ASIAN J. MATH.
Vol. 17, No. 2, pp. 321–334, June 2013

EXISTENCE OF COMPATIBLE CONTACT STRUCTURES ON G_2-MANIFOLDS*

M. FIRAT ARIKAN[†], HYUNJOO CHO[‡], AND SEMA SALUR[§]

Abstract. In this paper, we show the existence of (co-oriented) contact structures on certain classes of G_2-manifolds, and that these two structures are compatible in certain ways. Moreover, we prove that any seven-manifold with a spin structure (and so any manifold with G_2-structure) admits an almost contact structure. We also construct explicit almost contact metric structures on manifolds with G_2-structures.

Key words. (Almost) contact structures, G_2 structures.

AMS subject classifications. 53C38, 53D10, 53D15, 57R17.

1. Introduction. Let (M, g) be a Riemannian 7-manifold whose holonomy group $Hol(g)$ is the exceptional Lie group G_2 (or, more generally, a subgroup of G_2). Then M is naturally equipped with a covariantly constant 3-form φ and 4-form $*\varphi$. We can define (M, φ, g) as the G_2-manifold with G_2 structure φ.

We can also define a (co-oriented) contact manifold as a pair (N, ξ) where N is an odd-dimensional manifold and ξ, called a (co-oriented) contact structure, is a totally non-integrable (co-oriented) hyperplane distribution on N.

In dimension 7, so far contact geometry and G_2 geometry have been studied independently and each geometry has very distinguished characteristics which are rather different than those in the other. A basic example of such differences is the following: In contact geometry there are no local invariants, in other words, every contact 7-manifold is locally contactomorphic to $\mathbb{R}^7$ equipped with the standard contact structure. On the other hand, in G_2 geometry it is the G_2 structure itself that determines how local neighborhoods of points look like, and as a result, manifolds with G_2 structures can look the same only at a point, [7], [9].

The aim of this paper is to initiate a new interdisciplinary research area between contact and G_2 geometries. More precisely, we study the existence of (almost) contact structures on 7-dimensional manifolds with (torsion-free) G_2-structures.

The paper is organized as follows: After the preliminaries (Section 2), we show the existence of almost contact structures on 7-manifolds with spin structures in Section 3. In particular, we prove the following theorem:

THEOREM. *Every manifold with G_2-structure admits an almost contact structure.*

In Section 4, we define A- and B-compatibility between contact and G_2 structures, and also present the motivating example for $\mathbb{R}^7$. We also prove the nonexistence result:

THEOREM. *Let (M, φ) be a manifold with G_2-structure such that $d\varphi = 0$. If M is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on M which*

*Received February 9, 2012; accepted for publication April 3, 2012.
[†]Department of Mathematics, University of Rochester, Rochester NY, USA; Max Planck Institute for Mathematics, Bonn, Germany (arikan@math.rochester.edu; arikan@mpim-bonn.mpg.de). The first named author is partially supported by NSF FRG grant DMS-1065910.
[‡]Department of Mathematics, University of Rochester, Rochester NY, USA (cho@math.rochester.edu).
[§]Department of Mathematics, University of Rochester, Rochester NY, USA (salur@math.rochester.edu). The third named author is partially supported by NSF grant DMS-1105663.

is A-compatible with φ.

In Section 5, for any non-vanishing vector field R on a manifold M with G_2-structure φ, we construct explicit almost contact structure, denoted by $(J_R, R, \alpha_R, g_\varphi)$, and indeed prove the following theorems:

THEOREM. *Let (M, φ) be a manifold with G_2-structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on M for any non-vanishing vector field R on M. Moreover, such a structure exists on any manifold with G_2-structure.*

THEOREM. *Let (M, φ) be a manifold with G_2-structure. Suppose that ξ is a contact structure on M such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ. Then ξ is A-compatible.*

In Section 6, we define contact$-G_2-$structures on 7-manifolds and analyze their relations with A-compatible contact structures, the main results of that section are:

THEOREM. *Let (M, φ) be a manifold with G_2-structure. Assume that there are nowhere-zero vector fields X, Y and Z on M satisfying $\iota_Z\varphi = Y^\flat \wedge X^\flat$ where $X^\flat$ (resp. $Y^\flat$) is the covariant 1-form of X (resp. Y) with respect to the G_2-metric g_φ. Also suppose that $d(i_X i_Y \varphi) = i_X i_Y * \varphi$. Then the 1-form $\alpha := Z^\flat = g_\varphi(Z, \cdot)$ is a contact form on M and it defines an A-compatible contact structure* $\mathrm{Ker}(\alpha)$ *on (M, φ).*

THEOREM. *Let $(\varphi, R, \alpha, f, g)$ be a contact$-G_2-$structure on a smooth manifold M^7. Then α is a contact form on M. Moreover, $\xi = \mathrm{Ker}(\alpha)$ is an A-compatible contact structure on (M, φ). In particular, if M is closed, then it does not admit a contact$-G_2-$structure with $d\varphi = 0$.*

THEOREM. *Let (M, φ) be any manifold with G_2-structure. Then every A-compatible contact structure on (M, φ) determines a contact$-G_2-$structure on M.*

Finally, in Section 7, we present some examples of A-compatible structures and contact$-G_2-$structures.

2. Preliminaries.

2.1. G_2-structures and G_2-manifolds. A smooth 7-dimensional manifold M has a *G_2-structure*, if the structure group of TM can be reduced to G_2. The group G_2 is one of the five exceptional Lie groups which is the group of all linear automorphisms of the imaginary octonions $im\mathbb{O} \cong \mathbb{R}^7$ preserving a certain cross product. Equivalently, it can be defined as the subgroup of $GL(7, \mathbb{R})$ which preserves the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $(x_1, ..., x_7)$ are the coordinates on $\mathbb{R}^7$, and $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. As an equivalent definition, a manifold with a G_2-structure φ is a pair (M, φ), where φ is a 3-form on M, such that (T_pM, φ) is isomorphic to $(\mathbb{R}^7, \varphi_0)$ at every point p in M. Such a φ defines a Riemannian metric g_φ on M. We say φ is *torsion-free* if $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection of g_φ. A Riemannian manifold with a torsion-free G_2-structure is called a *G_2-manifold*. Equivalently, the pair (M, φ) is called a G_2-manifold if its holonomy group (with respect to g_φ) is a subgroup of G_2. As an another characterization, one can show that φ is torsion-free if and only if $d\varphi = d(*\varphi) = 0$ where "$*$" is the Hodge star operator defined by the metric g_φ.

The 3-form φ also determines the cross product and the orientation top (volume) form Vol on M. In fact, for any vector fields u, v, w on M, we have

$$\varphi(u, v, w) = g_\varphi(u \times v, w), \tag{1}$$

$$(\iota_u\varphi) \wedge (\iota_v\varphi) \wedge \varphi = 6g_\varphi(u, v) \text{ Vol}. \tag{2}$$

Also we will make use of the following formula as well:

$$u \times (u \times v) = -\|u\|^2 v + g_\varphi(u, v)u. \tag{3}$$

See [2], [3], [9] and [10] for more details on G_2 geometry.

2.2. Contact and almost contact structures. A *contact structure* on a smooth $(2n+1)$-dimensional manifold M is a global $2n$-plane field distribution ξ which is totally non-integrable. Non-integrability condition is equivalent to the fact that locally ξ can be given as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. If α is globally defined (in such a case, it is called a *contact form*), then one can define the *Reeb vector field* of α to be the unique global nowhere-zero vector field R on M satisfying the equations

$$\iota_R d\alpha = 0, \quad \alpha(R) = 1 \tag{4}$$

where "ι" denotes the interior product.

Using R, we can co-orient ξ and, as a result, the structure group of the tangent frame bundle can be reduced to $U(n) \times 1$. Such a reduction of the structure group is called an *almost contact structure* on M. Therefore, for the existence of a co-oriented contact structure on M, one should first ask the existence of an almost contact structure. We refer the reader to [1] and [7] for more on contact geometry.

Definition 2.1 ([8]). Let M^{2n+1} be a smooth manifold. If the structure group of its tangent bundles TM^{2n+1} reduces to $U(n) \times 1$, then M^{2n+1} is said to have *an almost contact structure.*

3. Almost contact structures on 7-manifolds with a spin structure. Although no explicit description is given, nevertheless the following result shows the existence of almost contact structures not only on manifolds with G_2-structures but also on a much wider family of 7-manifolds. Recall that if a manifold admits a spin structure, then its second Stiefel-Whitney class is zero.

Theorem 3.1. *Every 7-manifold with a spin structure admits an almost contact structure.*

Proof. Assume that M is a 7-manifold with a spin structure. By definition, M admits an almost contact structure if and only if the structure group of TM can be reduced to $U(3) \times 1$. Equivalently, the associated fiber bundle $TM[SO(7)/U(3)]$ with fiber $SO(7)/U(3)$ admits a cross-section [13]. If s is a cross section of fiber bundle over the the $(i-1)$-skeleton of M, then the cohomology class

$$o^i(s) \in H^i(M, \pi_{i-1}(SO(7)/U(3))$$

is the obstruction to extending s over the i-skeleton. Since we have

$$\pi_i(SO(7)/U(3)) = 0$$

unless $i = 2, 6$, the only obstructions to the existence of such a cross section arise in $H^i(M, \mathbb{Z})$ for $i = 3, 7$. In [11], Massey shows that these obstructions are the integral Stiefel-Whitney classes of the associated dimensions. Recall that the integral Stiefel-Whitney classes are defined as the images $\beta(w_i)$ of the Stiefel-Whitney classes under the Bockstein homomorphism. Here the Bockstein homomorphism is the connecting homomorphism $\beta : H^i(M, \mathbb{Z}/2\mathbb{Z}) \to H^{i+1}(M, \mathbb{Z})$ which arises from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Therefore, the obstructions o^3, o^7 to the existence of an almost contact structures on 7-manifolds are 2-torsion classes.

Now we know that $w_2(M) = 0$ (since M is spin), and hence the third integral Stiefel-Whitney class vanishes, i.e., $o^3 = W_3(M) = \beta(w_2) = 0$. Therefore, the only obstruction lies in the cohomology group $H^7(M)$.

We consider the following cases: First, if M is a closed manifold, then by Poincaré duality $H^7(M) \cong H_0(M) \cong \mathbb{Z}$ and hence the top-dimensional obstruction o^7 vanishes. Secondly, if M has a boundary, then (again by the duality) we have $o^7 \in H^7(M) \cong H_0(M, \partial M) \cong 0$. Now, if M is non-compact without a boundary, then the cohomology group $H^7(M) \cong (H^0_{cs}(M))^*$ where H_{cs} denotes the compactly supported cohomology. Hence, it is torsion-free. □

Since every manifold with G_2-structure is spin, we get

Corollary 3.2. *Every manifold with G_2-structure admits an almost contact structure.*

4. Compatibility and the motivating example. Assuming the existence of a contact structure on a manifold with a G_2-structure, we can also ask if and how these two structures are related. We define *two* different notions of *compatibility* between them as follows:

Definition 4.1. A (co-oriented) contact structure ξ on (M, φ) is said to be *A-compatible* with the G_2-structure φ if there exist a vector field R on M and a nonzero function $f : M \to \mathbb{R}$ such that $d\alpha = \iota_R\varphi$ for some contact form α for ξ and fR is the Reeb vector field of a contact form for ξ.

Definition 4.2. A (co-oriented) contact structure ξ on (M, φ) is said to be *B-compatible* with the G_2-structure φ if there are (global) vector fields X, Y on M such that $\alpha = \iota_Y\iota_X\varphi$ is a contact form for ξ.

In this paper, we will mainly consider A-compatible contact structures. We remark that if φ is torsion-free or at least $d\varphi = 0$, then Definition 4.1 makes sense only if M is noncompact or compact with boundary. Indeed, we can easily prove the following:

Theorem 4.3. *Let (M, φ) be a manifold with G_2-structure such that $d\varphi = 0$. If M is closed (i.e., compact and $\partial M = \emptyset$), then there is no contact structure on M which is A-compatible with φ.*

Proof. Suppose ξ is an A-compatible contact structure on (M, φ). Therefore, $d\alpha = \iota_R\varphi$ for some contact form α for ξ and some nonvanishing vector field R. Using the equation (2), we have

$$d\alpha \wedge d\alpha \wedge \varphi = (\iota_R\varphi) \wedge (\iota_R\varphi) \wedge \varphi = 6\|R\|^2 \text{ Vol}.$$

Since $d\varphi = 0$, we have $d\alpha \wedge d\alpha \wedge \varphi = d(\alpha \wedge d\alpha \wedge \varphi)$. Now by Stokes' Theorem,

$$0 \lneqq \int_M 6\|R\|^2 \text{ Vol} = \int_M d(\alpha \wedge d\alpha \wedge \varphi) = \int_{\partial M} \alpha \wedge d\alpha \wedge \varphi = 0$$

(as $\partial M = \emptyset$). This gives a contradiction. □

For another application of this argument on specific vector fields on manifolds with G_2 structures, see [5].

We now explore the relation between the standard contact structure ξ_0 and the standard G_2-structure φ_0 on $\mathbb{R}^7$. Indeed, the notion of A- and B-compatibility relies on this motivating example.

Fix the coordinates $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ on $\mathbb{R}^7$. In these coordinates,

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where e^{ijk} denotes the 3-form $dx_i \wedge dx_j \wedge dx_k$. Consider the standard contact structure ξ_0 on $\mathbb{R}^7$ as the kernel of the 1-form

$$\alpha_0 = dx_1 - x_3dx_2 - x_5dx_4 - x_7dx_6.$$

For simplicity, through out the paper we will denote $\partial/\partial x_i$ by ∂x_i (so we have $dx_i(\partial x_j) = \delta_{ij}$). Consider the vector fields

$$R = \partial x_1,\ X = \partial x_7 \text{ and } Y = -x_7\partial x_1 + x_5\partial x_3 - x_3\partial x_5 - \partial x_6 + f\partial x_7$$

where $f : \mathbb{R}^7 \to \mathbb{R}$ is any smooth function (in fact, it is enough to take $f \equiv 0$ for our purpose). By a straightforward computation, we see that

$$d\alpha_0 = \iota_R(\varphi_0), \quad \alpha_0 = \iota_Y\iota_X(\varphi_0).$$

Also observe that R is the Reeb vector field of α_0. Note that this contact structure is not unique A-compatible with φ_0. In fact we have various ways of choosing the contact structures by rotating indexes and signes. For example, the contact structure $\alpha = dx_2 + x_3dx_1 - x_6dx_4 + x_7dx_5$ with $R = \partial x_2$ is another A-compatible contact structure with φ_0 and by choosing two vectors $X = \partial x_7, Y = \partial x_5 - x_3\partial x_6 + x_6\partial x_3 - x_7\partial x_2 + f\partial x_7$ it is easily seen as being B-compatible with φ_0. Therefore, we have proved:

THEOREM 4.4. *There are contact structures ξ on $\mathbb{R}^7$ which are both A- and B-compatible with the standard G_2-structure φ_0.*

5. An explicit almost contact metric structure. We first give an alternative definition of an almost contact structure, and then construct an explicit almost contact structure on a manifold with G_2-structure. The reader is referred to [1] for the equivalence between the previous definition (Definition 2.1) and this new one.

DEFINITION 5.1 ([12]). An *almost contact structure* on a differentiable manifold M^{2n+1} is a triple (J, R, α) consists of a field J of endomorphisms of the tangent spaces, a vector field R, and a 1-form α satisfying

(i) $\alpha(R) = 1$,
(ii) $J^2 = -I + \alpha \otimes R$
where I denotes the identity transformation.

For completeness, we provide the proof of the following lemma.

LEMMA 5.2 ([12]). *Suppose that* (J, R, α) *is an almost contact structure on* M^{2n+1}. *Then* $J(R) = 0$ *and* $\alpha \circ J = 0$

Proof. Since $J^2(R) = -R + \alpha(R)R = -R + 1 \cdot R = 0$, we have either $J(R) = 0$ or $J(R)$ is nonzero vector field whose image is 0. Suppose $J(R)$ is nonzero vector field which is mapped to 0 by J. Then from

$$0 = J^2(J(R)) = -J(R) + \alpha(J(R)) \cdot R$$

we get $J(R) = \alpha(J(R)) \cdot R$, and so $\alpha(J(R)) \neq 0$ (as $J(R) \neq 0$). But then

$$J^2(R) = J(J(R)) = J(\alpha(J(R))R) = \alpha(J(R)) \cdot J(R) = [\alpha(J(R))]^2 \cdot R \neq 0$$

which contradicts to assumption that $J^2(R) = J(J(R)) = 0$. Hence, we conclude that $J(R) = 0$ must be the case.

Now for any vector X, we see that

$$J^3(X) = J(J^2(X)) = J((-X) + \alpha(X)R) = -J(X) + J(\alpha(X)R)$$

and also we have

$$J^3(X) = J^2(J(X)) = -J(X) + \alpha(J(X))R.$$

So combining these we compute

$$\begin{aligned}\alpha(J(X))R &= J^3(X) + J(X)\\ &= -J(X) + J(\alpha(X)R) + J(X) = J(\alpha(X)R).\end{aligned}$$

But using the fact $J(R) = 0$ we have

$$J(\alpha(X)R) = \alpha(X)J(R) = 0.$$

Therefore, $\alpha(J(X)) = 0$ as $R \neq 0$. Hence, $\alpha \circ J = 0$ for any vector X. □

We can also introduce a Riemannian metric into the picture as suggested in the following definition.

DEFINITION 5.3 ([12]). An *almost contact metric structure* on a differentiable manifold M^{2n+1} is a quadruple (J, R, α, g) where (J, R, α) is an almost contact structure on M and g is a Riemannian metric on M satisfying

$$g(Ju, Jv) = g(u, v) - \alpha(u)\alpha(v) \tag{5}$$

for all vector fields u, v in TM. Such a g is called a *compatible metric.*

REMARK 5.4. Every manifold with an almost contact structure admits a compatible metric (see [1], for a proof). Also setting $u = R$ in Equation (5) gives

$g(JR, Jv) = g(R, v) - \alpha(R)\alpha(v)$. Since $J(R) = 0$, an immediate consequence is that α is the covariant form of R, that is, $\alpha(v) = g(R, v)$.

DEFINITION 5.5 ([12]). Let M be an odd-dimensional manifold, and α be a contact form on M with the Reeb vector field R. Therefore, $d\alpha$ is a symplectic form on the contact structure (or distribution) $\xi = \mathrm{Ker}(\alpha)$. We say that the triple (J, R, α) is an *associated almost contact structure* for ξ if J is $d\alpha$-compatible almost complex structure on the complex bundle ξ, that is

$$d\alpha(JX, JY) = d\alpha(X, Y) \text{ and } d\alpha(X, JX) > 0 \text{ for all } X, Y \in \xi.$$

Furthermore, if g is a metric on M, we consider two equations:

$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y) \tag{6}$$

$$d\alpha(X, Y) = g(JX, Y) \tag{7}$$

for all $X, Y \in TM$. We say that (J, R, α, g) is an *associated almost contact metric structure* if two equations (6) and (7) hold. In this case, g is called an *associated metric.*

Suppose that (M, φ) is a manifold with G_2-structure. There might be many ways to construct almost contact metric structures on (M, φ). Here we give a particular way of constructing almost contact metric structures on (M, φ). Denote the Riemannian metric and the cross product (determined by φ) by $g_\varphi = \langle \cdot, \cdot \rangle_\varphi$ and $\times_\varphi$, respectively. Suppose that R is a nowhere vanishing vector field on M. By normalizing R using g_φ, we may assume that $\|R\| = 1$. Then using the metric, we define the 1-form α_R as the metric dual of R, that is,

$$\alpha_R(u) = g_\varphi(R, u) = \langle R, u \rangle_\varphi.$$

Moreover, using the cross product and R, we can define an endomorphism $J_R : TM \to TM$ of the tangent spaces by

$$J_R(u) = R \times_\varphi u.$$

Note that $J_R(R) = 0$, and so J_R, indeed, defines a complex structure on the orthogonal complement $R^\perp$ of R with respect to g_φ. With these, we have

THEOREM 5.6. *Let (M, φ) be a manifold with G_2-structure. Then the quadruple $(J_R, R, \alpha_R, g_\varphi)$ defines an almost contact metric structure on M for any non-vanishing vector field R on M. Moreover, such a structure exists on any manifold with G_2-structure.*

Proof. As before, we will assume that R is already normalized using g_φ. First, note that $\alpha_R(R) = g_\varphi(R, R) = \|R\|^2 = 1$. Also we have

$$J_R^2(u) = J_R(R \times_\varphi u) = R \times_\varphi (R \times_\varphi u) = -\|R\|^2 u + g_\varphi(R, u)R = -u + \alpha(u)R$$

where we made use of the identity (3). This shows that the endomorphism $J_R : TM \to TM$ satisfies the condition

$$J_R^2 = -I + \alpha \otimes R.$$

Therefore, the triple (J_R, R, α_R) is an almost contact structure on M. Next, we check g_φ is a compatible metric with this structure. Using (1) and (3), we compute

$$\begin{aligned} g_\varphi(J_R u, J_R v) &= g_\varphi(R \times_\varphi u, R \times_\varphi v) = \varphi(R, u, R \times_\varphi v) = -\varphi(R, R \times_\varphi v, u) \\ &= -g_\varphi(R \times_\varphi (R \times_\varphi v), u) = -g_\varphi(-\|R\|^2 v + g_\varphi(R, v)R, u) \\ &= -g_\varphi(-v + g_\varphi(R, v)R, u) = g_\varphi(v, u) - g_\varphi(\alpha_R(v)R, u) \\ &= = g_\varphi(u, v) - \alpha_R(v) \underbrace{g_\varphi(R, u)}_{\alpha_R(u)} = g_\varphi(u, v) - \alpha_R(u)\alpha_R(v) \end{aligned}$$

which holds for all vector fields u, v in TM. This proves that g_φ satisfies (5). Hence, $(J_R, R, \alpha_R, g_\varphi)$ is an almost contact metric structure on M.

For the last statement, we know that there always exists a nowhere vanishing vector field R on any 7-dimensional manifold. In particular, $(J_R, R, \alpha_R, g_\varphi)$ can be constructed on any manifold M with G_2-structure φ. □

THEOREM 5.7. *Let (M, φ) be a manifold with G_2-structure, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on M constructed as above. Suppose that ξ is a contact structure on M such that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ. Then ξ is A-compatible.*

Proof. By assumption $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure for ξ. Therefore, g_φ is an associated metric and satisfies

$$d\alpha_R(u, v) = g_\varphi(J_R(u), v) \quad \text{for all} \quad u, v \in TM.$$

But then using the equation defining J_R and (1), we obtain

$$d\alpha_R(u, v) = g_\varphi(R \times_\varphi u, v) = \varphi(R, u, v) = i_R\varphi(u, v), \quad \forall u, v \in TM.$$

Therefore, we have $d\alpha_R = i_R\varphi$. Also R is the Reeb vector field of α_R by assumption. Hence, ξ is A-compatible by definition. □

COROLLARY 5.8. *Let (M, φ) be a manifold with G_2-structure such that $d\varphi = 0$, and $(J_R, R, \alpha_R, g_\varphi)$ be an almost contact metric structure on M constructed as above. If M is closed, then there is no contact structure on M whose associated almost contact metric structure is $(J_R, R, \alpha_R, g_\varphi)$.*

Proof. On the contrary, suppose that $\xi = \mathrm{Ker}(\alpha_R)$ is a contact structure on a closed manifold M equipped with a G_2-structure φ and $d\varphi = 0$, and also that $(J_R, R, \alpha_R, g_\varphi)$ is an associated almost contact metric structure. Then, by Theorem 5.7, ξ is A-compatible, but this contradicts to Theorem 4.3. □

6. Contact$-G_2-$structures on 7-manifolds. Suppose that (M, φ) is a manifold with G_2-structure. Let us recall the decomposition of the space Λ^2 of 2-forms on M obtained from G_2-representation and some other useful formulas which we will use. A good source for these is [2] and also [9]. According to irreducible G_2-representation, $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$ where

$$\begin{aligned} \Lambda^2_7 &= \{i_v\varphi; v \in \Gamma(TM)\} \\ &= \{\beta \in \Lambda^2; *(\varphi \wedge \beta) = -2\beta\} \\ &= \{\beta \in \Lambda^2; *(*\varphi \wedge (*(*\varphi \wedge \beta))) = 3\beta\} \end{aligned} \tag{8}$$

$$\begin{aligned} \Lambda^2_{14} &= \{\beta \in \Lambda^2; *\varphi \wedge \beta = 0\} \\ &= \{\beta \in \Lambda^2; *(\varphi \wedge \beta) = \beta\} \end{aligned}$$

Also on any Riemannian n-manifold, for any k-form α and a vector field v, the following equalities hold:

$$i_v * \alpha = (-1)^k * (v^\flat \wedge \alpha) \quad \text{and} \tag{9}$$

$$i_v \alpha = (-1)^{nk+n} * (v^\flat \wedge *\alpha). \tag{10}$$

As a last one we recall a very useful equality: For any k-form λ, and any $(n+1-k)$-form μ and any vector field v on a smooth manifold of dimension n, we have

$$(\iota_v \lambda) \wedge \mu = (-1)^{k+1} \lambda \wedge (\iota_v \mu). \tag{11}$$

Now we are ready to prove:

THEOREM 6.1. *Let (M, φ) be a manifold with G_2-structure. Assume that there are nowhere-zero vector fields X, Y and Z on M satisfying*

$$\iota_Z \varphi = Y^\flat \wedge X^\flat \tag{12}$$

where $X^\flat$ (resp. $Y^\flat$) is the covariant 1-form of X (resp. Y) with respect to the G_2-metric g_φ. Also suppose that

$$d(i_X i_Y \varphi) = i_X i_Y * \varphi. \tag{13}$$

Then the 1-form $\alpha := Z^\flat = g_\varphi(Z, \cdot)$ is a contact form on M and it defines an A-compatible contact structure $\mathrm{Ker}(\alpha)$ *on (M, φ).*

Proof. From (8) we know that $\iota_Z \varphi$ is an element of Λ^2_7. Set $\iota_Z \varphi = \beta \in \Lambda^2_7$, and so we have $\iota_Z \varphi = \beta = Y^\flat \wedge X^\flat$ by (12). Also applying (9) twice gives

$$i_X i_Y * \varphi = -i_X(*(Y^\flat \wedge \varphi)) = - * (X^\flat \wedge Y^\flat \wedge \varphi) = *(Y^\flat \wedge X^\flat \wedge \varphi) = *(\beta \wedge \varphi)$$

from which we get

$$i_X i_Y * \varphi = -2\beta \tag{14}$$

where we use the second line in (8). Moreover, by (10) followed by (9),

$$i_X i_Y \varphi = i_X(*(Y^\flat \wedge *\varphi)) = - * (X^\flat \wedge Y^\flat \wedge *\varphi) = *(\beta \wedge *\varphi). \tag{15}$$

Now putting (14) and (15) into (13) gives us

$$d * (\beta \wedge *\varphi) = -2\beta = -2\, \iota_Z \varphi. \tag{16}$$

Recall the formula $(i_v \varphi) \wedge *\varphi = 3 * v^\flat$ which is true for any vector field v. By taking $v = Z$, we compute the left-hand side in (16) as

$$d * (\beta \wedge *\varphi) = d * (3 * Z^\flat) = 3\, dZ^\flat = 3\, d\alpha.$$

Combining these together we obtain

$$d\alpha = -\frac{2}{3}\, \iota_Z \varphi. \tag{17}$$

Next, consider the identity (11) by taking $\lambda = \varphi$, $v = Z$ and $\mu = \alpha \wedge (d\alpha)^2$: Using (17), we compute the left-hand side as

$$(\iota_Z\varphi) \wedge \alpha \wedge (d\alpha)^2 = -\frac{3}{2}\alpha \wedge (d\alpha)^3,$$

and the right-hand side as

$$\varphi \wedge \iota_Z(\alpha \wedge (d\alpha)^2) = \alpha(Z)\, \varphi \wedge d\alpha \wedge d\alpha = \frac{4}{9}\|Z\|^2\, \varphi \wedge (\iota_Z\varphi) \wedge (\iota_Z\varphi).$$

Therefore, by using the identity (2) in the right-hand side, we obtain

$$\alpha \wedge (d\alpha)^3 = -\frac{16}{9}\|Z\|^4 \text{ Vol}.$$

Hence, we conclude that $\alpha\wedge(d\alpha)^3$ is a volume form on M (as being a nonzero function multiple of the volume form Vol on M determined by the metric g_φ). Equivalently, α is a contact form on M. Moreover, it follows from (17) that $(1/\|Z\|^2)Z$ is the Reeb vector field of α, i.e., it satisfies (4). Hence, Ker(α) is an A-compatible contact structure on (M, φ). □

With the inspiration we get from the proof of Theorem 6.1, we define a new structure on 7-manifolds as follows:

Definition 6.2. Let M^7 be a smooth manifold. A *contact–G_2–structure* on M is a quintuple $(\varphi, R, \alpha, f, g)$ where φ is a G_2-structure, R is a nowhere-zero vector field, α is a 1-form on M, and $f, g : M \to \mathbb{R}$ are nowhere-zero smooth functions such that

(i) $\alpha(R) = f$
(ii) $d(g\,\alpha) = \iota_R\varphi$.

Observe that we have already seen an example of a contact–G_2–structure in the above proof (of course under the assumptions of Theorem 6.1) with $R = Z, \alpha = Z^\flat, f = \|Z\|^2, g \equiv -3/2$. The reason why we call the quintuple $(\varphi, R, \alpha, f, g)$ "contact–G_2–structure" is given by the following theorem.

Theorem 6.3. *Let $(\varphi, R, \alpha, f, g)$ be a contact–G_2–structure on a smooth manifold M^7. Then α is a contact form on M. Moreover, $\xi =$Ker(α) is an A-compatible contact structure on (M, φ). In particular, if M is closed, then it does not admit a contact–G_2–structure with $d\varphi = 0$.*

Proof. We first show that α is a contact form on M. Consider the 1-form

$$\alpha' := g\,\alpha.$$

Note that Ker(α) = Ker(α') as g is a nowhere-zero function. Therefore, if we show that α' is a contact form on M, then it will imply that so is α. The conditions in Definition 6.2 translate into

$$\alpha'(R) = fg \quad \text{and} \quad d\alpha' = \iota_R\varphi.$$

Also from the equation (2) we get

$$(d\alpha')^2 \wedge \varphi = (\iota_R\varphi) \wedge (\iota_R\varphi) \wedge \varphi = 6\|R\|^2 \text{ Vol}.$$

Now if we write the equation (11) by taking $\lambda = \varphi, \mu = \alpha' \wedge (d\alpha')^2$ and $v = R$, then the left-hand side gives

$$(\iota_R \varphi) \wedge \alpha' \wedge (d\alpha')^2 = (d\alpha') \wedge \alpha' \wedge (d\alpha')^2 = \alpha' \wedge (d\alpha')^3,$$

and from the right-hand side we have

$$\varphi \wedge \iota_R(\alpha' \wedge (d\alpha')^2) = \alpha'(R)\, \varphi \wedge (d\alpha')^2 = fg\, \varphi \wedge (d\alpha')^2 = 6\, fg\|R\|^2 \text{ Vol}.$$

Therefore, we conclude

$$\alpha' \wedge (d\alpha')^3 = 6\, fg\|R\|^2 \text{ Vol}$$

which implies that α' (and so α) is a contact form on M as $6\, fg\|R\|^2$ is a nowhere-zero function on M.

Next, we consider the vector field $R' = (1/fg)R$. Clearly, $\alpha'(R') = 1$. Also we compute

$$\iota_{R'} d\alpha' = (1/fg)\, \iota_R d\alpha' = (1/fg)\, \iota_R(\iota_R \varphi) = 0$$

as φ is skew-symmetric. Therefore, R' is the Reeb vector field of α', and so $\xi = \text{Ker}(\alpha')$ is an A-compatible contact structure on (M, φ) by definition. Finally, the last statement now follows from Theorem 4.3. □

The next result shows that we can go also in the reverse direction.

THEOREM 6.4. *Let (M, φ) be any manifold with G_2-structure. Then every A-compatible contact structure on (M, φ) determines a contact$-G_2-$structure on M.*

Proof. Let ξ be a given A-compatible contact structure on (M, φ). By definition, there exist a non-vanishing vector field R on M, a contact form α for ξ and a nowhere-zero function $h : M \to \mathbb{R}$ such that $d\alpha = \iota_R \varphi$ and hR is the Reeb vector field of some contact form (possibly different than α) for ξ. Being a Reeb vector field, hR is transverse to the contact distribution ξ. Therefore, R is also transverse to ξ because h is nowhere-zero on M. As a result, there must be a nowhere-zero function $f : M \to \mathbb{R}$ such that

$$\alpha(R) = f.$$

To check this, assume, on the contrary, that the function $M \to \mathbb{R}$ given by $x \mapsto \alpha_x(R_x)$ has a zero, say at p. So, we have $\alpha_p(R_p) = 0$ which means that $R_p \in \text{Ker}(\alpha_p) = \xi_p$. But this contradicts to the fact that R is everywhere transverse to ξ. Hence, we obtain a contact$-G_2-$structure $(\varphi, R, \alpha, f, 1)$. This finishes the proof. □

7. Some examples. In this final section, we give some examples of G_2-manifolds admitting A-compatible contact structures. In fact, by Theorem 6.4, in each example we will also have a corresponding contact$-G_2-$structure.

7.1. $CY \times S^1$ (or $CY \times \mathbb{R}$). Consider a well-known example of G_2-manifold $(CY \times S^1, \varphi)$ where we assume $CY(\Omega, \omega)$ is a 3-fold Calabi-Yau manifold which is either noncompact or compact with boundary. Assume Kähler form ω on CY is exact, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(CY)$ and set $\alpha = dt + \lambda$ where t is the coordinate on S^1. Then $\alpha \wedge (d\alpha)^3 = \omega^3 \wedge dt$ is a volume form, and so α is a contact 1-form on $CY \times S^1$. Moreover, ∂t is the Reeb vector field of α as $\iota_{\partial t} \alpha = 1$ and $\iota_{\partial t} d\alpha = \iota_{\partial t} \omega = 0$. Also observe that since $\varphi = Re(\Omega) + \omega \wedge dt$ (see [10], for instance), we compute

$\iota_{\partial t}\varphi = \iota_{\partial t}(Re(\Omega) + \omega \wedge dt) = \iota_{\partial t}Re(\Omega) + \iota_{\partial t}(\omega \wedge dt) = \omega \iota_{\partial t}dt = \omega = d\lambda = d\alpha.$

Thus, $\xi = \mathrm{Ker}(\alpha)$ is an A-compatible contact structure on $(CY \times S^1, \varphi)$, or in other words, $(\varphi, \partial t, \alpha, 1, 1)$ is a contact$-G_2-$structure on $CY \times S^1$. We note that, by considering t as a coordinate on $\mathbb{R}$, the above argument also gives a contact$-G_2-$structure on $CY \times \mathbb{R}$.

7.2. $W \times S^1$ **(or** $W \times \mathbb{R}$**).** We now give a special case of the above example. First, we need some definitions: A *Stein manifold* of complex dimension n is a triple (W^{2n}, J, ψ) where J is a complex structure on W and $\psi : W \to \mathbb{R}$ is a smooth map such that the 2$-$form $\omega_\psi = -d(d\psi \circ J)$ is non-degenerate (and so an exact symplectic form) on W. Indeed, (W, J, ω_ψ) is an exact Kähler manifold. We say that (M^{2n-1}, ξ) is *Stein fillable* if there is a Stein manifold (W^{2n}, J, ψ) such that ψ is bounded from below, M is a non-critical level of ψ, and $-(d\psi \circ J)$ is a contact form for ξ.

Next, consider a parallelizable Stein manifold (W, J, ψ) of complex dimension three. By a result of [6], we know that $c_1(W, J) = 0$, i.e., the first Chern class of (W, J) vanishes. Therefore, W admits a Calabi-Yau structure with associated Kähler form $\omega_\psi = -d(d\psi \circ J)$. Let Ω be the non-vanishing holomorphic 3-form on W corresponding to this Calabi-Yau structure. Then by the previous example, $(W \times S^1, \varphi)$ is a G_2-manifold with $\varphi = Re(\Omega) + \omega_\psi \wedge d\theta$ (where θ is the coordinate on S^1), $\alpha = d\theta - (d\psi \circ J)$ is a contact 1-form on $W \times S^1$ with the Reeb vector field $\partial\theta$, and $\xi = \mathrm{Ker}(\alpha)$ is an A-compatible contact structure on $(W \times S^1, \varphi)$. Again by considering θ as a coordinate on $\mathbb{R}$, we obtain an A-compatible contact structure on $(W \times \mathbb{R}, \varphi)$. Note that the corresponding contact$-G_2-$structure in both cases is $(\varphi, \partial\theta, \alpha, 1, 1)$.

Now consider the unit disk $\mathbb{D}^2 \subset \mathbb{C}$. Then $(W \times \mathbb{D}^2, J \times i, \psi + |z|^2)$ is a Stein manifold where i is the usual complex structure and $z = re^{i\theta}$ is the coordinate on $\mathbb{C}$. Let η be the induced contact structure on the boundary

$$\partial(W \times \mathbb{D}^2) = (\partial W \times \mathbb{D}^2) \cup (W \times S^1).$$

Then we remark that the restriction of the Stein fillable structure η on $W \times S^1$ is the contact structure ξ constructed above.

7.3. $\mathbb{R}^3 \times K^4$**.** Let K be a Kähler manifold with an exact Kähler form ω, i.e. $\omega = d\lambda$ for some $\lambda \in \Omega^1(K)$. Note that K is either noncompact or compact with boundary. Consider the G_2-manifold $\mathbb{R}^3 \times K^4$ with the G_2-structure

$$\varphi = dx_1 dx_2 dx_3 + \omega \wedge dx_1 + Re(\Omega) \wedge dx_2 - Im(\Omega) \wedge dx_3$$

where (x_1, x_2, x_3) are the coordinates on $\mathbb{R}^3$ (see [10]). Then $\alpha = dx_1 + x_2 dx_3 + \lambda$ is a contact 1-form as $\alpha \wedge (d\alpha)^3 = dx_1 dx_2 dx_3 \wedge \omega^2$ is a volume form on $\mathbb{R}^3 \times K^4$. One can easily check that ∂x_1 is the Reeb vector field of α. Furthermore,

$$\begin{aligned} i_{\partial x_1}\varphi &= i_{\partial x_1}(dx_1 dx_2 dx_3 + \omega \wedge dx_1 + \omega \wedge dx_2 + \omega \wedge dx_3) \\ &= dx_2 dx_3 + i_{\partial x_1}(\omega \wedge dx_1) = dx_2 dx_3 + \omega = d(x_2 dx_3 + \lambda) = d\alpha \end{aligned}$$

Hence, $\xi = \mathrm{Ker}(\alpha)$ is an A-compatible contact structure on $(\mathbb{R}^3 \times K^4, \varphi)$ and the corresponding contact$-G_2-$structure on $\mathbb{R}^3 \times K^4$ is $(\varphi, \partial x_1, \alpha, 1, 1)$.

7.4. $T^*M^3 \times \mathbb{R}$**.** Let M be any oriented Riemannian 3-manifold and T^*M denote the cotangent bundle of M. It is shown in [4] that $T^*M \times \mathbb{R}$ has a G_2-structure φ with $d\varphi = 0$. To describe φ, let (x_1, x_2, x_3) be local coordinates on M around a given point, and consider the corresponding standard local coordinates $(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$ on the cotangent bundle T^*M. These define the standard symplectic structure $\omega = -d\lambda$ on T^*M where $\lambda = \Sigma_{i=1}^{3} \xi_i dx_i$ is the tautological 1-form on T^*M. Let t denote the coordinate on $\mathbb{R}$. Then $\varphi = Re(\Omega) - \omega \wedge dt$ where $\Omega = (dx_1 + id\xi_1) \wedge (dx_2 + id\xi_2) \wedge (dx_3 + id\xi_3)$ is the complex-valued $(3, 0)$-form on M. On the other hand, the 1-form $\alpha = dt + \lambda$ is a contact form on $T^*M \times \mathbb{R}$ with the Reeb vector field ∂t. Now it is straightforward to check that $\xi = \mathrm{Ker}(\alpha)$ is an A-compatible contact structure on $(T^*M \times \mathbb{R}, \varphi)$ and also that $(\varphi, \partial t, \alpha, 1, 1)$ is the corresponding contact$-G_2-$structure on $T^*M \times \mathbb{R}$.

REFERENCES

[1] D. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics 509, Springer-Verlag, 1976.

[2] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math., 126 (1987), pp. 525–576.

[3] R. L. Bryant, *On the construction of some complete metrics with exceptional holonomy*, Duke math. J., 58:3 (1989), pp. 829–850.

[4] H. Cho, S. Salur, and A. J. Todd, *A note on closed G_2-structures and 3-manifolds*, arXiv:1112.0830v1.

[5] H. Cho, S. Salur, and A. J. Todd, *Diffeomorphisms of 7-Manifolds with Closed G_2-Structure* , arXiv:1112.0832v1.

[6] O. Forster, *Some remarks on parallelizable Stein manifolds*, Bull. Amer . Math. Soc., 73:5 (1967), pp. 712–716.

[7] H. Geiges, *Contact geometry*, Handbook of differential geometry. Vol. II, Elsevier/North-Holland, Amsterdam, (2006), pp. 315–382.

[8] J. W. Gray, *Some global properties of contact structures*, Ann. of Math., 69:2 (1959), pp. 421–450.

[9] F. R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta. Math., 148 (1982), pp. 47–157.

[10] D. Joyce, *The exceptional holonomy groups and calibrated geometry*, Proceedings of 11^{th} Gokova Geometry-Topology Conference, (2003), pp. 1–29.

[11] W. Massey, *Obstructions to the existence of almost compact structures*, Bull. Amer. Math. Soc., 67 (1961), pp. 559–564.

[12] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure I*, Tôhoku Math. J., 12 (1960), pp. 459–476.

[13] N. Steenrod, *The topology of fiber bundles*, Princeton Universtiy Press, 1951.