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POSTNIKOV-STABILITY VERSUS SEMISTABILITY OF SHEAVES*

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Abstract. We present a novel notion of stable objects in a triangulated category. This Postnikov-stability is preserved by equivalences. We show that for the derived category of a projective variety this notion includes the case of semistable sheaves. As one application we compactify a moduli space of stable bundles using genuine complexes via Fourier-Mukai transforms.

Key words. Stable complexes, derived category, moduli spaces, Postnikov stability.

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Introduction. Let X be a polarized, normal projective variety of dimension n over an algebraically closed field k. Our aim is to introduce a stability notion (called *Postnikov stability*) for complexes, i.e. for objects of $\mathrm{D}^{\mathrm{b}}(X)$, the bounded derived category of coherent sheaves on X.

We mention some applications of this theory: First, the purely homological definition of Postnikov stability for objects means stability is obviously conserved under equivalences (Theorem 1). Thus we get a more conceptual framework for what is called 'preservation of stability'. See Section 2 for details and Subsection 2.1 for a detailed example.

Second, the framework can produce new compactifications: while our initial Postnikov data $C_\bullet$ will always consist of sheaves, by applying an equivalence ϕ we obtain new Postnikov data $(\phi(C_\bullet))$ which are in general complexes. As an example, we mention elliptic K3 surfaces in Subsection 2.3; the Postnikov-stability approach has been used in [4] to answer a question of Friedman [8]. Subsection 2.4 deals with compactifications of moduli spaces of instantons.

Our first clue has been Faltings' observation that semistability on curves can be phrased as the existence of non-trivial orthogonal sheaves [7]. Additionally, there is a more recent result by Álvarez-Cónsul and King showing that every Gieseker semistable sheaf possesses a non-trivial orthogonal object, regardless of dimension [1]. This result together with the homological sheaf condition (Proposition 6) and the homological criterion for purity (Proposition 8) yields a purely homological condition (Theorem 12) for a complex to be isomorphic to a Gieseker semistable sheaf of given Hilbert polynomial.

It seems only fair to point out that the results of this article in all probability bear no connection with Bridgeland's notion of t-stability on triangulated categories (see [6]). His starting point about (semi)stability in the classical setting is the Harder-Narasimhan filtration whereas, as mentioned above, we are interested in the possibility to capture semistability in terms of Hom's in the derived category. Our approach is much closer to, but completely independent of, Inaba (see [15]).

On notation. We deviate slightly from common usage by writing e^i for the i-th cohomology sheaf of an object $e \in \mathrm{D}^{\mathrm{b}}(X)$. Derivation of functors is not denoted by

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a symbol: e.g. for a proper map $f\colon X \to Y$, we denote by $f_*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ the exact functor obtained by deriving $f_*\colon \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$.

Given objects a, b of a k-linear triangulated category, set $\mathrm{Hom}^i(a,b) := \mathrm{Hom}(a,b[i])$ and $\hom^i(a,b) := \dim_{\mathsf{k}} \mathrm{Hom}^i(a,b)$. For $e \in \mathrm{D^b}(X)$, we put $H^i(e) := \mathrm{Hom}^i(\mathcal{O}_X, e)$ and $h^i(e) := \dim H^i(e)$. The Hilbert polynomial of e is denoted by p_e; it is defined by $p_e(l) = \chi(e(l)) := \sum_i (-1)^i h^i(e \otimes \mathcal{O}_X(l))$. If $Z \subset X$ is a closed subset, then $e|_Z := e \otimes \mathcal{O}_Z$ denotes the derived tensor product. For a line bundle L on X, the notation L^n will mean the n-fold tensor product of L, except for the trivial bundle, where $\mathcal{O}_X^n$ denotes the free bundle of rank n.

P-stability. Let $\mathcal{T}$ be a k-linear triangulated category for some field k; we think of $\mathcal{T} = \mathrm{D^b}(X)$, the bounded derived category of a normal projective variety X, defined over an algebraically closed field k. A *Postnikov-datum* or just *P-datum* is a finite collection $C_a, C_{a-1}, \ldots, C_0, \ldots, C_{b+1}, C_b \in \mathcal{T}$ of objects together with nonnegative integers N^i_j (for $i, j \in \mathbb{Z}$) of which only a finite number are nonzero. We will write $(C_\bullet, N)$ for this.

Recall the notions of Postnikov system and convolution (see [9], [3], [23], [16]): given finitely many objects C_i (suppose $a \geq i \geq 0$) of $\mathcal{T}$ together with morphisms $d_i\colon C_i \to C_{i-1}$ such that $d^2 = 0$, a diagram of the form

$$\begin{tikzcd}
C_a \arrow[rr, "d_a"] \arrow[dr, equal] & & C_{a-1} \arrow[rr, "d_{a-1}"] \arrow[dr] & & C_{a-2} \arrow[r] & \cdots\cdots & & \cdots\cdots \arrow[r] & C_1 \arrow[rr, "d_1"] \arrow[dr] & & C_0 \arrow[dr] & \\
& T_a \arrow[ur] & & T_{a-1} \arrow[ll, "{[1]}"'] \arrow[ur] & & T_{a-2} \arrow[ll, "{[1]}"'] & \cdots\cdots \arrow[l] & T_2 \arrow[l] \arrow[ur] & & T_1 \arrow[ll, "{[1]}"'] \arrow[ur] & & T_0 \arrow[ll, "{[1]}"']
\end{tikzcd}$$

(where the upper triangles are commutative and the lower ones are distinguished) is called a *Postnikov system* subordinated to the C_i and d_i. The object T_0 is called the *convolution* of the Postnikov system.

Definition. An object $A \in \mathcal{T}$ is *P-stable with respect to* $(C_\bullet, N)$ if

(i) $\hom^i_{\mathcal{T}}(C_j, A) = N^i_j$ for all $j = a, \ldots, b$ and all i.

(ii) For $j > 0$, there are morphisms $d_j\colon C_j \to C_{j-1}$ such that $d^2 = 0$ and that the complex $(C_{\bullet \geq 0}, d_\bullet)$ admits a convolution T_0.

(iii) T_0 is orthogonal to A, i.e. $\mathrm{Hom}^*_{\mathcal{T}}(T_0, A) = 0$.

Remark.

(a) Convolutions in general do not exist, and if they do, there is no uniqueness in general, either. There are restrictions on the $\mathrm{Hom}^k(C_i, C_j)$'s which ensure the existence of a (unique) convolution. For example, if $\mathcal{T} = \mathrm{D^b}(X)$ and all C_j are sheaves, then the unique convolution is just the complex $C_\bullet$ considered as an object of $\mathrm{D^b}(X)$.

We are using Postnikov systems only to have a notion for abstract triangulated categories. If $\mathcal{T}$ is an algebraic triangulated category, i.e. comes from a dg category, then the total complex can be formed.

(b) Note that the objects C_j with $j < 0$ do not take part in forming the Postnikov system. We call the conditions enforced by these objects via (i) the *passive* stability conditions. They can be used to ensure numerical constraints, like fixing the Hilbert polynomial of sheaves. In some cases, it is useful to specify only some of the N^i_j. We will do this a few times — the whole theory runs completely parallel, with a slightly more cumbersome notation.

(c) In many situations there will be trivial choices that ensure P-stability. This should be considered as a defect of the parameters (like choosing non-ample line bundles when defining μ-stability) and not as a defect of the definition.

By the very definition of P-stability, the following statement about preservation of stability under fully faithful functors (e.g. equivalences) is immediate.

THEOREM 1. *Let $\Phi\colon \mathcal{T} \to \mathcal{S}$ be an exact, fully faithful functor between k-linear triangulated categories $\mathcal{T}$ and $\mathcal{S}$, and $(C_\bullet, N)$ a P-datum in $\mathcal{T}$. Then, an object $A \in \mathcal{T}$ is P-stable with respect to $(C_\bullet, N)$ if and only if $\Phi(A)$ is P-stable with respect to $(\Phi(C_\bullet), N)$.*

This shifts the viewpoint from preservation of stability to transformation of stability parameters under Fourier-Mukai transforms. See Proposition 13 for an example. The main result of this article is the following theorem: P-stability contains both Gieseker stability and μ-stability.

COMPARISON THEOREM. *Let X be a smooth projective variety and H a very ample divisor on X. Fix a Hilbert polynomial p. Then there is a P-stability datum $(C_\bullet, N)$ such that for any object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:*
(i) E is a μ-semistable sheaf with respect to H of Hilbert polynomial p
(ii) E is P-stable with respect to $(C_\bullet, N)$.
Likewise, there is a P-stability datum $(C'_\bullet, N')$ such that for any object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:
(i') E is a Gieseker semistable pure sheaf with respect to H of Hilbert polynomial p
(ii') E is P-stable with respect to $(C'_\bullet, N')$.

The proof of this theorem occupies the next section. The actual statements are slightly sharper; see Theorems 11 and 12. The case of surfaces was already treated in [12].

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1. Proof of the Comparison Theorem. The proof proceeds in the following steps:
1. Euler triangle and generically injective morphisms.
2. Homological conditions for a complex to be a sheaf.
3. Homological conditions for purity of a sheaf.
4. Homological conditions for semistability on curves.
5. P-stability implies μ-semistability.
6. P-stability implies Gieseker semistability.

1.1. The Euler triangle.

LEMMA 2. *Let U and W be k-vector spaces of finite dimension. Consider a morphism $\rho\colon U \otimes \mathcal{O}_{\mathbb{P}^n} \to W \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ with nonzero kernel $K = \ker(\rho)$. Then for any integer $m \geq (\dim(U) - 1)n$ we have $H^0(K(m)) \neq 0$.*

Proof. Write $u := \dim(U)$ and $w := \dim(W)$ in this proof. Denoting $I := \mathrm{im}(\rho)$ and $C := \mathrm{coker}(\rho)$, there are two short exact sequences $0 \to I \to W \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to C \to 0$ and $0 \to K \to U \otimes \mathcal{O}_{\mathbb{P}^n} \to I \to 0$. Their long cohomology sequences yield $h^0(I(m)) \leq h^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(m+1)) = w\binom{n+m+1}{n}$ and $h^0(K(m)) \geq h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(m)) - h^0(I(m))$.

First assume $w < u$. This implies $h^0(I(m)) \le (u-1)\binom{n+1+m}{n}$. Since $h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = u\binom{n+m}{n}$, we get $h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(m)) > h^0(I(m))$ for all $m \ge (u-1)n$. Thus, we obtain $h^0(K(m)) > 0$ for $m \ge (u-1)n$.

Now assume $w \ge u$. Then C has rank at least $w-u+1$, as the rank of K is positive by assumption. Hence there exists a subspace $W' \subset W$ of dimension $w-u+1$ such that the resulting morphism $W' \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to C$ is injective in the generic point, and hence injective. Thus, the image of the injective morphism $H^0(I(m)) \to H^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(m+1))$ is transversal to $H^0(W' \otimes \mathcal{O}_{\mathbb{P}^n}(m+1))$. This implies $h^0(I(m)) \le (u-1)\binom{n+1+m}{n}$ and we proceed as before. ∎

Construction 3. *The Euler triangle and objects* $S^m(V,a,b)$.

For any two objects a, b of a k-linear triangulated category $\mathcal{T}$ and some subspace $V \subset \mathrm{Hom}(a,b)$ of finite dimension we define a distinguished (Euler) triangle

$$S^m(V,a,b) \to \mathrm{Sym}^{m+1}(V) \otimes a \xrightarrow{\theta} \mathrm{Sym}^m(V) \otimes b \to S^m(V,a,b)[1]$$

where tensor products of vector spaces and objects are just finite direct sums, and θ is induced by the natural map $\mathrm{Sym}^{m+1}(V) \to \mathrm{Sym}^m(V) \otimes \mathrm{Hom}(a,b)$, $f_0 \vee \cdots \vee f_m \mapsto \sum_i (f_0 \vee \cdots \hat{f}_i \cdots \vee f_m) \otimes f_i$. If $\mathrm{Hom}(a,b)$ is finite-dimensional, we use the short hand $S^m(a,b) := S^m(\mathrm{Hom}(a,b),a,b)$. For any $c \in \mathcal{T}$, there is a long exact sequence

$$\begin{tikzcd}
\mathrm{Hom}^{k-1}(b,c) \otimes \mathrm{Sym}^m(V^\vee) \arrow[r] & \mathrm{Hom}^{k-1}(a,c) \otimes \mathrm{Sym}^{m+1}(V^\vee) \arrow[r] & \mathrm{Hom}^{k-1}(S^m(V,a,b),c) \arrow[dll] \\
\mathrm{Hom}^{k}(b,c) \otimes \mathrm{Sym}^m(V^\vee) \arrow[r] & \mathrm{Hom}^{k}(a,c) \otimes \mathrm{Sym}^{m+1}(V^\vee) \arrow[r] & \mathrm{Hom}^{k}(S^m(V,a,b),c).
\end{tikzcd}$$

Remark. In the special case where $\mathcal{T} = \mathrm{D^b}(\mathbb{P}^n_{\mathsf{k}})$ is the bounded derived category of the projective space $\mathbb{P}^n_{\mathsf{k}}$ and $a = \mathcal{O}_{\mathbb{P}^n}$, $b = \mathcal{O}_{\mathbb{P}^n}(1)$, $V = \mathrm{Hom}(a,b) = H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and $m = 0$, the above triangle comes from the Euler sequence $0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$.

Lemma 4. *Let $\mathcal{T}$ be a triangulated k-linear category with finite-dimensional Hom's, $a, b, c \in \mathcal{T}$ objects with $\mathrm{Hom}^{-1}(a,c) = 0$ and let $V \subset \mathrm{Hom}(a,b)$ be a subspace. Then the following conditions are equivalent:*

(i) The natural morphism $\varrho_v \colon \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,c)$ is injective for general $v \in V$.

(ii) $\mathrm{Hom}^{-1}(S^m(V,a,b),c) = 0$ holds for some $m \ge (\dim(V)-1)(\hom(b,c)-1)$.

Proof. We consider the morphism $\mathrm{Hom}(b,c) \to V^\vee \otimes \mathrm{Hom}(a,c)$. Together with the natural surjection $V^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \to \mathcal{O}_{\mathbb{P}(V^\vee)}(1)$, this gives a morphism of sheaves on $\mathbb{P}(V^\vee)$:

$$\varrho \colon \mathrm{Hom}(b,c) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \to \mathrm{Hom}(a,c) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1)\,.$$

The injectivity of ϱ is equivalent to the injectivity at all stalks, i.e. of $\varrho_v \colon \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,c)$ for all $v \in V$; since $\ker(\varrho)$ is a subsheaf of a torsion free sheaf, the injectivity for just one $v \in V$ is enough. By Lemma 2 this is equivalent to the injectivity of

$$H^0(\varrho \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m)) \colon H^0(\mathrm{Hom}(b,c) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m)) \to H^0(\mathrm{Hom}(a,c) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m+1))$$

for $m = (\dim(V)-1)(\hom(b,c)-1)$. Since $\mathrm{Hom}^{-1}(a,c) = 0$, the long exact cohomology sequence of the triangle from Construction 3 gives that the kernel of $H^0(\varrho \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m))$ is $\mathrm{Hom}^{-1}(S^m(V,a,b),c)$. ∎

1.2. Sheaf conditions. Let X be a projective variety over k (in this subsection, we only need k to be infinite) and $\mathcal{O}_X(1)$ a line bundle corresponding to the very ample divisor H. Let $V = H^0(\mathcal{O}_X(1))$ the space of global sections and $\mathbb{P} := \mathbb{P}(V^\vee) = |H|$ the complete linear system for H.

Our aim is to find conditions on a complex $e \in \mathrm{D^b}(X)$ in terms of the Hom's from finitely many test objects, ensuring that e is isomorphic to a sheaf, i.e. a complex concentrated in degree 0. These conditions only depend on the Hilbert polynomial p_e with respect to $\mathcal{O}_X(1)$.

The numerical data. Fix non-negative integers n and v. For a polynomial function $p \in \mathbb{Q}[t]$ with integer values, its derivative is defined as $p'(t) := p(t) - p(t-1)$. We also set $\mathrm{sym}_v(m) := \binom{m+v-1}{v-1}$, which is the dimension of $\mathrm{Sym}^m(V)$ for a v-dimensional vector space V.

Call a sequence $(m_1, \ldots, m_n)$ of integers (p,n)-*admissible* if $m_{k+1} \geq (p_k(-l) - 1)(v-1)$ for $l = 1, \ldots, n-k$ where the polynomials $p_0, \ldots, p_{n-1}$ are defined by $p_0 = p$ and $p_{k+1} = \mathrm{sym}_v(m_{k+1}) \cdot p_k' + \mathrm{sym}_{v-1}(m_{k+1}+1) \cdot p_k$. One can easily define a (p,n)-admissible sequence by recursion: set $m_{k+1} := \max\{(p_k(-l) - 1)(v-1) \mid l = 1, \ldots, n-k\}$, the polynomials being defined by the above formula in each step.

Suppose that $(m_1, \ldots, m_n)$ is a (p,n)-admissible sequence and that $p_k(-l) \geq 0$ for all $l, k \geq 0$ with $l + k \leq n$. Then $(m_1, \ldots, m_{n-1})$ is a $(p', n-1)$-admissible sequence, as follows from induction and unwinding the definitions. In this case, if the auxiliary polynomials for the (p,n)-sequence are denoted $p_0, \ldots, p_n$ as above, then those for the $(p', n-1)$-sequence are just $p_0', \ldots, p_{n-1}'$.

The vector bundles G_m and S_m and F_k. We denote the standard projections by $p\colon \mathbb{P} \times X \to \mathbb{P}$ and $q\colon \mathbb{P} \times X \to X$. The identity in $V \otimes V^\vee = H^0(\mathcal{O}_X(H)) \otimes H^0(\mathcal{O}_\mathbb{P}(1)) = \mathrm{Hom}(q^*\mathcal{O}_X(-H), p^*\mathcal{O}_\mathbb{P}(1))$ yields a natural morphism $\alpha\colon q^*\mathcal{O}_X(-H) \to p^*\mathcal{O}_\mathbb{P}(1)$. The cokernel $\mathcal{G}$ of α is the universal divisor, i.e. $\mathcal{G}|_{\{D\}\times X} = \mathcal{O}_D$ for all $D \in \mathbb{P}$. We can consider $q^*\mathcal{O}_X(-H)$ and $p^*\mathcal{O}_\mathbb{P}(1)$ and $\mathcal{G}$ as Fourier-Mukai kernels on $\mathbb{P} \times X$. Then we obtain, for any object $a \in \mathrm{D^b}(\mathbb{P})$, an exact triangle $\mathsf{FM}_{q^*\mathcal{O}_X(-H)}(a) \to \mathsf{FM}_{p^*\mathcal{O}_\mathbb{P}(1)}(a) \to \mathsf{FM}_\mathcal{G}(a)$. In particular, we set $G_m := \mathsf{FM}_\mathcal{G}(\mathcal{O}_\mathbb{P}(m))$. The projection formula and base change show that the above triangle reduces to the short exact sequence $0 \to \mathrm{Sym}^m(V^\vee) \otimes \mathcal{O}_X(-H) \to \mathrm{Sym}^{m+1}(V^\vee) \otimes \mathcal{O}_X \to G_m \to 0$. Hence in this case, $G_m = R^0q_*(\mathcal{G} \otimes p^*\mathcal{O}_\mathbb{P}(m))$ is a vector bundle and the higher direct images vanish. The exact sequence also yields $p_{G_m \otimes e} = \mathrm{sym}_v(m) \cdot p_e' + \mathrm{sym}_{v-1}(m+1) \cdot p_e$.

Let $S_m := G_m^\vee$; note that $S_m = S^m(V, \mathcal{O}_X, \mathcal{O}_X(1))$ using Construction 3. As a consequence of Lemma 4, we collect the next statement.

Corollary 5. *For $e \in \mathrm{D^b}(X)$ with $H^{-1}(e) = 0$ and $m \geq (v-1)(h^0(e(-1))-1)$, the following conditions are equivalent:*

(i) $H^{-1}(e|_D) = 0$ *for general $D \in |H|$ with $e|_D := e \otimes \mathcal{O}_D$ (derived tensor product)*
(ii) $\mathrm{Hom}^{-1}(S_m, e) = 0$
(iii) $H^{-1}(e \otimes G_m) = 0$.

Finally, we define another series of vector bundles by $F_0 := \mathcal{O}_X$ and $F_k := F_{k-1} \otimes G_{m_k}$.

Proposition 6. *Let X be a projective variety of dimension n and $\mathcal{O}_X(1)$ a very ample line bundle. Let $V = H^0(\mathcal{O}_X(1))$ and $v = \dim(V)$. Let $p \in \mathbb{Q}[t]$ be an integer valued polynomial with $\deg(p) \leq n$. Suppose that $(m_1, \ldots, m_n)$ is a (p,n)-admissible sequence with auxiliary polynomials $p_1, \ldots, p_n$.*

Assume that $e \in \mathrm{D^b}(X)$ is an object such that for all $l, k \geq 0$ with $l + k \leq n$ we have $h^0(F_k(-l) \otimes e) = p_k(-l)$ and $h^i(F_k(-l) \otimes e) = 0$ for all $i \neq 0$. Then $e \cong e^0$ is a sheaf with Hilbert polynomial p. Furthermore, e is 0-regular.

Proof. We proceed by induction on the dimension n. The start $n = 0$ is trivial. Let $n > 0$. We divide the proof into steps.

Step 1: *The complex $e|_D$ is a 0-regular sheaf for general $D \in |H|$.*
Let us begin by pointing out that $(m_1, \dots, m_{n-1})$ is a $(p', n-1)$-admissible sequence. Next, the graded vector spaces $H^*(F_k(-l) \otimes e)$ vanish by assumption outside of degree 0, where $k, l \geq 0$ and $k + l \leq n$. Hence, $H^*(F_k(-l) \otimes e|_D)$ can be nontrivial at most in degrees 0 and -1 (where $k + l < n$).

We have: $H^{-1}(F_k(-l) \otimes e) = 0$ and $H^{-1}(G_{m_{k+1}} \otimes F_k(-l) \otimes e) = 0$ by assumption on e (recall $G_{m_{k+1}} \otimes F_k = F_{k+1}$); also $m_{k+1} \geq (v-1)(p_k(-l-1)-1)$ with $p_k(-l-1) = h^0(F_k(-l-1) \otimes e)$. Hence we get $H^{-1}(F_k(-l) \otimes e|_D) = 0$ for general $D \in |H|$, using implication (iii)⇒(i) of Corollary 5. Thus, $H^*(F_k(-l) \otimes e|_D)$ is concentrated in degree 0 and of the correct dimension $h^0(F_k(-l) \otimes e|_D) = p_k(-l) - p_k(-l-1) = p'_k(-l)$. By induction, $e|_D$ is a 0-regular sheaf for a general divisor D.

Step 2: *The Eilenberg-Moore spectral sequence for $H^i(e)$.*
We fix a divisor D in the linear system $|H|$ such that $e|_D$ is a sheaf. We conclude that all homology sheaves e^i in degrees $i \neq 0$ either vanish or have 0-dimensional support. (Support of dimension one or higher would be detected by the ample divisor D.) Looking at the Eilenberg-Moore spectral sequence for $k = 0$ and $0 \leq l \leq n$

$$E_2^{p,q} = H^q(e^{-p}(-l)) \Rightarrow E_\infty^{p+q} = H^{p+q}(e(-l)),$$

we see that it has non-zero E_2 terms at most in the row $q = 0$ and the upper column $p = 0$, $q \geq 0$. By assumption, we know that the E_∞ terms vanish for all $p + q \neq 0$.

Step 3: *We have $H^i(e^0(-l)) = 0$ for all integers $i \geq 2$ and $l \leq 1$.*
We include Mumford's argument [22, §14] for completeness. Recall that a sheaf E is r-regular if $H^i(E(r-i)) = 0$ for all $i > 0$. This implies that $E(r)$ is globally generated. An r-regular sheaf is s-regular for $s \geq r$. Both properties will be used tacitly.

Consider the short exact sequence $0 \to e^0(-1) \to e^0 \to e^0|_D \to 0$. The sheaf $e^0|_D$ is 0-regular; this gives isomorphisms $H^i(e^0(-l-1)) \cong H^i(e^0(-l))$ for all $i \geq 2$, and $l \leq 1$. The claim follows from $H^i(e^0(-l)) = 0$ for $l \ll 0$.

Step 4: *The cohomology sheaves e^{-p} vanish for all integers $p \notin \{0, 2\}$.*
We apply the last step to the Eilenberg-Moore spectral sequence: all $E_2^{0,q}$ vanish except for $q \in \{0, 1\}$. From $E_\infty^{p+q} = 0$ unless $p + q = 0$ we conclude $H^0(e^{-p}) = 0$ unless $p \in \{0, 2\}$. However, the cohomology sheaves e^{-p} are 0-dimensional for all $p \neq 0$, so we get $e^{-p} = 0$ for $p \notin \{0, 2\}$.

Step 5: *We have an equality $h^1(e^0) = h^1(e^0(-1)) = h^0(e^{-2})$.*
We consider the d_2 map of our spectral sequence which gives the following exact sequence

$$0 = H^1(e(-l)) \to H^1(e^0(-l)) \xrightarrow{d_2} H^0(e^{-2}(-l)) \to H^2(e(-l)) = 0$$

for $l \in \{0, 1\}$. Therefore d_2 is an isomorphism. Since e^{-2} is 0-dimensional, we have $e^{-2} = e^{-2}(-1)$, hence $h^1(e^0) = h^0(e^{-2}) = h^0(e^{-2}(-1)) = h^1(e^0(-1))$.

Step 6: *We have $H^1(e^0) = 0 = H^1(e^0(-1))$.*
Again we can follow Mumford's proof on regularity [22, page 102]. Consider for all

$l \leq 0$ the following commutative diagram

$$\begin{tikzcd}
V \otimes H^0(e^0(-l)) \arrow[r, "\alpha_0"] \arrow[d, "\beta_0"] & V \otimes H^0(e^0(-l)|_D) \arrow[r, "\alpha_1"] \arrow[d, "\beta_1"] & V \otimes H^1(e^0(-l-1)) \arrow[d, "\beta_2"] \\
H^0(e^0(-l+1)) \arrow[r, "\gamma_0"] & H^0(e^0(-l+1)|_D) \arrow[r, "\gamma_1"] & H^1(e^0(-l))
\end{tikzcd}$$

where the horizontal maps are induced from the triangles $e^0(-l-1) \to e^0(-l) \to e^0(-l)|_D$ (top row) and $e^0(-l) \to e^0(-l+1) \to e^0(-l+1)|_D$ (bottom row) and the vertical maps correspond to composition with $V = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(1))$. The 0-regularity of $e^0|_D$ implies that β_1 is surjective for all $l \leq 0$.

By induction we now show $H^1(e^0(-l-1)) \cong H^1(e^0(-l))$ for $l \leq 0$. The start $l = 0$ was done in the previous step. Given such an isomorphism for some $-l$, we deduce $\alpha_1 = 0$. Hence α_0 is surjective, so γ_0 is as well. This implies $\gamma_1 = 0$, so the map $H^1(e^0(-l)) \to H^1(e^0(-l+1))$ is injective. Its cokernel $H^1(e^0(-l+1)|_D)$ vanishes by 0-regularity of $e^0|_D$, and the resulting isomorphism $H^1(e^0(-l)) \to H^1(e^0(-l+1))$ keeps the induction going.

We conclude $H^1(e^0(-1)) = H^1(e^0) = H^1(e^0(1)) = \cdots = H^1(e^0(-l)) = 0$ for $l \ll 0$.

Step 7: *Conclusion: e is a sheaf, and e^0 is 0-regular.*
¿From the last two steps we see $h^0(e^{-2}) = 0$. Thus by step 4, the only cohomology sheaf which is not zero is e^0. Now we can read off the 0-regularity by definition, since our spectral sequence degenerates and we have $h^i(e(-l)) = h^i(e^0(-l))$. □

1.3. Purity conditions. In this subsection, we formulate a homological purity condition for 0-regular sheaves on a projective variety X with very ample polarization $\mathcal{O}_X(1) = \mathcal{O}_X(H)$. Since this condition is needed only for the Gieseker stability part of the Comparison Theorem, the reader interested exclusively in slope stability may skip this subsection.

Our key result for detecting 0-dimensional subsheaves is:

LEMMA 7. *Let E be a sheaf on a projective variety X with very ample polarization $\mathcal{O}_X(1) = \mathcal{O}_X(H)$. Let $M = h^0(E)$ and denote by $E_0 \subset E$ the maximal subsheaf of dimension zero. Then, $E_0 = 0$ if and only if $h^0(E(-M)) = 0$.*

Proof. If $E_0 \neq 0$, then we have $h^0(E(k)) \neq 0$ for all $k \in \mathbb{Z}$. So we only need to show that $h^0(E(-M)) > 0$ implies $E_0 \neq 0$. We consider the decreasing sequence $M = h^0(E), h^0(E(-1)), \ldots, h^0(E(-M))$. If $h^0(E(-M)) > 0$, then there is an integer k with $h^0(E(-k)) = h^0(E(-k-1)) > 0$. Let E' be the image of the morphism $H^0(E(-k)) \otimes \mathcal{O}_X \to E(-k)$. The sheaf E' is globally generated and satisfies the condition $h^0(E'(-1)) = h^0(E')$. A general hyperplane $D \in |H|$ meets the associated locus of E' transversally, and thus yields a short exact sequence $0 \to E'(-1) \to E' \to E'|_D \to 0$. Since E' is globally generated, the sections of E' also generate $E'|_D$. However, all these sections come from $E'(-1)$. Thus $E'|_D = 0$. We conclude that the support of E' is of dimension zero. □

Now let E be a coherent sheaf on X with Hilbert polynomial $p = p_E$ of degree d. Assume that E is 0-regular, i.e. $H^i(E(-i)) = 0$ for $i > 0$. By [22], this implies that $E(l)$ is globally generated for $l \geq 0$ and also that $H^i(E(l)) = 0$ for all $i > 0$, $l \geq 0$. Set $M := p(0) = h^0(E)$. We consider the dimension filtration of E

$$0 = E_{-1} \subset E_0 \subset E_1 \subset \cdots \subset E_d = E \qquad \text{with } E_k/E_{k-1} \text{ pure of dimension } k.$$

As E is globally generated, it gives a closed point in a Quot scheme $Q := \mathrm{Quot}_X^p(\mathcal{O}_X^M)$ of finite type which parameterizes all 0-regular sheaves with Hilbert polynomial p.

In particular, given a coherent sheaf F on X, there exists a universal upper bound B (depending only on F, p and H) such that $B \geq h^1(F \otimes E \otimes \mathcal{O}_{H_1} \otimes \cdots \otimes \mathcal{O}_{H_m})$ for all $E \in Q$, $m \in \{0, \ldots, d-1\}$ and $H_i \in |H|$.

Proposition 8. *Let $\mathcal{O}_X(1) = \mathcal{O}_X(H)$ and p be as above. There exists a vector bundle F on X depending only on p and $\mathcal{O}_X(1)$ such that for any 0-regular sheaf E on X with Hilbert polynomial $p_E = p$ holds: E is pure if and only if* $\mathrm{Hom}(F, E) = 0$.

Proof. Restriction of E to a general hyperplane $H_i \in |H|$ commutes with the dimension filtration: $(E|_{H_i})_k = E_{k+1}|_{H_i}$. Coupled with Lemma 7, this shows that E is pure if and only if $H^0(E(-M) \otimes \mathcal{O}_{H_1} \otimes \cdots \otimes \mathcal{O}_{H_m}) = 0$ for all $m = 0, \ldots, d-1$ and general hyperplanes $H_i \in |H|$. This condition can be checked using Lemma 4, as done in the proof of Proposition 6: We define sequences of integers $(m_1, m_2, \ldots, m_{d-1})$ and of vector bundles $F_0, F_1, \ldots, F_{d-1}$ recursively by

(i) $F_0 := \mathcal{O}_X$
(ii) $\widetilde{m}_k \geq h^1(F_{k-1} \otimes E(-M-1) \otimes \mathcal{O}_{H_1} \otimes \cdots \otimes \mathcal{O}_{H_m})$ for all sheaves $[E] \in Q$, all $m \in \{0, \ldots d-1\}$, and all hyperplanes $H_i \in |H|$.
(iii) $m_k = (h^0(\mathcal{O}_X(1)) - 1)(\widetilde{m}_k - 1)$
(iv) $F_k = F_{k-1} \otimes G_{m_k}$ where G_{m_k} is the vector bundle from Subsection 1.2.

(We only need condition (ii) for generic hyperplanes. Note that for almost all choices of the H_i, the tensor product is underived, thus just a sheaf supported on an m-codimensional complete intersection.)

Proceeding as in the proof of Proposition 6, the vanishing of $H^0(E(-M) \otimes F_0), \ldots, H^0(E(-M) \otimes F_{d-1})$ is equivalent to the vanishing of $H^0(E(-M))$, $H^0(E(-M) \otimes \mathcal{O}_{H_1})$, $\ldots$, $H^0(E(-M) \otimes \mathcal{O}_{H_1} \otimes \cdots \otimes \mathcal{O}_{H_{d-1}})$ for general hyperplanes $H_1, H_2, \ldots H_{d-1}$ in the linear system $|H|$. By Lemma 7, the last condition is equivalent to $E_{d-1} = 0$. Setting $F := (F_0^\vee \oplus \cdots \oplus F_{d-1}^\vee) \otimes \mathcal{O}_X(M)$ yields the required vector bundle. ∎

1.4. Semistability on curves. Let X be a smooth projective curve of genus g over k. Fix integers $r > 0$ and d. Let $\mathcal{O}_X(1)$ be a fixed line bundle of degree one.

Theorem 9. *For a coherent sheaf E on X of rank r and degree d, the following conditions are equivalent:*

(i) *E is a semistable vector bundle.*
(ii) *There is a sheaf $0 \neq F$ with $E \in F^\perp$, i.e.* $\mathrm{Hom}(F, E) = \mathrm{Ext}^1(F, E) = 0$.
(iii) *There exists a sheaf F on X with* $\det(F) \cong \mathcal{O}_X(rd - r^2(g-1))$ *and* $\mathrm{rk}(F) = r^2$ *such that* $\mathrm{Hom}(F, E) = \mathrm{Ext}^1(F, E) = 0$.

Proof. The equivalence (i) $\iff$ (ii) is Faltings' characterization of semistable sheaves on curves [7]. One direction is easy: For $E' \subset E$ with $\mu(E') > \mu(E)$, we have $\mu(E' \otimes F^\vee) > \mu(E \otimes F^\vee)$, hence by Riemann-Roch $\chi(E' \otimes F^\vee) > \chi(E \otimes F^\vee) = 0$. But then $h^0(E' \otimes F^\vee) > 0$, contradicting $h^0(E \otimes F^\vee) = 0$. The refinement (i) $\iff$ (iii) is the content of Popa's paper [24, Theorem 5.3]. ∎

Based on this result, we can give two Postnikov data for semistable bundles on X. Introduce the slope $\mu := d/r$ and some further semistable vector bundles and integers:

$$A := \mathcal{O}_X(r^2(\lfloor\mu\rfloor - \mu - 2g - 1) + \lfloor\mu\rfloor - 3g), \qquad m_1 := r(r^2+1)\,(\mu - \lfloor\mu\rfloor + 2g + 1)\,,$$
$$B := \mathcal{O}_X^{r^2+1}(\lfloor\mu\rfloor - 3g), \qquad m_2 := (\hom(A, B) - 1)(m_1 - 1)\,.$$

Proposition 10. *Let X be a smooth projective curve, (r, d) two integers and A, B, m_1, and m_2 as above. For an object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:*

(i) *E is a semistable vector bundle of rank r and degree d.*

(ii) *The object E satisfies the following Postnikov conditions:*
  - *(1)* $\hom(A, E) = \hom(B, E) = m_1$, $\hom^i(A, E) = \hom^i(B, E) = 0$ *for $i \neq 0$.*
  - *(2)* *There is a cone $A \to B \to C$ in $\mathrm{D^b}(X)$ with* $\mathrm{Hom}^*(C, E) = 0$.

(iii) *The object E satisfies the following Postnikov conditions:*
  - *(1)* $\hom(A, E) = \hom(B, E) = m_1$, $\hom^i(A, E) = \hom^i(B, E) = 0$ *for $i \neq 0$.*
  - *(2)* $\mathrm{Hom}^{-1}(S^{m_2}(A, B), E) = 0$.

Proof. (i)⇒(ii) E is semistable of degree d and rank r, hence by Theorem 9 there exists a sheaf F with $\det(F) \cong \mathcal{O}_X(rd - r^2(g-1))$ and $\mathrm{rk}(F) = r^2$ such that $\mathrm{Hom}^*(F, E) = 0$. This implies that F is also a semistable bundle. Thus (see [11, Lemma 2.1]), it appears in a short exact sequence $0 \to A \to B \to F \to 0$. Since $\mu(E) - \mu(A) > 2g - 2$, we see that $\mathrm{Hom}^i(A, E) = 0$ for $i \neq 0$. Using the Riemann-Roch Theorem, we deduce that $\hom(A, E) = m_1$. The same works with B instead of A. We eventually conclude that (1) holds. Setting $C = F$ we obtain the object required in condition (2).

(ii)⇒(i) The conditions (1) and (2) imply that the morphism $A \to B$ is not zero. Since A is a line bundle, this morphism is injective; hence the distinguished triangle of (2) corresponds to a short exact sequence of sheaves $0 \to A \to B \to C \to 0$. As the global dimension of a smooth curve is one, we have $E \cong \bigoplus E^i[-i]$. The condition $\mathrm{Hom}^*(C, E) = 0$ implies that all the E^i are semistable of slope d/r. If $E^i \neq 0$, then $\mathrm{Hom}^i(A, E) \neq 0$. So from condition (1) we deduce that E is a sheaf object. As the slopes of A and B differ, we can read off the Hilbert polynomial of E^0 from the dimensions $\hom(A, E)$ and $\hom(B, E)$. Altogether, E^0 is of rank r and degree d.

(ii) $\iff$ (iii) Any morphism $\alpha\colon A \to B$ gives a distinguished triangle as in (ii). The total homomorphism space $\mathrm{Hom}^*(C, E)$ is zero if and only if $\mathrm{Hom}(B, E) \to \mathrm{Hom}(A, E)$ is a bijection. Because we work with finite-dimensional k-vector spaces, this is equivalent to the injectivity of $\mathrm{Hom}(B, E) \to \mathrm{Hom}(A, E)$. Thus, by Lemma 4 we are done. □

For a more detailed description and the relation to the Theta divisor and its base points see [11, Theorems 2.12 and 3.3] of the first author.

1.5. P-stability implies μ-semistability.

Theorem 11. (Comparison theorem for Mumford-Takemoto semistability) *For a polarized normal projective Gorenstein variety $(X, \mathcal{O}_X(1))$ and for a polynomial p of degree $n = \dim(X)$, there exist sheaves C_{-m}, C_{-m+1}, …, C_{2n-1} on X, and integers N_j^i such that for an object $E \in \mathrm{D^b}(X)$ the following two conditions are equivalent:*

- *(i)* *E is a μ-semistable sheaf concentrated in degree zero of Hilbert polynomial p.*
- *(ii)* $\hom^i(C_j, E) = N_j^i$ *for all $i = -m, \dots, -1$ and all j, and there exist homomorphisms $d_i \in \mathrm{Hom}(C_i, C_{i+n})$ for $i = 0, \dots, n-1$ such that for the complex $T_0 = \bigotimes_{i=0}^{n-1}(C_i \xrightarrow{d_i} C_{i+n})$ we have $\mathrm{Hom}^*(T_0, E) = 0$, that is $E \in T_0^{\perp}$.*

Remark. Condition (ii) of the theorem states that E is P-stable for the P-datum

$$(C_{-m},\dots,C_{-1},B_0,\dots,B_n,N) \text{ with } B_k = \bigoplus_{M\in S_{n,k}} \bigotimes_{i=0}^{n-1} C_{i+n\cdot\chi_M(i)},$$

where $S_{n,k}$ is the set of subsets of $\{0,\dots,n-1\}$ with k elements; and $\chi_M(i)=1$ when $i\in M$ and zero otherwise. However, we need the morphisms between the B_k to be of "tensor type" (i.e. as the total complex of the tensor double complex).

The sheaves $C_0,\dots,C_{2n-1}$ only depend on the numerical invariants whereas the homomorphisms $d_i\in \mathrm{Hom}(C_i,C_{i+n})$ depend on the sheaf E. However, semicontinuity implies that if a complex T_0 is orthogonal to E, then it is orthogonal to all semistable sheaves in a Zariski open subset of the moduli space. Also, with the construction given, the general complex T_0 is a sheaf supported on a curve, and this curve depends on E, too.

Proof. The objects $C_{-m},\dots,C_{2n-1}$ are defined in the proof of (i)$\Rightarrow$(ii), in a manner independent of E.

(i)$\Rightarrow$(ii) Suppose that E is a μ-semistable vector bundle with given Hilbert polynomial $p:=p_E$. As semistability implies that E appears in a bounded family, there is an integer l_1 (depending only on p) such that E is l_1-regular. Hence by Proposition 6 there are sheaves C_{-m}, $C_{-m+1},\dots,C_{-1}$ and integers N_j^i such that $\hom^i(C_j,E)=N_j^i$ forces E to be a l_1-regular sheaf of Hilbert polynomial p.

By Langer's effective restriction theorem [19, Theorem 5.2], there is a constant l_2 such that E is μ-semistable $\iff$ the restriction $E|_Y$ is semistable for $Y=H_1\cap H_2\cap\cdots\cap H_{n-1}$ a general complete intersection of hyperplanes $H_i\in|l_2H|$. By Bertini's theorem, we may take Y to be a smooth curve, embedded by $\iota\colon Y\to X$. We set $C_i:=\mathcal{O}_X(-l_2)$, and $C_{i+n}=\mathcal{O}_X$ for $i=1,\dots,n-1$. The morphisms $d_i\colon \mathcal{O}_X(-l_2)\to\mathcal{O}_X$ are picked such that their cokernels are $\mathcal{O}_{H_i}$. The semistability of ι^*E can be expressed (see Proposition 10 and its proof) by $\mathrm{Hom}^*(F,\iota^*E)=0$ for some coherent sheaf F on Y. Moreover, as shown by Popa (see Theorem 9 (iii)), we may choose the orthogonal object F to have any determinant of a certain degree d. As $F\in{}^\perp\iota^*E$ implies $F^{\oplus m}\in{}^\perp\iota^*E$ for any $m\neq 0$, and the latter direct sum has degree $m\cdot d$, we can assume that the determinant of the object orthogonal to ι^*E is the restriction of $\mathcal{O}_X(l)$ to Y for some integer l. By abuse of notation we write F for a sheaf on Y with $F\in{}^\perp\iota^*E$ and $\det(F)=i^*\mathcal{O}_X(l)$. Then the class of $[F]$ in the Grothendieck group $K_0(Y)$ can be selected from the image of $\iota^*\colon K_0(X)\to K_0(Y)$. Thus we find F in a short exact sequence

$$0\to\iota^*\mathcal{O}_X(l_5)\to\iota^*\mathcal{O}_X^{l_3}(l_4)\to F\to 0.$$

Note, that this short exact sequence lives on Y. However, we can replace the surjection $\iota^*\mathcal{O}_X^{l_3}(l_4)\to F$ with $\iota^*\mathcal{O}_X^{l_3}(l_4-k)\to F$ for any $k\in\mathbb{N}$ and doing so changes the kernel to $\iota^*\mathcal{O}_X(l_5-l_3\cdot k)$. Hence the difference l_4-l_5 can be chosen to be arbitrarily large. Eventually (and abusing notation again), we may require that $\iota^*\mathcal{O}_X(l_5)\to\iota^*\mathcal{O}_X^{l_3}(l_4)$ is the pull-back of a morphism $d_0'\colon\mathcal{O}_X(l_5)\to\mathcal{O}_X^{l_3}(l_4)$. Applying Serre duality (twice), adjunction and $\omega_Y=\iota^*(\omega_X(n-1))$, we get

$$\begin{aligned}
0&=\mathrm{Hom}_Y^*(F,\iota^*E)=\mathrm{Hom}_Y^*(\iota^*E,\omega_Y\otimes F)^*=\mathrm{Hom}^*(E,\iota_*(\omega_Y\otimes F))^*\\
&=\mathrm{Hom}_X^*(\iota_*(\omega_Y\otimes F)\otimes\omega_X^{-1},E)=\mathrm{Hom}_X^*(\iota_*(\iota^*(\omega_X(n-1))\otimes F)\otimes\omega_X^{-1},E)\\
&=\mathrm{Hom}_X^*(\iota_*F(n-1),E),
\end{aligned}$$

i.e. $F' := \iota_* F(n-1)$ is orthogonal to E. Setting $C_0 := \mathcal{O}_X(l_5+n-1)$, and $C_n := \mathcal{O}_X^{l_3}(l_4+n-1)$, and $d_0 := d_0' \otimes \mathcal{O}_X(n-1)\colon C_0 \to C_n$, we find that F' is represented by the tensor product complex $\bigotimes_{i=0}^{n-1}(C_i \xrightarrow{d_i} C_{i+n})$ as claimed.

(ii)⇒(i) If E is a complex satisfying the conditions of (ii), then E is a d_1-regular sheaf by Proposition 6 and the choice of $C_{-m}, \dots C_{-1}$. Assume that there exists a $T_0 = \bigotimes_{i=0}^{n-1}(C_i \xrightarrow{d_i} C_{i+n})$ such that $\mathrm{Hom}^*(T_0, E) = 0$. By semicontinuouity this holds also for the general choice of $d_i \in \mathrm{Hom}(C_i, C_{i+n})$. However, for such a general choice the tensor product is a vector bundle on a smooth complete intersection curve C such that the restriction $E|_C$ is a vector bundle. As we have seen before this implies that $E|_C$ is semistable on C. Since this holds for the general curve, E is μ-semistable. ∎

1.6. P-stability implies Gieseker semistability.

Theorem 12. (Comparison theorem for Gieseker semistability) *For a polarized projective variety $(X, \mathcal{O}_X(1))$ and for a given polynomial p there exist sheaves C_{-m}, C_{-m+1}, ..., C_0, C_1, and F on X, and integers N_j^i such that for an object $E \in \mathrm{D^b}(X)$ the following three conditions are equivalent:*

- *(i) E is concentrated in degree zero, and a Gieseker semistable sheaf of Hilbert polynomial p.*
- *(ii) $\hom^i(C_j, E) = N_j^i$, $\mathrm{Hom}(F, E) = 0$ and there exists a distinguished triangle $C \to C_0 \to C_1 \to C[1]$ in $\mathrm{D^b}(X)$ such that $\mathrm{Hom}^*(C, E) = 0$, that is $E \in C^\perp$.*
- *(iii) $\hom^i(C_j, E) = N_j^i$, $\mathrm{Hom}(F, E) = 0$ and $\mathrm{Hom}^{-1}(S^m(C_0, C_1), E) = 0$ for $m \gg 0$.*

Proof. (i) $\iff$ (ii) By Proposition 6 we can choose sheaves $C_{-m}, C_{-m+1}, \dots, C_{-1}$ and $N_j^i \in \mathbb{N}$ with $j = -m, \dots, -1$ such that any object E satisfying $\hom^i(C_j, E) = N_j^i$ is a sheaf with Hilbert polynomial p. By Proposition 8 there exists a sheaf F such that $\mathrm{Hom}(F, E) = 0$ is equivalent to the purity of E.

Assuming these conditions on E, [1, Theorem 7.2] implies that there are objects $C_0, C_1 \in \mathrm{D^b}(X)$ such that the existence of the above C is equivalent to the semistability of E.
(ii) $\iff$ (iii) Here we use that the sheaves C_0 and C_1 are direct sums of $\mathcal{O}_X(-N_i)$ for $N_i \gg 0$. So $\mathrm{Hom}^*(C_i, E)$ is concentrated in degree zero. Now we can argue as in the proof of (ii) $\iff$ (iii) in Proposition 10. ∎

Remark. The above system of sheaves $(F, C_{-m}, \dots, C_1)$ is a P-datum. It is worth pointing out that the active part only consists of a single morphism, by virtue of the theorem of Álvarez-Cónsul and King.

On the other hand, our treatment of the purity conditions in Subsection 1.3 can be used to improve the statement of [1], as their explicit hypothesis of 'pure' sheaf can be phrased in homological terms.

2. Preservation of stability. The classical approach to preservation of stability is this: let X and Y be smooth, projective varieties and consider a moduli space $\mathcal{M}_X(v)$ of semistable sheaves on X with fixed Mukai vector $v \in H^*(X)$. If furthermore we are given a Fourier-Mukai transform $\Phi\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$, then one might ask if a sheaf $E \in \mathcal{M}_X(v)$ is mapped under Φ to a shifted sheaf (i.e. the complex $\Phi(E) \in \mathrm{D^b}(Y)$ has cohomology only in a single degree i, in which case E is called WIT_i; the sheaf is called IT_i if the single cohomology sheaf is even locally free). Assuming this, one might next wonder if the resulting sheaf on Y is itself semistable with respect to suitable numerical constraints $v' \in H^*(Y)$ and some polarization on Y.

The hope is to produce (possibly birational) maps $\Phi\colon \mathcal{M}_X(v) \dashrightarrow \mathcal{M}_Y(v')$ — a hope that is often founded: if the Fourier-Mukai transform is of geometric origin (given by a universal bundle, for example), then there is a plethora of results stating that stability is preserved in this sense.

Our point is that the restriction to WIT sheaves is unnatural in the context of derived categories. It would be much more appealing if there was a notion of stability which is preserved by equivalences on general grounds. This would make the classical results about preservation of stability the special case where sheaves happen to be mapped to (shifted) sheaves again. Our notion of P-stability provides this. The Comparison Theorem shows that semistable sheaves in $\mathcal{M}_X(v)$ can be encoded via a P-datum; it is then tautological that the objects of $\Phi(\mathcal{M}_X(v))$ will be P-stable with respect to the transformed P-datum. Hence we shift our point of view to the following question: in which cases is the transformed P-datum of classical origin, i.e. induced by Gieseker or μ-semistability?

2.1. Abelian surfaces. Here is a typical example, see [2, Theorem 3.34]. Let (A, H) be a polarized Abelian surface, $\hat{A}$ the dual Abelian surface and $\mathcal{P} \in \mathrm{Pic}(A \times \hat{A})$ the Poincaré bundle. This bundle gives rise to the classical Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}}\colon \mathrm{D^b}(A) \xrightarrow{\sim} \mathrm{D^b}(\hat{A})$ of [21]. Then $\hat{H} = -c_1(\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_A(H)))$ is a polarization for $\hat{A}$.

Theorem 13. *If E is a μ-stable locally free sheaf on A with $\mu(E) = 0$ and rank $r > 1$, then E is IT_1 and $\mathsf{FM}_{\mathcal{P}}(E)[1]$ is a μ-semistable vector bundle with respect to $\hat{H}$.*

Proof. We are going to use the following characterization of μ-semistable sheaves on an abelian surface (cf. [10, Theorem 3.1])

$$\begin{aligned} E \text{ is } \mu\text{-semistable} &\iff E \otimes \mathcal{O}_C \text{ is semistable for } m \gg 0, \text{ and some } C \in |mH| \\ &\iff \mathrm{Hom}^*(E, F) = 0 \text{ for some coherent sheaf } F \text{ on } C \text{ as above.} \end{aligned}$$

The first equivalence is deduced from the restriction theorem of Mehta and Ramanathan (see [20] or also [13], or for effective bounds the results of Langer in [19]). The second equivalence follows from Theorem 9. For F, we can use a torsion sheaf supported on C with a resolution by prescribed vector bundles, as in the proof of Theorem 11. Since this C-vector bundle F is itself semistable it is an element in an irredicible moduli space. For the general element there we have $H^*(F) = 0$. Thus, we can assume $\mathrm{Hom}^*(E, F) = 0$ and $\mathrm{Hom}^*(\mathcal{O}_A, F) = 0$. This defines a P-datum on $\mathrm{D^b}(A)$. We will show that the image under $\mathsf{FM}_{\mathcal{P}}$ is a P-datum on $\mathrm{D^b}(\hat{A})$ containing μ-semistability for sheaves of degree 0.

For this, suppose that $\mathsf{FM}_{\mathcal{P}}(E)[1]$ is a sheaf. Fix a sheaf F as above such that $\mathrm{Hom}^*(E, F) = 0 = \mathrm{Hom}^*(\mathcal{O}_A, F)$. Then, $\mathsf{FM}_{\mathcal{P}}(F)[1]$ is a sheaf concentrated on a divisor in $|m\,\mathrm{rk}_C(F)\hat{H}|$. Since this sheaf is orthogonal to $\mathsf{FM}_{\mathcal{P}}(E)[1]$ this shows the μ-semistability of the latter object. Thus, the conditions $\mu(E) = 0$ and E μ-semistable force $\mathsf{FM}_{\mathcal{P}}(E)[1]$ to be μ-semistable with respect to the dual polarization $\hat{H}$.

It remains to show the vanishing of the cohomologies $\mathsf{FM}_{\mathcal{P}}(E)^0$ (step 1) and $\mathsf{FM}_{\mathcal{P}}(E)^2$ (step 2) of the complex $\mathsf{FM}_{\mathcal{P}}(E)$. After that we prove that $\mathsf{FM}_{\mathcal{P}}(E)^1$ is torsion free (step 3), and locally free (step 4).

Step 1: If $\mathsf{FM}_{\mathcal{P}}(E)^0 \neq 0$, then we have $\mathrm{Hom}(\mathcal{O}_{\hat{A}}(-m\hat{H}), \mathsf{FM}_{\mathcal{P}}(E)^0) \neq 0$ for $m \gg 0$. This implies $\mathrm{Hom}(\mathcal{O}_{\hat{A}}(-m\hat{H}), \mathsf{FM}_{\mathcal{P}}(E)) \neq 0$ (replace $\mathsf{FM}_{\mathcal{P}}(E)$ by a complex concentrated in non-negative degrees and use the Eilenberg-Moore spectral sequence). Applying the inverse Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}}^{-1}$, we get

$\mathrm{Hom}(\mathsf{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\hat{A}}(-m\hat{H})), E) \neq 0$. By [21, Theorem 2.2], the inverse is $\mathsf{FM}_{\mathcal{P}}^{-1} = (-1)^*\mathsf{FM}_{\mathcal{P}}[2]$. As $(-1)^*\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_{\hat{A}}(-m\hat{H}))[2]$ is a semistable vector bundle with positive first Chern class (see [21, Proposition 3.11]), $\mathsf{FM}_{\mathcal{P}}(E)^0 \neq 0$ would contradict the semistability of E.

Step 2: Now suppose $\mathsf{FM}_{\mathcal{P}}(E)^2 \neq 0$. We choose a point $P \in \mathrm{supp}(\mathsf{FM}_{\mathcal{P}}(E)^2)$ and obtain a morphism $\mathsf{FM}_{\mathcal{P}}(E)^2 \to \mathsf{k}(P)$. As before this gives a morphism $\mathsf{FM}_{\mathcal{P}}(E) \to \mathsf{k}(P)$, and a morphism $E \to L_P^{-1}$ on A where L_P is the line bundle parameterized by the point P. This morphism contradicts the μ-stability of E.

Step 3: By what was already proven, we know that $\mathsf{FM}_{\mathcal{P}}(E)[1]$ is μ-semistable. Thus, to show that this sheaf is torsion free, it is enough to exclude the existence of a subsheaf $T \subset \mathsf{FM}_{\mathcal{P}}(E)[1]$ with 0-dimensional support. If $T \neq 0$ we have $H^0(T) \neq 0$. We deduce $\mathrm{Hom}(\mathcal{O}_{\hat{A}}, \mathsf{FM}_{\mathcal{P}}(E)[1]) \neq 0$. Applying the inverse Fourier-Mukai transform we obtain $\mathrm{Ext}^1(\mathsf{k}(0), E) \neq 0$. However, this Ext group vanishes because E was locally free at $0 \in A$. So we derive that $T = 0$.

Step 4: Finally we show that the torsion free sheaf $\mathsf{FM}_{\mathcal{P}}(E)[1]$ is a vector bundle. If it was not locally free, there would be a proper inclusion $\mathsf{FM}_{\mathcal{P}}(E)[1] \xrightarrow{\iota} (\mathsf{FM}_{\mathcal{P}}(E)[1])^{\vee\vee}$. If $P \in \mathrm{supp}(\mathrm{coker}(\iota))$, then we have $\mathrm{Ext}^1(\mathsf{k}(P), \mathsf{FM}_{\mathcal{P}}(E)[1]) \neq 0$, or, after application of $\mathsf{FM}_{\mathcal{P}}^{-1}$, that $\mathrm{Hom}(L_P^{-1}, E) \neq 0$. But this contradicts the μ-stability of E. ∎

Remark. In the proof of the above theorem the μ-stability of E can be replaced by the following weaker condition: E is μ-semistable and for all line bundles L in $\mathrm{Pic}^0(A)$ we have $\mathrm{Hom}(L, E) = \mathrm{Hom}(E, L) = 0$.

Fix integers r and s and let $\mathcal{M}_A(r, 0, s)$ be the moduli space of μ-semistable sheaves E on A of rank $r > 1$ and $c_1(E) = 0$, $c_2(E) = s$. By Theorem 13, $\mathsf{FM}_{\mathcal{P}}(E)[1]$ is a μ-semistable (and in fact μ-stable) sheaf for μ-stable E. Hence, $\mathsf{FM}_{\mathcal{P}}$ provides an injective map $U \hookrightarrow \mathcal{M}_{\hat{A}}(s, 0, r)$ where $U \subset \mathcal{M}_A(r, 0, s)$ is the open subset of μ-stable sheaves. Using the inverse transform $\mathsf{FM}_{\mathcal{P}}^{-1}$ provides a derived compactification which in the case at hand is nothing but the standard compactification using μ-semistable sheaves.

2.2. Birational moduli spaces via reversed universal bundles. Let X and $\hat{X}$ be Fourier-Mukai partners, i.e. there is an equivalence $\Phi\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(\hat{X})$. Assume furthermore, that $M = M_X(v)$ is a fine moduli space of stable sheaves with given numerical invariants on X. By the Comparison Theorem we can phrase this stability in terms of a P-datum $(C_\bullet, N)$. Let $\hat{v}$ be the transform $\Phi(v)$ of the numerical invariants. We obtain equivalent conditions for a coherent sheaf E with numerical invariants v from our comparison theorems and since Φ is an equivalence:

$$E \text{ is semistable} \iff E \text{ is P-stable for } (C_\bullet, N) \iff \Phi(E) \text{ is P-stable for } (\Phi(C_\bullet), N).$$

The universal family $\mathcal{E}$ on $X \times M$ yields a universal family $\Phi(\mathcal{E}) \in \mathrm{D^b}(\hat{X} \times M)$. This way, M becomes also the fine moduli space $M_{\hat{X}}(\hat{v})$ with respect to the P-datum $(\Phi(C_\bullet), N)$. Note that a sheaf E is simple if and only if $\Phi(E)$ is simple. If $\Phi(E)$ corresponds to a sheaf for some $[E] \in M$, then this will hold on a open subset and we obtain a birational map $\Phi\colon M_X(v) \dashrightarrow M_{\hat{X}}(\hat{v})$ between moduli spaces of sheaves. To sum up:

Proposition 14. *Let $\Phi\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(\hat{X})$ be an equivalence, and $M_X(v)$ be a fine moduli space of stable sheaves on X with numerical invariants v. If for a point $[E] \in M_X(v)$ the complex $\Phi(E)$ is a sheaf, then Φ gives a birational morphism*

$M_X(v) \dashrightarrow M_{\hat{X}}(\hat{v})$ between moduli spaces of sheaves. For an open subset $U \subset M_X(v)$ the universal bundle $\mathcal{E}_U$ on $X \times U$ gives rise to a universal bundle $\hat{\mathcal{E}}_U = \Phi(\mathcal{E}_U)$.

2.3. Elliptic K3 surface. Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic K3 surface with a section $\sigma\colon \mathbb{P}^1 \to X$. Due to the presence of the section, the relative Jacobian of π is isomorphic to X itself. In particular, there is a relative Poincaré bundle $\mathcal{P}$ on $X \times_{\mathbb{P}^1} X$. We will use the associated Fourier-Mukai transform $\Phi := \mathsf{FM}_{\mathcal{P}}\colon \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ which is an equivalence by standard arguments [14] or [2].

We have two divisor classes at our disposal: the fiber $f = [\pi^{-1}(p)]$ (of any point $p \in \mathbb{P}^1$) and the section σ. They intersect as $f^2 = 0$, $f.\sigma = 1$ and $\sigma^2 = -2$; the latter because $\sigma \subset X$ is a smooth, rational curve.

The divisor $H = \sigma + 3f$ is big and effective, hence ample as X is a K3 surface. We consider two moduli spaces of μ-semistable sheaves (with respect to H) on X. One is the Hilbert scheme $\mathcal{M}_1 := \mathrm{Hilb}^2(X)$ of 0-dimensional subschemes of length 2 (or rather ideal sheaves of such); it is the moduli space of semistable sheaves of rank 1, $c_1 = 0$ and $c_2 = 2$. The other is the moduli space $\mathcal{M}_2 = \mathcal{M}_X(2, -\sigma, 0)$ of μ-semistable sheaves with prescribed Chern character. For a decomposable subscheme $Z \subset X$ of length 2 supported on distinct fibers, Φ maps the twisted ideal sheaf $\mathcal{O}_X(2\sigma) \otimes \mathcal{I}_Z$ to a μ-stable sheaf in $\mathcal{M}_2$; see [2, §6].

In this way, we obtain an isomorphism between the open set of points of $\mathrm{Hilb}^2(X)$ with support in different fibers and the locus $\mathcal{M}_2^s$ of stable sheaves. Φ also identifies the boundaries. An easy computation shows that for subschemes Z supported on a single fiber, $\Phi(\mathcal{O}_X(2\sigma) \otimes \mathcal{I}_Z)$ is a complex with nonzero cohomology in degrees 0 and 1. In other words, Φ provides a compactification of $\mathcal{M}_2^s$ using genuine complexes.

In this roundabout example, the compactification coming from $\mathcal{M}_1$ turns out to be the same as the classical one by coherent sheaves with a singular point. This point of view is treated much more thoroughly in [4]. In particular, it is shown there that birational moduli spaces $\mathcal{M}_X(2, \sigma - tf, 1)$ and $\mathrm{Hilb}^t(X)$ are *not* isomorphic for $t = 4$. This answers a question of Friedman who had shown in [8] that the spaces are isomorphic for $t = 1$ and $t = 2$, and posed the question for a general t.

2.4. Instanton compactifications. We follow the presentation of Bondal and Orlov [5, §2] to certain moduli spaces of instantons. Let Q_1 and Q_2 be two quadrics in $\mathbb{P}^5$ such that $X := Q_1 \cap Q_2$ is a smooth threefold. We consider the pencil of quadrics spanned by Q_1 and Q_2; this is a projective line $\mathbb{P}^1$ which contains six special points corresponding to degenerate quadrics. Let $C \to \mathbb{P}^1$ be the covering ramified over those six points. In [5] it is shown that C is the moduli space of odd spinor bundles and that the Fourier-Mukai transform with kernel the universal spinor bundle give rise to a fully faithful functor $\mathrm{D^b}(C) \hookrightarrow \mathrm{D^b}(X)$. The right orthogonal complement $\mathrm{D^b}(C)^\perp$ is generated by the exceptional objects $\mathcal{O}_X$ and $\mathcal{O}_X(1)$.

On the other hand, certain bundles of rank two and degree 0 on the curve C give via this transform instantons. Let $\mathcal{M}_C(2,0)$ denote the coarse moduli space of all such bundles and U the nonempty open subset parametrising instantons. As before, we choose a P-datum capturing μ-stability on C and transport it via $\mathrm{D^b}(C) \hookrightarrow \mathrm{D^b}(X)$ to X. We obtain a natural compactification of the space U of instantons by genuine complexes. Such compactifications are a subject of much research; our framework provides a homological approach.

This is related to Kuznetsov's work in [17]. A similar construction, leading to a fully faithful functor with $\mathrm{D^b}(C)$ replaced by a 2-Calabi-Yau category (sometimes, but not always, coming from a K3 surface) is given in [18]

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