ASIAN J. MATH.
Vol. 18, No. 1, pp. 029–052, January 2014
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002

A REMARK ON MIRZAKHANI'S ASYMPTOTIC FORMULAE*

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Abstract. We give a short proof of Penner-Grushevsky-Schumacher-Trapani's large genus asymptotics of Weil-Petersson volumes of moduli spaces of curves. We also study asymptotic expansions for certain integrals of pure ψ classes and answer a question of Mirzakhani on the asymptotic behavior of one-point volume polynomials of moduli spaces of curves.

Key words. Weil-Petersson volumes, moduli spaces of curves.

AMS subject classifications. 14H10, 14N10.

1. Introduction. We will follow Mirzakhani's notation in [25]. For $\mathbf{d} = (d_1, \cdots, d_n)$ with d_i non-negative integers and $|\mathbf{d}| = d_1 + \cdots + d_n < 3g-3+n$, let $d_0 = 3g-3+n-|\mathbf{d}|$ and define

$$[\tau_{d_1}\cdots\tau_{d_n}]_{g,n} = \frac{\prod_{i=1}^n (2d_i+1)!! 2^{2|\mathbf{d}|}(2\pi^2)^{d_0}}{d_0!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_1^{d_0}, \tag{1}$$

where κ_1 is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$ defined in [1]. Note that $V_{g,n} = [\tau_0, \cdots \tau_0]_{g,n}$ is the Weil-Peterson volume of $\overline{\mathcal{M}}_{g,n}$. Mirzakhani's volume polynomial is given by

$$V_{g,n}(2L) = \sum_{|\mathbf{d}|\le 3g-3+n} [\tau_{d_1}\cdots\tau_{d_n}]_{g,n} \frac{L_1^{2d_1}}{(2d_1+1)!}\cdots\frac{L_n^{2d_n}}{(2d_n+1)!}.$$

Let $S_{g,n}$ be an oriented surface of genus g with n boundary components. Let $\mathcal{M}_{g,n}(L_1,\dots,L_n)$ be the moduli space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1,\dots,L_n$. Then we know that the Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\dots,L_n))$ equals $V_{g,n}(L_1,\dots,L_n)$. In particular, when $n=1$, Mirzakhani's volume polynomial can be written as

$$V_g(2L) = \sum_{k=0}^{3g-2}\frac{a_{g,k}}{(2k+1)!}L^{2k},$$

where $a_{g,k} = [\tau_k]_{g,1}$ are rational multiples of powers of π

$$a_{g,k} = \frac{(2k+1)!! 2^{3g-2+2k}\pi^{6g-4-2k}}{(3g-2-k)!}\int_{\overline{\mathcal{M}}_{g,1}}\psi_1^k\kappa_1^{3g-2-k}. \tag{2}$$

Let γ be a separating simple closed curve on S_g and $S_g(\gamma) = S_{g_1,1}\times S_{g_2,1}$ the surface obtained by cutting S_g along γ. Then for any $L>0$, we have

$$\mathrm{Vol}(\mathcal{M}(S_g(\gamma),\ell_\gamma = L)) = V_{g_1}(L)\cdot V_{g_2}(L), \tag{3}$$

*Received April 26, 2012; accepted for publication August 29, 2012.
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where $\mathcal{M}(S_g(\gamma), \ell_\gamma = L)$ is the moduli space of hyperbolic structures on $S_g(\gamma)$ with the length of γ equal to L.

There is a large amount of work on the computations and asymptotics of Weil-Petersson volumes since the 90's. See e.g. [7, 26, 28, 32]. A great impetus comes from the celebrated Witten-Kontsevich theorem, which revolutionized intersection theory on moduli spaces of curves. In her thesis [24], Mirzakhani obtained a remarkable recursion formula of $V_{g,n}(L)$; as applications, she gave a new proof of the Witten-Kontsevich theorem and proved an asymptotic formula on the number of simple closed geodesics on Riemann surfaces. Mulase and Safnuk [27] showed that the integral formula of Mirzakhani was equivalent to the more explicit Virasoro constraint condition for the mixed integral of ψ and κ classes. The work is further clarified and generalized in [18, 19] to include higher degree κ classes. Eynard and Orantin [5] then realized that Mirzakhani's recursion formula fits in with the Eynard-Orantin recursion formalism whose spectral curve is the sine curve discovered in [27].

The Teichmüller metric was extensively studied by differential geometers. Liu, Sun and Yau [16, 17] proved the equivalence of the Teichmüller metric to the Kähler-Einstein metric on moduli spaces of curves, a conjecture proposed by Yau [34] more than 20 years ago. We refer the reader to [33] for a survey of recent works. Teichmuller geometry also appears naturally in physics. For example, D'Hoker and Phong [3] showd that the partition function of Polyakov's string theory can be expressed as certain integrals over Weil-Petersson measure.

In a recent paper [25], Mirzakhani proved some new results on large genus asymptotics of Weil-Petersson volumes, conjectured by Zograf [35], and found interesting applications in the geometry of random hyperbolic surfaces. Mirzakhani's work is reviewed in §6.

One of the main results of our paper is to give a short proof of the following large genus asymptotics of Weil-Petersson volumes originally due to Penner, Grushevsky, Schumacher and Trapani.

THEOREM 1.1. *For any fixed* $n \geq 0$. *There are constants* $0 < c < C$ *independent of* g *such that*

$$c^g (2g)! < V_{g,n} < C^g (2g)! \tag{4}$$

for large g.

The original proof of the above theorem consists of three papers: Penner [28] introduced the decorated Teichmüller space and invented a technique of integrating top degree differential forms on $\mathcal{M}_{g,n}$, which led to an estimate of lower bound of $V_{g,1}$ for large g. Grushevski [9] proved an upper bound for $V_{g,n}$ for fixed $n \geq 1$ and large g by elaborating on Penner's integration technique. By applying the intersection theory of certain effective divisors, Schumacher and Trapani [30] proved a lower bound for all $V_{g,n}$ from Penner's lower bound of $V_{g,1}$ and covered the case $n = 0$.

Inspired by work of Mirzakhani, we give a short proof of Theorem 1.1 in §2. Our proof first reduces Equation (4) to the case $n = 2$ through inequalities among $V_{g,n}$ derived from various recursion formulae due to Mirzakhani, Mulase, Safnuk, Do, Norbury and the authors. The $n = 2$ case of (4) will follow from a result of Hone, Joshi and Kitaev on asymptotics of solutions to the first Painlevé equation.

In [25, §1], Mirzakhani asked what is the asymptotics of $a_{g,k}/a_{g,k+1}$ for an arbitrary k (which can grow with g). The following result gives a partial answer to Mirzakhani's question, which should be compared with (62).

THEOREM 1.2. *For any given $k \geq 0$, there is a large genus asymptotic expansion*

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} = \frac{\pi^{2k}}{5^k k!}\left(1 + \frac{b_1(k)}{g} + \frac{b_2(k)}{g^2} + \cdots\right). \tag{5}$$

For each fixed $k \geq 0$, the series in the bracket of (5) *is a rational function of g. Moreover, for $r \geq 1$, $b_r(k)$ is a polynomial in k of degree $2r$ with the leading term $k^{2r}/(14^r r!)$. In particular,*

$$b_1(k) = \frac{k^2}{14} - \frac{4k}{7}, \qquad b_2(k) = \frac{k^4}{392} - \frac{107k^3}{2646} + \frac{85k^2}{441} + \frac{125k}{10584},$$
$$b_3(k) = \frac{k^6}{16464} - \frac{53k^5}{37044} + \frac{5441k^4}{407484} - \frac{75265k^3}{1629936} - \frac{7745k^2}{296352} + \frac{468319k}{3259872}.$$

It is interesting to note that polynomial structures also appear in Hurwitz numbers and Gromov-Witten invariants. In §3, we study asymptotics for certain integrals of pure ψ classes. Theorem 1.2 will be proved in §5.

Now we present a numerical test of (5). Denote by $Q_{k,g}$ the ratio of the left-hand side and the truncated right-hand side of (5).

$$Q_{k,g} = \frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} \cdot \frac{5^k k!}{\pi^{2k}} \Big/ \Big(1 + \frac{b_1(k)}{g}\Big). \tag{6}$$

Then we can see from Table 1.1 that $Q_{k,g}$ tends to 1 as g goes to infinity.

TABLE 1.1
Values of $Q_{k,g}$ (keep 6 decimal places)

k	$g=20$	$g=40$	$g=60$	$g=80$	$g=100$
1	1.000438	1.000106	1.000047	1.000026	1.000016
2	1.001334	1.000326	1.000144	1.000080	1.000051
3	1.002300	1.000563	1.000248	1.000139	1.000089
4	1.003090	1.000759	1.000335	1.000188	1.000120

Table 1.2 lists values of $a_{g,k}$, $0 \leq k \leq 3g-2$ and $g \leq 3$.

TABLE 1.2

$a_{1,0}$	$\frac{\pi^2}{12}$	$a_{1,1}$	$\frac{1}{2}$	$a_{2,0}$	$\frac{29\pi^8}{192}$	$a_{2,1}$	$\frac{169\pi^6}{120}$	$a_{2,2}$	$\frac{139\pi^4}{12}$
$a_{2,3}$	$\frac{203\pi^2}{3}$	$a_{2,4}$	210	$a_{3,0}$	$\frac{9292841\pi^{14}}{4082400}$	$a_{3,1}$	$\frac{8497697\pi^{12}}{388800}$	$a_{3,2}$	$\frac{8983379\pi^{10}}{45360}$
$a_{3,3}$	$\frac{127189\pi^8}{81}$	$a_{3,4}$	$\frac{94418\pi^6}{9}$	$a_{3,5}$	$\frac{166364\pi^4}{3}$	$a_{3,6}$	$\frac{616616\pi^2}{3}$	$a_{3,7}$	400400

Acknowledgements. The second author thanks Professor Mulase for helpful conversations and Professor M. Mirzakhani for helpful communications.

2. Asymptotics of Weil-Petersson volumes. We use the notation introduced in §1. For $n \geq 0$, define $a_n = \zeta(2n)(1 - 2^{1-2n})$.

LEMMA 2.1 ([25]). *$\{a_n\}_{n=1}^{\infty}$ is an increasing sequence. Moreover we have $\lim_{n\to\infty} a_n = 1$, and*

$$a_{n+1} - a_n \asymp 1/2^{2n}. \tag{7}$$

Here $f_1(n) \asymp f_2(n)$ means that there exists a constant $C > 0$ independent of n such that

$$\frac{1}{C} f_2(n) \le f_1(n) \le C f_2(n).$$

We have the following differential form of Mirzakhani's recursion formula [24, 27]. See also [5, 6, 19, 20, 29] for different proofs and generalizations.

$$\begin{aligned}
(8)\quad [\tau_{d_1}, \dots, \tau_{d_n}]_{g,n} &= 8\sum_{j=2}^{n}\sum_{L=0}^{d_0}(2d_j+1)a_L[\tau_{d_1+d_j+L-1}\prod_{i\neq 1,j}\tau_{d_i}]_{g,n-1}\\
&\quad + 16\sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L[\tau_{k_1}\tau_{k_2}\prod_{i\neq 1}\tau_{d_i}]_{g-1,n+1}\\
&\quad + 16\sum_{\substack{I\amalg J=\{2,\dots,n\}\\ 0\le g'\le g}}\sum_{L=0}^{d_0}\sum_{k_1+k_2=L+d_1-2} a_L\,[\tau_{k_1}\prod_{i\in I}\tau_{d_i}]_{g',|I|+1}\times[\tau_{k_2}\prod_{i\in J}\tau_{d_i}]_{g-g',|J|+1}.
\end{aligned}$$

LEMMA 2.2. *Given $\mathbf{d} = (d_1, \dots, d_n)$ and $g, n \ge 0$, the following recursive formulas hold*

(9)
$$[\tau_0\tau_1\prod_{i=1}^{n}\tau_{d_i}]_{g,n+2} = [\tau_0^4\prod_{i=1}^{n}\tau_{d_i}]_{g-1,n+4} + 6\sum_{\substack{g_1+g_2=g\\ \{1,\dots,n\}=I\amalg J}}[\tau_0^2\prod_{i\in I}\tau_{d_i}]_{g_1,|I|+2}[\tau_0^2\prod_{i\in J}\tau_{d_i}]_{g_2,|J|+2},$$

$$(10)\qquad (2g-2+n)[\prod_{i=1}^{n}\tau_{d_i}]_{g,n} = \frac{1}{2}\sum_{L\ge 0}(-1)^L(L+1)\frac{\pi^{2L}}{(2L+3)!}[\tau_{L+1}\prod_{i=1}^{n}\tau_{d_i}]_{g,n+1},$$

(11)
$$\sum_{j=1}^{n}(2d_j+1)[\tau_{d_j-1}\prod_{i\neq j}\tau_{d_i}]_{g,n} = \sum_{L\ge 0}\frac{(-\pi^2)^L}{4(2L+1)!}[\tau_L\prod_{i=1}^{n}\tau_{d_i}]_{g,n+1}.$$

Here (9) is a generalization of (25) (cf. [19, Prop. 3.3]). Equations (10) and (11) were proved in [2], and are respectively generalizations of the dilaton and string equations for the integrals of ψ classes. See [19] for more direct proofs and generalizations. As noted in [20], any recursion of integrals of ψ classes can be generalized to a recursion for integrals of mixed ψ, κ classes, since their generating functions differ only by a translation of parameters [26, 27].

Below we derive some inequalities of intersection numbers.

LEMMA 2.3. *When $d_1 > 0$, we have*

$$(12)\qquad [\tau_{d_1}\cdots\tau_{d_n}]_{g,n} < [\tau_{d_1-1}\tau_{d_2}\cdots\tau_{d_n}]_{g,n}.$$

Proof. We expand both sides of the inequalities using (8). Since each term in $\mathcal{A}^j_{\mathbf{d}}, \mathcal{B}_{\mathbf{d}}, \mathcal{C}_{\mathbf{d}}$ is positive, by comparing corresponding terms in the expansion, the inequality (12) follows from Lemma 2.1 that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence. ∎

Corollary 2.4. *For any fixed set* $\mathbf{d} = (d_1, \ldots, d_n)$ *of non-negative integers, we have*

$$[\tau_{d_1} \cdots \tau_{d_n}]_{g,n} \le V_{g,n}. \tag{13}$$

Lemma 2.5. *When* $3g + n - 2 > 0$, *we have*

$$V_{g,n+1} \le \frac{\pi^2}{6} [\tau_1 \tau_0^n]_{g,n+1}. \tag{14}$$

The equality holds only when $(g, n) = (0, 3)$ *or* $(1, 0)$.

Proof. First note that the coefficients in (11)

$$\left\{ \frac{\pi^{2L}}{4(2L+1)!} \right\}_{L \ge 1}$$

is a decreasing sequence. By Lemma 2.3, we know $[\tau_L \prod_{i=1}^n \tau_{d_i}]_{g,n+1}$ is a decreasing sequence in L.

Taking all $d_i = 0$ in (11), the left-hand side becomes 0. Writing down the first two terms of the right-hand side, we get

$$\frac{1}{4} V_{g,n+1} - \frac{2\pi^2}{2^4 \cdot 3} [\tau_1 \tau_0^n]_{g,n+1} < 0,$$

which is just (14). ∎

Remark 2.6. The inequality (14) can also be obtained by using (8). Let $f(x) = \zeta(2x)(1 - 2^{1-2x})$, we can check that $f''(x) < 0$ when $x \ge 1$. This implies that $\{a_{n+1} - a_n\}_{n \ge 1}$ is a decreasing sequence. By (8), we have

$$V_{g,n+1} - [\tau_1 \tau_0^n]_{g,n+1} \le \frac{a_1 - a_0}{a_1} V_{g,n+1}. \tag{15}$$

Substituting $a_0 = \frac{1}{2}$ and $a_1 = \frac{\pi^2}{12}$, we get $[\tau_1 \tau_0^n]_{g,n+1} \ge \frac{6}{\pi^2} V_{g,n+1}$.

Corollary 2.7. *For any* $g, n \ge 0$, *we have*

$$V_{g,n+1} > 12(2g - 2 + n) V_{g,n} \quad \text{and} \quad V_{g,n+1} < C(2g - 2 + n) V_{g,n}, \tag{16}$$

where $C = \frac{20\pi^2}{10 - \pi^2} = 1513.794\ldots$.

Proof. It is not difficult to see that the coefficients in (10)

$$\left\{ \frac{1}{2}(L+1) \frac{\pi^{2L}}{(2L+3)!} \right\}_{L \ge 0}$$

is a decreasing sequence.

Taking all $d_i = 0$ in (10) and keeping only the first term in the right-hand side, we get

$$(2g - 2 + n) V_{g,n} \le \frac{1}{12} [\tau_1 \tau_0^n]_{g,n+1} < \frac{1}{12} V_{g,n+1},$$

which is the first inequality in (16).

If we take first two terms in the right-hand side of (10) and apply Lemma 2.5, we get

$$\begin{aligned}(2g-2+n)V_{g,n} &\geq \frac{1}{12}[\tau_1\tau_0^n]_{g,n+1} - \frac{\pi^2}{120}[\tau_2\tau_0^n]_{g,n+1} \\ &> (\frac{1}{12} - \frac{\pi^2}{120})[\tau_1\tau_0^n]_{g,n+1} \\ &\geq \frac{10-\pi^2}{120}\cdot\frac{6}{\pi^2}V_{g,n+1} = \frac{10-\pi^2}{20\pi^2}V_{g,n+1},\end{aligned}$$

which is the second inequality in (16). □

We also need the following technical result due to Hone, Joshi and Kitaev [10].

LEMMA 2.8 ([10]). *For the nonlinear recursion relation*

$$\alpha_k = (k-1)^2\alpha_{k-1} + \sum_{m=2}^{k-2}\alpha_m\alpha_{k-m}, \quad k \geq 3 \tag{17}$$

and an arbitrary given $\alpha_2 > 0$, the limit $\lim_{k\to\infty}\alpha_k/(k-1)! < \infty$ exists.

The recursive equation (17) is related to the solution of the first Painlevé equation, which appears in the matrix model of two-dimensional quantum gravity. More precisely, if we define

$$\alpha_0 = -\frac{1}{2},\ \alpha_1 = \frac{1}{50},\ \alpha_2 = \frac{49}{2500}$$

and α_k, $k \geq 3$ are recursively given by (17), then the formal series (cf. [11])

$$y = -\sqrt{\frac{2}{3}}\sum_{k=0}^{\infty}\left(\frac{25}{8\sqrt{6}}\right)^k\alpha_k x^{\frac{1-5k}{2}}$$

is a solution of the first Painlevé equation:

$$\frac{d^2y}{dx^2} = 6y^2 - x.$$

We can now give a proof of Penner-Grushevsky-Schumacher-Trapani's large genus asymptotic estimates of Weil-Petersson volumes. The key observation is that Equation (27) is asymptotically similar to Equation (17).

Proof of Theorem 1.1. Taking $n = 0$ in (9), we get

$$[\tau_0\tau_1]_{g,2} = V_{g-1,4} + 6\sum_{i=1}^{g-1}V_{i,2}V_{g-i,2}. \tag{18}$$

Corollary 2.4 and Lemma 2.5 imply that

$$\frac{6}{\pi^2}V_{g,2} \leq [\tau_0\tau_1]_{g,2} \leq V_{g,2}. \tag{19}$$

Corollary 2.7 implies that exists contents $C_1, C_2 > 0$ independent of g such that

$$C_1(g-1)^2V_{g-1,2} \leq V_{g-1,4} + 12V_{1,2}V_{g-1,2} \leq C_2(g-1)^2V_{g-1,2}. \tag{20}$$

From (18), (19) and (20), we see that the solutions to the following two nonlinear recursion relations

$$A_2 = V_{2,2}, \qquad A_g = C_1(g-1)^2 A_{g-1} + 6\sum_{i=2}^{g-2} A_i A_{g-i}, \quad g \geq 3,$$

$$B_2 = V_{2,2}, \qquad \frac{6}{\pi^2} B_g = C_2(g-1)^2 B_{g-1} + 6\sum_{i=2}^{g-2} B_i B_{g-i} \quad g \geq 3$$

dominate $V_{g,2}$, namely $A_g \leq V_{g,2} \leq B_g$, $\forall g \geq 2$. Define $\widetilde{A}_g = \frac{6}{C_1^g} A_g$ and $\widetilde{B}_g = (\frac{6}{\pi^2 C_2})^g B_g$, then it is not difficult to see that both $\widetilde{A}_g$ and $\widetilde{B}_g$ satisfy the recursion relation (17). Thus by Lemma 2.8 we proved that there are constants $0 < c < C$ independent of g such that

$$c^g (2g)! < V_{g,2} < C^g (2g)!, \tag{21}$$

from which we deduce that (4) is an immediately consequence of Corollary 2.7.

Remark 2.9. Theorem 1.1 implies that for any fixed $n \geq 0$,

$$\lim_{g\to\infty} \frac{\log V_{g,n}}{g \log g} = 2,$$

which is weaker than Zograf's conjectural asymptotic formula (59) in §6.

3. Asymptotics of intersection numbers. In this section, we adopt Witten's notation

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m}. \tag{22}$$

For convenience, we will also use the normalized tau function denoted by

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g^{\mathrm{w}} := \prod_{i=1}^{n} (2d_i+1)!! \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g. \tag{23}$$

The celebrated Witten-Kontsevich theorem [31, 13] has several equivalent formulations, such as the DVV formula [4]

$$\begin{aligned}(2d_1+1)!!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g &= \sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \tau_{d_2} \cdots \tau_{d_j+d_1-1} \cdots \tau_{d_n} \rangle_g \\ &+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \sum_{\{2,\cdots,n\}=I\coprod J} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i\in J} \tau_{d_i} \rangle_{g-g'}\end{aligned} \tag{24}$$

which is equivalent to the Virasoro constraint.

We also have the following recursive formula from integrating the first KdV equation of the Witten-Kontsevich theorem.

$$(2g+n-1)\langle \tau_0 \prod_{j=1}^{n} \tau_{d_j} \rangle_g = \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^{n} \tau_{d_j} \rangle_{g-1} + \frac{1}{2} \sum_{\underline{n}=I\coprod J} \langle \tau_0^2 \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_0^2 \prod_{i\in J} \tau_{d_i} \rangle_{g-g'}. \tag{25}$$

Definition 3.1. The following generating function

$$F(x_1, \cdots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_i = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} x_i^{d_i}$$

is called the n-point function.

In particular, we have Witten's one-point function

$$F(x) = \frac{1}{x^2} \exp\left(\frac{x^3}{24}\right),$$

which is equivalent to $\langle \tau_{3g-2} \rangle_g = 1/(24^g g!)$.

The two-point function has a simple explicit form due to Dijkgraaf (cf. [8])

$$F(x_1, x_2) = \frac{1}{x_1 + x_2} \exp\left(\frac{x_1^3}{24} + \frac{x_2^3}{24}\right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2} x_1 x_2 (x_1 + x_2)\right)^k .$$

A general study of the n-point function can be found in [8, 18, 21].

From Dijkgraaf's two-points function, it is not difficult to see that for fixed $k \geq 0$,

$$\begin{aligned} \lim_{g \to \infty} \frac{\langle \tau_k \tau_{3g-1-k} \rangle_g}{g^k \langle \tau_{3g-2} \rangle_g} &= \lim_{g \to \infty} \frac{k!}{24^{g-k} (2k+1)! 2^k (g-k)!} \cdot \frac{24^g \cdot g!}{g^k} \\ &= \frac{k! 24^k}{(2k+1)! 2^k} = \frac{6^k}{(2k+1)!!}. \end{aligned}$$

In fact, we have the following more general result.

Proposition 3.2. *For any fixed set* $\mathbf{d} = (d_1, \ldots, d_n)$ *of non-negative integers, the limit of*

$$C(d_1, \cdots, d_n; g) = \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g}{(6g)^{|\mathbf{d}|} \langle \tau_{3g-2} \rangle_g} \prod_{i=1}^{n} (2d_i + 1)!! \tag{26}$$

when $g \to \infty$ *exists and we have* $\lim_{g\to\infty} C(d_1, \ldots, d_n; g) = 1$.

Proof. We use induction on $|\mathbf{d}|$. When $d_1 = \cdots = d_n = 0$, it is obviously true by the string equation.

From (25) and the string equation, we have that for any $\mathbf{k} = (k_1, \ldots, k_m)$ with $|\mathbf{k}| < |\mathbf{d}|$,

$$\begin{aligned} \langle \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-5+m-|\mathbf{d}|} \rangle_{g-1} &\leq \langle \tau_0^4 \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_{g-1} \\ &\leq 12(2g+m) \langle \tau_0 \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_g \\ &= O\left(g \cdot \langle \prod_{i=1}^{m} \tau_{k_i} \tau_{3g-1+m-|\mathbf{d}|} \rangle_g \right). \end{aligned} \tag{27}$$

Here $f_1(g) = O(f_2(g))$ means there exists a constant $C > 0$ independent of g such that

$$f_1(g) \leq C f_2(g).$$

Note that the last equation in (27) is obtained by induction, since $|\mathbf{k}| < |\mathbf{d}|$.

Let us expand $\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g$ using (24). From (27) and by induction, we see that the second term in the right-hand side of (24) has the estimate

$$\frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \frac{\langle \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \tau_{3g-2+n-|\mathbf{d}|} \rangle_{g-1}}{\langle \tau_{3g-2} \rangle_g} = O\left(g^{|\mathbf{d}|-1}\right). \tag{28}$$

Similarly, we can estimate the third term in the right-hand side of (24),

$$\sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \sum_{\{2,\cdots,n\}=I \coprod J} \frac{\langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \tau_{3g-2+n-|\mathbf{d}|} \rangle_{g-g'}}{\langle \tau_{3g-2} \rangle_g} = O\left(g^{|\mathbf{d}|-2}\right). \tag{29}$$

So by induction, we have

$$\lim_{g\to\infty} C(d_1,\dots,d_n;g) = \lim_{g\to\infty} \sum_{j=2}^n \frac{(2d_j+1)C(d_2,\cdots,d_j+d_1-1,\cdots d_n;g)}{6g} + \lim_{g\to\infty} \frac{(2d_1+2(3g-2+n-|\mathbf{d}|)-1)!!}{(2(3g-2+n-|\mathbf{d}|)-1)!!} \cdot \frac{C(d_2,\cdots,d_n;g)}{(6g)^{d_1}} = 1, \tag{30}$$

as claimed. ∎

Corollary 3.3. *We have the following large genus asymptotic expansion*

$$C(d_1,\dots,d_n;g) = 1 + \frac{C_1(d_1,\dots,d_n)}{g} + \frac{C_2(d_1,\dots,d_n)}{g^2} + \cdots, \tag{31}$$

where the coefficients $C_j(d_1,\dots,d_n;g)$ are determined recursively by induction on $|\mathbf{d}|$,

$$\begin{aligned} C(d_1,\dots,d_n;g) &= \frac{1}{6g}\sum_{j=2}^n (2d_j+1)C(d_2,\dots,d_j+d_1-1,\dots,d_n;g) \\ &+ \frac{\prod_{j=1}^{d_1}(g+\frac{2n-2|\mathbf{d}|+2j-5}{6})}{g^{d_1}} C(d_2,\dots,d_n;g) + \frac{(g-1)^{|\mathbf{d}|-2}}{3g^{|\mathbf{d}|-1}} \sum_{r+s=d_1-2} C(r,s,d_2,\dots,d_n;g-1) \\ &+ \sum_{\substack{r+s=d_1-2\\ I\coprod J=\{2,\cdots,n\}}} 24^{g'} 6^{|J|+1-n-3g'} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'}^{\mathbf{w}} \\ &\times \frac{(g-g')^{|J|+1-n+|\mathbf{d}|-3g'} \prod_{j=1}^{g'}(g+1-j)}{g^{|\mathbf{d}|}} C(s,d_J;g-g'), \end{aligned} \tag{32}$$

where d_J denote the set $\{d_i\}_{i\in J}$. Moreover, the expansion $C(d_1,\dots,d_n;g)$ in fact has only finite nonzero terms, i.e. $C_j(d_1,\dots,d_n;g) = 0$ when $j > |\mathbf{d}|$.

Proof. The recursive relation follows from the asymptotic expansions of the equations (28), (29) and (30). The last assertion will follow from Theorem 4.1. ∎

Remark 3.4. When $n = 0$ or $|\mathbf{d}| = 0$, we have

$$C(\emptyset;g) = C(0,\dots,0;g) = 1. \tag{33}$$

By the string and dilaton equations, we have

(34)
$$C(0,d_2,\dots,d_n;g)=\frac{1}{6g}\sum_{j=2}^{n}(2d_j+1)C(d_2,\dots,d_j-1,\dots,d_n;g)+C(d_2,\dots,d_n;g),$$

(35)
$$C(1,d_2,\dots,d_n;g)=(1+\frac{n-2}{2g})C(d_2,\dots,d_n;g).$$

So we may assume $d_i\ge 2,\forall i$ in $C(d_1,\dots,d_n;g)$.

REMARK 3.5. For any given number m, we have the large g expansion

$$\frac{1}{(g-m)^k}=\frac{1}{g^k(1-m/g)^k}=\left(\sum_{i=1}^{\infty}\frac{m^{i-1}}{g^i}\right)^k. \tag{36}$$

COROLLARY 3.6. *We have the following recursion for $C_r(d_1,\dots,d_n)$,*

$$\begin{aligned}(37)\quad C_r(d_1,\dots,d_n)=&\frac{1}{6}\sum_{j=2}^{n}(2d_j+1)C_{r-1}(d_2,\dots,d_j+d_1-1,\dots,d_n)\\
&+\sum_{k=0}^{\min(r,d_1)}C_{r-k}(d_2,\dots,d_n)\cdot[g^k]\prod_{j=1}^{d_1}\left(1+\frac{(2n-2|\mathbf{d}|+2j-5)g}{6}\right)\\
&+\frac{1}{3}\sum_{i=0}^{d_1-2}\sum_{k=0}^{r-1}C_{r-1-k}(i,d_1-2-i,d_2,\dots,d_n)\sum_{j=0}^{k}(-1)^{k-j}\binom{|\mathbf{d}|-2}{k-j}\binom{j+r-2-k}{j}\\
&+\sum_{i=0}^{d_1-2}\sum_{\{2,\cdots,n\}=I\coprod J}24^h6^{|J|+1-n-3h}\langle\tau_i\tau_{d_I}\rangle_h^{\mathbf{w}}\sum_{k=0}^{r-2h-n+|J|+1}C_{r-2h-n+|J|+1-k}(d_1-2-i,d_J)\\
&\times\sum_{j=0}^{k}(-h)^{k-j}\binom{|\mathbf{d}|-r-h+k}{k-j}[g^j]\prod_{\ell=2}^{h}(1+(1-\ell)g),\end{aligned}$$

where $[g^k]f$ denotes the coefficient of g^k in f and $[g^k]\prod_{i=1}^m(1+x_ig)=e_k(x_1,\dots,x_m)$, where $e_k(x_1,\dots,x_m)$ is the k-th elementary symmetric polynomial. The binomial $\binom{n}{k}$ is defined for all $n\in\mathbb{Z}$ and $k\ge 0$ by

$$\binom{n}{k}=\frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Proof. The recursion follows from (32) and (36) by a straightforward computation. In the third term of the right-hand side of (37), we used

$$[g^{-j}]\frac{1}{(1-1/g)^{r-1-k}}=\binom{j+r-2-k}{j}.$$

The last term of (37) is the coefficient of g^{-r} in the last term of (32), which is equal to

$$\sum_{i=0}^{d_1-2}\sum_{\{2,\cdots,n\}=I\coprod J}24^h 6^{|J|+1-n-3h}\langle\tau_i\tau_{d_I}\rangle_h^{\mathbf{w}}$$
$$\times[g^{-r}]\left(\frac{(1-h/g)^{|J|+1-n+|\mathbf{d}|-3h}\prod_{\ell=1}^h(1+(1-\ell)/g)}{g^{2h+n-|J|-1}}C(d_1-2-i,d_J;g-h)\right),$$

where the coefficient of g^{-r} in the bracket is equal to

$$\begin{aligned}
&[g^{-r}]\left(\frac{(1-h/g)^{|J|+1-n+|\mathbf{d}|-3h}\prod_{\ell=1}^h(1+(1-\ell)/g)}{g^{2h+n-|J|-1}}\right.\\
&\qquad\left.\times\sum_{k=0}^{r-2h-n+|J|+1}\frac{C_{r-2h-n+|J|+1-k}(d_1-2-i,d_J)}{g^{r-2h-n+|J|+1-k}(1-h/g)^{r-2h-n+|J|+1-k}}\right)\\
&\qquad=\sum_{k=0}^{r-2h-n+|J|+1}C_{r-2h-n+|J|+1-k}(d_1-2-i,d_J)\\
&\qquad\times[g^{-k}]\left((1-h/g)^{|\mathbf{d}|-r-h+k}\prod_{\ell=1}^h(1+(1-\ell)/g)\right)\\
&\qquad=\sum_{k=0}^{r-2h-n+|J|+1}C_{r-2h-n+|J|+1-k}(d_1-2-i,d_J)\\
&\qquad\qquad\times\sum_{j=0}^{k}(-h)^{k-j}\binom{|\mathbf{d}|-r-h+k}{k-j}[g^j]\prod_{\ell=2}^h(1+(1-\ell)g),
\end{aligned}$$

as claimed. ∎

LEMMA 3.7. (i) *Let $d_i\geq 0$ and $p=\#\{i\mid d_i=0\}$. Then*

$$C_1(d_1,\dots,d_n)=-\frac{|\mathbf{d}|^2}{6}+\frac{(n-1)|\mathbf{d}|}{3}+\frac{n^2-5n}{12}+\frac{5p-p^2}{12}. \tag{38}$$

(ii) *Let $d_i\geq 0$, $p_2=\#\{i\mid d_i=2\}$, $p_1=\#\{i\mid d_i=1\}$ and $p_0=\#\{i\mid d_i=0\}$. Then*

$$\begin{aligned}
C_2(d_1,\dots,d_n)=&\frac{|\mathbf{d}|^4}{72}-\frac{(3n-2)|\mathbf{d}|^3}{54}+\frac{n(3n+1)|\mathbf{d}|^2}{72}\\
&+\frac{(6n^3-48n^2+54n-11)|\mathbf{d}|}{216}+\frac{n(3n^3-50n^2+189n+14)}{864}-\frac{5p_2}{72}-\frac{17p_1}{72}\\
&+\frac{p_0^4}{288}-\frac{23p_0^3}{432}+\frac{p_0^2(4|\mathbf{d}|^2-8n|\mathbf{d}|+8|\mathbf{d}|-2n^2+22n-12p_1+47)}{288}\\
&+\frac{p_0(-30|\mathbf{d}|^2+60n|\mathbf{d}|-60|\mathbf{d}|+15n^2-165n+90p_1-7)}{432}.
\end{aligned}\tag{39}$$

Proof. Let $q = \#\{i \geq 2 \mid d_i = 0\}$. From (37) and by induction on n, we have

$$\begin{aligned}
C_1(d_1,\ldots,d_n) &= \frac{1}{6}\sum_{j=2}^{n}(2d_j+1) - \frac{q}{6}\,\delta_{d_1,0} + C_1(d_2,\ldots,d_n)\\
&\quad + \sum_{j=1}^{d_1}\frac{2n-2|\mathbf{d}|+2j-5}{6} + \frac{d_1-1}{3} + \frac{1}{3}\,\delta_{d_1,0}\\
&= -\frac{d_1^2}{6} - \frac{d_1}{3}\left(\sum_{j=2}^{n} d_j\right) + \frac{(n-1)d_1}{3} + \sum_{j=2}^{n}\frac{d_j}{3} + \frac{n}{6} - \frac{1}{2} + \frac{1}{3}\delta_{d_1,0} - \frac{q}{6}\delta_{d_1,0}\\
&\quad - \frac{(d_2+\cdots+d_n)^2}{6} + \frac{(n-2)(d_2+\cdots+d_n)}{3} + \frac{n^2-7n+6}{12} + \frac{5q-q^2}{12}\\
&= -\frac{|\mathbf{d}|^2}{6} + \frac{(n-1)|\mathbf{d}|}{3} + \frac{n^2-5n}{12} + \frac{5p-p^2}{12}.
\end{aligned}$$

Note that (38) obviously holds when $n = 1$, so we conclude the inductive proof of (38).

For (39), we may first assume that all $d_i \geq 2$ in $C_2(d_1,\ldots,d_n)$, which can be derived by solving a recursion relation from (37), although it is much more complicated than above. Then we use (35) and (34) to get $C_2(d_1,\ldots,d_n)$ for all $d_i \geq 0$. For example, assume there are exactly $k \geq 1$ zeros in d_i, $1 \leq i \leq n$, by (34),

$$C_2(\underbrace{0,\ldots,0}_{k}, d_{k+1},\ldots,d_n) = C_2(\underbrace{0,\ldots,0}_{k-1}, d_{k+1},\ldots,d_n) + f(n,k), \tag{40}$$

where $f(n,k)$ is given by

$$f(n,k) = \frac{1}{6}\sum_{j=k+1}^{n}(2d_j+1)C_1(\underbrace{0,\ldots,0}_{k-1}, d_{k+1},\ldots,d_j-1\ldots,d_n),$$

which can be computed by (38). So we get

$$C_2(\underbrace{0,\ldots,0}_{k}, d_{k+1},\ldots,d_n) = C_2(d_{k+1},\ldots,d_n) + \sum_{i=1}^{k} f(n+i-k,i).$$

Once we get (39), its verification is relatively straightforward. ∎

REMARK 3.8. One can prove inductively that each $C_r(d_1,\ldots,d_n)$ is a polynomial in $|\mathbf{d}|$ and n.

LEMMA 3.9. *For any fixed set* $\mathbf{d} = (d_1,\ldots,d_n)$ *of non-negative integers and* $r \geq 1$,

$$C_r(\underbrace{2,\ldots,2}_{k}, d_1,\ldots,d_n)$$

is a polynomial in k *of order* $2r$ *whose leading term* $k^{2r}/(12^r r!)$ *is independent of* $\mathbf{d}$.

In particular,

$$C_1(\underbrace{2,\ldots,2}_{k}) = \frac{k^2}{12} - \frac{13k}{12}, \quad C_2(\underbrace{2,\ldots,2}_{k}) = \frac{k^4}{288} - \frac{65k^3}{432} + \frac{23k^2}{32} - \frac{67k}{432}, \tag{41}$$

$$C_3(\underbrace{2,\ldots,2}_{k}) = \frac{k^6}{10368} - \frac{91k^5}{10368} + \frac{1373k^4}{10368} - \frac{4589k^3}{10368} + \frac{137k^2}{576} + \frac{35k}{432}, \tag{42}$$

$$C_1(\underbrace{2,\ldots,2}_{k},3) = \frac{k^2}{12} - \frac{5k}{4} - \frac{11}{6}, \quad C_1(\underbrace{2,\ldots,2}_{k},4) = \frac{k^2}{12} - \frac{19k}{12} - 3. \tag{43}$$

Proof. We will use induction on r. The polynomiality is obvious. When computing the leading term, we only need to compute the first two terms of the right-hand side of (37), since the last two terms belong to $O(k^{2r-2})$,

$$\begin{aligned} C_r(\underbrace{2,\ldots,2}_{k},d_1,\ldots,d_n) =& C_r(\underbrace{2,\ldots,2}_{k-1},d_1,\ldots,d_n) \\ &+ \frac{5k^{2r-1}}{6\cdot 12^{r-1}(r-1)!} + \frac{-4k^{2r-1}}{6\cdot 12^{r-1}(r-1)!} + O(k^{2r-2}), \end{aligned}$$

which implies that $C_r(\underbrace{2,\ldots,2}_{k},d_1,\ldots,d_n)$ has leading term $k^{2r}/(12^r r!)$. ∎

For the full expansion of $C(d_1,\ldots,d_n;g)$, let us look at some examples

$$\begin{gathered} C(1;g) = C(1,1;g) = 1 - \frac{1}{2g}, \quad C(0,1,1;g) = 1 + \frac{1}{2g}, \\ C(2;g) = 1 - \frac{1}{g} + \frac{5}{12g^2}, \quad C(3;g) = 1 - \frac{11}{6g} + \frac{95}{72g^2} - \frac{35}{72g^3}, \\ C(4;g) = 1 - \frac{3}{g} + \frac{83}{24g^2} - \frac{35}{16g^3} + \frac{35}{48g^4}, \quad C(2,2;g) = 1 - \frac{11}{6g} + \frac{17}{12g^2} - \frac{7}{12g^3}. \end{gathered}$$

In fact, we will prove in Theorem 4.1 that $C(d_1,\ldots,d_n;g)$ is a polynomial in $1/g$.

4. An integer-valued polynomial. Let $P_{d_1,\ldots,d_n}(g) = (6g)^{|\mathbf{d}|}C(d_1,\ldots,d_n;g)$. We will prove that $P_{d_1,\ldots,d_n}(g)$ is an integer-valued polynomial. By the recursive formula (32) in Corollary 3.3, we have

$$\begin{aligned} P_{d_1,\ldots,d_n}(g) =& \sum_{j=2}^{n}(2d_j+1)P_{d_2,\ldots,d_j+d_1-1,\ldots,d_n}(g) \\ &+ \prod_{j=1}^{d_1}(6g+2n-2|\mathbf{d}|+2j-5)P_{d_2,\ldots,d_n}(g) + 12g\sum_{r+s=d_1-2}P_{r,s,d_2,\ldots,d_n}(g-1) \\ &+ \sum_{\substack{r+s=d_1-2\\ I\coprod J=\{2,\cdots,n\}}} 24^{g'}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle^{\mathbf{w}}_{g'}\prod_{j=1}^{g'}(g+1-j)P_{s,d_J}(g-g'), \end{aligned} \tag{44}$$

which can be used to compute $P_{d_1,\ldots,d_n}(g)$ recursively.

The string and dilaton equations for $P_{d_1,\ldots,d_n}(g)$ are

$$P_{0,d_2,\ldots,d_n}(g) = \sum_{j=2}^{n}(2d_j+1)P_{d_2,\ldots,d_j-1,\ldots,d_n}(g) + P_{d_2,\ldots,d_n}(g), \tag{45}$$

$$P_{1,d_2,\ldots,d_n}(g) = (6g+3n-6)P_{d_2,\ldots,d_n}(g). \tag{46}$$

THEOREM 4.1. (i) *For any fixed set* $\mathbf{d}=(d_1,\ldots,d_n)$ *of non-negative integers,*

$$P_{d_1,\ldots,d_n}(g) = \frac{\langle \tau_{d_1}\cdots\tau_{d_n}\tau_{3g-2+n-|\mathbf{d}|}\rangle_g}{\langle \tau_{3g-2}\rangle_g}\prod_{i=1}^{n}(2d_i+1)!!$$

is a polynomial in g *with highest-degree term* $6^{|\mathbf{d}|}g^{|\mathbf{d}|}$. *Moreover,* $2^{\lfloor\frac{|\mathbf{d}|}{3}\rfloor}P_{d_1,\ldots,d_n}(g)\in\mathbb{Z}[g]$, *where* $\lfloor x\rfloor$ *is the greatest integer less than or equal to* x, *and* $P_{d_1,\ldots,d_n}(g)\in\mathbb{Z}$ *whenever* $g\in\mathbb{Z}$. *These polynomials* $P_{d_1,\ldots,d_n}(g)$ *are determined uniquely by the recursive relation* (44) *and* $P_\emptyset(g)=P_{0,\ldots,0}(g)=1$.

(ii) *The constant term of* $P_{d_1,\ldots,d_n}(g)$ *is equal to*

$$\prod_{j=1}^{|\mathbf{d}|}(n-j-1)\cdot\prod_{i=1}^{n}\frac{(2d_i+1)!!}{d_i!}. \tag{47}$$

Proof. From [22, Thm. 4.3 (iv) and Prop. 4.4], we know

$$2^{\mathrm{ord}(2,\,24^{g'}g'!)}\cdot\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}^{\mathbf{w}}\in\mathbb{Z}.$$

By induction using (44) and

$$\mathrm{ord}(2,g'!) \le \sum_{k\ge1}\Big\lfloor\frac{g'}{2^k}\Big\rfloor \le g' = \frac{r+\sum_{i\in I}d_i-|I|+2}{3},$$

$$\Big\lfloor\frac{r+\sum_{i\in I}d_i-|I|+2}{3}\Big\rfloor + \Big\lfloor\frac{\sum_{i\in J}d_i+s}{3}\Big\rfloor \le \Big\lfloor\frac{|\mathbf{d}|}{3}\Big\rfloor,$$

it is not difficult to see that $2^{\lfloor\frac{|\mathbf{d}|}{3}\rfloor}P_{d_1,\ldots,d_n}(g)$ are polynomials with integer coefficients. It is well-known that a polynomial of degree n is integer-valued if and only it takes integral values on $n+1$ consecutive integers. When $g\in\mathbb{N}$, it is easy to see from (44) that $P_{d_1,\ldots,d_n}(g)\in\mathbb{Z}$ since g' divides $\prod_{j=1}^{g'}(g+1-j)$, so $P_{d_1,\ldots,d_n}(g)$ is an integer-valued polynomial.

By (44), it is not difficult to prove that the constant term of $P_d(g)$ is equal to $(-1)^d(2d+1)!!$, and when $n\ge2$, $P_{d_1,\ldots,d_n}(0)=0$ unless $|\mathbf{d}|\le n-2$. Let us assume that $d_{k+1},\ldots,d_n\ge1$ and $|\mathbf{d}|\le n-2$. Then by (45), we have

$$P_{\underbrace{0,\ldots,0}_{k},d_{k+1},\ldots,d_n}(0) = \frac{(n-2)!}{(n-2-|\mathbf{d}|)!\prod_{i=k+1}^{n}d_i!}\prod_{i=k+1}^{n}(2d_i+1)!!,$$

which gives (47). On the other hand, it is easy to verify that (47) satisfies (45) and (46). $\square$

Corollary 4.2. *A positive integer* $k \geq 1$ *is a root of* $P_{d_1,\dots,d_n}(g)$ *if and only if* $k < \frac{|\mathbf{d}|-n+2}{3}$. *And* 0 *is a root of* $P_{d_1,\dots,d_n}(g)$ *if and only if* $2 \leq n \leq |\mathbf{d}|+1$.

Proof. The first assertion is obvious, since $\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g = 0$ if and only if $g < \frac{|\mathbf{d}|-n+2}{3}$. The second assertion follows from (47). ∎

Corollary 4.3. *The coefficient of* $1/g^{|\mathbf{d}|}$ *in* $C(d_1,\dots,d_n;g)$ *is equal to*

$$\frac{1}{6^{|\mathbf{d}|}} \prod_{j=1}^{|\mathbf{d}|} (n-j-1) \cdot \prod_{i=1}^{n} \frac{(2d_i+1)!!}{d_i!}.$$

Proof. It follows from (47), since the coefficient of $1/g^{|\mathbf{d}|}$ in $C(d_1,\dots,d_n;g)$ is equal to $P_{d_1,\dots,d_n}(0)/6^{|\mathbf{d}|}$. ∎

Corollary 4.4. *For any fixed set* $\mathbf{d} = (d_1,\dots,d_n)$ *of non-negative integers,*

$$P_{d_1,\dots,d_n}(1) = \frac{\prod_{j=0}^{|\mathbf{d}|-1}(n+1-j)\prod_{i=1}^{n}(2d_i+1)!}{\prod_{i=1}^{n} d_i!} \left(1 - \sum_{k=2}^{n+1} \frac{e_k(d_1,\dots,d_n,n+1-|\mathbf{d}|)}{k(k-1)\binom{n+1}{k}}\right),$$

where e_k *is the* k*-th elementary symmetric polynomial.*

Proof. Recall the following identity (cf. [14, §4.6]),

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle_1 = \frac{1}{24}\binom{n}{m_1,\dots,m_n}\left(1 - \sum_{k=2}^{n} \frac{(k-2)!(n-k)!}{n!} e_k(m_1,\dots,m_n)\right).$$

So we have

$$\begin{aligned} P_{d_1,\dots,d_n}(1) &= 24 \prod_{i=1}^{n} (2d_i+1)!! \langle \tau_{d_1} \cdots \tau_{d_n} \tau_{n+1-|\mathbf{d}|} \rangle_1 \\ &= \frac{\prod_{j=0}^{|\mathbf{d}|-1}(n+1-j)\prod_{i=1}^{n}(2d_i+1)!}{\prod_{i=1}^{n} d_i!} \left(1 - \sum_{k=2}^{n+1} \frac{e_k(d_1,\dots,d_n,n+1-|\mathbf{d}|)}{k(k-1)\binom{n+1}{k}}\right), \end{aligned}$$

as claimed. ∎

Corollary 4.5. *Let* $d \geq 0$ *be a nonnegative integer. Then*

$$\frac{P_d(g)}{(2d+1)!!} = \sum_{i=0}^{\lfloor \frac{d-1}{3} \rfloor} \sum_{k} \frac{\binom{k-1}{d-3i-k} 12^k k! \prod_{j=0}^{k+i-1}(g-j)}{i!(2k+1)!} + (-1)^{d \bmod 3} \binom{g-1}{\lfloor \frac{d}{3} \rfloor},$$

where the summation range of k *is* $\max(\lceil \frac{d-3i+1}{2} \rceil, 1) \leq k \leq d-3i$.

Proof. Since $P_d(g) = (2d+1)!!24^g g! \langle \tau_d \tau_{3g-1-d} \rangle_g$, so it follows from the following explicit formula of two-point tau functions (cf. [22, §4])

$$\begin{aligned} \langle \tau_d \tau_{3g-1-d} \rangle_g = &\sum_{i=0}^{\lfloor \frac{d-1}{3} \rfloor} \sum_{k} \binom{g-k}{i}\binom{k-1}{d-3i-k} \frac{k!}{(g-k)!24^{g-k}(2k+1)!2^k} \\ &+ \frac{(-1)^{d \bmod 3}}{g!24^g}\binom{g-1}{\lfloor \frac{d}{3} \rfloor}, \end{aligned}$$

where the summation range of k is $\max(\lceil\frac{d-3i+1}{2}\rceil, 1) \le k \le \min(g-i, d-3i)$. $\square$

REMARK 4.6. In general, $P_{d_1,\dots,d_n}(g) \notin \mathbb{Z}[g]$. For example,

$$\begin{aligned} P_6(g) &= 46656g^6 - 295488g^5 + 756216g^4 - 1024812g^3 + \frac{1668951}{2}g^2 - \frac{904365}{2}g + 135135 \\ &= \frac{27}{2}(g-1)(g-2)(3456g^4 - 11520g^3 + 14544g^2 - 9240g + 5005). \end{aligned}$$

Since $P_{d_1,\dots,d_n}(g)$ are integer-valued polynomials, there exist unique integers $\lambda_0, \dots, \lambda_{|\mathbf{d}|}$ such that

$$P_{d_1,\dots,d_n}(g) = \lambda_0 + \lambda_1 g + \lambda_2 \binom{g}{2} \cdots + \lambda_{|\mathbf{d}|}\binom{g}{|\mathbf{d}|}.$$

Denote by $I_{d_1,\dots,d_n} := (\lambda_0, \dots, \lambda_{|\mathbf{d}|})$ the sequence of integer coefficients. From (31), they satisfy the following recursion relation

$$\begin{aligned} (48)\quad \lambda_k = I_{d_1,\dots,d_n}(k) &= \sum_{j=2}^{n}(2d_j+1)I_{d_2,\dots,d_j+d_1-1,\dots,d_n}(k) \\ &+ \sum_{i=\max(0,k-d_1)}^{k} c_{k-i}(d_1, 2n-2|\mathbf{d}|+6i-5)\binom{k}{i} I_{d_2,\dots,d_n}(i) \\ +12k\sum_{r+s=d_1-2} I_{r,s,d_2,\dots,d_n}(k-1) &+ \sum_{\substack{r+s=d_1-2 \\ I\coprod J=\{2,\cdots,n\}}} 24^{g'}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}^{\mathbf{w}}\frac{k!}{(k-g')!}I_{s,d_J}(k-g'), \end{aligned}$$

where $c_t(d_1, m)$, $0 \le t \le d_1$ are the coefficients of

$$\prod_{j=1}^{d_1}(6x+m+2j) = c_0 + c_1 x + c_2\binom{x}{2} + \cdots + c_{d_1-1}\binom{x}{d_1-1} + c_{d_1}\binom{x}{d_1},$$

which can be determined recursively by

$$c_t = \prod_{j=1}^{d_1}(6t+m+2j) - \sum_{j=0}^{t-1} c_j\binom{t}{j}.$$

In particular, $c_0 = \prod_{j=1}^{d_1}(m+2j)$ and $c_{d_1} = 6^{d_1}d_1!$.

Below are some examples:

$$\begin{gathered} P_1(g) = 6g-3, \quad I_1 = (-3, 6), \\ P_2(g) = 36g^2 - 36g + 15, \quad I_2 = (15, 0, 72), \\ P_3(g) = 216g^3 - 396g^2 + 285g - 105, \quad I_3 = (-105, 105, 504, 1296), \\ P_{1,1}(g) = 36g^2 - 18g, \quad I_{1,1} = (0, 18, 72), \\ P_{1,2}(g) = 216g^3 - 216g^2 + 90g, \quad I_{1,2} = (0, 90, 864, 1296), \\ P_{2,2}(g) = 1296g^4 - 2376g^3 + 1836g^2 - 756g, \quad I_{2,2} = (0, 0, 7560, 32400, 31104), \\ I_6 = (135135, -135135, 135135, 1945944, 20015424, 48522240, 33592320). \end{gathered}$$

5. Proof of Theorem 1.2. First we introduce some notation. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i. When convenient, we also use $(1^{m(1)}2^{m(2)}\dots)$ to denote $\mathbf{m}$. Let $\mathbf{m}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$.

$$|\mathbf{m}| := \sum_{i\geq 1} m(i)\cdot i, \quad ||\mathbf{m}|| := \sum_{i\geq 1} m(i), \quad \binom{\mathbf{m}}{\mathbf{a}_1,\dots,\mathbf{a}_n} := \prod_{i\geq 1}\binom{m(i)}{a_1(i),\dots,a_n(i)}.$$

We denote by $\kappa(\mathbf{m}) := \prod_{i\geq 1}\kappa_i^{m(i)}$ a formal monomial of κ classes. The following remarkable identity was proved in [12].

(49)
$$\langle\prod_{j=1}^n \tau_{d_j}\kappa(\mathbf{m})\rangle_g = \sum_{p=0}^{||\mathbf{m}||}\frac{(-1)^{||\mathbf{m}||-p}}{p!}\sum_{\substack{\mathbf{m}=\mathbf{m}_1+\dots+\mathbf{m}_p\\ \mathbf{m_i}\neq\mathbf{0}}}\binom{\mathbf{m}}{\mathbf{m}_1,\dots,\mathbf{m}_p}\langle\prod_{j=1}^n\tau_{d_j}\prod_{j=1}^p\tau_{|\mathbf{m}_j|+1}\rangle_g,$$

from which we see that studying the asymptotics of integrals of ψ classes will be helpful in understanding the asymptotics of Weil-Petersson volumes. In a forthcoming paper [15], we will prove more asymptotic formulae for intersection numbers.

For any $k\geq 1$, by definition we have

$$\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} = \frac{(6g-3-2k)!!2^{6g-4-2k}(2\pi^2)^k\langle\tau_{3g-2-k}\kappa_1^k\rangle_g/k!}{g^k(6g-3)!!2^{6g-4}\langle\tau_{3g-2}\rangle_g}. \tag{50}$$

Using (49) to expand $\langle\tau_{3g-2-k}\kappa_1^k\rangle_g$ and taking limit as $g\to\infty$, we get by Proposition 3.2

$$\begin{aligned}\lim_{g\to\infty}\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} &= \lim_{g\to\infty}\frac{(6g-3-2k)!!(2\pi^2)^k\langle\tau_{3g-2-k}\tau_2^k\rangle_g}{g^k(6g-3)!!2^{2k}k!\langle\tau_{3g-2}\rangle_g}\\ &= \frac{\pi^{2k}}{5^k k!}\lim_{g\to\infty}\frac{15^k\langle\tau_{3g-2-k}\tau_2^k\rangle_g}{(6g)^{2k}\langle\tau_{3g-2}\rangle_g}\\ &= \frac{\pi^{2k}}{5^k k!}\lim_{g\to\infty}C(\underbrace{2,\dots,2}_{k};g) = \frac{\pi^{2k}}{5^k k!}.\end{aligned}$$

So we get the leading term in the right-hand side of (5).

Now we compute the coefficient of $1/g$ in the asymptotic expansion of $a_{g,3g-2-k}/(g^k a_{g,3g-2})$. We have

(51)
$$\begin{aligned}\frac{a_{g,3g-2-k}}{g^k a_{g,3g-2}} &= \frac{(6g-3-2k)!!\pi^{2k}\left(\langle\tau_{3g-2-k}\tau_2^k\rangle_g - \frac{k(k-1)}{2}\langle\tau_{3g-2-k}\tau_2^{k-2}\tau_3\rangle_g\right)}{g^k(6g-3)!!2^k k!\langle\tau_{3g-2}\rangle_g} + O\left(\frac{1}{g^2}\right)\\ &= \frac{\pi^{2k}}{5^k k!}\left(\frac{(6g)^k}{\prod_{j=1}^k(6g-2j-1)}C(\underbrace{2,\dots,2}_{k};g)\right.\\ &\left.-\frac{15}{14}k(k-1)\cdot\frac{(6g)^{k-1}}{\prod_{j=1}^k(6g-2j-1)}C(\underbrace{2,\dots,2}_{k-2},3;g)\right) + O\left(\frac{1}{g^2}\right).\end{aligned}$$

By (41), we get

$$b_1(k) = C_1(\underbrace{2,\ldots,2}_{k}) + \sum_{j=1}^{k} \frac{1+2j}{6} - \frac{15}{14}k(k-1) \times \frac{1}{6} = \frac{1}{14}k^2 - \frac{4}{7}k.$$

We can similarly compute $b_2(k)$ and $b_3(k)$.

Since there can only have a finite number of terms in the right-hand side of (49), it is not difficult to see that for each fixed $k \geq 1$, the series in the bracket of (5) is equal to

$$\frac{(6g-3-2k)!!\,15^k k!}{(6g-3)!!} \sum_{p=0}^{k} \frac{(-1)^{k-p}}{p!} \sum_{\substack{k=m_1+\cdots+m_p \\ m_i>0}} \frac{(6g)^p\, C(m_1+1,\ldots,m_p+1;g)}{\prod_{j=1}^{p} m_j!(2m_j+3)!!}, \tag{52}$$

which is a rational function of g, i.e. a division of two polynomials in $\mathbb{Z}[g]$. By Lemma 3.9, (52) implies that for any $r \geq 1$, $b_r(k)$ is a polynomial of k with degree $\leq 2r$. More explicitly,

$$\begin{aligned} b_r(k) = &\sum_{j=0}^{r} \sum_{\mu \vdash j} \frac{(-1)^j 15^{j+\ell(\mu)} \prod_{i=0}^{j+\ell(\mu)-1}(k-i)}{6^j |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)}(\mu_i+1)!(2\mu_i+5)!!} \\ &\times \sum_{i=0}^{r-j} \frac{s_i(k)}{6^i} C_{r-j-i}(\underbrace{2,\ldots,2}_{k-j-\ell(\mu)}, \mu_1+2,\ldots,\mu_{\ell(\mu)}+2), \end{aligned} \tag{53}$$

where $\mu = (\mu_1,\ldots,\mu_{\ell(\mu)})$ runs over all partitions of j and $\ell(\mu)$ is the length of μ. By convention, the empty partition is the unique partition of 0. For each fixed $i \geq 0$, the polynomial $s_i(k)$ is given by

$$s_i(k) = [g^{-i}] \left(\frac{1}{\prod_{j=1}^{k} (1-(2j+1)/g)} \right) = \frac{k^{2i}}{i!} + O(k^{2i-1}). \tag{54}$$

In particular, $s_0(k) = 1$, $s_1(k) = k^2 + 2k$, $s_2(k) = k^4/2 + 8k^3/3 + 4k^2 + 11k/6$.

By Lemma 3.9, (53) and (54), it is not difficult to see that the degree of $b_r(k)$ is no more than $2r$ and contributions to leading terms only come from partitions of maximum length $\ell(\mu) = j$. So the coefficient of k^{2r} in $b_r(k)$ is equal to

$$\sum_{j=0}^{r} \frac{(-1)^j\, 15^{2j}}{j!\, 6^j\, 210^j} \sum_{i=0}^{r-j} \frac{1}{i!\, 6^i\, 12^{r-j-i}(r-j-i)!} = \frac{1}{14^r\, r!},$$

which can be proved by showing that both sides satisfy the recursion $14(r+1)f(r+1) = f(r)$. Thus we conclude the proof of Theorem 1.2.

Example 5.1. When $k = 1$, we have

$$\begin{aligned} \frac{a_{g,3g-3}}{g a_{g,3g-2}} &= \frac{\pi^2}{5} \cdot \frac{6g}{6g-3} C(2;g) \\ &= \frac{\pi^2}{5} \cdot \frac{12g^2-12g+5}{6g(2g-1)} \\ &= \frac{\pi^2}{5} \left(1 - \frac{1}{2g} + \sum_{j=2}^{\infty} \frac{1}{3 \cdot 2^{j-1} g^j} \right). \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned}\frac{a_{g,3g-4}}{g^2 a_{g,3g-2}} &= \frac{\pi^4}{50}\left(\frac{(6g)^2}{(6g-3)(6g-5)}C(2,2;g) - \frac{15}{7}\cdot\frac{6g}{(6g-3)(6g-5)}C(3;g)\right)\\ &= \frac{\pi^4}{50}\cdot\frac{(g-1)(1008g^3-1200g^2+888g-175)}{84g^2(2g-1)(6g-5)}\\ &= \frac{\pi^4}{50}\left(1-\frac{6}{7g}+\frac{43}{84g^2}+\cdots\right).\end{aligned}$$

LEMMA 5.2. *Let $s_r(k)$ be the polynomial of k defined in (54). Then*

$$s_r(k) = (-2)^r \sum_{\mu\vdash r} \frac{(-1)^{\ell(\mu)}}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{B_{\mu_i+1}\left(-\frac{1}{2}-k\right) - B_{\mu_i+1}\left(-\frac{1}{2}\right)}{\mu_i(\mu_i+1)}, \tag{55}$$

where $B_m(x)$ is the Bernoulli polynomial.

Proof. By (54), we have

$$s_r(k) = 2^r\cdot[g^{-r}]\left(\frac{1}{\prod_{j=1}^{k}\left(1-(j+\frac{1}{2})/g\right)}\right) = 2^r\cdot[g^{k-r}]\frac{\Gamma\left(g-\frac{1}{2}-k\right)}{\Gamma\left(g-\frac{1}{2}\right)}. \tag{56}$$

As $g\to\infty$, we have the following asymptotic formula of Barnes,

$$\ln\Gamma(g+c) = \left(g+c-\frac{1}{2}\right)\ln g - g + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(c)}{j(j+1)g^j},$$

where c is an arbitrary constant.

So from (56) we get

$$\begin{aligned}s_r(k) &= 2^r\cdot[g^{-r}]\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\Big(B_{j+1}\big(-\frac{1}{2}-k\big) - B_{j+1}\big(-\frac{1}{2}\big)\Big)}{j(j+1)g^j}\right)\\ &= 2^r\sum_{\mu\vdash r}\frac{(-1)^{r+\ell(\mu)}}{|\mathrm{Aut}(\mu)|}\prod_{i=1}^{\ell(\mu)}\frac{B_{\mu_i+1}\left(-\frac{1}{2}-k\right)-B_{\mu_i+1}\left(-\frac{1}{2}\right)}{\mu_i(\mu_i+1)},\end{aligned}$$

as claimed. ☐

COROLLARY 5.3. *The coefficient of k in $s_r(k)$, $r\geq 1$ is equal to*

$$\begin{aligned}[k]s_r(k) &= \frac{1}{r}\sum_{j=0}^{r}\binom{r}{j}(-2)^j B_j = \frac{(-2)^r}{r}B_r\left(-\frac{1}{2}\right)\\ &= \frac{(-2)^r}{r}\sum_{m=0}^{r+1}\frac{1}{m+1}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(j-\frac{1}{2}\right)^r,\end{aligned}$$

where B_j is the j-th Bernoulli number. In particular, $[k]s_r(k) = 2$ when r is odd.

Proof. The equations follow from the well-known formula

$$
\begin{aligned}
B_n(x) &= \sum_{j=0}^{n} \binom{n}{n-j} B_j x^{n-j} \\
&= \sum_{m=0}^{n} \frac{1}{m+1} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x+j)^n.
\end{aligned}
$$

For the last assertion, we use

$$
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1},
$$

which implies that

$$
\begin{aligned}
\sum_{m=0}^{\infty} 2B_{2m+1}\left(-\frac{1}{2}\right) \frac{t^{2m+1}}{(2m+1)!} &= \frac{te^{-\frac{1}{2}t}}{e^t-1} - \frac{-te^{\frac{1}{2}t}}{e^{-t}-1} = \frac{te^{-\frac{1}{2}t} - te^{\frac{3}{2}t}}{e^t - 1} \\
&= \frac{te^{-\frac{1}{2}t}(1-e^{2t})}{e^t-1} = -te^{-\frac{1}{2}t}(e^t+1) = -t(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}) = -t \sum_{m=0}^{\infty} \frac{2\left(\frac{t}{2}\right)^{2m}}{(2m)!}.
\end{aligned}
$$

So we get $B_{2m+1}\left(-\frac{1}{2}\right) = \frac{-(2m+1)}{4^m}$, which implies $[k]s_{2m+1}(k) = 2$. ∎

For any given set of nonnegative integers $\mathbf{d} = (d_1, \ldots, d_l)$, define

$$
D_r(d_1, \ldots, d_l; k) = C_r(\underbrace{2, \ldots, 2}_{k}, d_1, \ldots, d_l), \tag{57}
$$

which is a polynomial in k of order $2r$ by Lemma 3.9. We also denote $D_r(k) := D_r(\emptyset; k)$.

COROLLARY 5.4. *The coefficient of k in $b_r(k)$, $r \geq 1$ is equal to*

$$
\begin{aligned}
\frac{[k]s_r(k)}{6^r} + [k]D_r(k) + \sum_{j=1}^{r} \sum_{\mu \vdash j} \frac{(-1)^{\ell(\mu)-1} 15^{j+\ell(\mu)} (j+\ell(\mu)-1)!}{6^j |\mathrm{Aut}(\mu)| \prod_{i=1}^{\ell(\mu)} (\mu_i+1)!(2\mu_i+5)!!} \\
\times D_{r-j}(\mu_1+2, \ldots, \mu_{\ell(\mu)}+2; -j-\ell(\mu)).
\end{aligned} \tag{58}
$$

Proof. It follows immediately from (53). ∎

COROLLARY 5.5. *For any nonnegative $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathbf{m} = (m(1), m(2), \ldots) \in N^\infty$, we have the following large genus expansion involving higher degree κ classes*

$$
\begin{aligned}
&\frac{\prod_{i=1}^{n}(2d_i+1)!! \prod_{j\geq 1}((2j+3)!!)^{m(j)} \langle \prod_{i=1}^{n} \tau_{d_i} \tau_{3g-2+n-|\mathbf{d}|-|\mathbf{m}|} \kappa(\mathbf{m}) \rangle_g}{(6g)^{|\mathbf{d}|+|\mathbf{m}|+||\mathbf{m}||} \langle \tau_{3g-2} \rangle_g} \\
&= 1 + \frac{1}{g}\Bigg(-\frac{|\mathbf{d}|^2}{6} + \frac{(n-|\mathbf{m}|-1)|\mathbf{d}|}{3} \\
&+ \frac{n^2 + (4|\mathbf{m}| + 6||\mathbf{m}|| - 5)n - 2|\mathbf{m}|^2 - 4|\mathbf{m}| + 3||\mathbf{m}||^2 - 9||\mathbf{m}||}{12} + \frac{5p-p^2}{12} \\
&- \sum_{i\geq 1} \frac{m(i)(m(i)-1)\big((2i+3)!!\big)^2}{12 \cdot (4i+3)!!} - \sum_{\substack{i,j\geq 1 \\ i\neq j}} \frac{m(i)m(j)(2i+3)!!(2j+3)!!}{12 \cdot (2i+2j+3)!!} \Bigg) + O\left(\frac{1}{g^2}\right),
\end{aligned}
$$

where $p = \#\{i \mid d_i = 0\}$.

Proof. The proof is a straightforward computation by using Proposition 3.2, Equations (49) and (38). ∎

Remark 5.6. Mirzakhani [25] proposed to study the asymptotic behavior of the sequences

$$V_g, V_{g-1,2}, \dots, V_{1,2g-2}, V_{0,2g}$$

and

$$V_{g,1} = [\tau_0]_g, [\tau_1]_g, \dots, [\tau_{3g-3}]_g, [\tau_{3g-2}]_g = \frac{(6g-3)!!}{16 \cdot 3^g g!}.$$

Note that for fixed $k \geq 0$, the large g asymptotics of $V_{k,2g-2k}$ is known (cf. [26]), as well as other three boundary cases (cf. (61), (62), (5))

$$V_{g-k+1,2k-2} \sim V_{g-k,2k}, \quad [\tau_k]_g \sim [\tau_{k+1}]_g, \quad \frac{[\tau_{3g-3-k}]_g}{[\tau_{3g-2-k}]_g} \sim \frac{\pi^2 g}{5(k+1)}.$$

Remark 5.7. Numerical computations suggest that the limit of each of the following ratios of one-point volumes

$$\frac{[\tau_{2g+1}]_g}{[\tau_{2g+2}]_g}, \ \frac{[\tau_{2g}]_g}{[\tau_{2g+1}]_g}, \ \frac{[\tau_{2g-1}]_g}{[\tau_{2g}]_g}, \ \frac{[\tau_g]_g}{[\tau_{g+1}]_g}, \ \dots$$

should exist when $g \to \infty$. So far, we do not know a proof. The method used in this paper seems not to be directly applicable.

6. On Zograf's conjecture. In [35], Zograf devised a fast algorithm for computing Weil-Petersson volumes and conjectured the following large genus asymptotic expansion based on numerical experiments.

Conjecture 6.1 (Zograf [35]). *For any fixed* $n \geq 0$

$$V_{g,n} = (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{1}{\sqrt{g\pi}}\left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right)\right) \tag{59}$$

as $g \to \infty$, *where* c_n *is a constant depending only on* n.

Note that the asymptotic expansion of $V_{g,n}$ for fixed g and large n has been determined by Manin and Zograf [26].

Lemma 6.2 ([25, Lem. 3.3]). *Let* $n_1, n_2 \geq 0$. *Then*

$$\sum_{\substack{g_1+g_2=g \\ g_2 \geq g_1 \geq 0}} V_{g_1,n_1+1}V_{g_2,n_2+1} = O\left(\frac{V_{g,n}}{g}\right), \quad g \to \infty, \tag{60}$$

where $n = n_1 + n_2$.

Using the above key lemma, Mirzakhani proved the following large genus asymptotic formulae of $V_{g,n}$ which were also conjectured by Zograf [35].

THEOREM 6.3 ([25]). *Let $n \geq 0$. Then we have*

$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + O(1/g) \quad \text{and} \quad \frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g). \tag{61}$$

Moreover the asymptotic expansions of $V_{g,n+1}/(2gV_{g,n})$ and $V_{g,n}/V_{g-1,n+2}$ exist.

REMARK 6.4. Furthermore, Mirzakhani showed that there exists $M > 0$ independent of n such that

$$(4\pi^2)^{2g+n-3}(2g-3+n)!\frac{g^{-M}}{\sqrt{g\pi}} < V_{g,n} < (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{g^{M}}{\sqrt{g\pi}},$$

which is stronger than (4).

Mirzakhani also proved the following asymptotic relations for coefficients of the one-point volume polynomial.

THEOREM 6.5 ([25]). *For given $i \geq 0$.*

$$\lim_{g\to\infty} \frac{a_{g,i+1}}{a_{g,i}} = 1 \quad \text{and} \quad \lim_{g\to\infty} \frac{a_{g,3g-2}}{a_{g,0}} = 0. \tag{62}$$

Theorem 6.3 and Lemma 2.3 of §2 immediately imply the following conjecture of Zograf [35] giving large genus ratio of Weil-Peterson volumes and intersection numbers involving ψ-classes.

THEOREM 6.6. *For any fixed $n > 0$ and a fixed set $\mathbf{d} = (d_1, \cdots, d_n)$ of non-negative integers, we have*

$$\lim_{g\to\infty} \frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1. \tag{63}$$

Proof. We use induction on $|\mathbf{d}|$. We may assume

$$\lim_{g\to\infty} \frac{[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1. \tag{64}$$

So in order to prove (63), we need only prove that

$$\lim_{g\to\infty} \left| \frac{[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n} - [\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} \right| = 0. \tag{65}$$

By comparing each term in (8) for $[\tau_{d_1-1}\tau_{d_2} \cdots \tau_{d_n}]_{g,n}$ and $[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}$, this actually follows from (13), (7), Theorem 6.3 and Lemma 6.2. The argument is similar to the proof of Theorem 3.5 in [25]. We omit the details. □

REMARK 6.7. By Stirling formula

$$k! \sim \frac{\sqrt{2\pi}k^{k+\frac{1}{2}}}{e^k}, \quad k \to \infty,$$

when $n = 2$, Zograf's conjecture (59) is equivalent to

$$V_{g,2} \sim 2^{6g-3}\pi^{4g-3}((g-1)!)^2, \tag{66}$$

which suggests that a plausible way of proving Zograf's conjecture is to have a detailed study of asymptotic approximations of Equation (18) in relation to the first Painlevé equation, the asymptotic expansion of whose solutions had been studied in [11]. Another possible approach to Zograf's conjecture is to have a complete understanding of the asymptotics of integrals of ψ classes in view of Equation (49).

Very recently, Mirzakhani and Zograf [26] made a striking advancement on Conjecture 6.1. They proved that there exists a universal constant $0 < C < \infty$ such that for any given $k \geq 1, n \geq 0$,

$$V_{g,n} = C\frac{(4\pi^2)^{2g+n-3}(2g-3+n)!}{\sqrt{g}}\left(1+\frac{c_n^{(1)}}{g}+\cdots+\frac{c_n^{(k)}}{g^k}+O\left(\frac{1}{g^{k+1}}\right)\right), \tag{67}$$

where each term $c_n^{(i)}$ is a polynomial in n of degree $2i$.

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