n_\beta.\end{aligned}$$

In the third line of the above computation, we have used the identity

$$h^0(d_\rho^\beta) - h^1(d_\rho^\beta) = d_\rho^\beta + 1, \tag{60}$$

which is Riemann Roch for $\mathbb{P}^1$.

Finally, the correlation function is defined as

$$\langle p\rangle_\beta = i_\beta\,(p\,F_\beta) \in H^*, \tag{61}$$

where i_β was defined immediately following the statement of Lemma 5.7. Following our discussion at the beginning of this section, if $p \in \mathrm{Sym}^d W$ with $d \neq c_1(X)\cdot\beta + \dim X$, we define $\langle p\rangle_\beta = 0$. By design, all correlation functions live in the same one-dimensional vector space H^*, so can be added over β. To formalize the sum, we recall one version of the Novikov ring.

Definition 5.8. *The* Novikov ring $\mathbb{C}[q^\beta]$ *of* X *is the ring generated over* $\mathbb{C}$ *by the formal expressions* q^β *for each* $\beta \in NE(X)$*, subject to the relations* $q^\beta q^{\beta'} = q^{\beta+\beta'}$.

For any $p \in \mathrm{Sym}^* W$ we define the correlation function

$$\langle p\rangle = \sum_\beta \langle p\rangle_\beta q^\beta \in H^* \otimes \mathbb{C}[q^\beta].$$

5.5. Quantum cohomology ring. In the quasi-topological sector of the half-twisted GLSM, as in all quantum field theories where correlation functions are independent of the insertion point, an operator $\mathcal{O}$ is the trivial operator iff

$$\langle \mathcal{O}, \mathcal{O}_1, \ldots, \mathcal{O}_k\rangle = 0$$

for all operators $\mathcal{O}_1, \ldots, \mathcal{O}_k$. Accordingly, the trivial operators with coefficients in the Novikov ring form an ideal in the ring $\mathrm{Sym}^* W \otimes \mathbb{C}[q^\beta]$, which we suggestively call the *quantum Stanley-Reisner ideal* and write as $QSR(X, \mathcal{E})$.

Definition 5.9. *The* quantum sheaf cohomology *of* $(X, \mathcal{E})$ *is the ring*

$$QH^*_{\mathcal{E}}(X) = \left(Sym^* W \otimes \mathbb{C}[q^\beta]\right)/QSR(X, \mathcal{E}).$$

By definition, if we set all $q^\beta = 0$, the correlation functions become the classical correlation functions described in Section 4 and the quantum Stanley-Reisner ideal $QSR(X, \mathcal{E})$ becomes the ordinary Stanley-Reisner ideal $SR(X, \mathcal{E})$. Thus, $QSR(X, \mathcal{E})$ is a deformation of $SR(X, \mathcal{E})$. Accordingly, we expect $QSR(X, \mathcal{E})$ to be generated by deformations of the generators Q_K of $SR(X,\mathcal{E})$. In fact, passing to the localization to make the comparison, the relations in [MM09] are of just this form. We will view these as predictions for the generators in our set-up and then prove that they are correct.

Fixing a primitive collection K, we rewrite (38) as

$$\sum a_\rho v_\rho = 0, \tag{62}$$

where $a_\rho = 1$ for each $\rho \in K$. Then there exists a unique $\beta_K \in H_2(X,\mathbb{Z})$ such that

$$d_\rho^{\beta_K} = D_\rho \cdot \beta_K = a_\rho \qquad \forall \rho.$$

Furthermore, the β_K generate the cone of effective curves. Details of these assertions can be found in [CLS11].

Then there is proposed a relation

$$\prod_{c\in[K]} Q_c = q^{\beta_K} \prod_{c\in[K^-]} Q_c^{-d_c^{\beta_K}}. \tag{63}$$

Since the left hand side of (63) is just Q_K, (63) says that the quantities

$$Q_K - q^{\beta_K} \prod_{c\in[K^-]} Q_c^{-d_c^{\beta_K}}$$

are in $QSR(X,\mathcal{E})$ and specialize to the generator Q_K of $SR(X,\mathcal{E})$ when the q^β are set to 0.

For $\mathbb{P}^1 \times \mathbb{P}^1$, by checking intersections, we see that for $K = \{\rho_1,\rho_2\}$ we have $\beta_K = \beta_2 := p \times \mathbb{P}^1$ while for $K' = \{\rho_3,\rho_4\}$ we have $\beta_{K'} = \beta_1 := \mathbb{P}^1 \times p$, where p is a point of $\mathbb{P}^1$. Furthermore, the relations among the generators are $v_1 + v_2 = 0$ for K and $v_3 + v_4 = 0$ for K'. It follows that the relations (63) are

$$Q = q^{\beta_2}, \qquad Q' = q^{\beta_1},$$

where Q and Q' have been defined in (41).

By definition, (63) is equivalent to the identity of correlation functions

$$\langle Y \prod_{c\in[K]} Q_c\rangle_{\beta+\beta_K} = \langle Y \prod_{c\in[K^-]} Q_c^{-d_c^{\beta_K}}\rangle_\beta \tag{64}$$

for any $Y \in \mathrm{Sym}^* W$ and $\beta \in H_2(X,\mathbb{Z})$.

THEOREM 5.10. *The quantum cohomology relations (63) hold for all primitive collections K.*

Proof. We show (64) for any $Y \in \mathrm{Sym}^* W$. Choosing β' dominating both β and $\beta + \beta_K$, we have to show the equality

$$R_{\beta',\beta+\beta_K} F_{\beta+\beta_K} Y \prod_{c\in[K]} Q_c = R_{\beta'\beta} F_\beta Y \prod_{c\in[K^-]} Q_c^{-D_c\cdot\beta_K} \tag{65}$$

as elements of the quotient ring

$$\mathrm{Sym}^* W / SR(X_{\beta'}, \mathcal{E}_{\beta'}).$$

In fact, we will see that (65) holds in Sym^*W. For this it suffices to show

$$R_{\beta',\beta+\beta_K}F_{\beta+\beta_K}\prod_{c\in[K]}Q_c = R_{\beta'\beta}F_\beta\prod_{c\in[K^-]}Q_c^{-D_c\cdot\beta_K}. \tag{66}$$

Both sides of (66) expand to products of powers of the Q_c, so we just have to check the exponents of each Q_c. We break this up into three cases, according to whether $d_c^{\beta_K}$ is positive, negative, or zero, or equivalently, $c\in[K]$, $c\in[K^-]$, or c in neither $[K]$ nor $[K^-]$. In any case, we note that

$$d_c^{\beta+\beta_K} = d_c^\beta + d_c^{\beta_K}. \tag{67}$$

If $d_c^{\beta_K}=0$, then the required equality of exponents is

$$h^0(d_c^{\beta'}) - h^0(d_c^{\beta+\beta_K}) + h^1(d_c^{\beta+\beta_K}) = h^0(d_c^{\beta'}) - h^0(d_c^\beta) + h^1(d_c^\beta).$$

However, in this case, $d_c^\beta = d_c^{\beta+\beta_K}$ by (67) and equality is clear.

If $d_c^{\beta_K}>0$ then $c\in[K]$ and the required equality of exponents is

$$h^0(d_c^{\beta'}) - h^0(d_c^{\beta+\beta_K}) + h^1(d_c^{\beta+\beta_K}) + 1 = h^0(d_c^{\beta'}) - h^0(d_c^\beta) + h^1(d_c^\beta).$$

The equality follows immediately from (67), $d_c^{\beta_K}=1$, and two applications of (60) (one time with β replaced by $\beta+\beta_K$).

Finally, if $d_c^{\beta_K}<0$, then $c\in[K^-]$ and we have to show

$$h^0(d_c^{\beta'}) - h^0(d_c^{\beta+\beta_K}) + h^1(d_c^{\beta+\beta_K}) = h^0(d_c^{\beta'}) - h^0(d_c^\beta) + h^1(d_c^\beta) - d_c^{\beta_K},$$

which is easily verified in the same way. □

From physics, we expect this result to generalize to complete intersections in toric varieties. Suppose that $Y\subset X$ is a complete intersection in a toric variety X. Then the tangent bundle of Y is the cohomology of a monad. The cohomology of a small deformation of this monad will be a vector bundle $\mathcal{E}$ on Y. Then Sym^*W generates a subalgebra of the polymology of $\mathcal{E}$, which we call the *toric polymology* of $\mathcal{E}$. We write

$$H^*_{\mathcal{E}}(X)^{\mathrm{toric}} = (\mathrm{Sym}^*W)/\mathrm{SR}(X,\mathcal{E}), \tag{68}$$

where for present purposes $\mathrm{SR}(X,\mathcal{E})$ is defined by (68).

Conjecture 5.11. *There is a* toric quantum sheaf cohomology ring $QH^*_{\mathcal{E}}(X)^{\mathrm{toric}}$ *which is of the form*

$$QH^*_{\mathcal{E}}(X)^{\mathrm{toric}} = (\mathrm{Sym}^*W)/\mathrm{QSR}(X,\mathcal{E}),$$

where $\mathrm{QSR}(X,\mathcal{E})$ *specializes to* $\mathrm{SR}(X,\mathcal{E})$ *after setting all the* q^β *to zero.*

Note that we have proven this conjecture for $Y=X$.

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