COHOMOGENEITY ONE SHRINKING RICCI SOLITONS: AN ANALYTIC AND NUMERICAL STUDY*

ANDREW S. DANCER†, STUART J. HALL‡, AND MCKENZIE Y. WANG§

Abstract. We use analytical and numerical methods to investigate the equations for cohomogeneity one shrinking gradient Ricci solitons. We show the existence of a winding number for this system around the subvariety of phase space corresponding to Einstein solutions and obtain some estimates for it. We prove a non-existence result for certain orbit types, analogous to that of Böhm in the Einstein case. We also carry out numerical investigations for selected orbit types.

Key words. Gradient Ricci solitons, shrinkers, winding number, non-existence, numerics.

AMS subject classifications. 53C25, 53C44.

0. Introduction. A Ricci soliton consists of a complete Riemannian metric g and a complete vector field X on a manifold which satisfy the equation:

$$(0.1) \qquad \mathrm{Ric}(g) + \frac{1}{2}\,\mathsf{L}_X g + \frac{\epsilon}{2}\,g = 0$$

where ϵ is a real constant and L denotes the Lie derivative. The soliton is called *steady* if $\epsilon = 0$, *expanding* if $\epsilon > 0$, and *shrinking* if $\epsilon < 0$. If X is a Killing vector field, the metric is actually Einstein, and the soliton is called *trivial*: we shall exclude these in what follows. For general background, see [Cetc] and [Ca2].

As well as being natural generalisations of Einstein metrics, Ricci solitons are of interest as generating particularly simple solutions to the Ricci flow, for they evolve under this flow just by its natural symmetries, that is, by diffeomorphisms and homothetic rescalings. Ricci solitons are of crucial importance in analysing, via dilations, singularities of the flow [Pe]. The Ricci flow is also of physical interest as an approximation to the renormalisation group flow for nonlinear sigma-models [F]. Indeed one of the earliest nontrivial Ricci solitons, the Hamilton cigar [Ha], was also discovered in a physical context by Witten [Wi].

Most known examples of Ricci solitons are of *gradient type*, that is, we have $X = \mathrm{grad}\, u$ for a smooth function u. Equation (0.1) then becomes

$$(0.2) \qquad \mathrm{Ric}(g) + \mathrm{Hess}(u) + \frac{\epsilon}{2}\,g = 0.$$

We shall use the abbreviation GRS for gradient Ricci solitons. In this case, the completeness of grad u actually follows from the completeness of g, by the recent work of Z. Zhang [Zh].

Noncompact complete solitons may be steady, expanding or shrinking, but nontrivial compact solitons must be shrinking and of gradient type [Pe]. In all cases Kähler examples have been constructed (see [Koi], [Ca1], [ACGT], [FIK], [DW1], [FuW] for example, among many other authors). In the expanding and steady cases

†Jesus College, Oxford University, OX1 3DW, United Kingdom (dancer@maths.ox.ac.uk).

‡Department of Applied Computing, University of Buckingham, Hunter Street, MK18 1EG, United Kingdom (stuart.hall@buckingham.ac.uk).

§Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada (wang@mcmaster.ca).

non-Kähler examples were constructed in [DW2] , [DW3], generalising examples of Bryant [Bry], Ivey [Iv], Gastel and Kronz [GK]. There are also many homogeneous non-Kähler expanders (necessarily of non-gradient type), constructed by Lauret [La] and others.

In contrast, so far the only known complete nontrivial shrinking solitons are Kähler, or else are rigid (i.e. obtained from a product of Einstein spaces and a Gaussian soliton [PW]). In this paper we begin a study of shrinking solitons which are not necessarily of Kähler type. We shall focus on solitons of cohomogeneity one, when the existence of a symmetry group with hypersurface principal orbits (or a suitable bundle ansatz) reduces the equations to a system of ordinary differential equations. This is a natural place to start, as homogeneous compact solitons are trivial. Moreover, the great majority of known complete nontrivial solitons are either homogeneous (in the case of Lauret's noncompact expanders) or obtained by a cohomogeneity one/bundle-type construction. (The main exceptions we are aware of are the existence result [WZh] for Kähler-Ricci solitons on toric varieties, and the generalisation [PS] to Fano toric bundles over a generalised flag variety).

In §1,2 we show that the general cohomogeneity one shrinking soliton equations yield a winding number, representing the winding of the flow around the subvariety of phase space corresponding to Einstein solutions. We obtain some estimates for the winding. We also introduce a functional which is monotone in a certain region of phase space. In §3 we prove a nonexistence result for compact cohomogeneity one solitons with certain orbit types, analogous to a result of Böhm in the Einstein case [B3]. In §4 we focus on solitons of multiple warped product type. In §5 we report on numerical investigations for some orbit types, including those where solutions were found in the Einstein case [B1]. The numerics indicate that the shrinking soliton equations on compact spaces exhibit a high degree of rigidity; in particular we discuss several spaces where Einstein metrics exist [B1] but where numerical searches did not produce any evidence of nontrivial solitons.

1. Cohomogeneity one Ricci soliton equations and a general winding number. Let us consider a cohomogeneity one gradient Ricci soliton: this means that the soliton is invariant under the action of a Lie group G with generic orbit (called the principal orbit type) of real codimension one. We write the principal orbit as G/K, where G is a compact Lie group and K is a closed subgroup. We refer to [DW1] for background on the soliton equations in the cohomogeneity one setting.

We denote by n the dimension of G/K, so the cohomogeneity one manifold M has dimension $n+1$. The open and dense subset M_0 of the manifold consisting of the principal orbits may be written as $I \times (G/K)$, where I is an open interval in $\mathbb{R}$ and the product structure results from choosing a geodesic which intersects all generic orbits orthogonally. On M_0 we may then write the soliton potential as $u(t)$ and the metric as

$$(1.1) \qquad\qquad\qquad \bar{g} := dt^2 + g_t$$

where, for each t, g_t is a G-invariant metric on G/K. If we let $\mathfrak{p}$ be an Ad_K-invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$, then the G-invariant metrics on G/K correspond to Ad_K-invariant inner products on $\mathfrak{p}$.

As described in [DW1] the cohomogeneity one gradient Ricci soliton equations on $I \times (G/K)$ are formulated in terms of endomorphisms of $\mathfrak{p}$. Recall that the shape operator L of the hypersurface orbits is given by $L(X) := \overline{\nabla}_X \frac{\partial}{\partial t}$ and is related to the metric by $\dot{g}_t = 2g_t L_t$. (Note that L_t is in general self-adjoint only with respect to g_t.) The soliton equations are then given by

$$(1.2) \qquad r_t - \dot{L} + (\dot{u} - \mathrm{tr}L)L + \frac{\epsilon}{2}\mathbb{I} = 0,$$

$$(1.3) \qquad -\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) + \ddot{u} + \frac{\epsilon}{2} = 0,$$

$$(1.4) \qquad d(\mathrm{tr}L) + \delta^\nabla L = 0,$$

where r_t denotes the Ricci endomorphism of g_t, and δ^∇ is the codifferential for $T^*(G/K)$-valued 1-forms. Note that the first equation represents the components of the equations tangent to G/K, the second is the equation in the normal direction, and the last equation represents the mixed directions.

The cohomogeneity one manifold can have up to two special orbits, corresponding to the endpoints of the interval I. In this paper we will assume that there is at least one special orbit, placed at $t = 0$. We may then assume that $I = (0, T)$ where T is an extended positive real number. If $T < +\infty$, M is compact and additional conditions must be imposed at 0 and T to ensure that the metric $\bar{g}$ and potential u extend smoothly to all of M. When there is only one special orbit, the cohomogeneity one manifold is non-compact, and there are further conditions which ensure that $\bar{g}$ is complete. (See [EW] and [DW1] for more details of the smoothness conditions.) Here we just like to highlight one result from [DW1] (cf Prop 3.19) which has practical consequences for the construction of gradient Ricci solitons.

Assume that there is a special orbit of dimension strictly smaller than n. Then if (1.2) holds for a sufficiently smooth (C^3) metric and potential, then (1.4) automatically holds. Furthermore, if the well-known *conservation law*

$$(1.5) \qquad \ddot{u} + (-\dot{u} + \mathrm{tr}\,L)\dot{u} - \epsilon u = C$$

holds for some fixed constant C, then (1.3) also holds. In other words, in constructing cohomogeneity one gradient Ricci solitons, it is enough to produce a solution to (1.2) satisfying the conservation law which is smooth up to order 3. Note that in [EW] examples of principal and special orbits are given for which the smoothness conditions for $\bar{g}$ have to be checked to arbitrarily high orders.

Using (1.3) and the trace of (1.2), the conservation law above can be written in the form

$$(1.6) \qquad S + \mathrm{tr}(L^2) - (-\dot{u} + \mathrm{tr}\,L)^2 + (n-1)\frac{\epsilon}{2} = C + \epsilon u,$$

where S is the scalar curvature of the principal orbits and the constant C is the same one as in (1.5). We note that this form of the conservation law may be viewed as the zero energy condition for a Hamiltonian constructed out of Perelman's energy and entropy functionals.

Two quantities appearing in (1.6) are of interest to us.
The first is

$$(1.7) \qquad \xi := -\dot{u} + \mathrm{tr}\,L,$$

which is the analogue of the mean curvature $\mathrm{tr}L$ of the principal orbits for the dilaton volume element $e^{-u}d\mu_{\bar{g}}$. By (1.3), ξ is strictly decreasing in the shrinking case ($\epsilon < 0$). Indeed we have the useful inequality

$$(1.8) \qquad\qquad \dot{\xi} \leq \frac{\epsilon}{2} < 0.$$

Furthermore, when there is a special orbit with codimension $k+1 \geq 2$ placed at $t = 0$, the smoothness conditions imply that $\dot{u}(0) = 0$ and $\mathrm{tr}\,L = \frac{k}{t} + O(t)$ as $t \to 0$. In the complete, noncompact case, the two facts above show that ξ tends to $+\infty$ as t tends to 0 and ξ tends to $-\infty$ as t tends to $+\infty$. In the compact case, an analogous calculation to that above shows that ξ tends to $-\infty$ as we approach the second special orbit at $t = T$. Hence in the shrinking case, if the metric is complete, ξ must decrease from $+\infty$ to $-\infty$ regardless of whether M is compact or non-compact.

Denote by t_* the unique zero of ξ, and introduce, as in [DW2], [DW3], the variable

$$(1.9) \qquad\qquad W := \xi^{-1} = \frac{1}{-\dot{u} + \mathrm{tr}L}$$

as well as a new independent variable s defined by

$$(1.10) \qquad\qquad \frac{d}{ds} := W\,\frac{d}{dt} = \frac{1}{(-\dot{u} + \mathrm{tr}L)}\,\frac{d}{dt}.$$

We shall use a prime $'$ to denote differentiation with respect to s. It is natural to let $s = -\infty$ correspond to $t = 0$. The zero t_* of ξ corresponds to the blow-up time of W, which occurs at a *finite* value s_* of s. To see this, note that equation (1.3) can be rewritten as

$$\dot{\xi} = -\mathrm{tr}(L^2) + \frac{\epsilon}{2},$$

or in terms of W as

$$(1.11) \qquad\qquad W' = W^3\left(\mathrm{tr}(L^2) - \frac{\epsilon}{2}\right).$$

Since $\epsilon < 0$, we have $W' \geq -\frac{\epsilon}{2}W^3$, which shows that W blows up in finite time.

We shall call a point where W blows up (or equivalently, where the generalised mean curvature ξ vanishes), a *turning point*. For the special case of Einstein trajectories, where u is constant, this reduces to Böhm's notion of turning point, that is, a point where the hypersurface becomes minimal [B1]. Note however that a turning point is not a singularity of the metric, just a coordinate singularity. One might hope to construct compact shrinkers by glueing together two trajectories at the turning point if the right matching conditions held.

The second interesting quantity is

$$(1.12) \qquad\qquad \mathcal{E} = C + \epsilon u,$$

which we view both as a smooth function on M and on the interval (orbit space) I. Since one may add an arbitrary constant to the soliton potential, (1.5) shows that this is compensated by a corresponding change in the constant C appearing in the conservation law. This is one of the reasons for interest in the quantity $\mathcal{E}$ rather than

the potential. It is then natural to introduce also $\mathcal{F} := \dot{u}$, as u is defined up to an arbitrary constant. Note that (1.5) may now be written as

$$(1.13) \qquad \ddot{\mathcal{E}} + \xi\dot{\mathcal{E}} - \epsilon\mathcal{E} = 0.$$

We now switch to derivatives with respect to s. We have

$$\mathcal{E}' = \epsilon u' = \epsilon W\mathcal{F}.$$

Differentiating $\mathcal{F}$ we obtain

$$\begin{aligned}
\mathcal{F}' &= W\ \ddot{u} \\
&= W\left(C + \epsilon u - \frac{\dot{u}}{W}\right) \\
&= W\left(\mathcal{E} - \frac{\mathcal{F}}{W}\right)
\end{aligned}$$

where we used (1.5) to get the second equality above. In other words, we have the associated system

$$(1.14) \qquad \begin{pmatrix} \mathcal{E} \\ \mathcal{F} \end{pmatrix}' = \begin{pmatrix} 0 & \epsilon W \\ W & -1 \end{pmatrix} \begin{pmatrix} \mathcal{E} \\ \mathcal{F} \end{pmatrix}.$$

along the solution trajectories as long as the independent variable s makes sense.

Note that the eigenvalues of the matrix are

$$\frac{-1 \pm \sqrt{1 + 4\epsilon W^2}}{2}$$

so in the shrinking case ($\epsilon < 0$) they form a complex conjugate pair with negative real part, provided $W > \frac{1}{2\sqrt{-\epsilon}}$.

We will now examine the system (1.14) more closely.

REMARK 1.15.   The origin in the $(\mathcal{E}, \mathcal{F})$-plane (which is invariant under the flow) represents trajectories with constant soliton potential u, that is, Einstein trajectories.

REMARK 1.16.   In the expanding case ϵ is positive, and hence the quadrants $\mathcal{E}, \mathcal{F} < 0$ and $\mathcal{E}, \mathcal{F} > 0$ are invariant under the flow (cf [DW3]). In [DW3] this fact was used to establish bounds on the flow for the case of expanding solitons of multiple warped product type.

In the shrinking case this is no longer the case. Indeed, $\mathcal{E}'$ has the opposite sign to $\mathcal{F}$, while $\mathcal{F}'$ takes the same sign as $\mathcal{E}$ on $\mathcal{F} = 0$. The phase plane diagram shows that no quadrant is invariant and indeed the flow exhibits rotational behaviour around the origin in the $(\mathcal{E}, \mathcal{F})$-plane.

In order to analyse the behaviour of the variables $\mathcal{E}$ and $\mathcal{F}$ in the shrinking case, it is convenient to take $\epsilon = -1$. We observe, from the equations (1.14), that

$$(\mathcal{E}^2 + \mathcal{F}^2)' = -2\mathcal{F}^2$$

so $\mathcal{E}^2 + \mathcal{F}^2$ is monotonic decreasing. (Note that for a non-trivial soliton, $\mathcal{F} = \dot{u}$ cannot be 0 along the flow.)

We also calculate

$$(1.17) \qquad \frac{d}{ds}\tan^{-1}(\mathcal{E}/\mathcal{F}) = \frac{\mathcal{E}'\mathcal{F} - \mathcal{F}'\mathcal{E}}{\mathcal{E}^2 + \mathcal{F}^2} = -W + \frac{\mathcal{E}\mathcal{F}}{\mathcal{E}^2 + \mathcal{F}^2}.$$

If we let $\theta = \tan^{-1}(\mathcal{E}/\mathcal{F})$, this equation becomes

$$(1.18) \qquad \frac{d\theta}{ds} = -W + \frac{1}{2}\sin(2\theta).$$

We are interested in estimating the winding angle around the origin of the flow in the $(\mathcal{E}, \mathcal{F})$-plane up to the (unique) turning point s_* (point where W blows up, cf the above discussion).

Let us take the $\mathcal{F}$-axis as horizontal and the $\mathcal{E}$-axis as vertical. We shall consider trajectories emanating from a point on the positive $\mathcal{E}$-axis (i.e., $\theta = \frac{\pi}{2}$) since at a special orbit we need $\dot{u} = 0$, that is, $\mathcal{F} = 0$. Our phase-plane diagram shows that the flow will initially move into the quadrant with $\mathcal{E}, \mathcal{F} > 0$ and can never re-cross the positive $\mathcal{E}$-axis from the right. The winding angle is therefore nonpositive. Analogous statements hold for trajectories emanating from a point on the negative $\mathcal{E}$-axis, where $\theta = \frac{3\pi}{2}$.

Observe further that the line $\theta = \frac{\pi}{4}$ is a reflecting barrier for the flow unless $W \geq \frac{1}{2}$. Also note that once W exceeds $\frac{1}{2}$ then θ is monotonic decreasing, i.e., the flow winds clockwise around the origin.

The winding angle is

$$\int_{-\infty}^{s_*} \theta'\,ds = \int_{-\infty}^{W^{-1}(\frac{1}{2})} \theta'\,ds + \int_{W^{-1}(\frac{1}{2})}^{s_*} \theta'\,ds,$$

where we have arranged for $s = -\infty$ to correspond to $t = 0$ (in view of (1.10) and the initial conditions). The value of the first integral is between zero and $-\frac{\pi}{4}$, from our above remarks, as until W reaches $\frac{1}{2}$ the flow is trapped between $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{4}$.

For the second integral, we can change variables to rewrite it (using (1.11) with $\mathcal{G} := W^2\mathrm{tr}(L^2)$ and (1.18)) as

$$\int_{\frac{1}{2}}^{\infty} -\frac{1}{\mathcal{G} + \frac{1}{2}W^2}\,dW + \int_{\frac{1}{2}}^{\infty} \frac{\frac{1}{2}\sin(2\theta)}{W(\mathcal{G} + \frac{1}{2}W^2)}\,dW.$$

These integrals are bounded in absolute value by 4 and 2 respectively, just using the nonnegativity of $\mathcal{G}$. We have therefore deduced

PROPOSITION 1.19. *For trajectories of the associated flow in the $(\mathcal{E}, \mathcal{F})$-plane starting from either the positive $\mathcal{E}$ or negative $\mathcal{E}$ axis, the winding angle around the origin up to the turning point is finite, nonpositive, and bounded below by $-(6 + \frac{\pi}{4})$.* $\square$

Of course one would like solutions with winding angle $-\pi$, so as to get $\mathcal{F} = 0$ at the turning point s_*, which is one of the conditions needed to get a symmetric solution. The above argument shows that one cannot achieve this by mimicking the Böhm techniques–we cannot make the winding angle arbitrarily large by varying some parameter.

REMARK 1.20.   We can relate $\mathcal{E}, \mathcal{F}$ to the quantities used in the paper on expanding solitons [DW3] as follows. Let us define

$$(1.21) \qquad \mathcal{H} := W\,\mathrm{tr}L \quad \text{and} \quad \mathcal{Q} := W^2\mathcal{E}.$$

We recall from [DW3] the calculation:

$$(1.22) \qquad u' = \frac{\dot{u}}{-\dot{u} + \mathrm{tr} L} = W \mathrm{tr} L - 1 = \mathcal{H} - 1$$

which is actually valid for general orbit types. It follows that $\mathcal{F} = \frac{\mathcal{H} - 1}{W}$.

We can then derive easily the equations:

$$(1.23) \qquad (\mathcal{H} - 1)' = \left(\mathcal{G} - 1 - \frac{\epsilon}{2} W^2 \right) (\mathcal{H} - 1) + \mathcal{Q},$$

$$(1.24) \qquad \mathcal{Q}' = \epsilon W^2 (\mathcal{H} - 1) + 2 \left(\mathcal{G} - \frac{\epsilon}{2} W^2 \right) \mathcal{Q},$$

which we saw in [DW3] in the special case where ϵ was positive and the cohomogeneity one metric was a warped product on m factors.

The associated system (1.14) described earlier may be regarded as a renormalised version of the above pair of equations where we replace $\mathcal{H} - 1$ and $\mathcal{Q}$ by their quotients by appropriate powers of the variable W. Note that the subset $\{\mathcal{H} = 1, \mathcal{Q} = 0\}$ is invariant under the flow of the GRS equations (at least while the variable s is defined), and consists of Einstein trajectories.

2. Some properties of complete shrinkers of cohomogeneity one. In this section we describe some properties of complete shrinking gradient Ricci solitons with cohomogeneity one.

It is known that the scalar curvature of a complete steady or shrinking gradient Ricci soliton must be non-negative. In the compact case, steady GRS are in fact Ricci flat and shrinking GRS must have positive scalar curvature (cf Proposition 1.13 of [Cetc]). For the non-compact case, the non-negativity follows from a theorem of B. L. Chen (Corollary 2.5 of [Ch]) on complete ancient solutions of the Ricci flow. (The non-negativity was previously known assuming sectional curvature bounds.) By the strong maximum principle, if equality holds at some point then the soliton metric is Ricci-flat. Recently, it was further shown in [PRS] (cf Theorem 1 there) that in this situation the manifold is actually isometric to Euclidean space.

For cohomogeneity one gradient Ricci solitons, the ambient scalar curvature $\overline{R}$ is given by

$$(2.1) \qquad \overline{R} = -2 \, \mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) - (\mathrm{tr} L)^2 + S,$$

where S denotes the scalar curvature of the principal orbits (cf §1 of [DW1]). Now we have

$$\begin{aligned} \overline{R} &= -S + (\mathrm{tr} L)^2 - \mathrm{tr}(L^2) - 2\dot{u} \, \mathrm{tr} L - \epsilon n \\ &= -C - \epsilon u - \dot{u}^2 - \frac{\epsilon}{2}(n+1) \\ &= -\mathcal{E} - \frac{\dot{\mathcal{E}}^2}{\epsilon^2} - \frac{\epsilon}{2}(n+1), \end{aligned}$$

where we used (1.2) and (1.6) respectively in the first two equalities. Now $\overline{R} \geq 0$ immediately leads to the upper bound

$$(2.2) \qquad \mathcal{E} \leq -\frac{\epsilon}{2}(n+1),$$

which is strict unless the soliton is flat.

We take this opportunity to emphasize that $\mathcal{E}$ can be viewed both as a function on M and a function on I, and we will frequently switch from one viewpoint to the other in the following.

REMARK 2.3.   For the steady case, the above inequality becomes $C \leq 0$ with equality iff $\dot{u}$ is identically zero, i.e., the soliton is trivial.

REMARK 2.4.   In the case of a complete non-trivial expanding GRS of cohomogeneity one, Corollary 2.3 of [Ch], applied to the Ricci flow generated by the expander, implies that $\overline{R} > -\frac{\epsilon}{2}(n+1)$. (Note that our ambient manifold has dimension $n+1$.) Since ϵ is now positive, the trace of (0.2) yields $\Delta \mathcal{E} < 0$, where Δ is the negative Laplacian of the cohomogeneity one metric. As $\mathcal{E}$ is a function of t only, $\Delta \mathcal{E} = \ddot{\mathcal{E}} + (\mathrm{tr}L)\dot{\mathcal{E}}$. Using the above facts in (1.13) we obtain $\mathcal{E} < 0$. In particular, if we assume without loss of generality that $u(0) = 0$, then the constant C in the conservation law must be negative.

Going back to the shrinking case, recall that the smoothness conditions at a special orbit placed at $t = 0$ imply that

$$(2.5) \qquad\qquad u = u(0) + \frac{1}{2}\,\ddot{u}(0)\,t^2 + \cdots$$

and

$$(2.6) \qquad\qquad \mathrm{tr}L = \frac{k}{t} + \cdots$$

for sufficiently small values of t. (Here the mean curvature $\mathrm{tr}L$ of the principal orbits is taken with respect to the "outward" normal $\frac{\partial}{\partial t}$.) Hence, in the case of a complete shrinking GRS of cohomogeneity one, the conservation law (1.5) and (2.2) give

$$(2.7) \qquad\qquad (k+1)\ddot{u}(0) = C + \epsilon u(0) = \mathcal{E}(t = 0) \leq -\frac{\epsilon}{2}(n+1).$$

So for the initial value $\ddot{u}(0)$ we have the upper bound

$$(2.8) \qquad\qquad \ddot{u}(0) \leq -\frac{\epsilon}{2}\left(\frac{n+1}{k+1}\right),$$

which must be strict at both endpoints in the case of compact shrinkers. This means that the trajectories of the associated flow in the $(\mathcal{E}, \mathcal{F})$-plane cannot start arbitrarily high on the positive $\mathcal{E}$-axis if they correspond to complete solutions, compact or not. By contrast, note that there is no such restriction on $\ddot{u}(0)$ from the initial value problem (cf [Bu]) around special orbits.

Note also that the upper bound $\mathcal{E}(t = 0) \leq -\frac{\epsilon}{2}(n+1)$ is attained by the conical Gaussian soliton on $\mathbb{R}^{n+1}$. Indeed, in this case $\mathcal{E} = -\frac{\epsilon}{2}(n+1) - \frac{\epsilon^2}{4}t^2$ (see Remark 4.12).

PROPOSITION 2.9.   *Let $(M, \bar{g})$ be a non-trivial compact shrinking GRS of cohomogeneity one with G-invariant potential function u. Let $[0, T]$ denote the closure of the interval I and $\mathcal{E} = C + \epsilon u$, where the constant C is that appearing in (1.6). Then, as a function of t,*

a. $\mathcal{E}$ must change sign and is a Morse-Bott function on M.

    b. $\mathcal{E} < -\frac{\epsilon}{2}(n+1)$.
    *c. $\mathcal{E}$ has at most 4 critical points in $(0, T)$. If $t_0 \in (0, T)$ is a critical point then
    it is either a local maximum if $\mathcal{E}(t_0) > 0$ or a local minimum if $\mathcal{E}(t_0) < 0$.*

    Proof. We note first that by (1.13) $\mathcal{E}$ can be viewed as a solution to the *linear* equation

$$\ddot{f} + \xi \dot{f} - \epsilon f = 0.$$

On $(0, T)$ the coefficient ξ is finite-valued and at least C^1. Since the soliton is non-trivial, $\mathcal{E}$ cannot vanish at an interior critical point t_0. Also, $\ddot{\mathcal{E}}(t_0) = \epsilon \mathcal{E}(t_0)$ holds at such a point, and the only non-vanishing component of the Hessian of $\mathcal{E}$ (as a function on M) is that in the $\frac{\partial}{\partial t}$ direction. So the critical submanifold of $\mathcal{E}$ at t_0, a principal orbit, is non-degenerate. This equation also implies the second part of c.

    Next we consider the situation at $t = 0$. (The situation at $t = T$ is completely analogous.) By smoothness, we have $\dot{\mathcal{E}}(0) = \epsilon \dot{u}(0) = 0$. If $\mathcal{E}(0) = 0$, our soliton would be trivial, as the origin $(\mathcal{E}, \mathcal{F}) = (0, 0)$ is invariant under the associated flow (1.14) and represents the Einstein trajectories. Now the computation (2.7) shows that $\ddot{u}(0) = \frac{1}{\epsilon}\ddot{\mathcal{E}}(0)$ has the same sign as $\mathcal{E}(0)$. In particular, the two singular orbits are also non-degenerate critical submanifolds of u, so that u is Morse-Bott. Since M is compact, $\mathcal{E}$ can only have a finite number of critical points, considered as a function on I. Because $6 + \frac{\pi}{4} < \frac{5}{2}\pi$, we may apply Proposition 1.19 separately to each of the intervals $[0, t_*]$ and $[t_*, T]$ to estimate the total number of critical points in $(0, T)$. (Here as usual t_* denotes the unique turning point). Indeed, when $\mathcal{E}(0)$ and $\mathcal{E}(T)$ have the same sign, the number of critical points is odd, and so is bounded above by 3. When $\mathcal{E}(0)$ and $\mathcal{E}(T)$ have opposite signs, then the number of critical points is even, and so is bounded above by 4.

    The inequality in part b is just (2.2) for the compact case.

    It remains to show that $\mathcal{E}$ must change sign. The (negative) Laplacian of $\mathcal{E}$ is

$$\Delta \mathcal{E} = \ddot{\mathcal{E}} + (\mathrm{tr}L)\dot{\mathcal{E}} = \epsilon \mathcal{E} + \frac{1}{\epsilon}\dot{\mathcal{E}}^2,$$

where we have used (1.13) in the second equality above. Hence the integral of $\mathcal{E}$ is negative if the soliton is non-trivial.

    Suppose that $\mathcal{E} \leq 0$ on $[0, T]$. By what we already proved, $\mathcal{E}$ must then be negative at $t = 0$ and $t = T$, and hence $\ddot{\mathcal{E}}$ must be positive at these endpoints. Therefore, $\mathcal{E}$ must have interior critical points. At each such point $\mathcal{E}$ is negative, so that $\ddot{\mathcal{E}}$ is positive. We obtain a contradiction at the first interior critical point, which must be a local maximum because $\mathcal{E}$ is convex for a little while after $t = 0$. $\square$

    REMARK 2.10.   At present the only known examples of nontrivial compact shrinking Ricci solitons are Kähler. These are generalizations of the Koiso examples (cf [ACGT], [DW1] and references therein), and $\mathcal{E}$ is monotone (no interior critical points) in these examples as a result of the Kähler condition. In particular, $\mathcal{E}$ is a perfect Morse-Bott function.

    In order to state the analogous result in the complete non-compact case, we need to recall a general result of Cao and Zhou ([CZ], Theorem 1.1) concerning the behaviour of the potential function. Adapted to the cohomogeneity one situation and with our notation, their theorem implies that, for sufficiently large t,

$$(2.11) \quad -\frac{\epsilon}{2}(n+1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} + c_2)^2 \leq \mathcal{E}(t) = C + \epsilon u(t) \leq -\frac{\epsilon}{2}(n+1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} - c_1)^2$$

where $c_1, c_2 > 0$ are constants which depend on $\dim M = n + 1$ and the geometry of the unit ball in M centred at a point on the singular orbit, e.g., the initial point of the solution trajectory associated to the soliton.

PROPOSITION 2.12. *Let $(M, \bar{g})$ be a non-trivial complete, non-compact shrinking GRS of cohomogeneity one with G-invariant potential function u. Let $\mathcal{E} = C + \epsilon u$, where the constant C is the one appearing in (1.6). Then, as a function of t,*

> *a. $\mathcal{E}$ must change sign and is a Morse-Bott function on M.*
> *b. $\mathcal{E}$ satisfies (2.11) above. In particular, $\mathcal{E} \leq -\frac{\epsilon}{2}(n+1)$ and eventually decreases monotonically to $-\infty$.*
> *c. $\mathcal{E}$ has at most 5 critical points in $(0, +\infty)$. If $t_0 \in (0, +\infty)$ is such a critical point then it is either a local maximum if $\mathcal{E}(t_0) > 0$ or a local minimum if $\mathcal{E}(t_0) < 0$.*

Proof. The local properties of $\mathcal{E}$ follow as in the compact case. The inequality (2.11) implies that $\mathcal{E}$ tends to $-\infty$. Suppose $\mathcal{E} \leq 0$ everywhere. Since $\mathcal{E}(0) < 0$ and $\mathcal{E}$ is convex for a while after $t = 0$, there must be a critical point at some $t_0 > 0$. At the first such critical point we have a contradiction as in the compact case. So $\mathcal{E}$ changes sign.

Next choose t_1 such that $\mathcal{E} < 0$ for all $t \geq t_1$. So any critical point $t_2 > t_1$ must have $\ddot{\mathcal{E}}(t_2) > 0$ and hence must be a minimum. Since $\mathcal{E}$ tends to $-\infty$, we now see that there are actually no critical points in $[t_1, +\infty)$, and so the number of critical points is finite. It also follows that $\dot{\mathcal{E}} < 0$ for $t \geq t_1$.

To get a more precise bound on the number of interior critical points, we will modify the arguments in §1 leading up to Proposition 1.19. (Note that here we do not need the work of Cao-Zhou.) Recall that ξ has a unique zero at $t = t_*$. For the portion of the trajectory with $t \leq t_*$, the proof of Proposition 1.19 gives a lower bound of $-(6 + \frac{\pi}{4})$ for the winding angle. For the portion with $t \geq t_*$, we can introduce the independent variable

$$(2.13) \qquad \sigma = s_* - \int_{t_*}^t \xi(x) \; dx, \quad \text{i.e., } d\sigma = -\xi dt.$$

Differentiating $\mathcal{E}, \mathcal{F}$ with respect to σ (with $\epsilon = -1$), we now have the following analogue of (1.18):

$$\frac{d\theta}{d\sigma} = W - \frac{1}{2}\sin(2\theta).$$

Note that the angle θ is well-defined and smooth as t passes through t_* and as $t \to +\infty$, so does σ.

We need to estimate the integral

$$\int_{s_*}^{\infty} \frac{d\theta}{d\sigma} \; d\sigma.$$

Recall that W is now negative and increases strictly to 0 as $\sigma \to +\infty$ and satisfies the equation

$$\frac{dW}{d\sigma} = -W\left(\frac{1}{2}W^2 + \mathcal{G}\right).$$

Let σ_1 be the value such that $W(\sigma_1) = -\frac{1}{2}$. As in §1, the integral on $[s_*, \sigma_1]$ is bounded below by -6. On $[\sigma_1, +\infty)$ the net change in θ is bounded below by $-\pi$

because whatever $\theta(\sigma_1)$ equals, one encounters a reflecting barrier of the flow (in the clockwise direction) at or before $\theta(\sigma_1) - \pi$. Hence the integral over the entire trajectory is bounded below by $-(6 + \frac{\pi}{4}) - (6 + \pi) > -(5.1)\pi$. This gives the desired bound for the number of critical points. $\square$

REMARK 2.14.   So far the only examples of complete non-compact shrinking Ricci solitons are the Kähler examples of [FIK] and their generalizations (cf [DW1]), and (up to covers) the Gaussian soliton and its product with a finite number of Einstein spaces. In all these cases, $\mathcal{E}$ is monotone decreasing. Critical points of the soliton potential for complete non-compact shrinkers have been studied in [Na] (cf Corollary 2.1) assuming Ricci bounds.

Next we introduce a function which has the Lyapunov property for the cohomogeneity one GRS equations in a region of phase space. This generalises an analogous function due to C. Böhm [B2] for the Einstein case. Let

$$(2.15) \qquad \mathcal{F} := v^{\frac{2}{n}}\left(S + \mathrm{tr}((L^{(0)})^2)\right)$$

where $v(t) = \sqrt{\det g_t}$, S is the scalar curvature of the principal orbit, and $L^{(0)} := L - \frac{\mathrm{tr}L}{n}\mathbb{I}$ is the traceless part of the shape operator. Modulo the conservation law (1.6), $\mathcal{F}$ can also be written as

$$(2.16) \qquad \mathcal{F} = v^{\frac{2}{n}}\left(\frac{n-1}{n}(\mathrm{tr}L)^2 + \dot{u}(\dot{u} - 2\,\mathrm{tr}L) + \mathcal{E} - \frac{\epsilon}{2}(n-1)\right).$$

PROPOSITION 2.17. *Along the trajectory of a cohomogeneity one GRS of class C^3 we have*

$$\dot{\mathcal{F}} = -2v^{\frac{2}{n}}\,\mathrm{tr}((L^{(0)})^2)\left(\xi - \frac{1}{n}\mathrm{tr}L\right).$$

Hence $\mathcal{F}$ is non-decreasing wherever $\xi \leq \frac{1}{n}\mathrm{tr}L$.

Proof. If we differentiate (2.16), use $\dot{v} = (\mathrm{tr}L)v$ and (1.3), we obtain after some simplification

$$\dot{\mathcal{F}} = \frac{2}{n}\,v^{\frac{2}{n}}(\mathrm{tr}L)\left(\frac{n-1}{n}(\mathrm{tr}L)^2 + \dot{u}^2 - 2\dot{u}(\mathrm{tr}L) + \mathcal{E} - \frac{\epsilon}{2}(n-1)\right)$$
$$+ \; v^{\frac{2}{n}}(\mathrm{tr}L)\left(\frac{n-1}{n}\,\epsilon - \frac{2}{n}\ddot{u} + 2\mathrm{tr}(L^2)\left(\frac{\dot{u}}{\mathrm{tr}L} - \frac{n-1}{n}\right)\right).$$

Splitting up $\mathrm{tr}(L^2)$ as $\mathrm{tr}((L^{(0)})^2) + \frac{1}{n}(\mathrm{tr}L)^2$ and simplifying gives

$$\dot{\mathcal{F}} = v^{\frac{2}{n}}(\mathrm{tr}L)\left(\frac{2(n-1)}{n^2}(\mathrm{tr}L)^2 - \frac{2}{n}(\ddot{u} - \dot{u}^2 + 2\dot{u}(\mathrm{tr}L) - \mathcal{E})\right.$$
$$\left. + \; 2\left(\frac{\dot{u}}{\mathrm{tr}L} - \frac{n-1}{n}\right)\mathrm{tr}((L^{(0)})^2) + \frac{2}{n}\dot{u}(\mathrm{tr}L) - \frac{2(n-1)}{n^2}(\mathrm{tr}L)^2\right)$$
$$= v^{\frac{2}{n}}(\mathrm{tr}L)\left(2\mathrm{tr}((L^{(0)})^2)\left(\frac{\dot{u}}{\mathrm{tr}L} - 1 + \frac{1}{n}\right)\right)$$
$$= -2v^{\frac{2}{n}}\mathrm{tr}((L^{(0)})^2)\left(\xi - \frac{1}{n}(\mathrm{tr}L)\right),$$

as required. Note that we used the conservation law (1.5) in the penultimate equality.

□

REMARK 2.18. The C^3 regularity assumption is natural since for a GRS the conservation law (1.5) is usually derived from the equation for the Laplacian of du, cf Remark 3.8 in [DW1]. Conversely, as discussed in §1, in constructing a complete GRS, this hypothesis allows us to work with (1.2) and (1.5) as long as there is a singular orbit of dimension strictly smaller than that of a principal orbit.

3. Non-existence results. Fundamental results of Perelman [Pe] imply that on a closed manifold all Ricci solitons are of gradient type and all non-trivial solitons are shrinkers. If the soliton metric is invariant under a compact group of isometries, then by averaging over the group one obtains a new soliton potential that is an invariant function. In particular the metric of a compact homogeneous Ricci soliton must be Einstein. Starting from dimensions ≥ 12, examples of compact (simply connected) homogeneous spaces without homogeneous Einstein metrics are known, cf [WZ] and [BK]. Hence these homogeneous spaces do not admit any homogeneous soliton structures, trivial or not.

We consider next the compact cohomogeneity one case. In this case, the orbit space is either a circle or a closed interval. In the former case, the fundamental group of the manifold is infinite, so by the work of [De], [FG] no cohomogeneity one Ricci soliton structures exist. Let us then assume that the orbit space is a closed interval and the singular orbits have dimension strictly smaller than that of the principal orbits. Böhm showed in [B3] that for certain principal orbit types G/K there are no compact cohomogeneity one Einstein metrics. His hypotheses are that there exists a G-invariant distribution on the principal orbits G/K such that:

(i) the tracefree Ricci of any G-invariant metric on G/K is negative definite on the distribution,
(ii) the distribution is irreducible,
(iii) the distribution has trivial intersection with the vertical (collapsing) space of the two special orbits.

Many examples of such orbit types were given in [B3], §7.

It turns out that Böhm's argument also works in the Ricci soliton case, as the extra Hessian terms $\dot{u}L$ and $\ddot{u}$ in (1.2), (1.3) do not play an essential role in the proof. For the convenience of the reader, we give a self-contained proof for the soliton case below. As a consequence, one concludes that the examples in [B3] also do not admit any non-trivial G-invariant Ricci soliton structures.

Let $\overline{M}$ be a closed manifold with a cohomogeneity one smooth action by a compact Lie group G such that the orbit space is a finite closed interval and no singular orbit is exceptional. Let $(\bar{g}, u)$ be a gradient Ricci soliton structure where both the metric and potential are G-invariant. We may choose a unit speed geodesic γ that intersects all principal orbits orthogonally. Then there are closed subgroups $K \subset H_j \subset G$, $j = 1, 2$, such that G/K is the principal orbit type along the interior of γ and G/H_j are the singular orbits at the endpoints of γ. Assume that the domain of γ is $[0, \tau]$, which may be identified with the orbit space of the cohomogeneity one G-action.

Let us choose an Ad_K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that in this section we are not making any assumptions on the multiplicities of the irreducible K-summands in $\mathfrak{p}$. For each of the singular orbits G/H_j, $j = 1, 2$, we have a decomposition $\mathfrak{p} = \mathfrak{s}_j \oplus \mathfrak{q}_j$, where $\mathfrak{h}_j \approx \mathfrak{s}_j \oplus \mathfrak{k}$, $\mathfrak{s}_j$ is Ad_K-invariant, and $\mathfrak{q}_j$ is Ad_{H_j}-invariant. In order to address smoothness issues at G/H_1 it is convenient to fix a G-invariant

background metric β on G/K such that on $\mathfrak{s}_1$ it induces the constant curvature 1 metric on the sphere H_1/K and on $\mathfrak{q}_1$ it induces a fixed G-invariant metric on G/H_1. (Such a background metric will not in general come from a bi-invariant metric on G.) Now on the open subset consisting of principal orbits, one can write the soliton metric $\bar{g}$ in the form $dt^2 + g_t$ where g_t is a one-parameter family of invariant metrics on the principal orbit G/K, regarded as Ad_K- invariant endomorphisms of $\mathfrak{p}$ which are symmetric with respect to the fixed background metric β. On the other hand, the Ricci endomorphisms r_t and shape operators L_t are symmetric with respect to g_t but not necessarily with respect to β.

We next decompose $\mathfrak{p}$ as a sum

$$(3.1) \qquad\qquad \mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$$

where each $\mathfrak{p}_i$ is itself a sum of *equivalent* Ad_K-irreducible summands, and, for all $i \neq j$, no summand of $\mathfrak{p}_i$ is equivalent to a summand of $\mathfrak{p}_j$. Such a decomposition is unique up to permutation, and the summands $\mathfrak{p}_i$ are orthogonal with respect to β as well as g_t. Note that any Ad_K-invariant g_t-symmetric endomorphism of $\mathfrak{p}$ must map $\mathfrak{p}_i$ into $\mathfrak{p}_i$ and we will denote by tr_i its trace on the summand $\mathfrak{p}_i$. Its full trace on $\mathfrak{p}$ will be denoted by tr as usual.

Recall that the relative volume $v(t) = \sqrt{\det g_t}$ of the principal orbits satisfies the equation $\dot{v} = (\mathrm{tr}L)v$. Consider the conformally related metrics

$$\tilde{g} = v^{-\frac{2}{n}}\, g.$$

One easily computes that

$$(3.2) \qquad\qquad \dot{\tilde{g}} = 2\tilde{g}L^{(0)}$$

and

$$(3.3) \qquad\qquad \ddot{\tilde{g}} = \dot{\tilde{g}}\tilde{g}^{-1}\dot{\tilde{g}} + 2\tilde{g}\dot{L^{(0)}},$$

where $L^{(0)} = L - \left(\frac{\mathrm{tr}L}{n}\right)\mathbb{I}$ is the trace-free part of L.

Consider the quantities

$$(3.4) \qquad\qquad F_i := \frac{1}{2}\, \mathrm{tr}_i(\tilde{g}^2).$$

Using (3.2) and (3.3) we obtain

$$(3.5) \qquad\qquad \dot{F}_i = \mathrm{tr}_i(\tilde{g}\dot{\tilde{g}}) = 2\, \mathrm{tr}_i(\tilde{g}^2 L^{(0)}),$$

$$(3.6) \qquad\qquad \ddot{F}_i = \mathrm{tr}_i(\dot{\tilde{g}}^2) + \mathrm{tr}_i(\tilde{g}\dot{\tilde{g}}\tilde{g}^{-1}\dot{\tilde{g}}) + 2\, \mathrm{tr}_i(\tilde{g}^2\dot{L^{(0)}}).$$

To compute further, we need to break the terms in Eq. (1.2) into their trace and trace-free parts. Recall that the generalised mean curvature ξ is given by $-\dot{u} + \mathrm{tr}L$. Then one obtains, using (1.2),

$$\dot{L^{(0)}} = r^{(0)} - \xi L^{(0)} + \left(\frac{S}{n} + \frac{\epsilon}{2} - \frac{\mathrm{tr}\dot{L}}{n} - \left(\frac{\mathrm{tr}L}{n}\right)\xi\right)\mathbb{I}$$

where $r^{(0)}$ is the trace-free part of r and S is its trace, i.e., the scalar curvature. Note that the trace of Eq. (1.2) yields the relation

$$S - \text{tr}(\dot{L}) - \xi \text{ tr} L + \frac{n\epsilon}{2} = 0.$$

Hence we obtain

(3.7) $$\dot{L^{(0)}} + \xi L^{(0)} - r^{(0)} = 0.$$

Substituting this into (3.6) and using (3.5) we finally get

(3.8) $$\ddot{F}_i + \xi \dot{F}_i = \text{tr}_i(\dot{\tilde{g}}^2) + \text{tr}_i(\dot{\tilde{g}}\tilde{g}^{-1}\dot{\tilde{g}}\tilde{g}) + 2 \text{ tr}_i(\tilde{g}^2 r^{(0)}),$$

which is an analogue of the formula in Proposition 3.2 in [B3].

We are now ready to prove

THEOREM 3.9. *Let $\overline{M}$ be a closed cohomogeneity one G-manifold as described above. Assume that some summand $\mathfrak{p}_{i_0}$ in (3.1) is actually Ad_K-irreducible and that for any G-invariant metric on G/K, the restriction to $\mathfrak{p}_{i_0}$ of its traceless Ricci tensor is always negative definite. Assume further that $\mathfrak{p}_{i_0} \cap \mathfrak{s}_j = \{0\}$ for $j = 1, 2$.*

Then there cannot be any G-invariant gradient Ricci soliton structure on $\overline{M}$.

Proof. (cf [B3]) Consider the function F_{i_0} given by (3.4). We claim that as t approaches 0, $F_{i_0}(t)$ approaches $+\infty$. This follows since v tends to zero (the singular orbit G/H_1 has strictly smaller dimension than the principal orbits), and the irreducibility of $\mathfrak{p}_{i_0}$ together with the fact that $\mathfrak{p}_{i_0} \cap \mathfrak{s}_1 = \{0\}$ imply that the endomorphism $g|\mathfrak{p}_{i_0}$ tends to a multiple of the identity. Similarly, we see that $F_{i_0}(t)$ tends to $+\infty$ as t tends to τ. (Note that F_{i_0} depends on the background metric, but the functions defined using different backgrounds differ only by positive constants, as $\mathfrak{p}_{i_0}$ is irreducible.)

It follows that F_{i_0} has a global minimum at some point $t_* \in (0, \tau)$. By (3.5), $\dot{\tilde{g}}(t_*)$ restricts to zero on $\mathfrak{p}_{i_0}$, and using this in (3.8) with the negative definiteness of $r^{(0)}$ on $\mathfrak{p}_{i_0}$, we arrive at a contradiction. ◻

REMARK 3.10. Examples of cohomogeneity one manifolds which satisfy the hypotheses of Theorem 3.9 include $S^{k+1} \times (G'/K') \times M_3 \times \cdots \times M_r$ where $M_3, \cdots, M_r$ are arbitrary compact isotropy irreducible homogeneous spaces, $G'/K' = \text{SU}(\ell + m)/(\text{SO}(\ell)\text{U}(1)\text{U}(m))$ (a bundle over a complex Grassmannian with a symmetric space as fibre), and $\ell \geq 32, m = 1, 2, k = 1, \ldots, [\ell/3]$ (cf [B3]). The significance of the spaces G'/K' is that they do not admit any G'-invariant Einstein metrics (cf [WZ]). More information about the structure of such G'/K' can be found in [B4], cf Theorem B in particular.

4. Multiply warped principal orbit types. In the final section §5 we are going to conduct numerical investigations of the soliton equations for certain relatively simple orbit types, which in the Einstein case were studied by Böhm [B1]. In the current section we shall therefore see how the soliton equations simplify under certain assumptions. We shall in particular make some remarks about the case of multiple warped products, which have some special features (some of the Böhm examples are warped products with two factors).

Let us first assume that the isotropy representation is *multiplicity free*, that is, $\mathfrak{p}$ decomposes into pairwise inequivalent real K-modules $\mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_m$. We let d_i denote

the real dimension of $\mathfrak{p}_i$ and let $n = d_1 + \ldots + d_m$. Note that the summands $\mathfrak{p}_i$ have a different meaning than that in §3.

Under the multiplicity free assumption, both the metric and shape operator diagonalise with respect to the decomposition $\mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_m$. We write the components as g_i^2 and $L_i = \frac{\dot{g}_i}{g_i}$ respectively. Furthermore, the Ricci term r_t also diagonalises with components r_i.

In addition to the variable W defined by (1.9), we introduce variables

$$(4.1) \qquad X_i = \frac{\sqrt{d_i}}{(-\dot{u} + \mathrm{tr}L)} \frac{\dot{g}_i}{g_i},$$

$$(4.2) \qquad Y_i = \frac{\sqrt{d_i}}{g_i} \frac{1}{(-\dot{u} + \mathrm{tr}L)}.$$

Note that the non-negative quantity $\mathcal{G} = W^2 \mathrm{tr}(L^2)$ at the end of §1 becomes $\sum_{i=1}^m X_i^2$ and $\mathcal{H} = W \mathrm{tr}L$ becomes $\sum_{i=1}^m \sqrt{d_i} X_i$.

The gradient Ricci soliton system (1.2)-(1.3) becomes the following equations

$$(4.3) \qquad X_i' = X_i \left(\sum_{j=1}^m X_j^2 - 1 \right) + \sqrt{d_i}\, r_i W^2 + \frac{\epsilon}{2} \left(\sqrt{d_i} - X_i \right) W^2,$$

$$(4.4) \qquad Y_i' = Y_i \left(\sum_{j=1}^m X_j^2 - \frac{X_i}{\sqrt{d_i}} - \frac{\epsilon}{2} W^2 \right),$$

$$(4.5) \qquad W' = W \left(\sum_{j=1}^m X_j^2 - \frac{\epsilon}{2} W^2 \right).$$

Note that (1.4) is automatically satisfied under the multiplicity free hypothesis (cf Remark 2.21 in [DW1]).

The term involving r_i in (4.3) can be expressed as rational functions of the Y_j. Without loss of generality we can take W to be positive (before the blow-up time).

In the above variables, the conservation law (1.6) becomes:

$$(4.6) \qquad \mathcal{L} + \left(\frac{n-1}{2} \right) \epsilon W^2 = (C + \epsilon u) W^2 \quad (= \mathcal{E} W^2)$$

where C is a constant and

$$(4.7) \qquad \mathcal{L} := \frac{\mathrm{tr}(L^2)}{(-\dot{u} + \mathrm{tr}L)^2} + \frac{\mathrm{tr}(r_t)}{(-\dot{u} + \mathrm{tr}L)^2} - 1.$$

In the steady case ($\epsilon = 0$), we recall that $\mathcal{L}$ is a Lyapunov function for the flow, but this fails for ϵ nonzero. The first term in $\mathcal{L}$ is just $\mathcal{G} = \sum_{i=1}^m X_i^2$.

Let us now focus on the multiple warped product situation (see [DW2] for the steady and [DW3] for the expanding case). Here the hypersurface is a product of Einstein manifolds with positive Einstein constants λ_i. Now $r_i = \frac{\lambda_i}{g_i^2}$ so the term involving r_i in the equation for X_i is just Y_i^2 (up to a positive multiplicative constant).

Moreover, we have $\mathrm{tr}(r_t) = S > 0$ and it follows from (4.1) and (4.2) that

$$\mathcal{L} = \sum_{i=1}^m (X_i^2 + Y_i^2) - 1,$$

so $\mathcal{L}$ is bounded below. This can be used to give more precise information about the flow, especially in the neighbourhood of a turning point. By contrast, for more general orbit types, the $\mathrm{tr}(r_t)$ term and the functional $\mathcal{L}$ involve $\frac{Y_k^2 Y_p^2}{Y_j^2}$ terms. Moreover $\mathcal{L}$ is indefinite.

In the multiple warped product case we can sharpen the analysis of the functional $\mathcal{F}$ that was introduced in §2. As we are dealing with shrinking solitons, ϵ is negative.

PROPOSITION 4.8. *Let $\gamma(t)$ denote a solution trajectory of the shrinking GRS equations in the multiple warped product case. Assume that the soliton is complete. Then the quantity $\eta := \xi - \frac{1}{n}\mathrm{tr}\,L$ eventually becomes and stays negative. Hence $\mathcal{F}$ eventually is non-decreasing, and is then strictly increasing unless the shape operator L becomes a multiple of the identity or we reach a second singular orbit.*

Proof. We shall assume that our soliton is non-compact; otherwise, $\dot{u}$ is close to zero and $\mathrm{tr}\,L$ is close to $-\infty$ near $t = T$, so that all the assertions hold by Proposition 2.17.

Let us consider the region $\{\eta < 0\}$. We claim that it is invariant under the forward flow of the GRS equations. By abuse of notation suppose $\gamma(t)$ is a trajectory of the flow starting at t_0 with $\eta(\gamma(t_0)) < 0$. Let $t_1 > t_0$ be the first time when γ leaves the region $\{\eta < 0\}$. Then

$$\xi(t_1) = \frac{1}{n}(\mathrm{tr}\,L)|_{t=t_1}.$$

On the other hand

$$\dot{\eta} = \dot{\xi} - \frac{1}{n}\mathrm{tr}(\dot{L})$$
$$= -\mathrm{tr}(L^2) - \frac{S}{n} + \frac{1}{n}\,\xi\,\mathrm{tr}\,L,$$

where we have used (1.3) and the trace of (1.2). At $t = t_1$ we have

$$\dot{\eta}(t_1) = -\frac{1}{n}S(t_1) + \left(\frac{(\mathrm{tr}\,L)^2}{n^2} - \mathrm{tr}(L^2)\right)(t_1)$$
$$\leq -\frac{1}{n}\left(S + \frac{n-1}{n}(\mathrm{tr}\,L)^2\right)(t_1) < 0$$

since $S(t_1) > 0$. This means that $\eta(\gamma(t))$ must be positive slightly to the left of $t = t_1$, a contradiction to the choice of t_1.

It remains to see that $\eta(\gamma(t))$ is negative somewhere. The previous paragraph implies that once $\eta(\gamma(t_1))$ is zero for some t_1, it must be negative to the right of t_1. So suppose that $\eta(\gamma(t)) > 0$ for all t. Then by (1.3),

$$\frac{\dot{v}}{nv} = \frac{1}{n}(\mathrm{tr}\,L) < \xi \leq \frac{\epsilon}{2}t + a$$

for some positive constant a for large values of t. It follows that the cohomogeneity one manifold has finite volume, contradicting a theorem of H.-D. Cao and X.-P. Zhu ([Ca3], Theorem 3.1). $\square$

REMARK 4.9.   In the above proof the only time we used the hypothesis that the hypersurface is of multiple warped product type is when we asserted $S(t_1) > 0$.

PROPOSITION 4.10. *Let the hypersurface (principal) orbit be a product $(M_1, h_1) \times \cdots \times (M_m, h_m)$ of Einstein manifolds with dimension d_i and $\mathrm{Ric}(h_i) = \lambda_i h_i, \lambda_i > 0$. Assume further that $M_1 = S^{k+1}$ and h_1 is the constant curvature 1 metric. Then the Lyapunov function $\mathcal{F}$ satisfies*

$$\mathcal{F} \geq n \prod_{i=1}^{m} \lambda_i^{d_i/n}.$$

Equality holds for a trajectory of the GRS equations iff $g_i/\sqrt{\lambda_i}$ is independent of i $(1 \leq i \leq m)$ and the traceless shape operator $L^{(0)}$ is 0.

Proof. By (2.15), we have

$$\mathcal{F} \geq v^{\frac{2}{n}} \, S = \left(\prod_{i=1}^{m} g_i^{\frac{2d_i}{n}} \right) \sum_{j=1}^{m} \frac{d_j \lambda_j}{g_j^2}$$

$$\geq \left(\prod_{i=1}^{m} g_i^{\frac{2d_i}{n}} \right) n \prod_{i=1}^{m} \left(\frac{\lambda_i}{g_i^2} \right)^{\frac{d_i}{n}}$$

$$= n \prod_{i=1}^{m} \lambda_i^{d_i/n},$$

where we used the inequality (cf [HLP], p. 17)

$$\prod_{i=1}^{m} a_i^{\frac{\rho_i}{\sum \rho_j}} \leq \frac{\sum \rho_j a_j}{\sum \rho_j}$$

for $a_i, \rho_i \geq 0$. Equality holds in this inequality iff all the a_i are equal, which translates into our condition since we took $a_j = \lambda_j/g_j^2$ and $\rho_j = d_j$. $\square$

Let us consider the 3-dimensional subset of phase space

$$\mathcal{D} := \{(g, L, \dot{u}) : L^{(0)} = 0, \ g_i/\sqrt{\lambda_i} \text{ independent of } i\}.$$

By Proposition 4.10, $\mathcal{D}$ is the set on which the function $\mathcal{F}$ attains its (global) minimum value. It is invariant under the flow of the GRS equations. In fact, if we parametrize this set by $x = g_i \sqrt{\frac{n-1}{\lambda_i}}, y = \dot{x}/x$, and $z = \dot{u}$, then the induced flow on $\mathcal{D}$ is precisely the GRS equations for the special case $m = 1, d_1 = n$, which was analysed by Kotschwar [Kot]. Recall that Kotschwar showed that only two trajectories in $\mathcal{D}$ represented complete smooth solitons with at least one singular orbit (actually a point). However, when $m > 1$, no trajectory lying in $\mathcal{D}$ represents a smooth soliton. When $m > 1$, the non-compact solution acquires a conical singularity and will be referred to as the *conical* Gaussian soliton. The compact solution has two conical singularities and will be called the *spherical Einstein cone*. On the other hand, there is a smooth non-compact complete solution which is the product of the shrinking Gaussian soliton on $\mathbb{R}^{d_1+1}$ with the remaining Einstein factors. (This does not represent a trajectory in $\mathcal{D}$). We will call this solution the *smooth Gaussian soliton*, which is *rigid* in the terminology of Petersen-Wylie [PW]. For comparison purposes we list below some information about these special solutions.

EXAMPLE 4.11. (*smooth Gaussian* on $\mathbb{R}^{d_1+1} \times M_2 \times \cdots \times M_m$) The metric is given by

$$dt^2 + t^2 h_1 + \left(\frac{2\lambda_2}{-\epsilon}\right) h_2 + \cdots + \left(\frac{2\lambda_m}{-\epsilon}\right) h_m$$

and $u(t) = -\frac{\epsilon}{4}t^2$. We further have

$$\mathrm{tr}L = \frac{d_1}{t}, \quad \xi = \frac{\epsilon}{2}t + \frac{d_1}{t}, \quad \mathcal{E} = -\frac{\epsilon^2}{4}t^2 - \frac{\epsilon}{2}(d_1 + 1).$$

EXAMPLE 4.12. (*conical Gaussian*) The metric is given by

$$dt^2 + t^2 \left(\frac{\lambda_1}{n-1} h_1 + \cdots + \frac{\lambda_m}{n-1} h_m\right)$$

and $u(t) = -\frac{\epsilon}{4}t^2$. A conical singularity occurs at $t = 0$ except when $m = 1$. The metric is Ricci-flat and complete at infinity. We further have

$$\mathrm{tr}L = \frac{n}{t}, \quad \xi = \frac{\epsilon}{2}t + \frac{n}{t}, \quad \mathcal{E} = -\frac{\epsilon^2}{4}t^2 - \frac{\epsilon}{2}(n + 1).$$

EXAMPLE 4.13. (*spherical Einstein cone*) The metric, which is Einstein, is given by

$$dt^2 + \left(\frac{\sin^2(\alpha t)}{\alpha^2(n-1)}\right) (\lambda_1 h_1 + \cdots + \lambda_m h_m)$$

where $\alpha = \sqrt{-\epsilon/2n}$. We also have $\xi = \mathrm{tr}L = \alpha n \cot(\alpha t)$. When $m > 1$ there are conical singularities at $t = 0$ and $t = \pi/\alpha$.

Let us now further specialise to the case $m = 2$. This is of particular importance because it includes some of the examples for which Böhm [B1] was able to produce infinite families of Einstein metrics.

We have found, in §1, a general winding number for the flow, representing winding round the submanifold of Einstein trajectories. We shall now, in the case of warped products on 2 factors, investigate another winding number, similar to the one used by Böhm in the Einstein case (cf [B1]). This counts winding around the subset $\mathcal{D}$ of phase space, which we introduced above. We saw that trajectories in $\mathcal{D}$ represent solutions which are equivalent to warped products on *one* factor.

For convenience, we shall take the Einstein constants λ_i to be $d_i - 1$. We recall that $\mathcal{D}$ is defined by

$$\frac{X_1}{\sqrt{d_1}} = \frac{X_2}{\sqrt{d_2}} \quad : \quad \frac{Y_1}{\sqrt{d_1}} = \frac{Y_2}{\sqrt{d_2}}.$$

Note in particular that the intersection of $\mathcal{D}$ with the region $\{\mathcal{H} = 1, \mathcal{Q} = 0\}$ (cf. Remark 1.20) consists (with ϵ normalised to $-2n$) of the curve parametrised by

$$X_i = \frac{\sqrt{d_i}}{n}, \quad Y_i = \frac{\sqrt{d_i}\sqrt{n-1}}{n} \sec t, \quad W = \frac{1}{n} \tan t.$$

This is the spherical Einstein cone solution in (4.13), with $\alpha = 1$.

The flow-invariant subvariety $\mathcal{Z}$ of phase space given by $\mathcal{H} = 1, \mathcal{Q} = 0, W = 0$ is transverse to $\mathcal{D}$. More precisely, this subvariety meets $\mathcal{D}$ only at the points $P_{\pm}$ given by

$$X_i = \frac{\sqrt{d_i}}{n} \quad : Y_i = \pm\frac{\sqrt{d_i}\sqrt{n-1}}{n}.$$

Note that P_+ is the initial point (i.e. the value at $t = 0$) of the spherical cone and Gaussian soliton solutions. In fact P_+ is a critical point of our equations (recall $t = 0$ corresponds to $s = -\infty$).

Let us now linearise our equations around P_+ in the invariant subvariety $\mathcal{Z}$. This means we only consider vectors tangent to $\mathcal{Z}$, so we have

$$w = 0 : \sqrt{d_1}x_1 + \sqrt{d_2}x_2 = 0 = \sqrt{d_1}y_1 + \sqrt{d_2}y_2$$

(the final equation follows from the condition $d\mathcal{Q} = 0$). Parametrising the tangent space by x_1, y_1, we obtain the system

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}' = \begin{pmatrix} \frac{1-n}{n} & \frac{2\sqrt{n-1}}{n} \\ -\frac{\sqrt{n-1}}{n} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

The eigenvalues are roots of the quadratic

$$\lambda^2 + \frac{n-1}{n}\lambda + \frac{2(n-1)}{n^2}$$

which has discriminant $\frac{(n-1)(n-9)}{n^2}$. We deduce

PROPOSITION 4.14. P_+ is a focus of the flow on the invariant variety $\mathcal{Z}$ transverse to $\mathcal{D}$ if and only if $2 \leq n \leq 8$.

So this winding number behaves in just the same way as its analogue in the Einstein case in [B1], where it was used to produce symmetric metrics (cf. remarks after equation (1.11)). However, to produce soliton analogues of these Böhm metrics one would need to control $\dot{u}$ as well, probably by means of the winding number of §1, and as we have seen this has quite different behaviour. In the next section we shall investigate these and other examples of Böhm numerically in the soliton case.

5. Numerical investigations. The non-existence results of §3 rule out certain special principal orbit types but leave the existence problem open for most other orbit types. In particular they do not apply to the examples considered by Böhm in [B1], who produced infinite families of cohomogeneity one Einstein metrics on certain manifolds with low dimensions ($5 \leq n + 1 \leq 9$).

In this section we report on some attempts to investigate the existence of cohomogeneity one shrinking Ricci solitons using numerical methods, focusing on the Böhm spaces and some related examples. The numerical methods employed are relatively simple, but do show some interesting results.

We first consider an example that *is* known to admit a non-trivial (i.e. non-Einstein) shrinking Ricci soliton, namely $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. The soliton in this case is a $U(2)$-invariant Kähler metric that was constructed independently by Koiso [Koi] and Cao [Ca1]. This metric provides an important test case for any "soliton-hunting" programs, and we shall henceforth refer to it as the Koiso-Cao soliton.

For the more complicated spaces considered by Böhm we are able to replicate his Einstein metrics from the numerical search, but do not find any examples admitting non-trivial shrinking solitons. We also examine some examples of Böhm type but with dimension above the range where his Einstein existence results apply. In this case too we do not find any solitons. The results seem to hint that the cohomogeneity one soliton equations in the shrinking case exhibit a high degee of rigidity and instability.

Let us now turn to the analysis of the equations. We are interested in solutions to equation (0.2). Following Böhm [B1], we consider situations where the space of invariant metrics on the hypersurface is 2-dimensional. The cohomogeneity one metric therefore involves two functions, and the soliton equations, being a system of second-order equations in the metric and the soliton potential, are a dynamical system in $\mathbb{R}^6$. As remarked in §1, the conservation law (1.5) may be viewed as a constraint and plays a vital role.

More precisely, we can consider a principal orbit G/K and take one of the special orbits to be G/H, where

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$$

with $\mathfrak{p}_1, \mathfrak{p}_2$ being inequivalent irreducible K-modules and $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1$. As in §4, we denote by d_i the dimension of $\mathfrak{p}_i$. To ensure smoothness at the special orbit we need H/K to be a sphere $\mathbb{S}^{d_1}$. Note also that $Q := G/H$ is isotropy irreducible and hence Einstein; we shall denote its Einstein constant by $C_Q > 0$. Our cohomogeneity one metric is now given, with respect to an invariant background inner product B on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, by

$$dt^2 + g_t = dt^2 + f(t)^2 B|_{\mathfrak{p}_1} + h(t)^2 B|_{\mathfrak{p}_2}.$$

Note that the fibration $H/K \to G/K \to G/H$ now becomes a Riemannian submersion with respect to g_t on G/K and the metrics given by $f(t)^2 B|_{\mathfrak{p}_1}$ and $h(t)^2 B|_{\mathfrak{p}_2}$ on H/K and G/H respectively.

Examples of this situation include the case when G/K is a product of two isotropy irreducible spaces with one being a sphere. Another case, familiar in the Einstein situation from the work of Bérard Bergery [BB], is when $d_1 = 1$, so G/K is a circle bundle over G/H.

As usual, the same equations arise in certain more general situations which are not strictly of cohomogeneity one type. For example in the Bérard Bergery situation, the hypersurface can be a circle bundle over an arbitrary Fano Kähler-Einstein base, not necessarily homogeneous.

The Ricci soliton equations (1.3) and (1.2) specialize to:

$$(5.1) \qquad -d_1 \frac{\ddot{f}}{f} - d_2 \frac{\ddot{h}}{h} + \ddot{u} + \frac{\epsilon}{2} = 0,$$

$$(5.2) \qquad -\frac{\ddot{f}}{f} + (1-d_1)\frac{\dot{f}^2}{f^2} - d_2 \frac{\dot{f}\dot{h}}{fh} + \frac{d_1 - 1}{f^2} + \frac{d_2}{d_1}\|\mathcal{A}\|^2 \frac{f^2}{h^4} + \frac{\dot{u}\dot{f}}{f} + \frac{\epsilon}{2} = 0,$$

$$(5.3) \qquad -\frac{\ddot{h}}{h} + (1-d_2)\frac{\dot{h}^2}{h^2} - d_1 \frac{\dot{f}\dot{h}}{fh} + \frac{C_Q}{h^2} - 2\|\mathcal{A}\|^2 \frac{f^2}{h^4} + \frac{\dot{u}\dot{h}}{h} + \frac{\epsilon}{2} = 0,$$

where the quantity $\|\mathcal{A}\|^2$ is the norm-squared of the O'Neill tensor for the Riemanian submersion $H/K \to G/K \to G/H$ mentioned above. It is a constant that depends

only on the topology of the bundle. When G/K is a product of the two factors $H/K = \mathbb{S}^{d_1}$ and G/H then $\|\mathcal{A}\|^2$ is zero. We recall the conservation law (1.5), given in this setting by

$$\ddot{u} + \dot{u}\left(d_1\frac{\dot{f}}{f} + d_2\frac{\dot{h}}{h}\right) - \dot{u}^2 - \epsilon u = C. \tag{5.4}$$

Note that by changing the value of $u(0)$ we may fix the constant C to be 0, and by a homothety of the metric we may alter the value of ϵ.

To carry out our numerical study we shall consider the system (5.2)-(5.4) and introduce new variables $(z_1, z_2, z_3, z_4, z_5, z_6) := (f, \dot{f}, h, \dot{h}, u, \dot{u})$. We then obtain the system

$$\dot{z}_1 = z_2, \tag{5.5}$$

$$\dot{z}_2 = -(d_1 - 1)\frac{z_2^2}{z_1} - d_2\frac{z_2 z_4}{z_3} + \frac{d_1 - 1}{z_1} + \frac{d_2}{d_1}\|\mathcal{A}\|^2\frac{z_1^3}{z_3^4} + z_2 z_6 + \frac{\epsilon}{2}z_1, \tag{5.6}$$

$$\dot{z}_3 = z_4, \tag{5.7}$$

$$\dot{z}_4 = -(d_2 - 1)\frac{z_4^2}{z_3} - d_1\frac{z_2 z_4}{z_1} + \frac{C_Q}{z_3} - 2\|\mathcal{A}\|^2\frac{z_1^2}{z_3^3} + z_4 z_6 + \frac{\epsilon}{2}z_3, \tag{5.8}$$

$$\dot{z}_5 = z_6, \tag{5.9}$$

$$\dot{z}_6 = -z_6\left(d_1\frac{z_2}{z_1} + d_2\frac{z_4}{z_3}\right) + z_6^2 + \epsilon z_5. \tag{5.10}$$

The algorithm used to solve the above system numerically is the Runge-Kutta method. A good account of it can be found in [But], and we include a brief discussion here for completeness. Given an ODE $\dot{y}(t) = F(t, y)$ with initial condition $y(0) = y_0$, the Runge-Kutta sequence of approximations is given by:

$$y_{n+1} = y_n + \frac{1}{6}(\mathsf{k}_1 + 2\mathsf{k}_2 + 2\mathsf{k}_3 + \mathsf{k}_4),$$
$$t_{n+1} = t_n + \mathsf{h},$$

where

$$\mathsf{k}_1 = F(t_n, y_n),$$
$$\mathsf{k}_2 = F(t_n + \frac{1}{2}\mathsf{h}, y_n + \frac{1}{2}\mathsf{h}\mathsf{k}_1),$$
$$\mathsf{k}_3 = F(t_n + \frac{1}{2}\mathsf{h}, \frac{1}{2}\mathsf{h}\mathsf{k}_2),$$
$$\mathsf{k}_4 = F(t_n + \mathsf{h}, \mathsf{h}\mathsf{k}_3).$$

The error at each step is $O(|\mathsf{h}|^5)$ which gives an accumulated error of $O(|\mathsf{h}|^4)$.

The smoothness conditions at the singular orbit require that $(z_1, z_2, z_3, z_4, z_5, z_6) = (0, 1, \bar{h}, 0, \bar{u}, 0)$ where $\bar{h}$ is an arbitrary positive constant and $\bar{u}$ is an arbitrary constant $\geq -(n+1)/2$ (by (2.2)). Note that this means that the initial Runge-Kutta step is not defined. We get around this problem using the standard trick of constructing a power-series solution about a small neighbourhood and then running the algorithm from this solution.

5.1. Numerical results for $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. The metrics we consider here are $U(2)$-invariant and the principal orbits are Berger spheres $S^3 = U(2)/U(1)$, so we have $d_1 = 1$ and $Q = \mathbb{S}^2$ with $C_Q = 1$. The constant $\|\mathcal{A}\|^2 = 1$. As remarked in the introduction, there is a non-trivial Kähler-Ricci soliton on this manifold due to Koiso and Cao. There is also a Hermitian, non-Kähler, Einstein metric due to Page [Pa]. The Page metric is constructed explicitly in our variables in [Kod], and in order to check our algorithm we use the same normalisation and so take $\epsilon = -7.46562$.

In order to define a smooth metric on the blowup of $\mathbb{CP}^2$, we need the S^1 fibre of the Berger sphere to collapse at both ends. We are looking for a $t = t_{sol} > 0$ such that

$$(f(t_{sol}), \dot{f}(t_{sol}), h(t_{sol}), \dot{h}(t_{sol}), u(t_{sol}), \dot{u}(t_{sol})) = (0, -1, \tilde{h}, 0, \tilde{u}, 0)$$

for some $\tilde{h} > 0$ and some $\tilde{u} > -2$. We output initial conditions where

$$SOL(t) = f^2(t) + (\dot{f}(t) + 1)^2 + \dot{h}^2(t) + \dot{u}^2(t)$$

satisfies $SOL < 0.005$ for some $t > 0$. The plot below indicates the output of the program. The parameters searched are $0 \le \bar{h} \le 3$, $-2 \le \bar{u} \le 2$ and the stepsize is 0.005.

Output of soliton hunting program on CP2 blown-up at 1 point
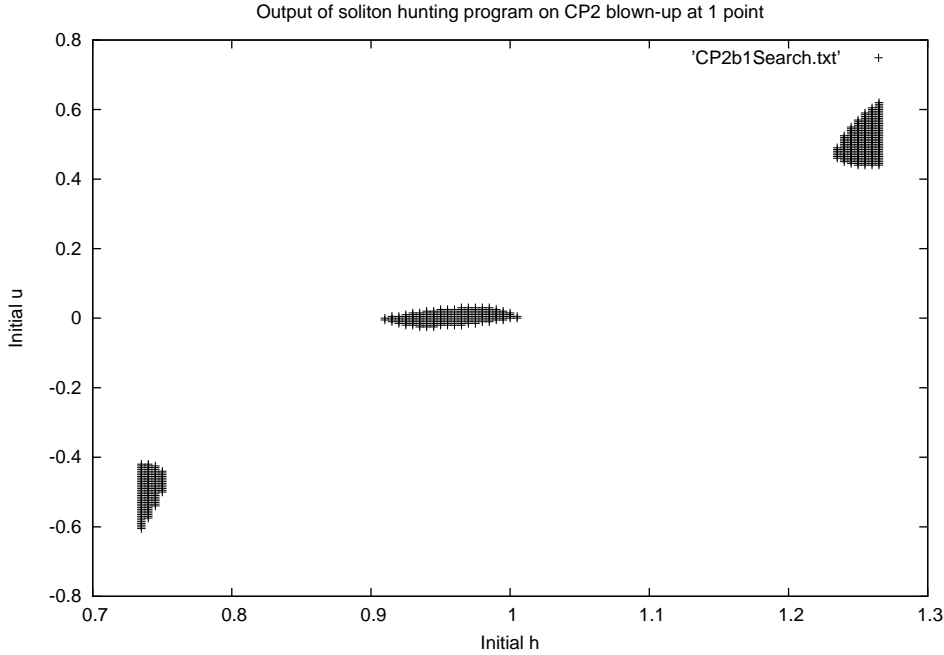
The cluster about $(\bar{h}, \bar{u}) = (0.7319, -0.5276)$ corresponds to the Koiso-Cao soliton, the cluster about $(\bar{h}, \bar{u}) = (0.9595, 0)$ is the Page metric, and the third cluster is the Koiso-Cao soliton with the conjugate complex structure.

We investigated for the soliton the behaviour of the winding angle of §1 numerically, and found that the angle decreases monotonically along the flow.

5.2. Numerical results on $\mathbb{S}^5$. Here $G = SO(3) \times SO(3)$ and the principal orbits are $\mathbb{S}^2 \times \mathbb{S}^2$. As we have mentioned before, Böhm [B1] found infinitely many cohomogeneity one Einstein metrics on this manifold. We choose the normalisation

$\frac{\epsilon}{2} = -0.04$ so that the standard round metric has initial conditions $(h(0), u(0)) = (10, 0)$.
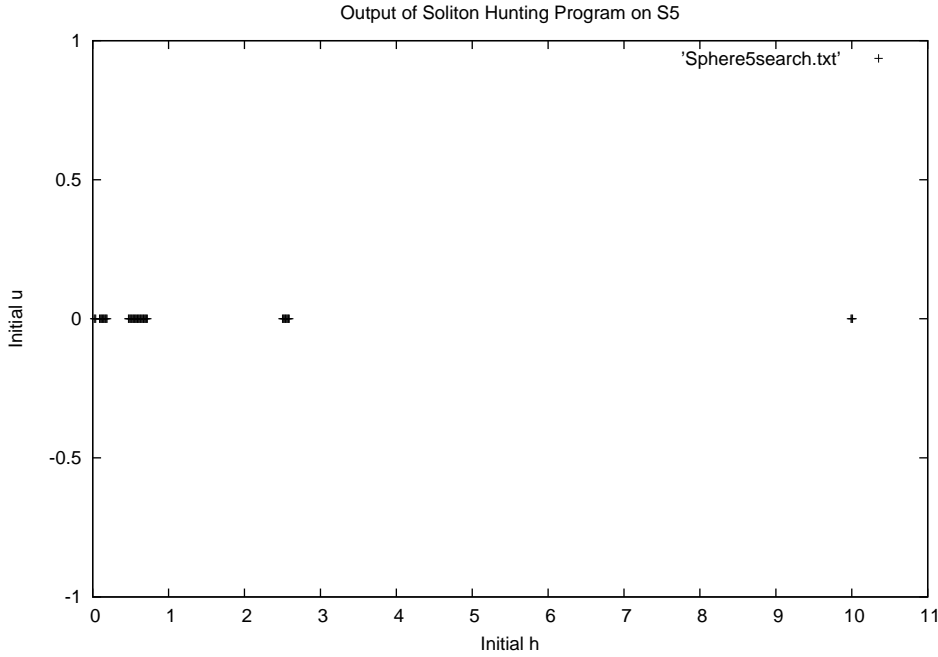
We need one of the $\mathbb{S}^2$ factors to collapse at one end, and the other factor to collapse at the other end. We are looking for a $t = t_{sol} > 0$ such that

$$(f(t_{sol}), \dot{f}(t_{sol}), h(t_{sol}), \dot{h}(t_{sol}), u(t_{sol}), \dot{u}(t_{sol})) = (\tilde{f}, 0, 0, -1, \tilde{u}, 0)$$

for some $\tilde{f} > 0$ and some $\tilde{u}$. We output initial conditions where

$$SOL(t) = \dot{f}^2(t) + h^2(t) + (\dot{h}(t) + 1)^2 + \dot{u}^2(t)$$

satisfies $SOL < 0.005$ for some $t > 0$.

The only values the program plotted are on the 'Einstein axis', $\bar{u} = 0$. We can see a cluster of points around $\bar{h} = 10$ which correspond to the standard metric, a cluster around 2.5 which is the first Böhm metric ($\bar{h} \approx 2.53554$), then a final large cluster around 0 (of course there should be infinitely many clusters between 2.5 and 0 but obviously our numerics cannot detect all these). The authors are yet to find a cluster away from the Einstein axis. It is interesting to restrict the algorithm to $\bar{u} = 0$ and output the value of SOL closest to 0 for each $\bar{h}$. This plot gives us

Closest point on trajectory to closing up on S5
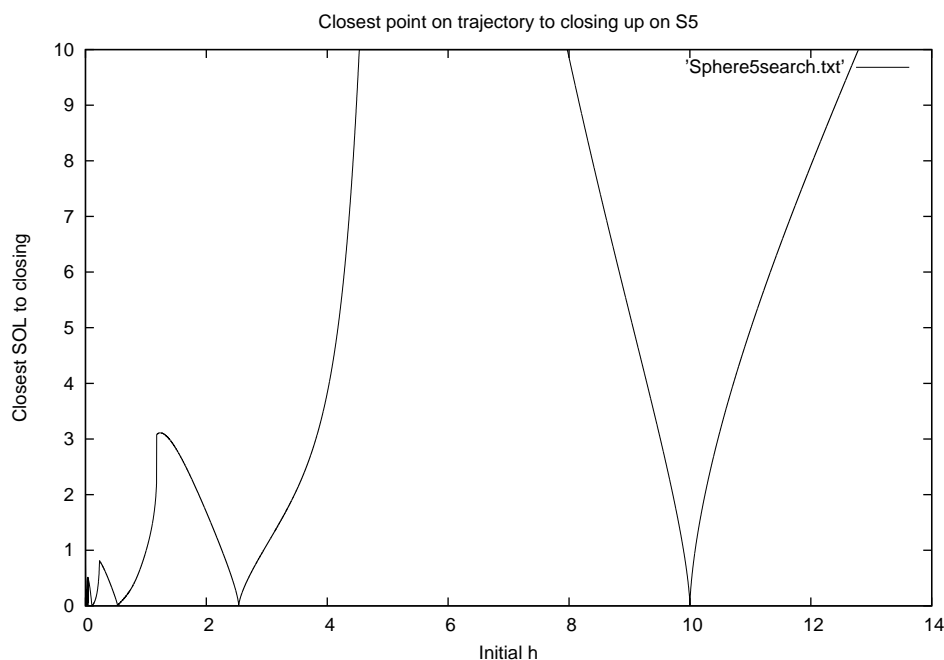
We can clearly pick out the first three values of $\bar{h}$ corresponding to the Böhm metrics $\bar{h} \approx 10, 2.5354, 0.53054$. We also see that the behaviour gets increasingly complicated as $\bar{h} \to 0$.

5.3. Numerical results on $\mathbb{S}^2 \times \mathbb{S}^3$. As in the $\mathbb{S}^5$ case, the group acting on the manifold is $G = SO(3) \times SO(3)$ and the principal orbits are $\mathbb{S}^2 \times \mathbb{S}^2$.

We now need the same $\mathbb{S}^2$ factor to collapse at each end. We are looking for a $t = t_{sol} > 0$ such that

$$(f(t_{sol}), \dot{f}(t_{sol}), h(t_{sol}), \dot{h}(t_{sol}), u(t_{sol}), \dot{u}(t_{sol})) = (0, -1, \tilde{h}, 0, \tilde{u}, 0)$$

for $\tilde{h} > 0$ and some $\tilde{u}$. As in the $\mathbb{S}^5$ case, we do not find any non-trivial solitons. The hunting program does output clusters of points corresponding to the Böhm-Einstein metrics on this manifold. As it is more illuminating we output the smallest values of SOL along some trajectories with $\bar{u} = 0$.

Closest point on trajectory to closing up on S3xS2
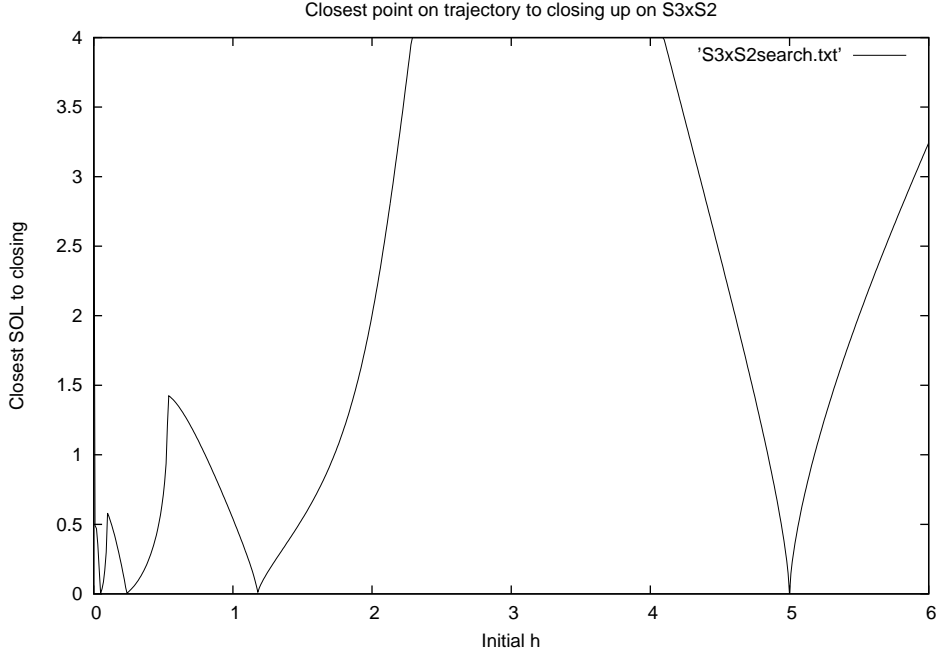
So we see we can pick out the standard product metric on this space $\bar{h} = 5$ and then the Böhm metrics corresponding to the initial values $\bar{h} \approx 1.1779$ and 0.23571.

As mentioned earlier, the analytical methods used by Böhm to construct Einstein metrics only work in the dimension range $5 \leq \dim M \leq 9$. It is not clear, however, for which orbit types this restriction is essential.

It is also conceivable that soliton solutions might exist on manifolds where Einstein metrics do not. This is certainly the situation in the Kähler case.

We therefore examine next an example in higher dimensions, namely $\mathbb{S}^{11}$.

5.4. Numerical results on $\mathbb{S}^{11}$. In this example the group is $G = SO(6) \times SO(6)$ and the principal orbits are $\mathbb{S}^5 \times \mathbb{S}^5$.

The only cluster found by this search is one around $(10, 0)$ corresponding to the standard round metric on $\mathbb{S}^{11}$.

5.5. Numerical results on $\mathbb{HP}^{n+1} \sharp \overline{\mathbb{HP}}^{n+1}$. Another interesting case is that of the connected sum of two copies of quaternionic projective space with opposite orientations. In this case the groups are $G = Sp(1) \times Sp(n+1)$, $K = \Delta Sp(1) \times Sp(n)$ and $H = Sp(1) \times Sp(1) \times Sp(n)$. The principal orbits are $G/K = \mathbb{S}^{4n+3}$, and we have $d_1 = 3, d_2 = 4n$ and $C_Q = 4n + 8$. The special orbits are $\mathbb{HP}^n$, and the constant $\|\mathcal{A}\|^2 = 3$.

In the case $n = 1$ Böhm [B1] was able to prove the existence of a cohomogeneity one Einstein metric on $\mathbb{HP}^2 \sharp \overline{\mathbb{HP}}^2$. He was also able to give numerical evidence for the existence of two cohomogeneity one Einstein metrics in many cases where $n \geq 2$, although these examples are outside the dimension range where his analytical methods work (some of these examples were also found numerically by Page and Pope [PP]).

We also looked for solitons in these cases. We ran the hunting program for a variety of values of n, using the initial values $\bar{h} \in (0.1, 10)$ and $\bar{u} \in (-\frac{4n+4}{2}, \frac{4n+4}{2})$. In order to increase accuracy, we used a step size of 0.001 in the Runge-Kutta algorithm.

We could not find any non-trivial solitons. When $n = 1$ we found the Böhm metric at $\bar{h} \approx 1.060793$. We include a plot of SOL below for this case. It demonstrates a certain instability that all the Einstein metrics found on these manifolds shared, namely that one needs to be close to the exact conditions in order to get the trajectories to come close to closing up.

Closest point on trajectory to closing up on connect sum of HP2
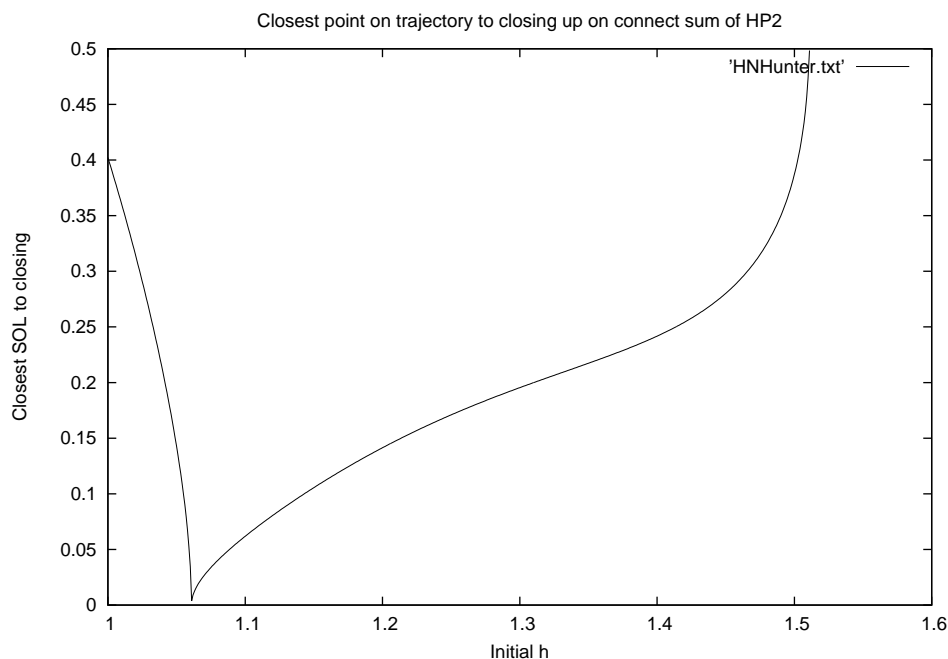
We also include the same plot when $n = 2$. Notice we recover metrics that Böhm found numerically with $\bar{h} \approx 1.0856062$ and $\bar{h} \approx 0.2184791$.

Closest point on trajectory to closing up on connect sum of HP3
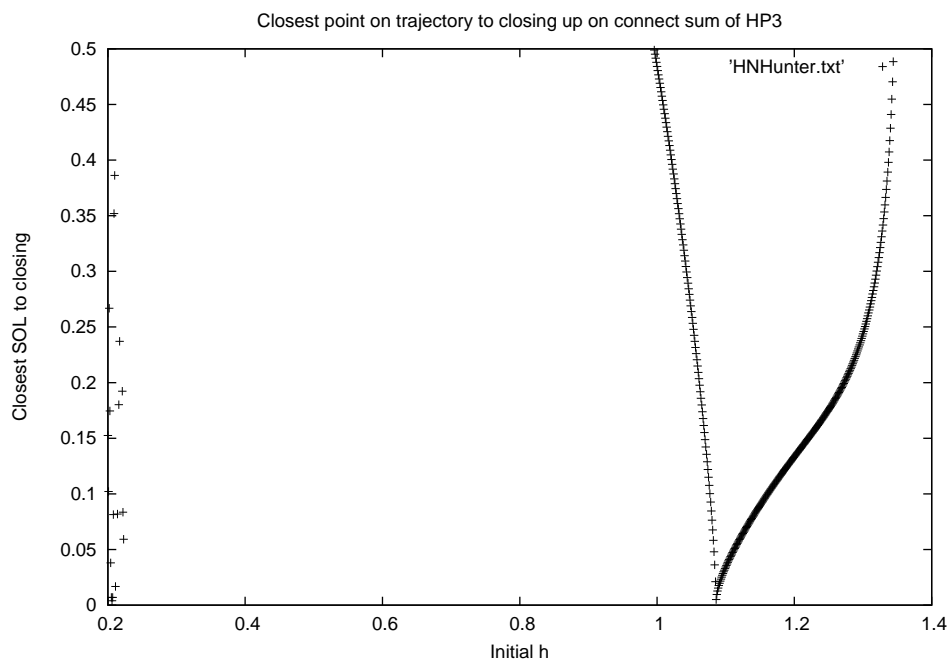
5.6. Numerical results for F^{n+1} . We also considered the space F^{n+1} from [B1]. For this space $G = Sp(n+1), K = U(1) \times Sp(n)$ and $H = Sp(1) \times Sp(n)$, so the principal orbit is $\mathbb{CP}^{2n+1}$, and the special orbits are $\mathbb{HP}^n$. The fibration $H/K \to G/K \to G/H$ is the twistor fibration for $\mathbb{HP}^n$, and we have $d_1 = 2$ and $d_2 = 4n$, as well as $C_Q = 4n + 8$ and $\|\mathcal{A}\|^2 = 8$.

Again we can recover Einstein metrics that Böhm and Gibbons-Page-Pope [GPP] found numerically, but we did not find any nontrivial solitons. We include below the results for $n = 1$ where we can see there is evidence for an Einstein metric at $\bar{h} \approx 0.866$.

Closest point on trajectory to closing up on F2
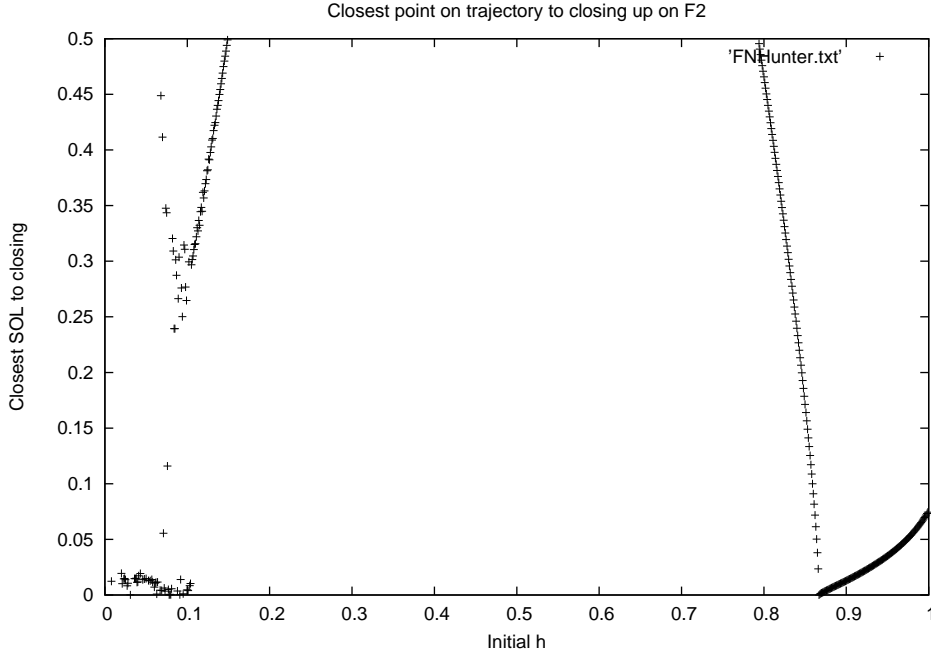
5.7. Numerical results on $Ca\mathbb{P}^2 \sharp \overline{Ca\mathbb{P}}^2$. In the case of the connected sum of two copies of the Cayley plane with opposite orientations, we have $G = Spin(9), K = Spin(7)$ and $H = Spin(8)$, so the principal orbit is $\mathbb{S}^{15}$, and the special orbits are $\mathbb{S}^8$. Now $d_1 = 7$ and $d_2 = 8$, with $Ric_Q = 28$ and $\|\mathcal{A}\|^2 = 7$.

We ran the soliton hunting program but did not find any evidence of Einstein metrics or solitons.

5.8. Numerical results on the non-trivial sphere bundle over $\mathbb{S}^2$. We have also investigated numerically the soliton equations on the non-trivial sphere bundle $\mathbb{S}^d \to \mathbb{S}^2$. These manifolds are cohomogeneity one with principal orbit $\mathbb{S}^3 \times \mathbb{S}^{d-2}$ and the special orbits are $\mathbb{S}^2 \times \mathbb{S}^{d-2}$ and $\mathbb{S}^3$. Due to work of Hashimoto, Sakaguchi and Yasui [HSY] it is known that these manifolds admit a cohomogeneity one Einstein metric (this metric is the Page metric on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ in the case $d = 2$). The numerical investigations did not provide any evidence of a non-Einstein soliton when $d > 2$.

5.9. A Non-compact example–the Gaussian. We conclude by discussing the smooth non compact Gaussian of Example 4.11. We consider the case when the principal orbits are $\mathbb{S}^2 \times \mathbb{S}^2$. With our normalisations that $C = 0$ in equation (5.4)

and $\frac{\epsilon}{2} = -4$, the Gaussian soliton is described by

$$dt^2 + t^2 dS_2^2 + \frac{1}{2} dS_2^2$$

where dS_2^2 is the usual Einstein metric on $\mathbb{S}^2$ with Einstein constant 1. The soliton potential function is then given by $u(t) = 2t^2 - \frac{3}{2}$. When we input the initial condition $\bar{h} = 0.5$ and $\bar{u} = -1.5$ and integrate we indeed recover the Gaussian soliton for a short time. However, as the numerical errors accumulate along the trajectory the numerical solution deviates from the Gaussian and very quickly becomes singular. Initial conditions close to the Gaussian also display this behaviour. This suggests instability of the soliton, so that it might be difficult to produce complete non-compact solutions close to the Gaussian. It should be noted that there are some gap theorems for the Gaussian soliton on Euclidean space, due to O. Munteanu-M.-T. Wang [MuW] and Yokota [Y].

REFERENCES

[ACGT] V. Apostolov, D. Calderbank, P. Gauduchon, and C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry IV: weakly Bochner-flat Kähler manifolds*, Comm. Anal. Geom., 16 (2008), pp. 91–126.

[BB]   L. Bérard Bergery, *Sur des nouvelles variétés Riemanniennes d'Einstein*, Publications de l'Institut Elie Cartan, Nancy, (1982).

[B1]   C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces,* Invent. Math., 134 (1998), pp. 145–176.

[B2]   C. Böhm, *Non-compact cohomogeneity one Einstein manifolds*, Bull. Soc. Math. France, 127 (1999), pp. 135–177.

[B3]   C. Böhm, *Nonexistence of cohomogeneity one Einstein metrics*, Math. Ann., 314 (1999), pp. 109–125.

[B4]   C. Böhm, *Non-existence of homogeneous Einstein metrics*, Comm. Math. Helv., 80 (2005), pp. 123–146.

[BK]   C. Böhm and M. Kerr, *Low-dimensional homogeneous Einstein manifolds*, Trans. A. M. S., 358:4 (2005), pp. 1455–1468.

[Bry]   R. Bryant, unpublished.

[But]   J. C. Butcher, *Numerical methods for ordinary differential equations*, John Wiley & Sons, Ltd., Chichester, 2003.

[Bu]   M. Buzano, *Initial value problem for cohomogeneity one gradient Ricci solitons*, J. Geom. Phys., 61 (2011), pp. 1033–1044.

[Ca1]   H. D. Cao, *Existence of gradient Ricci solitons*, Elliptic and Parabolic Methods in Geometry, A. K. Peters, (1996), pp.1–16.

[Ca2]   H. D. Cao, *Recent progress on Ricci solitons,* Adv. Lect. Math., 11 (2009), pp. 1–38.

[Ca3]   H. D. Cao, *Geometry of complete gradient shrinking Ricci solitons,* Geometry and Analysis, Vol. 1, Adv. Lect. Math., 17 (2011), pp. 227–246.

[CZ]   H. D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons,* J. Diff. Geom., 85 (2010), pp. 175–185.

[Ch]   B. L. Chen, *Strong uniqueness of the Ricci flow*, J. Diff. Geom., 82 (2009), pp. 363–382.

[Cetc] B. Chow, S. C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, part I: geometric aspects*, Mathematical Surveys and Monographs Vol. 135, American Math. Soc., (2007).

[DW1]   A. Dancer and M. Wang, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom., 39 (2011), pp. 259–292.

[DW2]   A. Dancer and M. Wang, *New examples of non-Kähler Ricci solitons,* Math. Res. Lett., 16 (2009), pp. 349–363.

[DW3]   A. Dancer and M. Wang, *Non-Kähler expanding Ricci solitons,* Int. Math. Research Notices, 2009:6 (2009), pp. 1107–1133.

[De]   A. Derdzinski, *A Myers-type theorem and compact Ricci solitons,* Proc. A. M. S., 134 (2006), pp. 3645–3648.

[EW] J. ESCHENBURG AND M. WANG, *The Initial value problem for cohomogeneity one Einstein metrics*, J. Geom. Anal., 10 (2000), pp. 109–137.

[F] D. FRIEDAN, *Nonlinear models in $2 + \epsilon$ dimensions*, Ann. Phys., 163 (1985), pp. 318–419.

[FIK] M. FELDMAN, T. ILMANEN, AND D. KNOPF, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Diff. Geom., 65 (2003), pp. 169–209.

[FG] M. FERNÁNDEZ-LÓPEZ AND E. GARCÍA-RÍO, *A remark on compact Ricci solitons,* Math. Ann., 340 (2008), pp. 893–896.

[FuW] A. FUTAKI AND M.-T. WANG, *Constructing Kähler-Ricci solitons from Sasaki-Einstein manifolds*, Asian J. Math., 15 (2011), pp. 33–52.

[GK] A. GASTEL AND M. KRONZ, *A family of expanding Ricci solitons*, Variational Problems in Riemannian Geometry, Prog. Nonlinear Differential Equations Appl., 59, Birkhäuser, Basel (2004), pp. 81–93.

[GPP] G. W. GIBBONS, D. PAGE, AND C. N. POPE, *Einstein metrics on $\mathbb{S}^3, \mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, Commun. Math. Phys., 127 (1990), pp. 529–553.

[Ha] R. S. HAMILTON, *The Ricci Flow on Surfaces*, in Mathematics and General Relativity (Santa Cruz, CA, 1986), Contemp Math., 71, Amer. Math. Soc., (1988), pp. 237–262.

[HLP] G. HARDY, J. E. LITTLEWOOD, AND G. POLYA, *Inequalities*, Cambridge University Press, (1988).

[HSY] Y. HASHIMOTO, M. SAKAGUCHI, AND Y. YASUI, *New infinite series of Einstein metrics on sphere bundles from AdS black holes*, Commun. Math. Phys., 257 (2005), pp. 273–285.

[Iv] T. IVEY, *New examples of complete Ricci solitons*, Proc. AMS, 122 (1994), pp. 241–245.

[Kod] T. KODA, *A remark on the manifold $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ with Bérard-Bergery's metric*, Ann. Global Anal. Geom., 11 (1993), pp. 323–329.

[Koi] N. KOISO, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Adv. Studies Pure Math., 18-I, Academic Press, (1990), pp. 327–337.

[Kot] B. KOTSCHWAR, *On rotationally invariant shrinking Ricci solitons*, Pacific J. Math., 236 (2008), pp. 73–88.

[La] J. LAURET, *Ricci soliton homogeneous nilmanifolds*, Math. Ann., 319 (2001), pp. 715–733.

[MuW] O. MUNTEANU AND M.-T. WANG, *The curvature of gradient Ricci solitons*, Math. Res. Lett., 18:6 (2011), pp. 1051–1069.

[Na] A. NABER, *Noncompact shrinking four solitons with nonnegative curvature*, J. reine angew. Math., 645 (2010), pp. 125–153.

[Pa] D. PAGE, *A compact rotating gravitational instanton,* Phys. Lett., 79B (1978), pp. 235–238.

[PP] D. PAGE AND C. N. POPE, *Einstein metrics on quaternionic line bundles*, Class. Quant. Gravity, 3 (1986), pp. 249–259.

[Pe] G. PERELMAN, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:/0211.1159.

[PW] P. PETERSEN AND W. WYLIE, *Rigidity of gradient Ricci solitons,* Pacific J. Math., 241 (2009), pp. 329–345.

[PRS] S. PIGOLA, M. RIMOLDI, AND A. SETTI, *Remarks on non-compact gradient Ricci solitons*, Math. Z., 268 (2011), pp. 777–790.

[PS] F. PODESTÀ AND A. SPIRO, *Kähler-Ricci solitons on homogenous toric bundles*, J. reine angew. Math., 642 (2010), pp. 109–127.

[WZ] M. WANG AND W. ZILLER, *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math., 84 (1986), pp. 177–194.

[WZh] X.-J. WANG AND X. ZHU, *Kähler-Ricci solitons on toric manifolds with positive first Chern class,* Adv. Math., 188 (2004), pp. 87–103.

[Wi] E. WITTEN, *String theory and black holes*, Phys. Rev. D, 44 (1991), pp. 314–324.

[Y] T. YOKOTA, *Perelman's reduced volume and gap theorem for the Ricci flow*, Comm. Anal. Geom., 17 (2009), pp. 227–263.

[Zh] Z.-H. ZHANG, *On the completeness of gradient Ricci solitons*, Proc. A. M. S., 137 (2009), pp. 2755–2759.