EMBEDDINGS OF FIELDS INTO SIMPLE ALGEBRAS OVER GLOBAL FIELDS[*]

SHENG-CHI SHIH[†], TSE-CHUNG YANG[‡], AND CHIA-FU YU[§]

Abstract. Let F be a global field, A a central simple algebra over F and K a finite (separable or not) field extension of F with degree $[K : F]$ dividing the degree of A over F. An embedding of K into A over F exists implies an embedding exists locally everywhere. In this paper we give detailed discussions about when the converse (i.e. the local-global principle in question) may hold.

Key words. Central simple algebras, embeddings, the Hasse principle, Galois cohomology.

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1. Introduction. The topic on central simple algebras over global fields is a central theme in number theory. One way to extract information of a central simple algebra is to study its maximal subfields. For example, this information would be useful when one attempts to analyze the terms in the geometric side of trace formulas. Indeed, in the geometric side of the trace formula for the multiplicative group of a given central simple algebra, certain major terms (elliptic ones) are described by conjugacy classes of these maximal subfields. This leads to a very important approach, due to Eichler [6], to calculating the class number (of an open compact level subgroup), as the class number can be written as the trace of the characteristic function supported on this level subgroup. In his pioneer work [6] Eichler established results on the trace formula for computing the class numbers and the type numbers of a maximal order or a hereditary order in a given quaternion algebra over any global field. He reduced some of the problem of computing class numbers to computing the so called *optimal embeddings* among orders in maximal subfields and those in the quaternion algebras, which are of purely local nature. See Eichler [6] and Vignéras [15] for more details.

A useful tool to study maximal subfields is the Hasse principle, which enables us to describe the properties of these subfields from local information. A naive attempt along the same direction is figuring out whether one can describe subfields of a central simple algebra over a global field F from local information. In this paper we study embeddings of a finite field extension of F, which does not necessarily have the maximal degree, into a central simple algebra over F, and attempted to clarify related problems in this situation. More precisely, consider a finite-dimensional central simple algebra A over a global field F and a finite (separable or not) field extension K of F whose degree divides the *degree* $\deg A$ of A over K, that is, $[K : F] \mid \sqrt{[A : F]} = \deg(A)$. Naturally one considers the following basic questions:

(Q1) What is the necessary and sufficient condition for which the field K can be F-linearly embedded into A?

(Q2) What is the necessary and sufficient condition for which the F_v-algebra K_v can be F_v-linearly embedded into the F_v-algebra A_v? Here v is a place of F,

F_v denotes the completion of F at v, $K_v := K \otimes_F F_v$ and $A_v := A \otimes_F F_v$.

(Q3) Does the local-global principle for embedding the field K into A as F-algebras
hold? That is, if K_v can be F_v-linearly embedded in A_v for all places v of F,
then can K be F-linearly embedded in A?

We say that the Hasse principle for a triple (K, A, F) as above holds if the question (Q3) for K and A over a global field F has a positive answer. When the field extension K has the maximal degree $\deg A$, this result is well known and useful; see Pierce [8, Section 18.4], Prasad-Rapinchuk [9, Proposition A.1] and [10, Proposition 2.7]. However, the case when K does not have the maximal degree is not explored in the literature.

We now outline the contents of this paper. In the first part of this paper we answer the questions (Q1), (Q2) and (Q3). In [17], the third named author of the present paper studies the problem of embeddings of one semi-simple algebra into another one over an arbitrary ground field. In Section 2 we recall some results of embeddings obtained in [17] and provide some proofs of them for the reader's convenience. These give rise to a numerical criterion for the question (Q1) over an arbitrary field; see Lemma 2.4 for the precise statement. In Section 3 we apply results of Section 2 to the case where the base field is either a local field or a global field. This yields a more explicit numerical criterion for (Q1) and (Q2), respectively. Using these criteria, we then show that for any global field F, there is a Galois extension K over F of degree 8 and a central simple algebra A over F of degree 24 for which the Hasse principle does not hold; see an example in Section 3.4. Also see Proposition 1.4 below for other cases for which the Hasse principle fails .

We now describe the results in this part. Let K and A be as above. Let $A = \mathrm{Mat}_n(\Delta)$, where Δ is the division part of A and $n := \mathrm{c}(A)$ is called the *capacity* of A (see Definition 2.1). Recall the degree $\deg(\Delta)$ of Δ is $\sqrt{[\Delta : F]}$. For any place v of F, let $K_v := K \otimes_F F_v = \prod_{w|v} K_w$ be the product of local fields. Let V^F and V^K denote the sets of places of F and K, respectively. Let d_v be the order of the class $[A_v]$ in the Brauer group of F_v, which is also the index of A_v. By an F-embedding of K into A we mean an F-linear embedding of K into A. The notion of F_v-embedding of K_v into A_v is defined similarly.

THEOREM 1.1. *Let the notation be as above.*

1. *The set of $A_v^\times$-conjugacy classes of F_v-embeddings of K_v into A_v is in bijection with the following set*

$$\mathcal{E}_{F_v}(K_v, A_v) := \left\{ (x_w)_{w|v} \mid x_w \in \mathbb{N}, \ \sum_{w|v} \ell_w x_w = \mathrm{c}(A_v) \right\},$$

where

$$\ell_w := [K_w : F_v]/\gcd([K_w : F_v], \deg(\Delta)).$$

In particular, there is an F_v-embedding of K_v into A_v if and only if the finite set $\mathcal{E}_{F_v}(K_v, A_v)$ is non-empty.

2. *There is an F-embedding of K into A if and only if*

$$[K : F] \mid n \, \mathrm{c}(\Delta \otimes_F K),$$

where $\mathrm{c}(\Delta \otimes_F K)$ is the capacity of the central simple algebra $\Delta \otimes_F K$ over K.

3. The number $c(\Delta \otimes_F K)$ *can be computed as follows:*

(i) The degree δ_0 of Δ is the least common multiple of all d'_vs for $v \in V^F$ (the Hasse-Brauer-Noether theorem).

(ii) For each $w|v$, the order d'_w of the class $[A_v \otimes_{F_v} K_w]$ in the Brauer group of K_w is given by

$$d'_w = d_v / \gcd([K_w : F_v], d_v).$$

(iii) The order δ' of the class $[A \otimes_F K]$ in the Brauer group of K is the least common multiple of d'_w for all $w \in V^K$ (the Hasse-Brauer-Noether theorem).

(iv) The capacity $c(\Delta \otimes_F K)$ is given by $c(\Delta \otimes_F K) = \delta_0/\delta'$.

The proof of Theorem 1.1 is given in Sections 3.1 and 3.2; see especially Propositions 3.2, 3.3 and 3.4.

In the last part of Section 3, we give a necessary and sufficient condition for a pair (K, A) so that the Hasse principle in question holds. Here we fix a global field F. We associate to each pair (K, A) an element

$$\bar{\mathbf{x}} = (\bar{\mathbf{x}}_w)_{w \in V^K} \in \bigoplus_{w \in V^K} \mathbb{Q}/\mathbb{Z}$$

as follows. For any $w \in V^K$, put

$$\mathbf{x}_w := \frac{c(A_v) \cdot \gcd([K_w : F_v], d_v)}{[K : F]} \in \mathbb{Q}_{>0}$$

and let $\bar{\mathbf{x}}_w$ be the class of $\mathbf{x}_w$ in $\mathbb{Q}/\mathbb{Z}$, where v is the place of F below w.

We show that the element $\bar{\mathbf{x}}$ is an obstruction class to the Hasse principle for (K, A), i.e. if this class does not vanish, then the Hasse principle for (K, A) fails. We also show that the vanishing of $\bar{\mathbf{x}}$ is the only obstruction to the Hasse principle. Namely, we have the following result (Theorem 3.6).

THEOREM 1.2. *Notations as above. An F-embedding of K into A exists if and only if an F_v-embedding of K_v into A_v exists for all $v \in V^F$ and that the element $\bar{\mathbf{x}}$ vanishes.*

The second part of this paper deals with the Hasse principle for a family of pairs (K, A) over a fixed global field F with both degrees $[K : F]$ and $\deg(A)$ constant. Let (k, δ) be a pair of positive integers with $k \mid \delta$. We say that the Hasse principle for the pair (k, δ) holds if for any finite field extension K over F of degree k and any central simple algebra A over F of degree δ, the Hasse principle for the pair (K, A) holds.

We describe the results in this part. Let (k, δ) be a pair of positive integers as above. For each partition λ of k

$$\lambda = (k_1, \ldots, k_t), \quad k_1 \leq \cdots \leq k_t, \quad \sum_{i=1}^{t} k_i = k$$

and a positive divisor $s|\delta$, we define a finite set

$$\mathcal{E}(\lambda, s) := \{(x_i) \in \mathbb{N}^t \mid \sum_{i=1}^{t} \ell_i x_i = s\},$$

where $\ell_i := k_i/\gcd(k_i, d)$, and a vector $\mathbf{x}(\lambda, s)$ in $\mathbb{Q}^t$, where

$$\mathbf{x}(\lambda, s) := (\mathbf{x}_i), \quad \mathbf{x}_i = \mathbf{x}_i(\lambda, s) := s \gcd(k_i, d)/k, \quad \forall\, i = 1, \ldots, t.$$

Let $\mathrm{LD}(k, \delta)$ be the set consisting of all pairs (λ, s), where
- λ is a partition of k, and
- s is a positive integer dividing δ,

such that the set $\mathcal{E}(\lambda, s)$ is non-empty. ("LD" stands for all possible local decompositions). The following result gives a simple way to check the Hasse principle for a pair (k, δ).

THEOREM 1.3. *Let (k, δ) be a pair of positive integers with $k \mid \delta$.*
(1) *Suppose for all elements $(\lambda, s) \in \mathrm{LD}(k, \delta)$, the vector $\mathbf{x}(\lambda, s)$ has integral components. Then the Hasse principle for (k, δ) holds.*
(2) *Suppose there is an element $(\lambda, s) \in \mathrm{LD}(k, \delta)$ such that $\mathbf{x}_i(\lambda, s) \notin \mathbb{N}$ for some i. Then there is a finite separable field extension K of degree k over F and a central division algebra A of degree δ over F so that the Hasse principle for (K, A) fails. In particular, the Hasse principle for the pair (k, δ) fails.*

The proof of Theorem 1.3 uses Theorem 1.2, the Hilbert irreducibility theorem and the global class field theory. Based on Theorem 1.3, we prove the following result (Proposition 4.6 and Corollary 4.7), which also shows that (Q3) has a negative answer in general.

PROPOSITION 1.4.
1. *Let $\delta = p_1^{n_1} \cdots p_r^{n_r}$, and let k be a positive integer with $k > 1$ and $k \mid \delta$, where p_i's are distinct prime divisors of δ. Assume that $k \leq \delta/p_i^{n_i}$ and $p_i \mid k$ for some $1 \leq i \leq r$ (so $r \geq 2$). Then the Hasse principle for the pair (k, δ) does not hold.*
2. *Let δ be a positive integer divisible by at least two primes. Then there is a positive integer k with $k \mid \delta$ such that the Hasse principle for the pair (k, δ) does not hold.*

Clearly Proposition 1.4 (2) follows from Proposition 1.4 (1). Proposition 1.4 tells us that when δ is divisible by two primes and k is "small" comparable to δ, the Hasse principle for (δ, k) fails.

In the last section we discuss the Hasse principle for a geometric orbit of the variety of embeddings of K into A that is defined over F, where K is a *finite etale commutative F-algebra.* Let X be the F-scheme of embeddings of K into A (see Section 5). The group $H = \mathrm{GL}_1(A)$ acts on X by conjugation. We have the following result.

THEOREM 1.5 (The Hasse principle). *Let $\overline{X}_0$ be a geometric orbit of $H \otimes_F F^{\mathrm{sep}}$ that is defined over F and let X_0 be the F-subscheme of X whose base extension to F^{sep} is $\overline{X}_0$. If $X_0(F_v) \neq \emptyset$ for all $v \in V^F$, then $X_0(F) \neq \emptyset$.*

The proof relies on results of Springer [14], Douai and Borovoi [1] about the second non-abelian Galois cohomology and neutral classes. Among early results in this area one can mention the validity of the Hasse principle for projective homogeneous spaces due to Harder [7] and for symmetric homogeneous spaces of absolutely simple simply-connected groups due to Rapinchuk [11]. Later, Borovoi in a series of papers developed cohomological methods for analyzing the Hasse principle for homogeneous spaces with connected stabilizers, of an arbitrary connected group

whose maximal semi-simple subgroup is simply-connected; see Borovoi [2, 3]. In [10, Appendix A] Prasad and Rapinchuk used the cohomological method to study the local-global principle for embeddings of maximal subfields into central simple algebras over global fields *with involutions.*

This paper is organized as follows. In Section 2 we collect and show some general embedding results over any field based on [17]. In Section 3, we give a more detailed study about embeddings of a field extension K into a central simple algebra A over a global field. We determine for which pair (K, A) the Hasse principle holds. In particular, we answer the questions (Q1), (Q2) and (Q3). In Section 4, we work on the Hasse principle for a given pair (k, δ) of positive integers with k dividing δ. In Section 5 we give the proof of Theorem 1.5 following a referee's suggestion.

2. General embedding results.

2.1. Setting. Let F denote the ground field, which is arbitrary in this section. All F-algebras considered in this paper are assumed to be finite-dimensional as F-vector spaces and have the identity. As the standard convention, an F-algebra homomorphism is a ring homomorphism over F; in particular, it sends the identity of the source to the identity of the target.

We recall the following definition for central simple algebras; see [12].

DEFINITION 2.1. The *degree*, *capacity*, and *index* of a central simple algebra A over F are defined as

$$\deg(A) := \sqrt{[A : F]}, \quad c(A) := n, \quad i(A) := \sqrt{[\Delta : F]},$$

if $A \cong \mathrm{Mat}_n(\Delta)$, where Δ is a division algebra over F, which is uniquely determined by A up to isomorphism. The algebra Δ is also called the *division part* of A.

For the convenience of discussion, we introduce the following definition and notations.

DEFINITION 2.2.
1. Let V be a finite-dimensional vector space over F, and A a finite-dimensional arbitrary F-algebra. We say that V *admits an A-module structure* if there is a right (or left) A-module structure on V. If B is any F-subalgebra of A and V is already a right (resp. left) B-module, then by saying V admits an A-module structure we mean that the right (resp. left) A-module structure on V is required to be compatible with the underlying B-module structure on V.
2. For any two F-algebras A_1, A_2, let $\mathrm{Hom}_F(A_1, A_2)$ denote the set of F-algebra homomorphisms from A_1 to A_2, and let

$$\mathrm{Hom}_F^*(A_1, A_2) \subset \mathrm{Hom}_F(A_1, A_2)$$

be the subset consisting of embeddings of A_1 into A_2. For two maps $\varphi_1, \varphi_2 \in \mathrm{Hom}_F(A_1, A_2)$, we say φ_1 and φ_2 are *equivalent* if there is an element $b \in A_2^\times$ such that $\varphi_2 = \mathrm{Int}(b) \circ \varphi_1$. That is, $\varphi_2(a) = b\,\varphi_1(a)\,b^{-1}$ for all $a \in A_1$. Then $A_2^\times \backslash \mathrm{Hom}_F(A_1, A_2)$ is the set of equivalence classes of F-algebra homomorphisms from A_1 to A_2. Write

$$\mathcal{O}_{A_1, A_2} := A_2^\times \backslash \mathrm{Hom}_F(A_1, A_2) \quad \text{and} \quad \mathcal{O}_{A_1, A_2}^* := A_2^\times \backslash \mathrm{Hom}_F^*(A_1, A_2).$$

for the orbits spaces.

We often simply write $A_1 \otimes A_2$ for $A_1 \otimes_F A_2$ if the ground field F is understood.

2.2. General embedding lemmas. Let A be a central simple algebra over F. We realize A as $\operatorname{End}_\Delta(V)$, the endomorphism algebra of V, where Δ is the division part of A and V is a right Δ-module of finite rank. Let A' be another simple F-algebra with center K. Then there is an F-algebra homomorphism $\varphi : A' \to A$ if and only if there is an (A', Δ)-bimodule structure on V, or V admits a right $\Delta \otimes_F A'^o$-module structure. Here A'^o denotes the opposite algebra of A'. Suppose $n := \dim_\Delta V$ and

$$\Delta \otimes_F A'^o \simeq (\Delta \otimes_F K) \otimes_K A'^o \simeq \operatorname{Mat}_c(\Delta'),$$

where Δ' is the division part of the central simple algebra $\Delta \otimes_F A'^o$ over K. We have

$$(2.1) \qquad [\Delta : F][A' : F] = c^2[\Delta' : F] \quad \text{and} \quad \dim_F V = n[\Delta : F].$$

Then V admits a right $\Delta \otimes_F A'^o$-module structure if and only if

$$(2.2) \qquad c[\Delta' : F] \mid \dim_F V.$$

This is equivalent to $[\Delta : F][A' : F] \mid cn[\Delta : F]$, or equivalently

$$(2.3) \qquad [A' : F] \mid nc.$$

We have proved the following result.

LEMMA 2.3. *Let A be a central simple F-algebra and A' a simple F-algebra. Suppose $A \simeq \operatorname{Mat}_n(\Delta)$. Then there exists an F-embedding of A' into A if and only if*

$$(2.4) \qquad [A' : F] \mid n \cdot \mathrm{c}(\Delta \otimes_F A'^o).$$

Next we consider the case where A' is a *semi-simple* F-algebra instead. Write $A' = \prod_{i=1}^s A_i'$ into simple factors and let K_i be the center of A_i'. The existence of an F-embedding of A' into A is equivalent to that there is an (A', Δ)-bimodule structure on V. This means that there is a decomposition of V into Δ-submodules

$$(2.5) \qquad V = V_1 \oplus \cdots \oplus V_s$$

so that each V_i is a *non-zero* (A_i', Δ)-bimodule. Put $n_i := \dim_\Delta V_i$ and let c_i be the capacity of the central simple algebra

$$\Delta \otimes_F A_i'^o = (\Delta \otimes_F K_i) \otimes_{K_i} A_i'^o$$

over K_i. Then we get the conditions

$$(2.6) \qquad n = \sum_{i=1}^s n_i \quad \text{and} \quad [A_i' : F] \mid n_i c_i, \quad \forall i = 1, \ldots, s.$$

This yields the following criterion for embeddings.

LEMMA 2.4. *Let A and $A' = \prod_{i=1}^s A_i'$ be as above. Then there is an embedding of the F-algebra A' into A if and only if there are positive integers n_i for $i = 1, \ldots, s$ such that*

$$n = \sum_{i=1}^s n_i \quad \text{and} \quad [A_i' : F] \mid n_i c_i, \quad \forall i = 1, \ldots, s,$$

where c_i the capacity of the central simple algebra $\Delta \otimes_F A_i'^o$ over K_i.

LEMMA 2.5. *Let A and $A' = \prod_{i=1}^s A_i'$ be as above. Let φ, φ' be two maps in $\mathrm{Hom}_F(A', A)$, and let V_φ and $V_{\varphi'}$ be the associated (A', Δ)-bimodules underlying the space V. Then φ and φ' are equivalent if and only if the (A', Δ)-bimodules V_φ and $V_{\varphi'}$ are isomorphic.*

Proof. See [17, Lemma 3.2] ◻

2.3. Maximal degree field extension case. We apply Lemmas 2.3 and 2.4 to the case where the semi-simple algebra A' is commutative and obtain the following well-known result. This is also a consequence of a result of Chuard-Koulmann and Morales [5, Proposition 4.3].

LEMMA 2.6.
1. *Let A be a central simple algebra over F and K is a field extension of F with $[K : F] = \deg(A)$. Then there exists an F-embedding of K into A if and only if K splits A.*
2. *Let A be a central simple algebra over F and $K = \prod_{i=1}^s K_i$ is commutative semi-simple F-algebra with $[K : F] = \deg(A)$. Then there exists an F-embedding of K into A if and only if each K_i splits A.*

Proof. (1) By Lemma 2.3, the set $\mathrm{Hom}_F(K, A)$ is non-empty if and only if $[K : F] \,|\, nc$, where $c := \mathrm{c}(\Delta \otimes K)$. If K splits Δ, then $c = \deg(\Delta)$ and hence $[K : F] = n \deg(\Delta) = nc$. Therefore, $\mathrm{Hom}_F(K, A)$ is non-empty. Suppose $[K : F]|nc$. Then $\deg(\Delta)|c$ and $\deg(\Delta) = c$. This shows that K splits Δ.

(2) Suppose there is an F-embedding of K into A. Then there are positive integers n_i with $n = \sum n_i$ and there is an embedding of K_i into $\mathrm{Mat}_{n_i}(\Delta)$. Since $[K_i : F] \,|\, n_i \deg(\Delta)$, it follows from

$$[K : F] = \deg(A) = \sum_i n_i \deg(\Delta) \geq \sum_i [K_i : F] = [K : F]$$

that $[K_i : F] = n_i \deg(\Delta)$ for each $i = 1, \ldots, s$. Therefore, K_i splits Δ. Conversely, if K_i splits Δ for each i, then $[K_i : F] = m_i \deg(\Delta)$ for a positive integer m_i and $c_i := \mathrm{c}(\Delta \otimes_F K_i) = \deg(\Delta)$. Then we have

$$n = \sum_i m_i, \quad \text{and} \quad [K_i : F] \,|\, m_i c_i, \quad \forall i = 1, \ldots, s.$$

It follows from Lemma 2.4 that there is an embedding of K into A. ◻

3. Answers to (Q1) and (Q2) by numerical invariants. In this section we study F-embeddings of K into A over F, where F is either a local or global field, K is a commutative semi-simple algebra over F and A is a central simple algebra.

3.1. Local results. Let F denote a local field.

LEMMA 3.1. *Let $A = \mathrm{End}_\Delta(V) = \mathrm{Mat}_n(\Delta)$ be a central simple algebra over F.*
1. *Let K be a finite field extension of F. The following statement are equivalent:*
   (a) *There exists an embedding of K into A over F.*
   (b) *$[K : F] \,|\, n \cdot \mathrm{c}(\Delta \otimes_F K)$.*
   (c) *$[K : F] \,|\, n \deg(\Delta)$.*

2. *Let $K = \prod_{i=1}^{s} K_i$ be a commutative semi-simple algebra over F. Then there exists an embedding of K into A over F if and only if there are positive integers n_i for $i = 1, \ldots, s$ such that*

$$(3.1) \qquad n = \sum_{i=1}^{s} n_i \quad and \quad [K_i : F] \mid n_i \deg(\Delta), \quad \forall\, i = 1, \ldots, s.$$

It follows from Lemma 2.4 that the statements (a) and (b) are equivalent. The implication $(b) \implies (c)$ is trivial. Put $\delta := \deg(\Delta)$ and $k := [K : F]$. If $\mathrm{inv}(\Delta) = a/\delta$ with $\gcd(a, \delta) = 1$, then (see [13])

$$\mathrm{inv}(\Delta \otimes_F K) = [K : F]\,\mathrm{inv}(\Delta) = \frac{ak}{\delta} = \frac{a'}{\delta'}, \quad \text{with} \quad \gcd(a', \delta') = 1,$$

where $\delta = \delta' c$, $ak = a'c$, and $c := \gcd(k, \delta)$. It follows that

$$(3.2) \qquad\qquad c(\Delta \otimes_F K) = \gcd([K : F], \deg(\Delta)).$$

Note that $\gcd(\delta', k) = 1$, so we have

$$k \mid n\delta \iff k \mid nc\delta' \iff k \mid nc.$$

The statement Lemma 3.1 (2) follows from Lemma 2.4 and Lemma 3.1 (1).

Now consider the case where $K = \prod_{i=1}^{s} K_i$ is a commutative semi-simple F-algebra. Put

$$(3.3) \qquad\qquad c_i := \gcd([K_i : F], \deg(\Delta)) \quad \text{and} \quad \ell_i := [K_i : F]/c_i.$$

For any positive integer n_i, we have

$$(3.4) \qquad\qquad [K_i : F] \mid n_i \deg(\Delta) \iff \ell_i \mid n_i.$$

Put

$$(3.5) \qquad \mathcal{E}_F(K, A) := \{ x = (x_1, \ldots, x_s) \in \mathbb{N}^s \mid \sum_{i=1}^{s} \ell_i x_i = \dim_\Delta V \}.$$

If a tuple $\mathbf{n} = (n_1, \ldots, n_s)$ is a solution to (3.1), then the tuple $x = (x_1, \ldots, x_s)$, where $x_i := n_i/\ell_i$, is an element in $\mathcal{E}_F(K, A)$. Conversely, any element x in $\mathcal{E}_F(K, A)$ gives a solution $\mathbf{n}$ to (3.1) by setting $n_i = \ell_i x_i$. Recall that $A^\times \backslash \mathrm{Hom}_F^*(K, A)$ is the set of equivalence classes of embeddings of F-algebras from K into A.

PROPOSITION 3.2. *There is a natural bijection*

$$(3.6) \qquad\qquad e : A^\times \backslash \mathrm{Hom}_F^*(K, A) \overset{\sim}{\longrightarrow} \mathcal{E}_F(K, A).$$

Proof. Let φ and φ' be two maps in $\mathrm{Hom}_F^*(K, A)$, and let V_φ and $V_{\varphi'}$ be the induced (K, Δ)-bimodule structures on V. Write

$$V_\varphi = V_1 \oplus \cdots \oplus V_s \quad \text{and} \quad V_{\varphi'} = V_1' \oplus \cdots \oplus V_s',$$

where V_i and V_i' are (K_i, Δ)-bimodules. We have shown (Lemma 2.5) that φ and φ' are equivalent if and only if V_φ and $V_{\varphi'}$ are isomorphic as (K, Δ)-bimodules, equivalently, $V_i \simeq V_i'$ as (K_i, Δ)-bimodules for $i = 1, \ldots, s$. Since each $\Delta \otimes_F K_i$ is simple, the latter is the same as the condition $\dim_\Delta V_i = \dim_\Delta V_i'$ for $i = 1, \ldots, s$. One associates to φ a tuple

$$\mathbf{n} = (\dim_\Delta V_1, \ldots, \dim_\Delta V_s)$$

which satisfies the condition (3.1) and determines the map φ up to equivalence. As such tuples are in one-to-one correspondence with elements in $\mathcal{E}_F(K, A)$. Then we show a bijection map $e : A^\times \backslash \operatorname{Hom}_F^*(K, A) \to \mathcal{E}_F(K, A)$ which is given by

$$(3.7) \qquad e(\varphi) = (\dim_\Delta V_1/\ell_1, \ldots, \dim_\Delta V_s/\ell_s).$$

This completes the proof of the proposition. $\square$

3.2. Global results. In the remaining of this article let F be a global field. Let A be a central simple algebra over F and K a finite field extension over F. We use the following notations.

- $A = \operatorname{End}_\Delta(V)$, where Δ is the division part of A, and V is a finite right Δ-module of rank n.
- $k := [K : F]$ and $\delta_0 := \deg(\Delta)$.
- For any place v of F, denote by F_v the completion of F at v. Put

$$K_v := K \otimes F_v = \prod_{w|v} K_w, \quad A_v := A \otimes F_v, \quad \Delta_v = \Delta \otimes F_v = \operatorname{Mat}_{s_v}(D_v),$$

  where D_v is the division part of the central simple algebra Δ_v (we do not use the letter D as an algebra over F in this section; do not confuse D_v as the completion of D) and s_v is the capacity of Δ_v.
- $k_w := [K_w : F_v]$ and $d_v := \deg(D_v)$, where w is a place of K over v.
- $\Delta \otimes_F K = \operatorname{Mat}_c(\Delta')$ and $\delta' := \deg(\Delta')$, where Δ' is the division part of the central simple algebra $\Delta \otimes K$ over K, and c is its capacity. One has

$$(3.8) \qquad \delta_0 = \delta' c.$$

- For any place w of K, put

$$\Delta_w' := \Delta' \otimes_K K_w = \operatorname{Mat}_{t_w}(D_w'), \quad d_w' := \deg(D_w'),$$

  where D_w' is the division part of the central simple algebra Δ_w' and t_w is the local capacity of Δ' at w.
- $c_w := \operatorname{c}(D_v \otimes_{F_v} K_w)$, i.e. $D_v \otimes_{F_v} K_w = \operatorname{Mat}_{c_w}(D_w')$. One has

$$(3.9) \qquad d_v = d_w' c_w.$$

It follows from

$$\Delta \otimes_F K_w = (\Delta \otimes F_v) \otimes_{F_v} K_w = \Delta_v \otimes K_w = \operatorname{Mat}_{s_v c_w}(D_w') \quad \text{and}$$
$$\Delta \otimes_F K_w = (\Delta \otimes_F K) \otimes_K K_w = \operatorname{Mat}_c(\Delta') \otimes_K K_w = \operatorname{Mat}_{c t_w}(D_w')$$

that

$$(3.10) \qquad s_v c_w = c t_w.$$

- For any rational number $a \in \mathbb{Q}$, we write $\mathbf{d}(a)$ for the positive denominator of a in its reduced form, and $\mathbf{n}(a)$ for its numerator.
- For each place v of F, write

$$\mathrm{inv}_v(\Delta) = \frac{a_v}{\delta_0} = \frac{a'_v \, s_v}{d_v \, s_v} = \frac{a'_v}{d_v}, \quad \gcd(a'_v, d_v) = 1 \text{ and } s_v = \gcd(a_v, \delta_0).$$

One has, by the Grunwald-Wang theorem

(3.11) $\qquad \delta_0 = \mathrm{lcm}\{d_v\}_{v \in V^F} \quad \text{and} \quad (\gcd\{a_v\}_{v \in V^F}, \delta_0) = 1,$

where V^F denotes the set of all places of F.

- For each place w of K, write

$$\mathrm{inv}_w(\Delta') = \frac{b_w}{\delta'} = \frac{b'_w \, t_w}{d'_w \, t_w} = \frac{b'_w}{d'_w}, \quad \gcd(b'_w, d'_w) = 1 \text{ and } t_w = \gcd(b_w, \delta').$$

One has

(3.12) $\qquad \delta' = \mathrm{lcm}\{d'_w\}_{w \in V^K} \quad \text{and} \quad (\gcd\{b_w\}_{w \in V^K}, \delta') = 1,$

where V^K denotes the set of all places of K.

- It follows from $\mathrm{inv}(D'_w) = \mathrm{inv}(D_v)[K_w : F_v]$ (see [13]) that

(3.13) $\qquad\qquad\qquad\qquad c_w = \gcd(d_v, k_w).$

Given K and A, we have, for each place v of F,
- a tuple $(k_w)_{w|v}$ of positive integers, and
- a rational number $\mathrm{inv}_v(\Delta) = a'_v/d_v$

satisfying the following conditions:
(a) $\sum_{w|v} k_w = k$ for all $v \in V^F$,
(b) (i) $d_v = 1$ if v is a complex place,
     (ii) $d_v \in \{1, 2\}$ if v is a real place,
     (iii) $d_v = 1$ for almost all v, and
     (iv) (Global class field theory) one has

$$\sum_{v \in V^F} \frac{a'_v}{d_v} = 0 \quad (\text{ in } \mathbb{Q}/\mathbb{Z}).$$

We compute all other numerical invariants δ_0, c_w, d'_w, δ' and c as follows.
(i) The (global) degree δ_0 of Δ can be computed by (3.11).
(ii) Then one computes the local capacity c_w of $D_v \otimes_{F_v} K_w$ and the (local) degree d'_w of D'_w by (3.13) and (3.9), respectively.
(iii) Using (3.12) we compute the (global) degree δ' of Δ' and then compute the (global) capacity c of $\Delta \otimes K$ using (3.8).
We define the following condition (G stands for "global")

(**G**) $k \mid n \, c.$

PROPOSITION 3.3. *The set* $\operatorname{Hom}_F(K, A)$ *is non-empty if and only if the condition* (**G**) *holds.*

Proof. This follows from Lemma 2.3. □

Now we formulate the corresponding local conditions. Note that

$$K_v = \prod_{w|v} K_w \quad \text{and} \quad A_v = \operatorname{Mat}_{ns_v}(D_v).$$

Put

(3.14) $$\mathcal{E} := A^\times \backslash \operatorname{Hom}_F(K, A);$$

the Noether-Skolem theorem says that if this set is non-empty then it has one element. For each place v of F, define a set (c.f. (3.5))

(3.15) $$\mathcal{E}_v := \mathcal{E}_{F_v}(K_v, A_v) = \{(x_w)_{w|v} \mid x_w \in \mathbb{N}, \ \sum_{w|v} \ell_w x_w = ns_v \},$$

where $\ell_w := k_w / c_w$. Define the following condition (L stands for "local")

(**L**) The set $\mathcal{E}_v$ is non-empty for all $v \in V^F$.

PROPOSITION 3.4. *There is an embedding of K_v in A_v over F_v if and only if the set $\mathcal{E}_v$ is non-empty.*

Proof. This follows from Proposition 3.2. □

We have the following implication

the condition (**G**) holds $\implies$ the condition (**L**) holds.

The local-global principle then asks whether the converse is also true.

3.3. Special vectors and the local-global principle. Let

$$e_v : A_v^\times \backslash \operatorname{Hom}^*_{F_v}(K_v, A_v) \xrightarrow{\sim} \mathcal{E}_v$$

be the corresponding bijection obtained in Proposition 3.2. Let us suppose first that the set $\operatorname{Hom}_F(K, A) = \operatorname{Hom}^*_F(K, A)$ of embeddings of K into A over F is *non-empty*. For any element φ in $\operatorname{Hom}_F(K, A)$, let $\varphi_v \in \operatorname{Hom}_{F_v}(K_v, A_v)$ be the extension of φ by F_v-linearity, and let $[\varphi_v]$ be its equivalence class. Then one defines an element $\mathbf{x}_v \in \mathcal{E}_v$ by

$$\mathbf{x}_v := e_v([\varphi_v]).$$

The association $\varphi \mapsto \mathbf{x}_v$ induces a well-defined map, which we denote again by e_v,

$$e_v : \mathcal{E} \to \mathcal{E}_v.$$

The non-emptiness of $\operatorname{Hom}_F(K, A)$ implies the existence of such a vector $\mathbf{x}_v$ in $\mathcal{E}_v$ for each place $v \in V^F$. We now calculate these special vectors explicitly.

The map φ gives rise to a (K, Δ)-bimodule structure on V. Since V is free K-module of rank $n\delta_0^2/k$, its completion $V \otimes_F F_v$ is also a free K_v-module of same rank. Therefore, one has the decomposition

$$V \otimes F_v = \bigoplus_{w|v} V_w,$$

where each factor V_w is a $(K_w, \mathrm{Mat}_{s_v}(D_v))$-bimodule of K_w-rank $n\delta_0^2/k$ (recall that $\Delta_v = \mathrm{Mat}_{s_v}(D_v)$). Using the Morita equivalence, the module V_w is isomorphic to $W_w^{\oplus s_v}$ for a (K_w, D_v)-bimodule W_w of D_v-rank ns_vk_w/k. Using the formula (3.7), the w-component $\mathbf{x}_w$ of the vector $\mathbf{x}_v$ is given by

$$(3.16) \qquad \mathbf{x}_w := \dim_{D_v} W_w/\ell_w = ns_vc_w/k,$$

which is a positive integer. Recall that $c_w = \gcd(k_w, d_v)$ and $\ell_w = k_w/c_w$.

Therefore, this leads us to the following definition of *special vectors* no matter the set $\mathcal{E}$ is non-empty or not. For each place v of F we define a vector (still denoted by) $\mathbf{x}_v = (\mathbf{x}_w)_{w|v} \in \prod_{w|v} \mathbb{Q}_{>0}$ by (3.16), and we call them *special vectors*. The above calculation shows if the set $\mathcal{E}$ is non-empty, then the vector $\mathbf{x}_v$ is the image of the map e_v.

PROPOSITION 3.5. *Notations as above. If the set $\mathcal{E}$ is non-empty, then one has*

$$\mathbf{x}_v \in \mathcal{E}_v, \quad \forall\, v \in V^F,$$

or equivalently, each vector $\mathbf{x}_v$ lies in $\prod_{w|v} \mathbb{N}$ for all $v \in V^F$.

If we denote by $\bar{\mathbf{x}}_w$ the class of $\mathbf{x}_w$ in $\mathbb{Q}/\mathbb{Z}$, then we associate to the pair (K, A) an element

$$(3.17) \qquad \bar{\mathbf{x}} = (\bar{\mathbf{x}}_w)_{w \in V^K} \in \bigoplus_{w \in V^K} \mathbb{Q}/\mathbb{Z}.$$

Then Proposition 3.5 states that the vanishing of the class $\bar{\mathbf{x}}$ is a necessary condition for the set $\mathcal{E}$ to be non-empty. The following result states that this is the only obstruction to the local-global principle.

THEOREM 3.6. *Notations as above. We have*

$$\mathrm{Hom}_F^*(K, A) \neq \emptyset \iff \bar{\mathbf{x}} = 0.$$

Proof. Note that the condition $\bar{\mathbf{x}} = 0$ implies $\mathbf{x}_v \in \mathcal{E}_v$ and hence $\mathrm{Hom}_{F_v}^*(K_v, A_v)$ is non-empty for all $v \in V^F$. The implication $\Longrightarrow$ is already proved. To show the other direction, we must show that the condition (**G**) $k \mid nc$ holds. Using $c = \delta_0/\delta'$ and $\delta' = \mathrm{lcm}\{d_w'\}$, we rewrite the condition (**G**) as

$$(3.18) \qquad k \mid (n\delta_0/d_w'), \quad \forall\, w \in V^K.$$

Using (3.9) and (3.16), we have

$$\mathbf{x}_w = ns_vc_w/k = ns_vd_v/kd_w' = n\delta_0/kd_w' \in \mathbb{N}$$

for all $w \in V^K$. This verifies the condition (**G**) and hence proves the theorem. $\qed$

3.4. An example. We will show an example of a pair (K, A), where K/F is a Galois extension and A is a central simple F-algebra so that
- the set $\mathrm{Hom}_{F_v}^*(K_v, A_v)$ is non-empty for all $v \in V^F$, and
- the set $\mathrm{Hom}_F(K, A)$ is empty.

This particularly shows that the question (Q3) has a negative answer. Let K/F be a Galois extension of degree 8. Choose two places v_1 and v_2 of F so that $k_w = 2$ for all $w|v_1$ or $w|v_2$. Such places exist by the Chebotarev density theorem. Let A be a central simple algebra over F of degree 24 that is ramified exactly at the two places v_1 and v_2, and

$$A_{v_1} = \mathrm{Mat}_6(D_{v_1}), \quad \text{and} \quad A_{v_2} = \mathrm{Mat}_6(D_{v_2}),$$

where D_{v_1} and D_{v_2} are central division algebras of degree 4. The existence of such an A follows from the global class field theory. One has $c_w = 2$ and $\ell_w = 1$. The sets $\mathcal{E}_v$ for unramified places v are non-empty. For $v = v_i$, $i = 1, 2$, one see that

$$\mathcal{E}_{v_i} = \{(x_i) \in \mathbb{N}^4 | \sum_{i=1}^{4} x_i = 6\} \neq \emptyset.$$

By Proposition 3.4, the sets $\mathrm{Hom}^*_{F_v}(K_v, A_v)$ are non-empty for all $v \in V^F$. On the other hand, $\mathbf{x}_w = 6c_w/k = 12/8$ for any $w|v_1$ or $w|v_2$, which is not an integer. By Theorem 3.6, the set $\mathrm{Hom}_F(K, A)$ is empty.

The same argument shows that there is a Galois extension K/F of degree p^m, and a central simple algebra A/F of degree $p^m q$, where p and q are primes, with $m \geq 2$ and $p < q$, so that the local-global principle for embedding K into A fails. One takes $k_w = p$ and $d_v = p^2$. Then $c_w = p$ and $\ell_w = 1$, We see that $\mathcal{E}_{v_i}$ is non-empty from the inequality $p^{m-1} \leq p^{m-2}q$. However, $\mathbf{x}_w = p^{m-2}qp/p^m \notin \mathbb{N}$.

4. The local-global principle. Keep the notation in § 3.2 and 3.3. In this section, we study on the local-global principle in detail.

For the convenience of discussion, we make the following definitions.

DEFINITION 4.1.
(1) Let K be a finite field extension over a global field F and A be a central simple algebra over F. We say that the Hasse principle for the pair (K, A) holds if the following properties are equivalent:

$$\mathrm{Hom}^*_{F_v}(K_v, A_v) \neq \emptyset, \quad \forall v \in V^F \Longleftrightarrow \mathrm{Hom}^*_F(K, A) = \mathrm{Hom}_F(K, A) \neq \emptyset.$$

In other words, the conditions (**G**) and (**L**) in § 3.2 are equivalent.
(2) Let F be a global field. Let (k, δ) be a pair of two positive integers with $k \mid \delta$. We say that the Hasse principle for the pair (k, δ) holds if for any finite field extension K over F of degree k and for any central simple algebra A over F of degree $\deg(A) = \delta$, the Hasse principle for the pair (K, A) holds.

Theorem 3.6 gives us an effective way to check the Hasse principle. Namely, one only needs to check

$$(4.1) \qquad \mathbf{x}_w = n s_v c_w / k \overset{?}{\in} \mathbb{N}, \quad \forall w \in V^K$$

for a given pair (K, A).

4.1. Basic positive results.

PROPOSITION 4.2. *Let A be a central simple algebra over F with index $\mathrm{i}(A) = \delta_0$ and degree $\deg A = n\delta_0$, and let K be a finite field extension over F of degree $[K : F] \mid \deg A$. Suppose one of the following properties holds:*

(1) $[K : F] = \deg A$.

(2) K splits A.

(3) For any place v, the algebra A_v is either a division algebra or a matrix algebra over F_v.

Then the Hasse principle for the pair (K, A) holds.

Proof.

(1) This is a well-known result; see Pierce [8, § 18.4] or Prasad-Rapinchuk [9, Proposition A.1].

(2) Since K splits A, we have $d'_w = 1$ for all $w \in V^K$ and hence $\delta' = \operatorname*{lcm}_w \{d'_w\} = 1$ and $c = \delta_0$. Thus,

$$k \mid n\delta_0 \Longleftrightarrow k \mid nc.$$

This verifies the condition (**G**) and hence $\operatorname{Hom}_F(K, A) \neq \emptyset$.

(3) Suppose there exists an embedding of K_v into A_v for all $v \in V^F$. If A_v is a central division algebra, then $K_v = K_w$ is a field extension over F_v with $[K_w : F_v] = k$ and $c_w = k$. Then we have

$$\mathbf{x}_w = n s_v c_w / k = n \in \mathbb{N}, \quad \forall\, w \mid v.$$

For the other case that A_v is a matrix algebra over F_v, we have $s_v = \delta_0$. Thus, by the assumption $k \mid n\delta_0$, we have

$$\mathbf{x}_w = n s_v c_w / k = n \delta_0 c_w / k \in \mathbb{N}, \quad \forall\, w \mid v.$$

By Theorem 3.6, the set $\operatorname{Hom}_F(K, A)$ is not empty. $\square$

4.2. Construction of counterexamples. Let K be a finite field extension over F with $[K : F] = k = p_1^{m_1} \cdots p_r^{m_r}$, and A be a central division algebra over F of degree $\delta = p_1^{n_1} \cdots p_r^{n_r}$, where p_i is a prime number, $n_i \in \mathbb{N}$, and $m_i \in \mathbb{Z}_{\geq 0}$ with $m_i \leq n_i$ for $i = 1, \ldots, r$.

We shall construct examples for which the Hasse principle does not hold. Recall (Definition 4.1) that the Hasse principle for a pair (k, δ) does not hold if there exists a pair (K, A) such that $\operatorname{Hom}^*_{F_v}(K_v, A_v) \neq \emptyset$ for all $v \in V^F$ but $\operatorname{Hom}_F(K, A) = \emptyset$.

The construction is to set local data s_v, d_v, and a partition $k = \sum_{w \mid v} k_w$ of the integer k for some place v of F so that some w-component $\mathbf{x}_w$ of the special vector $\mathbf{x}_v$ is not integral. Then we apply Theorem 3.6 to conclude that the Hasse principle for certain pair (K, A) does not hold. To ensure that the data $k = \sum_{w \mid v} k_w$ come from a global field K, we need the following result.

LEMMA 4.3 (c.f. [16, Lemma 3.2]). *Let S be a finite subset of V^F. Let L_v, for each $v \in S$, be any etale algebra over F_v of same degree $[L_v : F_v] = n$. Then there is a finite separable field extension K over F of degree n such that $K \otimes_F F_v \simeq L_v$ for all $v \in S$.*

Moreover, one also needs to ensure that the data s_v and d_v come from a central division algebra A over F. For this, we need the global class field theory; see [12, § 32] or [4].

THEOREM 4.4. *Let S be a finite subset of V^F. For any positive integer δ, suppose we are given any set of rational numbers $\{a_v / d_v\}_{v \in S}$ with $\gcd(a_v, d_v) = 1$ such that*

(1) $\operatorname{lcm}_{v \in S}\{d_v\} = \delta$.

(2) $\sum_{v \in S} a_v/d_v = 0 \in \mathbb{Q}/\mathbb{Z}$.

(3) $d_v = 1$ *if v is complex.*

(4) $d_v = 1$ *or 2 if v is real.*

Then up to isomorphism, there is a unique central division algebra A over F with $\deg A = \delta$ *such that* $\operatorname{inv} A_v = a_v/d_v \in \mathbb{Q}/\mathbb{Z}$ *for all $v \in S$ and* $\operatorname{inv} A_v = 0 \in \mathbb{Q}/\mathbb{Z}$ *for all $v \notin S$.*

Let (k, δ) be a pair of positive integers with $k \mid \delta$. For any partition λ of k

$$\lambda = (k_1, \ldots, k_t), \quad k_1 \leq \cdots \leq k_t, \quad \sum_{i=1}^{t} k_i = k$$

and a positive integer s dividing δ, we set

$$\mathcal{E}(\lambda, s) := \{(x_i) \in \mathbb{N}^t \mid \sum_{i=1}^{t} \ell_i x_i = s\},$$

where $\ell_i := k_i / \gcd(k_i, d)$, and define a vector $\mathbf{x}(\lambda, s) \in \mathbb{Q}^t$, where

$$\mathbf{x}(\lambda, s) := (\mathbf{x}_i), \quad \mathbf{x}_i = \mathbf{x}_i(\lambda, s) := s \gcd(k_i, d)/k, \quad \forall\, i = 1 \ldots, t.$$

Let $\mathrm{LD}(k, \delta)$ be the set of pairs (λ, s) of partitions λ of k and divisors s of δ such that $\mathcal{E}(\lambda, s)$ is non-empty ("LD" stands for all possible local decompositions).

We transform the problem of checking the Hasse principle for the pair (k, δ) in purely combinatorial terms.

THEOREM 4.5. *Let F be a global field and let (k, δ) be a pair of positive integers with $k \mid \delta$.*

(1) Suppose for all elements $(\lambda, s) \in \mathrm{LD}(k, \delta)$, the vector $\mathbf{x}(\lambda, s)$ has integral components. Then the Hasse principle for (k, δ) holds.

(2) Suppose there is an element $(\lambda, s) \in \mathrm{LD}(k, \delta)$ such that $\mathbf{x}_i(\lambda, s) \notin \mathbb{N}$ for some i. Then there are a finite separable field extension K of degree k over F and a central division algebra A of degree δ over F so that the Hasse principle for (K, A) fails. In particular, the Hasse principle for the pair (k, δ) fails.

Proof. (1) Let K be any finite field extension of F of degree k and A be any central simple algebra over F of degree δ. For any place v of F, we get a partition $\lambda = (k_w)_{w|v}$ of k and a positive divisor s of δ as the capacity of $A_v = A \otimes F_v$. The assumption $\operatorname{Hom}^*_{F_v}(K_v, A_v) \neq \emptyset$ assures that $\mathcal{E}(\lambda, s) = \mathcal{E}_v$ is non-empty. Therefore, the pair (λ, s) is an element in $\mathrm{LD}(k, \delta)$. Then the assumption gives that $\mathbf{x}_v = \mathbf{x}(\lambda, s)$ has integral components. This works for all $v \in V^F$. By Theorem 3.6, we have $\operatorname{Hom}^*_F(K, A) \neq \emptyset$.

(2) Let $(\lambda, s) \in \mathrm{LD}(k, \delta)$, where $\lambda = (k_1, \ldots, k_t)$ and $s|\delta$, be an element such that $\mathbf{x}(\lambda, s)$ is not an integral vector. Choose any finite set $S = \{v_1, v'_1, v_2, v'_2\}$ of 4 finite places of F. Let A be the central division algebra over F of degree δ with following local invariants (Theorem 4.4):

- $\operatorname{inv}_{v_1}(A) = -\operatorname{inv}_{v'_1}(A) = 1/d$ and $\operatorname{inv}_{v_2}(A) = -\operatorname{inv}_{v'_2}(A) = 1/\delta$.
- $\operatorname{inv}_v(A) = 0$ for all $v \notin S$.

Let L_{v_2} and $L_{v'_2}$ be any finite separable *field* extensions of degree k over F_{v_2} and $F_{v'_2}$, respectively. Let $L_{v_1} = \prod_{i=1}^t E_i$ and $L_{v'_1} = \prod_{i=1}^t E'_i$ where E_i (resp. E'_i) is a separable field extension of F_{v_1} (resp. of $F_{v'_1}$) of degree k_i for all i. By Lemma 4.3, there exists a finite separable field extension K of F of degree k such that $K \otimes F_v \simeq L_v$ for all $v \in S$. It follows from $(\lambda, s) \in \mathrm{LD}(k, \delta)$ that $\mathcal{E}_{v_1}$ and $\mathcal{E}_{v'_1}$ are non-empty. Also the sets $\mathcal{E}_{v_2}$ and $\mathcal{E}_{v'_2}$ are non-empty as K_{v_2} and $K_{v'_2}$ are fields. Since A_v is a matrix algebra for $v \notin S$, these sets $\mathcal{E}_v$ are non-empty, too. Finally, since $\mathbf{x}(\lambda, s)$ is not integral, the special vector $\mathbf{x}_{v_1} = \mathbf{x}(\lambda, s)$ is not integral. Therefore, we have constructed a pair (K, A) for which the Hasse principle fails. $\square$

PROPOSITION 4.6. *Let $\delta = p_1^{n_1} \cdots p_r^{n_r}$, and let k be a positive integer with $k > 1$ and $k \mid \delta$, where p_i's are prime divisors of δ. Assume that $k \leq \delta/p_i^{n_i}$ and $p_i \mid k$ for some $1 \leq i \leq r$ (so $r \geq 2$). Then the Hasse principle for the pair (k, δ) does not hold.*

Proof. Without loss of generality, we may assume $i = 1$. We write $k = p_1^{m_1} k'$ for some prime-to-p_1 integer k'. Let
  (i) $s = \delta/p_1^{n_1} = p_2^{n_2} \cdots p_r^{n_r}$.
  (ii) $\lambda = (1, \ldots, 1) =: (1^{\oplus k})$.
Then one has

$$d = p_1^{n_1}, \quad c_i := \gcd(k_i, d) = 1, \quad \ell_i := k_i/c_i = 1,$$

and

$$\mathbf{x}_i = s/k = p_2^{n_2} \cdots p_r^{n_r} / p_1^{m_1} k' \notin \mathbb{N}, \quad \forall i.$$

Note that the condition $k \leq \delta/p_1^{n_1} = s$ implies the equation $X_1 + \cdots + X_k = s$ has a positive integral solution, i.e. the set $\mathcal{E}(\lambda, s)$ is non-empty. By Theorem 4.5, the Hasse principle does not hold. $\square$

COROLLARY 4.7. *Let δ be a positive integer divisible by at least two primes. Then there is a positive integer k with $k \mid \delta$ such that the Hasse principle for the pair (k, δ) does not hold.*

Proof. Write $\delta = p_1^{n_1} \cdots p_r^{n_r}$, $r \geq 2$, and assume that p_1 is the smallest prime divisor. Let $k = p_1$. Then the corollary follows from Proposition 4.6. $\square$

COROLLARY 4.8. *Let δ be a positive integer divisible by at least two primes. Then there are a central simple algebra A of degree δ and a finite field extension of F of degree $[K : F] \mid \delta$ so that the Hasse principle for the pair (K, A) does not hold.*

5. Hasse principle for homogeneous spaces. The space of all embeddings of an etale commutative algebra K into A over F forms an variety X over F. It is naturally equipped with an action by the algebraic group $A^\times$ through conjugation. Therefore, the space X is decomposed into several orbits and each orbit is a homogeneous space of $A^\times$. In the previous section, we see some counterexamples for the Hasse principle in a more combinatorial way. A closer look at these counterexamples would find out that some of local points actually lie in different orbits. This is the main reason that causes the failure of the Hasse principle. It turns out this is the only reason that fails the Hasse principle. Namely, if we requires that all local points lie in the same orbit Y, then Y indeed has an F-rational point. This the main result of this section; see Theorem 5.4.

5.1. F-kernels and H^2. We recall the definition and basic properties of the second non-abelian Galois cohomology H^2. Our references are Springer [14] and Borovoi [1].

Let F be a field and let F^{sep} denote a separable closure of F. Put $\Gamma_F :$ $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let $\overline{G}$ be a linear algebraic group over F^{sep} and let $p : \overline{G} \to \mathrm{Spec}\, F^{\mathrm{sep}}$ be the structure morphism.

For $\sigma \in \Gamma_F$, let $\sigma_\natural : \mathrm{Spec}\, F^{\mathrm{sep}} \to \mathrm{Spec}\, F^{\mathrm{sep}}$ denote the isomorphism induced by the map σ^{-1}. We have $(\sigma\tau)_\natural = \sigma_\natural \tau_\natural$ for $\sigma, \tau \in \Gamma_F$.

Let $\sigma \in \Gamma_F$. We say that an automorphism of schemes $s : \overline{G} \to \overline{G}$ is a σ-*semialgebraic automorphism of* $\overline{G}$ if the following diagram

$$
\begin{array}{ccc}
\overline{G} & \xrightarrow{\;s\;} & \overline{G} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\mathrm{Spec}\, F^{\mathrm{sep}} & \xrightarrow{\;\sigma_\natural\;} & \mathrm{Spec}\, F^{\mathrm{sep}}
\end{array}
$$

commutes and the induced morphism $s' : \overline{G} \to \overline{G}^\sigma$ is an isomorphism of algebraic groups over $\mathrm{Spec}\, F^{\mathrm{sep}}$, where $\overline{G}^\sigma := \overline{G} \times_{\sigma_\natural} \mathrm{Spec}\, F^{\mathrm{sep}}$ is the base change deduced by the morphism $\sigma_\natural$. An *semialgebraic automorphism s of* $\overline{G}$ is a σ-semialgebraic automorphism of $\overline{G}$ for some $\sigma \in \Gamma_F$. Define $\gamma(s) := \sigma$ if s is a σ-semialgebraic automorphism of $\overline{G}$.

Let $\mathrm{SAut}\,\overline{G}$ denote the group of semialgebraic automorphisms of $\overline{G}$. The map $\gamma : \mathrm{SAut}\,\overline{G} \to \Gamma_F$ is a homomorphism. Let $\mathrm{Aut}\,\overline{G}$ denote the group of algebraic automorphisms of $\overline{G}$ over F^{sep}. We have an embedding $\mathrm{Aut}\,\overline{G} \to \mathrm{SAut}\,\overline{G}$ and an exact sequence

$$
1 \longrightarrow \mathrm{Aut}\,\overline{G} \longrightarrow \mathrm{SAut}\,\overline{G} \xrightarrow{\;\gamma\;} \Gamma_F.
$$

Let $\mathrm{Int}\,\overline{G}$ denote the group of inner automorphisms of $\overline{G}$. Set $\mathrm{Out}\,\overline{G} := \mathrm{Aut}\,\overline{G}/\mathrm{Int}\,\overline{G}$ and $\mathrm{SOut}\,\overline{G} := \mathrm{SAut}\,\overline{G}/\mathrm{Int}\,\overline{G}$. We have an exact sequence

$$
(5.1) \qquad 1 \longrightarrow \mathrm{Out}\,\overline{G} \longrightarrow \mathrm{SOut}\,\overline{G} \xrightarrow{\;q\;} \Gamma_F.
$$

DEFINITION 5.1. An F-*kernel* is a pair $L = (\overline{G}, \kappa)$, where $\overline{G}$ is an F^{sep}-group and $\kappa : \Gamma_F \to \mathrm{SOut}\,\overline{G}$ is a homomorphism satisfying the following two conditions:
(a) κ is a splitting of (5.1);
(b) κ can be lifted to a continuous map $f : \Gamma_F \to \mathrm{SAut}\,\overline{G}$; here "continuous" means that the stabilizer in Γ_F of any regular function $\phi \in F^{\mathrm{sep}}[\overline{G}]$ is open.

For an F^{sep}-kernel $L = (\overline{G}, \kappa)$, the second Galois cohomology set $H^2(F, L) = H^2(F, \overline{G}, \kappa)$ is defined as follows (also see [1, p. 221]). A 2-cocycle is a pair (f, u) of continuous maps

$$
f : \Gamma_F \to \mathrm{SAut}\,\overline{G}, \quad u : \Gamma_F \times \Gamma_F \to \overline{G}(F^{\mathrm{sep}})
$$

such that for any $\sigma, \tau, v \in \Gamma_F$, the following holds:

$$
(5.2) \qquad
\begin{aligned}
\mathrm{int}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau &= f_{\sigma\tau}, \\
u_{\sigma,\tau v} \cdot f_\sigma(u_{\tau,v}) &= u_{\sigma\tau,v} \cdot u_{\sigma,\tau}, \\
f_\sigma \bmod \mathrm{Int}\,\overline{G} &= \kappa(\sigma).
\end{aligned}
$$

Let $Z^2(F, L)$ denote the set of 2-cocycles with coefficients in L. The group $C(F, \overline{G})$ of continuous maps $c : \Gamma_F \to \overline{G}(F^{\text{sep}})$ acts on $Z^2(F, L)$ on the left by

$$c \cdot (f, u) = (f', u')$$

where

(5.3) $$f'_\sigma = \text{int}(c_\sigma) \circ f_\sigma, \quad u'_{\sigma,\tau} = c_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(c_\tau)^{-1} \cdot c_\sigma^{-1}.$$

The quotient set $H^2(F, L) := C(F, \overline{G}) \backslash Z^2(F, L)$ is called the *second Galois cohomology set of F with coefficients in L*. If $(f, u) \in Z^2(F, L)$, we write $\text{Cl}(f, u)$ for the cohomology class of (f, u) in $H^2(F, L)$. A *neutral 2-cocycle* is a cocycle of the form $(f, 1)$. A *neutral cohomological class* in $H^2(F, L)$ is the class of a neutral cocycle.

Let G be an F-group. Set $\overline{G} := G \otimes_F F^{\text{sep}}$. For $\sigma \in \Gamma_F$, let $\sigma_* : \overline{G} = \overline{G}^\sigma \to \overline{G}$ be the morphism induced by the fiber product

$$
\begin{array}{ccc}
\overline{G} & \xrightarrow{\;\sigma_*\;} & \overline{G} \\
p \downarrow & & \downarrow p \\
\text{Spec}\, F^{\text{sep}} & \xrightarrow{\;\sigma_\natural\;} & \text{Spec}\, F^{\text{sep}}.
\end{array}
$$

Then σ_* is a σ-semialgebraic automorphism of $\overline{G}$. We have $(\sigma\tau)_* = \sigma_*\tau_*$ for $\sigma, \tau \in \Gamma_F$. We obtain a continuous homomorphism

$$f_G : \Gamma_F \to \text{SAut}\,\overline{G}, \quad \sigma \mapsto \sigma_*$$

which splits the exact sequence (5.1). Composing f_G with the homomorphism $\text{SAut}\,\overline{G} \to \text{SOut}\,\overline{G}$, we obtain a homomorphism

$$\kappa_G : \Gamma_F \to \text{SOut}\,\overline{G}, \quad \kappa_G(\sigma) = \sigma_* \bmod \text{Int}\,\overline{G}$$

and, thus, an F-kernel $L_G = (\overline{G}, \kappa_G)$. Set $H^2(F, G) := H^2(F, L_G)$. When G is commutative, $H^2(F, G)$ is the group cohomology $H^2(\Gamma_F, G(F^{\text{sep}}))$ and it is an abelian group. In $H^2(F, G)$, we have a neutral class $\text{Cl}(f_G, 1)$, which we denote by $n(G)$.

Suppose $H^2(F, L)$ of an F-kernel $L = (\overline{G}, \kappa)$ contains a neutral class $\text{Cl}(f, 1)$. Then by Weil's descent theorem there is an F-form G of $\overline{G}$, unique up to F-isomorphism, such that $f = f_G$ and $\kappa = \kappa_G$. Let

$$\psi : \Gamma_F \to (G/Z)(F^{\text{sep}}) = \text{Int}\,\overline{G}$$

be a cocycle, where Z is the center of G. The map ψ defines an inner form $G' =_\psi G$ of G and gives rise to a homomorphism $f' := f_{G'}$. We have $f'_\sigma = \psi_\sigma f_\sigma$ for all $\sigma \in \Gamma_F$ and $\kappa_{G'} = \kappa$. This defines another neutral class $n(G') = \text{Cl}(f', 1)$ in $H^2(F, L)$. Conversely, any neutral class arises in this way. Indeed, one puts $\psi_\sigma := f'_\sigma f_\sigma^{-1}$ and then the map $\psi : \Gamma_F \to \text{Int}\,\overline{G}$ is a cocycle.

5.2. Homogeneous spaces. Let H be a connected reductive group over F. Let X be a right homogeneous space of H. The homogeneous space X gives rise to an F-kernel L as follows. Let $x \in X(F^{\text{sep}})$ be an F^{sep}-point and let $\overline{G}$ be the stabilizer subgroup of x in $\overline{H} := H \otimes_F F^{\text{sep}}$. For $\sigma \in \Gamma_F$, write

$$^\sigma x = x \cdot h_\sigma$$

where $h_\sigma \in H(F^{\mathrm{sep}})$. As Γ_F acts continuously on the discrete set $X(F^{\mathrm{sep}})$, we have a continuous map $h : \Gamma_F \to H(F^{\mathrm{sep}})$. The σ-semialgebraic automorphism of $\overline{H}$

$$f_\sigma := \mathrm{int}(h_\sigma) \circ \sigma_*$$

takes $\overline{G}$ into itself. To see this, if $g \in \overline{G}$, then we have $x \cdot g = x$ and $^\sigma x \cdot {}^\sigma g = {}^\sigma x$. So $x \cdot h_\sigma {}^\sigma g h_\sigma^{-1} = x$ and hence $\kappa(\sigma)(g) = h_\sigma {}^\sigma g h_\sigma^{-1} \in \overline{G}$. We regard f_σ as a σ-semialgebraic automorphism of $\overline{G}$. Then $f : \Gamma_F \to \mathrm{SAut}\,\overline{G}$ is a a continuous map, the composition

$$\kappa : \Gamma_F \to \mathrm{SAut}\,\overline{G} \to \mathrm{SOut}\,\overline{G}$$

is a homomorphism and this defines an F-kernel $L = (\overline{G}, \kappa)$.

A *principal homogeneous space of H over X* is a pair (P, α), where P is a right principal homogeneous space of H and $\alpha : P \to X$ is an H-equivariant F-morphism. The etale descent shows that α is necessarily faithfully flat.

If X has an F-rational point x_0, then there exists a principal homogeneous space (P, α) over X. Indeed, one takes $P = H$ and $\alpha(h) = x_0 \cdot h$. Conversely, if there exists a principal homogeneous space (P, α) over X and P has an F-rational point p_0, then X has an F-rational point $\alpha(p_0)$.

In [14, 1.20] (also see [1, 7.7]), Springer defined a cohomology class $\eta(X) \in H^2(F, L)$ associated to X. Springer [14, 3.7] and Borovoi [1, 7.7, p. 235] proved the following result.

PROPOSITION 5.2. *Notations as above. The class $\eta(X)$ is neutral if and only if there exists a principal homogeneous space (P, α) over X.*

In particular, if $H^1(F, H)$ is trivial, then $\eta(X)$ is neutral if and only if $X(F)$ is non-empty.

5.3. Homogeneous spaces of embeddings of K into A. We return to the problem of embeddings of an etale algebra K into a central simple algebra A. We remain F a global field as in the previous sections. Let X be the F-scheme that represents the functor

$$X(R) = \mathrm{Hom}^*_{R-alg}(K \otimes_F R, A \otimes_F R)$$

for any commutative F-algebra R. The group $H = A^\times$ of multiplicative group of A, viewed as an algebraic group over F, acts naturally on X on the left. We make the right H-action on X by setting $x \cdot h := h^{-1} \cdot x$.

The geometric orbits of X under H are in one-to-one correspondence with the elements in the orbit set $\mathcal{O}_{K \otimes F^{\mathrm{sep}}, A \otimes F^{\mathrm{sep}}}$. Therefore each geometric orbit can be represented by a function $f : \Sigma \to \mathbb{N}$ on Σ, where $\Sigma := \mathrm{Hom}_F(K, F^{\mathrm{sep}})$. Moreover, this correspondence is Γ_F-equivariant. We have the decomposition of geometric orbits

$$\overline{X} := X \otimes_F F^{\mathrm{sep}} = \bigcup_f \overline{X}_f,$$

where each $\overline{X}_f$ is a homogeneous space of $\overline{H} := H \otimes_F F^{\mathrm{sep}}$ corresponding to the function f. As the correspondence $f \mapsto \overline{X}_f$ is Γ_F-equivariant, the subvariety $\overline{X}_f$ is defined over F if and only if the function f is Γ_F-invariant.

Let $\overline{X}_f$ be a geometric orbit which is defined over F. Let X_f be the F-subscheme of X whose base extension to F^{sep} is $\overline{X}_f$; this is a homogeneous space of H over

F. As f is Γ_F-invariant, there is a tuple $(m_1, \ldots, m_r) \in \mathbb{N}^r$ so that for any point $\bar{\epsilon} \in X_f(F^{\mathrm{sep}})$

$$\bar{\epsilon} : K \otimes_F F^{\mathrm{sep}} \hookrightarrow \overline{A} := A \otimes F^{\mathrm{sep}} = \mathrm{End}_{F^{\mathrm{sep}}}(\overline{V}),$$

the induced $K \otimes F^{\mathrm{sep}}$-module $\overline{V}$ is isomorphic to

$$(K_1 \otimes F^{\mathrm{sep}})^{m_1} \oplus \cdots \oplus (K_r \otimes F^{\mathrm{sep}})^{m_r}.$$

Fix an element $\bar{\epsilon}_0$ in $X_f(F^{\mathrm{sep}})$. The centralizer $\overline{B}$ of the subalgebra $\bar{\epsilon}_0(K \otimes F^{\mathrm{sep}})$ in $\overline{A}$ is isomorphic to

$$\mathrm{Mat}_{m_1}(K_1 \otimes F^{\mathrm{sep}}) \times \cdots \times \mathrm{Mat}_{m_r}(K_r \otimes F^{\mathrm{sep}}),$$

and the stabilizer subgroup $\overline{G}$ of $\bar{\epsilon}_0$ is equal to the multiplicative group $\overline{B}^{\times}$ (viewed as an algebraic group over F^{sep}). Let $\overline{Z}$ be the center of $\overline{G}$. The center $Z(\overline{B})$ of $\overline{B}$ is equal to $\bar{\epsilon}_0(K \otimes F^{\mathrm{sep}})$ and we have $\overline{Z} = Z(\overline{B})^{\times}$. The homomorphism $\kappa : \Gamma_F \to \mathrm{SOut}\,\overline{Z} = \mathrm{SAut}\,\overline{Z}$ associated to X_f gives an action on the center

$$(5.4) \qquad \overline{Z}(F^{\mathrm{sep}}) = Z(\overline{B})^{\times} = \bar{\epsilon}_0(K \otimes F^{\mathrm{sep}})^{\times}.$$

LEMMA 5.3.
1. *The center Z of $(\overline{G}, \kappa)$ is isomorphic to $\prod_{i=1}^{r} \mathrm{Res}_{K_i/F}\,\mathbb{G}_{\mathrm{m}}$.*
2. *Any F-form G of $\overline{G}$ with $\kappa_G = \kappa$ is of the form $\mathrm{GL}_1(A_1) \times \cdots \times \mathrm{GL}_1(A_r)$ for some central simple K_i-algebras A_i of degree m_i.*

Proof. (1) For any $\sigma \in \Gamma_F$, we have $^{\sigma}\bar{\epsilon}_0 = \mathrm{int}(h_{\sigma}^{-1}) \circ \bar{\epsilon}_0$ for some element $h_{\sigma} \in \overline{A}^{\times}$. We have the commutative diagram:

$$(5.5) \qquad \begin{array}{ccc} K \otimes F^{\mathrm{sep}} & \xrightarrow{\bar{\epsilon}_0} & A \otimes F^{\mathrm{sep}} \\ {\scriptstyle \mathrm{id}_K \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes \sigma} \\ K \otimes F^{\mathrm{sep}} & \xrightarrow{^{\sigma}\bar{\epsilon}_0} & A \otimes F^{\mathrm{sep}}. \end{array}$$

As $\kappa(\alpha) = \mathrm{int}(h_{\sigma}) \circ (\mathrm{id}_A \otimes \sigma)$, we have the following commutative diagram:

$$(5.6) \qquad \begin{array}{ccc} K \otimes F^{\mathrm{sep}} & \xrightarrow{\bar{\epsilon}_0} & A \otimes F^{\mathrm{sep}} \\ {\scriptstyle \mathrm{id}_K \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \kappa(\sigma)} \\ K \otimes F^{\mathrm{sep}} & \xrightarrow{\bar{\epsilon}_0} & A \otimes F^{\mathrm{sep}}. \end{array}$$

This shows that $\kappa(\sigma)$ acts on $\bar{\epsilon}_0(K \otimes F^{\mathrm{sep}})$ on the second factor and hence $Z(F) = \bar{\epsilon}_0(K)^{\times}$. The F-subalgebra generated by $Z(F)$ is equal to $\bar{\epsilon}_0(K)$. Therefore, Z is the subtorus $\bar{\epsilon}_0(K)^{\times}$ and is isomorphic to $\prod_{i=1}^{r} \mathrm{Res}_{K_i/F}\,\mathbb{G}_{\mathrm{m}}$. .

(2) Since the $Z(F)$ generates the F-subalgebra $\bar{\epsilon}_0(K)$, the F-subalgebra generated by $G(F)$ in $\overline{A}$ is a semi-simple F-algebra containing $\bar{\epsilon}_0(K)$, which is of the form $A_1 \times \cdots \times A_r$, where each A_i is a central simple algebra over $\bar{\epsilon}_0(K_i)$. Therefore, G is isomorphic to $\mathrm{GL}_1(A_1) \times \cdots \times \mathrm{GL}_1(A_r)$. This completes the proof of the lemma. $\square$

THEOREM 5.4 (The Hasse principle). *Let X_f be a homogeneous space of H defined over F and $(m_1, \ldots, m_r)$ be the corresponding tuple as above. If $X_f(F_v) \neq \emptyset$ for all $v \in V^F$, then $X_f(F) \neq \emptyset$.*

Proof. Since H is an inner form of GL_m, one has $H^1(F, H) = 1$ and $H^1(F_v, H) = 1$ by the Hilbert 90 theorem. By Proposition 5.2, the obstruction class $\eta(X_f)$ (resp. $\eta_v(X_f)$) is neutral if and only if $X_f(F) \neq \emptyset$ (resp. $X_f(F_v) \neq \emptyset$). Choose the quasi-split form $G_0 = \prod_{i=1}^r \mathrm{Res}_{K_i/F} \mathrm{GL}_{m_i}$ of the isotropic subgroup $\overline{G}$ with $\kappa_{G_0} = \kappa$, and let $n(G_0)$ denote the neutral class $\mathrm{Cl}(f_{G_0}, 1) \in H^2(F, L)$. As $H^2(F, L)$ is non-empty, it is a principal homogeneous space under $H^2(F, Z)$; see [14, 1.17, p. 170] . The base class $n(G_0)$ gives a natural bijection $H^2(F, Z) \simeq H^2(F, L)$. As $H^2(F, Z) = \mathrm{Br}(K_1) \times \cdots \times \mathrm{Br}(K_r)$, the class $\eta(X_f) \in H^2(F, L)$ can be represented by $([A_1], \ldots, [A_r]) \in H^2(F, Z)$, where each A_i is a central simple algebra over K_i. Moreover, the class $\eta(X)$ is neutral if and only if the class $([A_1], \ldots, [A_r])$ lies in the image of the boundary map $H^1(F, G_0/Z) \to H^1(F, Z)$ (see [1, Proposition 2.3, p. 224]), that is, the condition

$$\mathrm{ind}_{K_i}(A_i)|m_i, \quad i = 1, \ldots, r$$

holds. Now under the assumption $X_f(F_v) \neq \emptyset$ for all $v \in V^F$ we get the condition

$$\mathrm{ind}_{K_{i,w}}(A_i \otimes_{K_i} K_{i,w})|m_i$$

for all $i = 1, \ldots, r$ and $w \in V^{K_i}$. It follows from the Hasse-Brauer-Noether theorem that $\mathrm{ind}_{K_i}(A_i)|m_i$, for $i = 1, \ldots, r$ and hence $X_f(F) \neq \emptyset$. This completes the proof of the theorem. $\square$

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