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CALABI-YAU MANIFOLDS AND GENERIC HODGE GROUPS*

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Abstract. We study the generic Hodge groups $\mathrm{Hg}(\mathcal{X})$ of local universal deformations $\mathcal{X}$ of Calabi-Yau 3-manifolds with onedimensional complex moduli, give a complete list of all possible choices for $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ and determine the latter real groups for known examples.

Key words. Calabi-Yau manifold, variation of Hodge structures.

AMS subject classifications. 14D07.

Introduction. Let X be a Calabi-Yau 3-manifold with $h^{2,1}(X) = 1$. Moreover let $f : \mathcal{X} \to B$ denote the local universal deformation of X and Q denote the symplectic form on $H^3(X,\mathbb{Q})$ given by the cup product. In the generic Hodge group $\mathrm{Hg}(\mathcal{X})$ information about the arithmetic of the fibers, the variation of Hodge structures and the monodromy groups of the families containing X as fiber is encoded. Similar computations are made for the Lie algebras of monodromy groups of families of $K3$ surfaces in the appendix of [15]. Since the monodromy group $\mathrm{Mon}^0(\mathcal{X})$ is a normal subgroup of the derived Hodge group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})$, this is related to our results. Here we classify the possible generic Hodge groups of $\mathcal{X}$, which is also a natural problem by itself.

In the case of a Calabi-Yau 3-manifold with $h^{2,1}(X) = 1$ we consider a Hodge structure on $H^3(X,\mathbb{Q})$, which is a vector space of dimension 4. We have much information about the variation of Hodge structures (VHS) of families of Calabi-Yau 3-manifolds. For example by Bryant, Griffiths [2], we have a classical description of the VHS of such families. By using the Hodge structure on $H^3(X,\mathbb{Q})$, one can construct the associated Weil- and the Griffiths intermediate Jacobians and their corresponding Hodge structures as introduced by C. Borcea [1]. These latter Hodge structures are given by the representations h_W and h_G of the circle group S^1 on $H^3(X,\mathbb{Q})$. In particular the centralizers $C(h_G(i))$ and $C(h_W(i))$ in $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ will be helpful. By using these techniques, the theory of bounded symmetric domains [6], the theory of Shimura varieties [3], [4], [7], [9] and some intricate computations, we obtain the result:

Theorem 0.1. *Let $\mathcal{X}$ denote the local universal deformation of a Calabi-Yau 3-manifold X with $h^{2,1}(X) = 1$. Then one of the following cases holds true:*

1.

$$\mathrm{Hg}(\mathcal{X}) = \mathrm{Sp}(H^3(X,\mathbb{Q}),Q)$$

2.

$$\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$$

3. The representation of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ on $H^3(X,\mathbb{R})$ is isomorphic to the natural representation of $\mathrm{SL}_{\mathbb{R}}(2)$ on $\mathrm{Sym}^3(\mathbb{R}^2)$.

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In the case (2) we will also give an explicite description of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$, which tells us that one has a reducible representation in this case. At present there does not exist any example of a family of Calabi-Yau 3-manifolds known to the author, which has a generic Hodge group satisfying (3). Nevertheless we will determine the generic Hodge groups of known examples of Calabi-Yau 3-manifolds and see that there exists a Calabi-Yau like variation of Hodge structures satisfying (3).

1. Facts and conventions. Here a Calabi-Yau 3-manifold X is a compact Kähler manifold of complex dimension 3 such that

$$H^{1,0}(X) = H^{2,0}(X) = 0 \text{ and } \omega_X \cong \mathcal{O}_X.$$

We will only study Calabi-Yau 3-manifolds X with $h^{2,1}(X) = 1$ here. Let $f : \mathcal{X} \to B$ denote the local universal deformation of $X \cong \mathcal{X}_0$, where $0 \in B$.

Moreover recall the algebraic groups

$$S^1 = \mathrm{Spec}(\mathbb{R}[x, y]/x^2 + y^2 - 1) \text{ and } \mathbb{S} = \mathrm{Spec}(\mathbb{R}[t, x, y]/t(x^2 + y^2) - 1),$$

where

$$S^1(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \right\} \cong \{z \in \mathbb{C} : |z| = 1\}$$

and

$$\mathbb{S}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\} \cong \mathbb{C}^*.$$

The group $\mathbb{S}$ is the Deligne torus given by the Weil restriction $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ and S^1 is a subgroup of $\mathbb{S}$. Let V be a real vector space. By the eigenspace decompositions of $V_{\mathbb{C}}$ with respect to the characters $z^p \bar{z}^q$ for $p, q \in \mathbb{Z}$ of $\mathbb{S}$, the real representations $h : \mathbb{S} \to \mathrm{GL}(V)$ correspond to the Hodge structures on V (see [4], 1.1.1). If there is some fixed k such that all characters $z^p \bar{z}^q$ with non-trivial associated eigenspace satisfy $p + q = k$, one says that the Hodge structure has weight k. There exists an embedding $w : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ given by

$$\mathbb{G}_m(\mathbb{R}) \cong \{\mathrm{diag}(a, a) \in \mathrm{GL}_2(\mathbb{R})\} \overset{\mathrm{id}}{\hookrightarrow} \mathbb{S}(\mathbb{R}).$$

The Hodge structure h has weight k, if and only if the weight homomorphism $h \circ w$ satisfies

$$r \to \mathrm{diag}(r^k, \ldots, r^k) \quad (\forall \ r \in \mathbb{R}^* = \mathbb{G}_m(\mathbb{R}))$$

(see [10], Remark 1.1.4). Hodge structures of some given weight k are determined by the restricted representation $h|_{S^1}$. For example the integral Hodge structure on $H^3(X, \mathbb{Z})$ of weight 3 corresponds to the representation

$$h_X : S^1 \to \mathrm{GL}(H^3(X, \mathbb{R})), \quad h_X(z)v = z^p \bar{z}^q v \quad (\forall v \in H^{p,q}(X) \text{ with } p + q = 3).$$

We also denote h_X by h for short. The Hodge group $\mathrm{Hg}(H^3(X, \mathbb{Q}), h) \subset \mathrm{GL}(H^3(X, \mathbb{Q}))$ is the smallest $\mathbb{Q}$-algebraic group $G \subset \mathrm{GL}(H^3(X, \mathbb{Q}))$ with $h(S^1) \subset G_{\mathbb{R}}$. Assume without loss of generality that B is contractible. Thus for each $b \in B$ one has a canonical isomorphism

$$H^3(\mathcal{X}_b, \mathbb{Q}) \cong R^3 f_*(\mathbb{Q})(B) \cong H^3(\mathcal{X}_0, \mathbb{Q}) = H^3(X, \mathbb{Q}).$$

By using this isomorphism, a subgroup of $\mathrm{GL}(H^3(\mathcal{X}_b,\mathbb{Q}))$ can be considered as a subgroup of $\mathrm{GL}(H^3(X,\mathbb{Q}))$. This allows to define an inclusion relation for the Hodge groups of the several fibers, which we use now. The generic Hodge group $\mathrm{Hg}(\mathcal{X})$ of $\mathcal{X}$ is given by the generic Hodge group of the rational variation of Hodge structures (VHS) of weight 3 of $\mathcal{X}$. Recall that the generic Hodge group of a VHS is the maximum of the Hodge groups of all occurring Hodge structures. In an analogue way one can define the Mumford-Tate group $\mathrm{MT}(H^3(X,\mathbb{Q}),h)$ and the generic Mumford-Tate group $\mathrm{MT}(\mathcal{X})$ by using $h(\mathbb{S})$ instead of $h(S^1)$. One has that $\mathrm{MT}(H^3(\mathcal{X}_b,\mathbb{Q}),h_b)=\mathrm{MT}(\mathcal{X})$ over the complement of countably many proper analytic subsets of the basis (follows from [9], 1.2). Since

$$\mathrm{Hg}(H^3(\mathcal{X}_b,\mathbb{Q}),h)=(\mathrm{MT}(H^3(\mathcal{X}_b,\mathbb{Q}),h)\cap\mathrm{SL}(H^3(\mathcal{X}_b,\mathbb{Q})))^0$$

(see [10], Lemma 1.3.17), one has also that $\mathrm{Hg}(H^3(\mathcal{X}_b,\mathbb{Q}),h_b)=\mathrm{Hg}(\mathcal{X})$ over the complement of countably many proper analytic subsets of the basis.

1.1. We consider only algebraic groups over fields K of characteristic zero. A group G over K is a torus, if $G_{\bar{K}}\cong\mathbb{G}^{\ell}_{m,\bar{K}}$. Moreover a group G is simple, if it does not contain any proper connected normal subgroup. We say that G is semisimple, if its maximal connected normal solvable subgroup is trivial.

A group G is reductive, if it is the almost direct product of a torus and a semisimple group. In this situation the torus can be given by the connected component of identity of the center $Z(G)$ of G and the semisimple group can be given by the derived subgroup G^{der} generated by the commutators (follows from [12], page 9).

Let ad denote the adjoint representation. For a reductive group G, we have the exact sequence

$$1\to Z(G)\to G\to G^{\mathrm{ad}}\to 1$$

and the adjoint group G^{ad} and G^{der} are isogenous.

We say that a semisimple group is adjoint, if its center is trivial. It is a well-known fact that connected semisimple adjoint $\mathbb{R}$-algebraic groups are direct products of simple subgroups.

It is a well-known fact that $\mathrm{Hg}(\mathcal{X})^0_{\mathbb{R}}$ and $\mathrm{MT}(\mathcal{X})^0_{\mathbb{R}}$ is defined over $\mathbb{Q}$. Moreover

$$h(S^1)\subset\mathrm{Hg}(\mathcal{X})^0_{\mathbb{R}}\ \text{ and }\ h(\mathbb{S})\subset\mathrm{MT}(\mathcal{X})^0_{\mathbb{R}}.$$

Thus

$$\mathrm{Hg}(\mathcal{X}),\ \ \mathrm{Hg}(\mathcal{X})_{\mathbb{R}},\ \ \mathrm{MT}(\mathcal{X})\ \text{ and }\ \mathrm{MT}(\mathcal{X})_{\mathbb{R}}$$

are Zariski connected. Moreover Hodge groups and Mumford-Tate groups of polarized rational Hodge structures are reductive (for example see [10], Theorem 1.3.16 and Corollary 1.3.20). From this fact and the definition of reductive groups one concludes that

$$\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}},\ \ \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}},\ \ \mathrm{MT}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}\ \text{ and }\ \mathrm{MT}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$$

are also Zariski connected.

By knowing the associated Lie groups of $\mathbb{R}$-valued points, one can determine the isomorphism classes of some algebraic groups of our interest:

LEMMA 1.2. *Assume that G and H are $\mathbb{R}$-algebraic connected semisimple adjoint groups, where $H(\mathbb{R})$ is a connected Lie group. Moreover let $h : G(\mathbb{R})^+ \to H(\mathbb{R})$ be an isomorphism of Lie groups. Then G and H are isomorphic as $\mathbb{R}$-algebraic groups.*

Proof. From the assumptions we conclude that there is an isomorphism $dh_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$. Note that $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are also semisimple as real Lie algebras and that for an arbitrary real Lie algebra $\mathfrak{g}'$ one can define its adjoint Lie group $\mathrm{Int}(\mathfrak{g}')$ (see [6], **II**. §5). Due to the assumption that G and H are semisimple adjoint, the adjoint representation yields isomorphisms

$$G(\mathbb{C})^+ \cong \mathrm{Int}(\mathfrak{g}_{\mathbb{C}}) \quad \text{and} \quad H(\mathbb{C})^+ \cong \mathrm{Int}(\mathfrak{h}_{\mathbb{C}}).$$

Moreover for a real semisimple Lie algebra $\mathfrak{g}'$ the connected component of identity of the Lie group given by the automorphism group of $\mathfrak{g}'$ coincides with $\mathrm{Int}(\mathfrak{g}')$ (see [6], **II**. Corollary 6.5). Thus one concludes that $G(\mathbb{C})^+$ and $H(\mathbb{C})^+$ are the connected components of identity of the Lie groups given by the automorphism groups of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$. Therefore one obtains a holomorphic isomorphism $h_{\mathbb{C}} : G(\mathbb{C})^+ \to H(\mathbb{C})^+$. By [12], **I**. Proposition 3.5, the semisimple Lie groups $G(\mathbb{C})^+$ and $H(\mathbb{C})^+$ are the groups of $\mathbb{C}$-valued points of $\mathbb{C}$-algebraic groups and the homomorphism $h_{\mathbb{C}}$ is a $\mathbb{C}$-algebraic regular map given by some polynomials $f_1, \dots, f_k$ over $\mathbb{C}$. Since $h_{\mathbb{C}}|_{G(\mathbb{R})^+}$ coincides with $h : G(\mathbb{R})^+ \to H(\mathbb{R})$, one concludes that $\Im f_1, \dots, \Im f_k$ vanish on the Zariski closure of $G(\mathbb{R})^+$. The Zariski closure of $G(\mathbb{R})^+$ is G, since we assume that G is Zariski connected. Thus the isomorphism h is $\mathbb{R}$-algebraic. □

1.3. Let G be a connected $\mathbb{R}$-algebraic group and θ be an involutive automorphism of G. We say that θ is a Cartan involution, if the Lie subgroup

$$G^{\theta}(\mathbb{R}) = \{g \in G(\mathbb{C}) | g = \theta(\bar{g})\}$$

of $G(\mathbb{C})$ is compact. An $\mathbb{R}$-algebraic group G has a Cartan involution, if and only if G is reductive (see [10], Proposition 1.3.10). In the case of a compact connected $\mathbb{R}$-algebraic group K we have the Cartan involution id_K (see [10], Example 1.3.11). Thus all compact connected $\mathbb{R}$-algebraic groups are reductive.

The Griffiths intermediate Jacobian J_G resp., the Weil intermediate Jacobian J_W is the torus corresponding to the weight 1 Hodge structure given by

$$F^1_G(H^3(X,\mathbb{C})) = F^2(H^3(X,\mathbb{C})) \ \text{ resp.,} \ F^1_W(H^3(X,\mathbb{C})) = H^{3,0}(X) \oplus H^{1,2}(X).[1]$$

Let $h_G : S^1 \to \mathrm{GL}(H^3(X,\mathbb{R}))$ and $h_W : S^1 \to \mathrm{GL}(H^3(X,\mathbb{R}))$ denote the corresponding representations. It is a well-known fact that weight 1 Hodge structures correspond to complex structures. We will use the complex structures

$$h_G(i) \quad \text{and} \quad h_W(i) = -h_X(i).$$

Moreover $h_W(z)$ and $h_G(z)$ commute and

$$h(z) = h_G^2(z) h_W(z).$$

[1]Note that in [1] one has

$$F^1_W(H^3(X,\mathbb{C})) = H^{0,3}(X) \oplus H^{2,1}(X) \ \text{ instead of } \ F^1_W(H^3(X,\mathbb{C})) = H^{3,0}(X) \oplus H^{1,2}(X).$$

But this is only a matter of the chosen conventions and personal preferences.

Let Q denote the symplectic form on $H^3(X,\mathbb{Q})$ given by the cup product. For the rest of this article let us fix $v_{p,3-p} \in H^{p,3-p}(X) \setminus \{0\}$ with

$$\bar{v}_{p,3-p} = v_{3-p,p} \quad \text{and} \quad Q(iv_{3,0}, v_{0,3}) = Q(-iv_{2,1}, v_{1,2}) = 1.$$

There exist unique vectors satisfying these properties because of the well-known form of the polarization of $H^3(X,\mathbb{C})$ (see [14], 7.1.2) and the given Hodge numbers in our case. Thus our alternating form Q on $H^3(X,\mathbb{C})$ is given by

$$(1)\qquad Q(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}) = (v_1, v_2, v_3, v_4) \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

with respect to the basis $\{v_{3,0}, v_{1,2}, v_{2,1}, v_{0,3}\}$.

The reader can easily check that each $M \in \mathrm{GL}(H^3(X,\mathbb{R}))$ is given by a matrix

$$M = \begin{pmatrix} v_1 & w_1 & \bar{w}_4 & \bar{v}_4 \\ v_2 & w_2 & \bar{w}_3 & \bar{v}_3 \\ v_3 & w_3 & \bar{w}_2 & \bar{v}_2 \\ v_4 & w_4 & \bar{w}_1 & \bar{v}_1 \end{pmatrix}, \quad \text{where} \quad v_1, \ldots, v_4, w_1, \ldots, w_4 \in \mathbb{C}$$

with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$ by using the $\mathbb{R}$-vector space isomorphism given by the trace map

$$F^2(H^3(X,\mathbb{C})) \to H^3(X,\mathbb{R}), \quad w \to w + \bar{w}.$$

In a similar way on can easily check that the matrices with complex entries, which will occur in this paper, are in fact real.

Remark 1.4. The conjugation by elements of $h_X(S^1)(\mathbb{R})$ is given by

$$\begin{pmatrix} \xi^3 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \bar{\xi} & 0 \\ 0 & 0 & 0 & \bar{\xi}^3 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} \bar{\xi}^3 & 0 & 0 & 0 \\ 0 & \bar{\xi} & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi^3 \end{pmatrix}$$
$$= \begin{pmatrix} a_{1,1} & \xi^2 a_{1,2} & \xi^4 a_{1,3} & \xi^6 a_{1,4} \\ \bar{\xi}^2 a_{2,1} & a_{2,2} & \xi^2 a_{2,3} & \xi^4 a_{2,4} \\ \bar{\xi}^4 a_{3,1} & \bar{\xi}^2 a_{3,2} & a_{3,3} & \xi^2 a_{3,4} \\ \bar{\xi}^6 a_{4,1} & \bar{\xi}^4 a_{4,2} & \bar{\xi}^2 a_{4,3} & a_{4,4} \end{pmatrix}$$

with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$. Moreover the conjugation by the elements of $h_W(S^1(\mathbb{R}))$ is given by:

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \bar{\xi} & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \bar{\xi} \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} \bar{\xi} & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \bar{\xi} & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a_{1,1} & \xi^2 a_{1,2} & a_{1,3} & \xi^2 a_{1,4} \\ \bar{\xi}^2 a_{2,1} & a_{2,2} & \bar{\xi}^2 a_{2,3} & a_{2,4} \\ a_{3,1} & \xi^2 a_{3,2} & a_{3,3} & \xi^2 a_{3,4} \\ \bar{\xi}^2 a_{4,1} & a_{4,2} & \bar{\xi}^2 a_{4,3} & a_{4,4} \end{pmatrix}$$

Remark 1.5. The centralizer $C(h(S^1))$ of $h(S^1)$ in $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ is given by matrices $\mathrm{diag}(\xi,\zeta,\bar{\zeta},\bar{\xi})$ with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$ as one concludes by the description of the conjugation by elements of $h(S^1)(\mathbb{R})$ in Remark 1.4. Moreover by explicit computations using (1), one concludes $|\xi| = |\zeta| = 1$. Thus

$C(h(S^1)) \cong S^1 \times S^1$. The group of real symplectic automorphisms in $C(h(S^1))$, whose order is atmost 4, is generated by $\mathrm{diag}(1,i,-i,1)$ and $\mathrm{diag}(i,1,1,-i)$. Thus $C(h(S^1))$ contains only the complex structures

$$(2) \qquad \pm h_W(i) = \pm\mathrm{diag}(i,-i,i,-i) \text{ and } \pm h_G(i) = \pm\mathrm{diag}(i,i,-i,-i).$$

Moreover $C(h(S^1))$ is generated by $h_W(S^1)$ and $h_G(S^1)$. The kernel of the natural homomorphism

$$h_W(S^1) \times h_G(S^1) \to C(h)$$

obtained from multiplication is given by $\{(1,1),(-1,-1)\}$.

Let $C(h_G(i))$ and $C(h_W(i))$ denote the respective centralizers of $h_G(i)$ and $h_W(i)$ in $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$. The centralizer $C(h(i))$ of $h(i)$ in $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ coincides with $C(h_W(i))$, since $h_W(i) = -h(i)$. Let H denote the Hermitian form

$$H = iQ(\cdot,\bar{\cdot}).$$

Since $h(i)$ is a Hodge isometry of the real Hodge structure on $H^3(X,\mathbb{R})$, one concludes from the definition of H as in [10], Section 4.3 and [11], Lemma 3.4:

PROPOSITION 1.6. *The group* $C(h_G(i))$ *is given by* $\mathrm{diag}(M,\bar{M})$, *where*

$$M \in \mathrm{U}(F^2(X), H|_{F^2(X)})(\mathbb{R}) \cong \mathrm{U}(1,1)(\mathbb{R})$$

and $\bar{M}$ *acts on* $\bar{F}^2(X)$.

In an analogue way one concludes:[2]

PROPOSITION 1.7. *The group* $C(h_W(i))$ *is given by* $\mathrm{diag}(M,\bar{M})$, *where*

$$M \in \mathrm{U}(F^2(X), H|_{H^{3,0}(X)\oplus H^{1,2}(X)})(\mathbb{R}) \cong \mathrm{U}(2)(\mathbb{R})$$

and $\bar{M}$ *acts on* $H^{0,3}(X) \oplus H^{2,1}(X)$.

Thus the unitary groups $\mathrm{U}(1,1)$ and $\mathrm{U}(2)$ will be important:

REMARK 1.8. One can describe $\mathrm{U}(1,1)$ and $\mathrm{U}(2)$ explicitly. The special unitary group $\mathrm{SU}(1,1)$ is given by the matrices

$$M_1 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ with } |\alpha|^2 - |\beta|^2 = 1 \text{ and } \alpha,\beta \in \mathbb{C}$$

(see [12], page 59) and $\mathrm{SU}(2)$ is given by the matrices

$$M_2 = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ with } |\alpha|^2 - |\beta|^2 = 1 \text{ and } \alpha,\beta \in \mathbb{C}$$

as one concludes from the fact that $\bar{M}_2^t = M_2^{-1}$. Since

$$\bar{M}_1^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } M_1 \in \mathrm{U}(1,1)(\mathbb{R}),$$

[2]It should be pretty clear to the experts that the conjugacy class of $h_W(S^1)$ in $\mathrm{Sp}(H^3(X,\mathbb{Q}),Q)$ yields the upper half plane $\mathfrak{h}_2$, which is also a way to conclude that $C(h_W(i)) \cong \mathrm{U}(2)$.

one has $|\det M_1| = 1$. Thus the reductive group $\mathrm{U}(1,1)$ is the almost direct product of the simple group $\mathrm{SU}(1,1)$ and its center $Z(\mathrm{U}(1,1))$ isomorphic to S^1, where

$$\mathrm{SU}(1,1) \cap Z(\mathrm{U}(1,1)) \cong \{\pm 1\}.$$

Moreover recall that

$$\mathrm{SU}(1,1) \cong \mathrm{Sp}_{\mathbb{R}}(2) \cong \mathrm{SL}_{\mathbb{R}}(2).$$

In an analogue way one concludes that the reductive group $\mathrm{U}(2)$ is the almost direct product of the simple group $\mathrm{SU}(2)$ and its center $Z(\mathrm{U}(2))$ isomorphic to S^1, where

$$\mathrm{SU}(2) \cap Z(\mathrm{U}(2)) \cong \{\pm 1\}.$$

Moreover we will need an explicit description of the Lie algebra of $\mathrm{SU}(1,1)$:

Remark 1.9. One has that

$$M_1 = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(1,1)(\mathbb{R})$$

is unipotent, if and only if

$$2\Re(a) = tr(M_1) = 2.$$

Since each nontrivial unipotent $M_1 \in \mathrm{SU}(1,1)(\mathbb{R})$ has only one Jordan block of length 2, one computes that

$$\log M_1 = M_1 - E_2 = \begin{pmatrix} i\Im(a) & b \\ \bar{b} & -i\Im(a) \end{pmatrix}.$$

This yields 2 linearly independent vectors of $\mathfrak{su}(1,1)$ given by $\log(M_1) = M_1 - E_2$ for some unipotent M_1. By appending

$$\log(\mathrm{diag}(a,\bar{a})) = \mathrm{diag}(iy, -iy)$$

for $|a| = 1$ and $y \in \mathbb{R}$, one obtains a basis of the threedimensional algebra $\mathfrak{su}(1,1)$. Thus for each $N \in \mathfrak{su}(1,1)(\mathbb{R})$ there are suitable $u, v, y \in \mathbb{R}$ such that

$$N = \begin{pmatrix} iy & u+iv \\ u-iv & -iy \end{pmatrix}.$$

Remark 1.10. Since the centralizer $C(h_G(i)) \cong \mathrm{U}(1,1)$ of $h_G(i)$ is not compact, the conjugation by $h_G(i)$ does not yield a Cartan involution of $\mathrm{Sp}(H^3(X,\mathbb{R}), Q)$.

Lemma 1.11. *The conjugation by $h_W(i)$ and the conjugation by $h_X(i)$ yield the same Cartan involutions on $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ resp., $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$. The conjugation by $\mathrm{ad}(h_W(i))$ yields a Cartan involution on $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$.*

Proof. Note that the conjugation by a complex structure

$$J \in \mathrm{Sp}(H^3(X,\mathbb{R}), Q)(\mathbb{R}) \quad \text{with} \quad Q(J\cdot, \cdot) > 0$$

yields a Cartan involution of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ (see [7], page 67). Since $Q(h_W(i)\cdot,\cdot) > 0$ as one can verify by using (1) and (2), the conjugation by $h_W(i)$ yields a Cartan involution of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)(\mathbb{R})$. Due to the fact that $h_W(i) \in \mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$, the conjugation by $h_W(i)$ yields a Cartan involution of the subgroup $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \subset \mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ (follows from [12], **I**. Theorem 4.2). Since $h_W(i) = -h_X(i)$, the conjugation by $h_X(i)$ yields the same involution.

Due to the fact that the reductive group $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ is the almost direct product of $Z(\mathrm{Hg}(\mathcal{X}))^0$ and its derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$, one concludes $h_W(i) = J_C \cdot J_D$, where $J_C \in Z(\mathrm{Hg}(\mathcal{X}))(\mathbb{C})^0$ and $J_D \in \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{C})$. Thus

$$\begin{aligned} h_W(i)\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R})h_W(i)^{-1} &= J_C J_D \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R}) J_D^{-1} J_C^{-1} \\ &= J_C J_C^{-1} J_D \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R}) J_D^{-1} \\ &= J_D \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R}) J_D^{-1} = \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R}). \end{aligned}$$

Therefore the conjugation by $h_W(i)$ yields a Cartan involution of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$. This Cartan involution corresponds clearly to a Cartan involution on $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ given by the conjugation by $\mathrm{ad}(h_W(i))$. ∎

Let K be a maximal compact subgroup of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$. Since all maximal compact subgroups of a reductive group are conjugate, we assume without loss of generality that K is the subgroup fixed by the Cartan involution obtained from conjugation by $h_W(i)$. Let $C((\mathrm{ad} \circ h)(i))$ denote the centralizer of $(\mathrm{ad} \circ h)(i)$ in $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})$.

Lemma 1.12.

$$C((\mathrm{ad} \circ h)(i)) = \mathrm{ad}(K) = \mathrm{ad}(K \cap \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}).$$

Proof. One has clearly

$$C((\mathrm{ad} \circ h)(i)) \supseteq \mathrm{ad}(K) \supseteq \mathrm{ad}(K \cap \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}).$$

Thus it remains to prove

$$C((\mathrm{ad} \circ h)(i)) \subseteq \mathrm{ad}(K \cap \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}).$$

Since $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ and $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ are isogenous, we have a correspondence between their maximal compact subgroups. The maximal compact subgroups K_G of real algebraic reductive groups G are the subgroups of G satisfying

$$K_G = \{g \in G \mid \theta(g) = g\}$$

for some Cartan involution θ (follows from [12], **I**. Corollary 4.3 and Corollary 4.5). Recall that the conjugation by $h(i)$ yields a Cartan involution on $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ and the conjugation by $(\mathrm{ad} \circ h)(i)$ yields a Cartan involution on $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$. Thus one concludes that the centralizer of $(\mathrm{ad} \circ h)(i)$ is given by $\mathrm{ad}(K \cap \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}})$. ∎

2. Computation of the adjoint Hodge group. In this section we prove the following theorem:

Theorem 2.1. *The group* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ *is isomorphic to* $\mathrm{PU}(1,1)$ *or* $\mathrm{Sp}^{\mathrm{ad}}_{\mathbb{R}}(4)$.

For the proof of Theorem 2.1 we need to understand K first:

Lemma 2.2. *The group* K^0 *is a torus or* $K = C(h_W(i))$.

Proof. Since K^0 is compact, K^0 is reductive (see 1.3). One has without loss of generality

$$K \subseteq C(h_W(i)) \cong \mathrm{U}(2).$$

If K^0 is a torus, we are done. Otherwise K^0 has a nontrivial semisimple subgroup

$$K^{\mathrm{der}} \subseteq C^{\mathrm{der}}(h_W(i)) \cong \mathrm{SU}(2)$$

(see 1.1). Since SU(2) does not contain any simple proper subgroup, $K^{\mathrm{der}} = C^{\mathrm{der}}(h_W(i))$. From the facts that $h(S^1)$ is not contained in $C^{\mathrm{der}}(h_W(i))$, but contained in $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ and commutes with $h_W(i) = h(-i)$, we conclude $K = C(h_W(i))$ in this case. □

LEMMA 2.3. *The centralizer of $C^{\mathrm{der}}(h_W(i))$ in $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ is given by the center $Z(C(h_W(i)))$ of $C(h_W(i))$.*

Proof. Recall the description of $C^{\mathrm{der}}(h_W(i)) \cong \mathrm{SU}(2)$ in Proposition 1.7 and the description of SU(2) in Remark 1.8. Thus $N \in C^{\mathrm{der}}(h_W(i))(\mathbb{R})$ is given by

$$N = \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & \bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad |a|^2 + |b|^2 = 1$$

with respect to the basis $\{v_{3,0}, v_{1,2}, v_{2,1}, v_{0,3}\}$. Now let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(H^3(X,\mathbb{R}),Q)(\mathbb{R})$$

commute with each $N \in C^{\mathrm{der}}(h_W(i))(\mathbb{R})$ for some suitable $A, B, C, D \in \mathrm{GL}_2(\mathbb{C})$. Thus M commutes with $\mathrm{diag}(i,-i,i,-i)$ and one computes that A, B, C, D are diagonal matrices. Moreover one has that M has to commute with

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

From this fact and the assumptions that M is a real matrix and commutes with each element of $C^{\mathrm{der}}(h_W(i))(\mathbb{R})$, one concludes

$$M = \begin{pmatrix} z & 0 & \bar{y} & 0 \\ 0 & z & 0 & -\bar{y} \\ -y & 0 & \bar{z} & 0 \\ 0 & y & 0 & \bar{z} \end{pmatrix}.$$

Moreover one computes that

$$\begin{aligned} M^tQM &= \begin{pmatrix} z & 0 & -y & 0 \\ 0 & z & 0 & y \\ \bar{y} & 0 & \bar{z} & 0 \\ 0 & -\bar{y} & 0 & \bar{z} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z & 0 & \bar{y} & 0 \\ 0 & z & 0 & -\bar{y} \\ -y & 0 & \bar{z} & 0 \\ 0 & y & 0 & \bar{z} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2iyz & 0 & i|y|^2 - i|z|^2 \\ 2iyz & 0 & i|y|^2 - i|z|^2 & 0 \\ 0 & -i|y|^2 + i|z|^2 & 0 & -2i\bar{y}\bar{z} \\ -i|y|^2 + i|z|^2 & 0 & 2i\bar{y}\bar{z} & 0 \end{pmatrix}. \end{aligned}$$

Hence $M \in \mathrm{Sp}(H^3(X,\mathbb{R}),Q)$, only if $y=0$ and $|z|=1$. Thus $M \in Z(C(h_W(i)))$. ☐

LEMMA 2.4. *$\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be compact.*

Proof. Assume that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ would be compact. Thus one concludes that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = K$ is a torus or $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_W(i))$ (see Lemma 2.2). In the first case one concludes $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \subseteq C(h(S^1))$, which contains only 4 complex structures (see Remark 1.5). In the second case the Cartan involution obtained from conjugation by $h_{\mathcal{X}_b}(i) \in C(h_W(i))$ fixes each element of the compact group $\mathrm{Hg}(\mathcal{X})(\mathbb{R}) = C(h_W(i))(\mathbb{R})$ for each $b \in B$. Hence each $h_{\mathcal{X}_b}(i)$ has to be contained in the center of $C(h_W(i))$. Note that $Z(C(h_W(i)))$ has only the two complex structures $\pm h_W(i)$. Thus in any case $h(i) = h_{\mathcal{X}_b}(i)$ for each $b \in B$, since the VHS is continuous and for each $b \in B$ one obtains

$$H^{3,0}(\mathcal{X}_b) \subset \mathrm{Eig}(h_{\mathcal{X}_b}(i), -i) = \mathrm{Eig}(h_X(i), -i) = \mathrm{Span}(v_{3,0}, v_{1,2}).$$

But this contradicts the fact that $\omega(0)$ and $\nabla_{\frac{\partial}{\partial b}}\omega(0)$ generate $F^2(X)$, where ω denotes a generic section of the F^3-bundle in the VHS (see [2]). ☐

Now we change for a moment to the language of semisimple adjoint Lie groups. Connected semisimple adjoint Lie groups are direct products of their normal simple subgroups (see [10], Lemma 1.3.8). The group $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ is an example of a connected semisimple adjoint Lie group.

PROPOSITION 2.5. *There does not exist any nontrivial direct factor F of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ such that*

$$Z(K)(\mathbb{R})^+ \subset \ker(pr_F \circ \mathrm{ad}).$$

Proof. Assume that F is a direct factor of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ with

$$Z(K)(\mathbb{R})^+ \subset \ker(pr_F \circ \mathrm{ad}).$$

We show that F is trivial. Since $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be compact (see Lemma 2.4), the maximal compact subgroup K associated to the Cartan involution obtained from conjugation by $h(i)$ is a proper subgroup. Thus $h(i)$ is not contained in the center of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$. Since $h(S^1)(\mathbb{R})$ is connected, $h(S^1)(\mathbb{R}) \subset Z(K)(\mathbb{R})^+$, which implies that $h(i) \in Z(K)(\mathbb{R})^+$. Thus from our assumption we conclude that F is contained in the maximal compact subgroup associated to the Cartan involution obtained from conjugation by $(\mathrm{ad} \circ h)(i)$. Consider the projection map $pr_F : \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+ \to F$. Since

$$(\mathrm{ad} \circ h)(i) \in G := \ker(pr_F) \subset \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+,$$

one concludes that G is non-trivial semisimple adjoint. Note that

$$\ker(pr_G) = F \quad \text{and} \quad \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+ = F \times G,$$

since connected semisimple adjoint Lie groups are direct products of thier normal simple subgroups (see [10], Lemma 1.3.8). Let

$$F' = \ker(pr_G \circ \mathrm{ad}|_{\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R})})^+ \quad \textit{and} \quad G' = \ker(pr_F \circ \mathrm{ad}|_{\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})(\mathbb{R})})^+.$$

Since $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ and $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ are isogenous, one concludes that F' and G' commute. Since F' is a semisimple group fixed by the Cartan involution obtained from conjugation by $h(i)$ and $C^{\mathrm{der}}(h(i))(\mathbb{R}) \cong \mathrm{SU}(2)(\mathbb{R})$ contains no semisimple proper subgroup, one concludes

$$F' = C^{\mathrm{der}}(h(i))(\mathbb{R}) \quad \text{or} \quad F' = \{e\}.$$

Only the torus $Z(C(h(i)))$ commutes with $C^{\mathrm{der}}(h(i))$ (see Lemma 2.3). Thus from the fact that G' is nontrivial semisimple and commutes with F', we conclude $F' = \{e\}$. Thus F is trivial. □

The connected semisimple adjoint Lie group $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ is a direct product of connected simple adjoint subgroups. Let F be one of these direct factors. The maximal compact subgroup of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ is given by

$$\mathrm{ad}(K(\mathbb{R})) \cap \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$$

(follows from Lemma 1.12). Thus for the maximal compact subgroup K_F of F one concludes that $K_F^+ = (pr_F \circ \mathrm{ad})(K(\mathbb{R})^+)$. Due to the fact that $Z(K)(\mathbb{R})^+$ is not contained in $\ker(pr_F \circ \mathrm{ad})$ and not discrete as one concludes from Lemma 2.2, the maximal compact subgroup K_F has a nondiscrete center. Since F has a trivial center, $K_F \neq F$ and one concludes:

Corollary 2.6. *The connected adjoint Lie group* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$ *is a direct product of noncompact simple adjoint subgroups, whose maximal compact subgroups have nondiscrete centers.*

Note that each Hermitian symmetric domain is a direct product of irreducible Hermitian symmetric domains (for the definition and more details about Hermitian symmetric domains see [6]). If G is a connected simple adjoint noncompact Lie group and K_G is a maximal compact subgroup of G with nondiscrete center, the quotient G/K_G has the structure of a uniquely determined irreducible Hermitian symmetric domain ([6], **XIII**. Theorem 6.1,). Hence one concludes from Corollary 2.6:

Proposition 2.7. *The quotient*

$$D = \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+/\mathrm{ad}(K(\mathbb{R})) \cap \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$$

has the structure of an Hermitian symmetric domain.

Since $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \subset \mathrm{Sp}(H^3(X,\mathbb{R}),Q)$, the associated Hermitian symmetric domain of $\mathrm{Sp}(H^3(X,\mathbb{Q}),Q)(\mathbb{R})$ is $\mathfrak{h}_2$ and $\dim_{\mathbb{C}} \mathfrak{h}_2 = 3$, the Hermitian symmetric domain D has dimension 1, 2 or 3. By using these conditions, we obtain some candidates for $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})^+$. Since these candidates are the Lie groups of real valued points of $\mathbb{R}$-algebraic semisimple adjoint groups, we obtain not only connected Lie groups, but $\mathbb{R}$-algebraic groups in our cases by using Lemma 1.2. Moreover we will exclude all of these candidates except of the candidates stated in Theorem 2.1.

Lemma 2.8. *If D has dimension one, we obtain*

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1).$$

Proof. Assume that D has dimension one. By consulting the list of irreducible Hermitian symmetric domains ([6], **X**, Table **V**), one concludes $D = \mathbb{B}_1$. Thus from the fact that there are no direct compact factors (see Corollary 2.6) one concludes

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1). \qquad \square$$

Lemma 2.9. *If D has dimension two, we obtain*

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,2), \quad \textit{or} \quad \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1) \times \mathrm{PU}(1,1).$$

Proof. By consulting the list of irreducible Hermitian symmetric domains ([6], **X**, Table **V**), the only possible Hermitian symmetric domains of dimension two are up to isomorphisms given by $\mathbb{B}_1 \times \mathbb{B}_1$ and $\mathbb{B}_2$. Thus we obtain the stated result. □

Lemma 2.10. *One obtains* $\mathrm{Hg}(\mathcal{X}) = \mathrm{Sp}(H^3(X,\mathbb{Q}), Q)$*, if D has the dimension 3.*

Proof. We show that $\mathfrak{h}_2$ contains no bounded symmetric domain of dimension 3 except of itself. In order to do this we check the list of Hermitian Symmetric Domains (compare [6], **X**, Table **V**). The domain D cannot be the direct product of 3 copies of $\mathbb{B}_1$, since in this case the centralizer of $(\mathrm{ad} \circ h_X)(i)$ would be a torus of dimension 3. But the centralizer of $h_X(i)$ is isomorphic to $\mathrm{U}(2)$, which contains a maximal torus of dimension 2. Since each point $p \in \mathbb{B}_1 \times \mathbb{B}_2$ has a centralizer $S^1 \times \mathrm{U}(2)$ of dimension 5 and $C(h(i)) \cong \mathrm{U}(2)$ has dimension 4, one concludes that D cannot be isomorphic to $\mathbb{B}_1 \times \mathbb{B}_2$. In the case of $\mathbb{B}_3$ the stabilizer is $\mathrm{U}(3)$ and hence it is to large. The same holds true in the case of $\mathrm{SO}^*(6)/\mathrm{U}(3)$. Moreover the associated bounded symmetric domain of $\mathrm{SO}(2,3)^+(\mathbb{R})$ is isomorphic to $\mathfrak{h}_2$. Thus we obtain the stated result. □

By the previous lemmas, the following adjoint semisimple groups are possible candidates for $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$:

$$\mathrm{PU}(1,1), \quad \mathrm{PU}(1,1) \times \mathrm{PU}(1,1), \quad \mathrm{PU}(1,2), \quad \mathrm{Sp}^{\mathrm{ad}}_{\mathbb{R}}(4).$$

Now we exclude $\mathrm{PU}(1,2)$ and $\mathrm{PU}(1,1) \times \mathrm{PU}(1,1)$.

Proposition 2.11. *The group* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ *cannot be isomorphic to* $\mathrm{PU}(1,2)$.

Proof. Assume that $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ would be isomorphic to $\mathrm{PU}(1,2)$. In this case the centralizer $C((\mathrm{ad} \circ h)(i)) \subset \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ of the complex structure $(\mathrm{ad} \circ h)(i)$ is isomorphic to $\mathrm{U}(2)$. Hence $C((\mathrm{ad} \circ h)(i))$ has dimension 4. One has that $C((\mathrm{ad} \circ h)(i))$ is isogenous to $C(h(i)) \cap \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$. Since $C(h(i))$ has already dimension 4 and $h(S^1) \subset C(h(i))$, one concludes

$$C(h(i)) \subset \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}} \quad \text{and} \quad \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}} = \mathrm{Hg}(\mathcal{X})_{\mathbb{R}}.$$

Note that

$$C^{\mathrm{der}}(h(i)) \cong \mathrm{SU}(2).$$

Moreover ad yields a homomorphism

$$g := \mathrm{ad}|_{C^{\mathrm{der}}(h(i))} : C^{\mathrm{der}}(h(i)) \to C(\mathrm{ad} \circ h(i)),$$

whose kernel consists of $\{\pm \mathrm{id}\}$. Since

$$C^{\mathrm{der}}(h(i))/\{\pm \mathrm{id}\} \cong \mathrm{PU}(2)$$

is semisimple, one has

$$(g(C^{\mathrm{der}}(h(i))))^{\mathrm{der}} = g(C^{\mathrm{der}}(h(i))).$$

Hence

$$g(C^{\mathrm{der}}(h(i))) \subset C^{\mathrm{der}}(\mathrm{ad} \circ h(i)) \cong \mathrm{SU}(2).$$

Recall that

$$\mathrm{SU}(2)(\mathbb{R}) = \{M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1\}.$$

Each matrix $M(\alpha, \beta) \in \mathrm{SU}(2)(\mathbb{R})$ with $\alpha \in i\mathbb{R}$ has the characteristic polynomial

$$x^2 + 1 = (x - i)(x + i),$$

which implies that $M(\alpha, \beta)$ is a complex structure. Therefore $C^{\mathrm{der}}(h(i))(\mathbb{R}) \cong \mathrm{SU}(2)(\mathbb{R})$ contains infinitely many complex structures. Since $\ker(g) = \{\pm \mathrm{id}\}$, all these complex structures are mapped to infinitely many elements of order 2 in $C^{\mathrm{der}}(\mathrm{ad} \circ h(i))$. Since each 2×2 matrix M of order 2 has a minimal polynomial dividing the polynomial $x^2 - 1$, the matrix M is either given by $\mathrm{diag}(-1, -1)$ or one has an eigenspace with respect to 1 and one eigenspace with respect to -1. In the second case $\det(M) = -1$. Thus $\mathrm{diag}(-1, -1)$ is the only element of order 2 in $\mathrm{SU}(2)(\mathbb{R})$. On the other hand there are infinitely many complex structures in $C^{\mathrm{der}}(h(i))(\mathbb{R})$, which are mapped by g to infinitely many elements of order 2 in $C((\mathrm{ad} \circ h)(i))(\mathbb{R}) \cong \mathrm{SU}(2)(\mathbb{R})$. Thus we have a contradiction. □

Let H denote the centralizer of $h_G(i)h_W(i)$ in $\mathrm{Sp}(H^3(X, \mathbb{R}), Q)$. Note that

$$h_G(i)h_W(i) = \mathrm{diag}(-1, -1, 1, 1)$$

with respect to the basis $\{v_{3,0}, v_{0,3}, v_{2,1}, v_{1,2}\}$. Thus $H(\mathbb{R})$ is given by the matrices

$$(3) \qquad M_1 = \begin{pmatrix} a & b & 0 & 0 \\ \bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & \bar{d} & \bar{c} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix} \in \mathrm{SU}(1,1)(\mathbb{R})$$

with respect to the basis $\{v_{3,0}, v_{0,3}, v_{2,1}, v_{1,2}\}$. One can easily verify this fact by explicit computations using the description of the symplectic form Q in (1). The group H will play an important role due to the following lemma:

LEMMA 2.12. *The group* $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ *cannot be a subgroup of* H.

Proof. Assume that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ would be a subgroup of H. Since for each $b \in B$ the conjugation by $h_W(i)_b$ yields a Cartan involution of $\mathrm{Sp}(H^3(X, \mathbb{R}), Q)$, which can be restricted to an involution of H in this case, the conjugation by $h_W(i)_b$ yields a Cartan involution of H (compare [12], **I**. Theorem 4.2). Due to the fact $H \cong \mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$, the corresponding maximal compact subgroup is a torus of dimension 2 containing $h_b(S^1)$. By Remark 1.5, the centralizer $C(h_b(S^1))$ is already a torus of dimension 2. Hence

$$h_G(i)_b \in C(h_b(S^1)) \subset H.$$

Thus from the description of H in (3) and the fact that $h_G(i)_b, h_W(i)_b \in H$ are real complex structures, one concludes that

$$\mathrm{Eig}(h_G(i)_b, i) = \mathrm{Span}(v_1, v_3), \quad \mathrm{Eig}(h_W(i)_b, i) = \mathrm{Span}(v_2, v_4)$$

with

$$v_1, v_2 \in \mathrm{Span}(v_{3,0}, v_{0,3}), \quad v_3, v_4 \in \mathrm{Span}(v_{2,1}, v_{1,2}). \tag{4}$$

For each $b \in B$ one has the onedimensional vector space

$$H^{3,0}(\mathcal{X}_b) = \mathrm{Eig}(h_G(i)_b, i) \cap \mathrm{Eig}(h_W(i)_b, i).$$

Hence $\{v_1, \ldots, v_4\}$ is not linearly independent and one concludes from the description of H in (4) that $H^{3,0}(\mathcal{X}_b)$ is either contained in $\mathrm{Span}(v_{3,0}, v_{0,3})$ or contained in $\mathrm{Span}(v_{2,1}, v_{1,2})$.[3] Since the period map is continuous, one has for each $b \in B$

$$H^{3,0}(\mathcal{X}_b) \subset \mathrm{Span}(v_{3,0}, v_{0,3}).$$

This contradicts the fact that $\omega(0)$ and $\nabla_{\frac{\partial}{\partial b}}\omega(0)$ generate $F^2(X)$, where ω denotes a generic section of the F^3-bundle in the VHS (see [2]). Thus $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be a subgroup of H. □

Proposition 2.13. *One cannot have*

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1) \times \mathrm{PU}(1,1).$$

Proof. Assume that $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1) \times \mathrm{PU}(1,1)$. Without loss of generality the only possible Cartan involution of $\mathrm{PU}(1,1) \times \mathrm{PU}(1,1)$ is given by the conjugation by

$$([\mathrm{diag}(i,-i)], [\mathrm{diag}(i,-i)]) \in \mathrm{PU}(1,1) \times \mathrm{PU}(1,1).$$

Moreover in $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1) \times \mathrm{PU}(1,1)$ the maximal compact subgroup of elements fixed by the Cartan involution is given by a torus of dimension 2. Thus there is a torus $T \subset \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ of dimension two, whose elements are fixed by the Cartan involution. Assume without loss of generality that the Cartan involution of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ is obtained from conjugation by $h(i)$. Thus T is a maximal torus of $C(h(i)) \cong \mathrm{U}(2)$, since T has dimension 2. Therefore the center of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ is discrete and one concludes from 1.1 that

$$\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}} = \mathrm{Hg}(\mathcal{X})_{\mathbb{R}}.$$

From the fact that each element of $h(S^1)$ commutes with $h(i)$, one concludes $h(S^1) \subset T$. Since T is a torus of dimension 2 containing $h(S^1)$, one concludes from Remark 1.5 that $T = C(h(S^1))$. Thus $h_G(i) \in T$ and $h_G(S^1) \subset T$. Note that $h_G(i)$ cannot be contained in the center of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$, since $h_G(i) \in Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$ would imply that $h_G(S^1) \subset Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$ as one can easily conclude from the fact that

$$h_G(S^1)(\mathbb{R}) = \{a \cdot \mathrm{id} + b \cdot h_G(i) \ \mid \ a^2 + b^2 = 1\}.$$

This contradicts our conclusion that $Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$ is discrete. Since $h_G(i)$ has order 4 and

$$h_G(i)^2 = -\mathrm{id} \in \ker(\mathrm{ad}),$$

[3] This is only an exercise in linear algebra.

one concludes that $\mathrm{ad}(h_G(i))$ yields an element of order two in $\mathrm{ad}(T)$. Note that $\mathrm{ad}(T)$ has only the three elements

$$([\mathrm{diag}(i,-i)],[\mathrm{diag}(1,1)]),\quad ([\mathrm{diag}(i,-i)],[\mathrm{diag}(i,-i)]) \text{ and } ([\mathrm{diag}(1,1)],[\mathrm{diag}(i,-i)])$$

of order 2. Thus we have two cases: In the first case $\mathrm{ad}(h_G(i))$ is without loss of generality given by

$$([\mathrm{diag}(i,-i)],[\mathrm{diag}(1,1)]).$$

Let pr_i $(i=1,2)$ denote the projection of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1)\times\mathrm{PU}(1,1)$ to the respective copy of $\mathrm{PU}(1,1)$. One has that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ contains $\ker(pr_1\circ\mathrm{ad})^0$ and $\ker(pr_2\circ\mathrm{ad})^0$. Since the groups $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ and $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}} = \mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ are isogenous, $\ker(pr_1\circ\mathrm{ad})^0$ and $\ker(pr_2\circ\mathrm{ad})^0$ are also isogenous to $\mathrm{PU}(1,1)$ and commute also. Moreover since $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}$ and $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ are isogenous, $(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}\cap C(h_G(i)))^0$ is also isogenous to $C((\mathrm{ad}\circ h_G)(i))$. Since $\ker(pr_1)$ commutes with $\mathrm{ad}(h_G(i))$, one concludes that $\ker(pr_1\circ\mathrm{ad})^0$ is a nontrivial simple subgroup of $C(h_G(i))$. Since the only nontrivial simple subgroup of $C(h_G(i))$ is $C^{\mathrm{der}}(h_G(i))$, one gets

$$\ker(pr_1\circ\mathrm{ad})^0 = C^{\mathrm{der}}(h_G(i)).$$

By analogue arguments, one concludes

$$\ker(pr_2\circ\mathrm{ad})^0 \subset H := C(h_G(i)h_W(i)).$$

We obtain the desired contradiction by showing that $\ker(pr_1\circ\mathrm{ad})^0$ and $\ker(pr_2\circ\mathrm{ad})^0$ cannot commute here. One has that $C^{\mathrm{der}}(h_G(i))(\mathbb{R})$ is given by matrices of the form

$$M_2 = \begin{pmatrix} \alpha & 0 & \beta & 0\\ 0 & \bar\alpha & 0 & \bar\beta\\ \bar\beta & 0 & \bar\alpha & 0\\ 0 & \beta & 0 & \alpha \end{pmatrix} \quad\text{with}\quad |\alpha|^2-|\beta|^2=1$$

with respect to the basis

$$\{v_{3,0}, v_{0,3}, v_{2,1}, v_{1,2}\}$$

as the reader can easily verify by the description of $C(h_G(i))(\mathbb{R})\cong\mathrm{U}(1,1)$ in Proposition 1.6 and the description of $\mathrm{SU}(1,1)$ in Remark 1.8. Moreover by explicit computations using (3), one checks that in $H(\mathbb{R})$ only the diagonal matrices of the kind $\mathrm{diag}(\xi,\bar\xi,\xi,\bar\xi)$ commute with each element of $C^{\mathrm{der}}(h_G(i))(\mathbb{R})$. This contradicts our previous conclusion that H contains a subgroup isogenous to $\mathrm{PU}(1,1)$, which commutes with $C^{\mathrm{der}}(h_G(i))(\mathbb{R})$. Hence the first case cannot hold true.

In the second case $\mathrm{ad}(h_G(i))$ is given by

$$([\mathrm{diag}(i,-i)],[\mathrm{diag}(i,-i)]) \in \mathrm{PU}(1,1)\times\mathrm{PU}(1,1).$$

This implies that $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X}) = \mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ is contained in the subgroup of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ on which both involutions obtained from conjugation by $h_W(i)$ and $h_G(i)$ coincide. One has that

$$h_W(i) = \mathrm{diag}(i,-i,-i,i) \text{ and } h_G(i) = \mathrm{diag}(i,-i,i,-i)$$

with respect to the basis

$$\{v_{3,0}, v_{0,3}, v_{2,1}, v_{1,2}\}.$$

Thus H is the subgroup of $\mathrm{Sp}(H^3(X,\mathbb{R}))$ on which both involutions obtained from conjugation by $h_W(i)$ and $h_G(i)$ coincide as one can easily compute by using the description of H in (3). But by Lemma 2.12, the group H cannot contain $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$. Thus the second case cannot occur. $\square$

3. The case of a onedimensional period domain. In this section we will assume that the period domain D has dimension 1 unless stated otherwise. In the previous section we saw that $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1)$, if $D = 1$. Since

$$\mathrm{Hg}(\mathcal{X}) = (\mathrm{SL}(H^3(X,\mathbb{Q})) \cap \mathrm{MT}(\mathcal{X}))^0$$

(follows from [10], Lemma 1.3.17), one concludes

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X}) = \mathrm{MT}^{\mathrm{ad}}(\mathcal{X}).$$

Recall the definition of Shimura data:

Definition 3.1. Let G be a reductive $\mathbb{Q}$-algebraic group and $h : \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism. Then the pair (G, h) is a Shimura datum, if:

1. The group G^{ad} has no nontrivial direct compact factor over $\mathbb{Q}$.
2. The conjugation by $h(i)$ is a Cartan involution.
3. The representation $\mathrm{ad} \circ h$ of $\mathbb{S}$ on $\mathrm{Lie}(G_{\mathbb{R}})$ is a Hodge structure of type

$$(1,-1), (0,0), (-1,1).$$

We will show that the pair $(\mathrm{MT}(\mathcal{X}), h_X)$ is a Shimura datum. Moreover we will determine the center of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ and $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ in the case of a nondiscrete center. In addition we describe the monodromy in the latter case and give some examples.

Proposition 3.2. *The center of* $\mathrm{Hg}(\mathcal{X})(\mathbb{R})$ *is given by diagonal matrices* $\mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi})$ *for* $|\xi| = 1$ *with respect to the basis* $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$.

Proof. Each element Z in the center of $\mathrm{Hg}(\mathcal{X})(\mathbb{R})$ commutes in particular with $h_X(S^1)(\mathbb{R})$. This holds only true, if Z is a diagonal matrix with respect to $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$ as the conjugation by elements of $h(S^1)(\mathbb{R})$ in Remark 1.4 shows. The subgroup of the matrices in $\mathrm{Sp}(H^3(X,\mathbb{R}), Q)$, which are diagonal with respect to $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$, is contained in $C(h_W(i)) \cong \mathrm{U}(2)$ and therefore compact. By Lemma 2.4, the group $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be compact. Thus $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ contains elements, which are not given by diagonal matrices with respect to $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$. Since Z has to be real and to commute with the matrices in $\mathrm{Hg}(\mathcal{X})(\mathbb{R})$, which are not diagonal, one concludes that

$$Z = \pm\mathrm{diag}(\xi, 1, 1, \bar{\xi}), \quad Z = \pm\mathrm{diag}(1, \xi, \bar{\xi}, 1),$$

$$Z = \mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi}) \quad \text{or} \quad Z = \mathrm{diag}(\xi, \bar{\xi}, \xi, \bar{\xi})$$

with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$. Moreover one has $|\xi| = 1$, since $Z^t Q Z = Q$. For $Z = \pm\mathrm{diag}(\xi, 1, 1, \bar{\xi})$ with $\xi \neq \pm 1$ the centralizer $C(Z)$ of Z in $\mathrm{Sp}(H^3(X,\mathbb{R}), Q)$

is given by the group of matrices

$$(5) \qquad M = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \bar{\beta} & \bar{\alpha} & 0 \\ 0 & 0 & 0 & \bar{\zeta} \end{pmatrix} \quad \text{with } |\zeta| = 1 \text{ and } \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$$

as one concludes by computations using (1). Thus one concludes that $C(Z) \subset H$ from the description of H in (3).

Moreover for $Z = \pm\mathrm{diag}(1, \xi, \bar{\xi}, 1)$ with $\xi \neq \pm 1$ the centralizer $C(Z)$ is given by

$$M = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \bar{\zeta} & 0 \\ \bar{\beta} & 0 & 0 & \bar{\alpha} \end{pmatrix} \quad \text{with } |\zeta| = 1 \text{ and } \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1),$$

which is also a subgroup of H as one concludes from analogue arguments. By Lemma 2.12, the group $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be a subgroup of H. Since the matrices of the form

$$\pm\mathrm{diag}(\xi, 1, 1, \bar{\xi}), \quad \pm\mathrm{diag}(1, \xi, \bar{\xi}, 1) \quad \text{with} \quad \xi \neq \pm 1$$

have centralizers contained in H, these matrices are not contained in the center of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$.

One can also not have that

$$Z = \pm\mathrm{diag}(1, -1, -1, 1) \in Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}),$$

too, since in this case the centralizer of Z in $\mathrm{Sp}(H^3(X, \mathbb{R}), Q)$ is H.

Hence one has

$$Z = \mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi}) \quad \text{or} \quad Z = \mathrm{diag}(\xi, \bar{\xi}, \xi, \bar{\xi}).$$

The matrix $\mathrm{diag}(\xi, \bar{\xi}, \xi, \bar{\xi})$ commutes only with elements in $C(h_X(i)) \cong \mathrm{U}(2)$, if $\xi \neq \pm 1$. Recall that $\mathrm{U}(2)$ is compact. Moreover

$$\mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi}) = \mathrm{diag}(\xi, \bar{\xi}, \xi, \bar{\xi})$$

for $\xi = \pm 1$. Again we use the fact that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be compact and conclude that $Z = \mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi})$. ◻

Since

$$h_X(\xi) \in Z(\mathrm{Hg}(\mathcal{X})) \Rightarrow \mathrm{diag}(\xi, \xi, \bar{\xi}, \bar{\xi}) = \mathrm{diag}(\xi^3, \xi, \bar{\xi}, \bar{\xi}^3) \Leftrightarrow \xi^3 = \xi \Leftrightarrow \xi^2 = 1 \Leftrightarrow \xi = \pm 1$$

and $h_X(-1) = -E_4$, one concludes from the previous proposition:

Corollary 3.3. *The kernel of the representation* $\mathrm{ad} \circ h$ *consists of* $\{\pm 1\}$.

Corollary 3.4. *One has* $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$, *if and only if* $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ *has a nondiscrete center.*

Proof. Due to the fact that $C(h_G(i)) \cong \mathrm{U}(1,1)$ has a nondiscrete center, it is clear that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ has a nondiscrete center, if $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$. Conversely, if the center $Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$ is nondiscrete, $\dim Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}) \geq 1$. Moreover the $\mathbb{R}$-valued points of $Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$ are a subgroup of the group of diagonal matrices $\mathrm{diag}(\xi,\xi,\bar{\xi},\bar{\xi})$ for $|\xi| = 1$ with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$ (see Proposition 3.2). Since the latter group is given by the onedimensional group $h_G(S^1)(\mathbb{R})$, one concludes that $Z(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}) \supseteq h_G(S^1)$. Thus $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \subseteq C(h_G(S^1))$. Recall that reductive groups are almost direct products of their centers and their derived subgroups (see 1.1). Moreover note that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot commutative. Otherwise it would be a subgroup of the compact torus

$$C(h(S^1)) \cong S^1 \times S^1$$

(compare Remark 1.2), which contradicts the fact that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ cannot be compact (see Lemma 2.4). Thus $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ has a nontrivial derived subgroup. Due to the fact that

$$C^{\mathrm{der}}(h_G(S^1)) = C^{\mathrm{der}}(h_G(i)) \cong \mathrm{SU}(1,1)$$

contains no semisimple proper subgroup and does not contain $h_G(S^1)$, one concludes $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$. □

Proposition 3.5. *The pair* $(\mathrm{MT}(\mathcal{X}), h_X)$ *is a Shimura datum, if* $D \cong \mathbb{B}_1$.

Proof. By our previous results and assumptions,

$$\mathrm{MT}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} = \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1).$$

Thus $\mathrm{MT}^{\mathrm{ad}}(\mathcal{X})$ is simple and noncompact. Moreover $\mathrm{ad}(h(i))$ yields a Cartan involution (see Lemma 1.11). Due to the fact that the conjugation by a diagonal matrix $\mathrm{diag}(a,\ldots,a)$ is the identity map, the weight homomorphism of the Hodge structure $\mathrm{ad}_{\mathrm{MT}(\mathcal{X})_{\mathbb{R}}} \circ h$ is given by $\mathbb{G}_{m,\mathbb{R}} \to \{e\}$. Thus the Hodge structure $\mathrm{ad}_{\mathrm{MT}(\mathcal{X})_{\mathbb{R}}} \circ h$ has weight zero and all characters of the representation $\mathrm{ad}_{\mathrm{MT}(\mathcal{X})_{\mathbb{R}}} \circ h$ are given by $(z/\bar{z})^k$ with $k \in \mathbb{Z}$. By Corollary 3.3, the kernel of $\mathrm{ad} \circ h|_{S^1}$ consists of $\{\pm 1\}$. Since $\dim(\mathrm{MT}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}}) = 3$, this implies that the representation $\mathrm{ad}_{\mathrm{MT}(\mathcal{X})_{\mathbb{R}}} \circ h$ is a Hodge structure of type $(1,-1), (0,0), (-1,1)$. Thus we have a Shimura datum as claimed. □

The variation $\mathcal{V}$ of weight 3 Hodge structures of a nonisotrivial family $\mathcal{Y} \to \mathcal{Z}$ of Calabi-Yau 3-manifolds has an underlying local system $\mathcal{V}_{\mathbb{Z}}$ corresponding to an up to conjugation unique monodromy representation

$$\rho : \pi_1(\mathcal{Z}, z) \to \mathrm{GL}(H^3(\mathcal{Y}_z, \mathbb{Z})).$$

Let $\mathcal{Y}_z \cong X$. The algebraic group $\mathrm{Mon}^0(\mathcal{Y})$ denotes the connected component of identity of the Zariski closure of $\rho(\pi_1(\mathcal{Z}, z))$ in $\mathrm{GL}(H^3(X,\mathbb{Q}))$. The group $\mathrm{Mon}^0(\mathcal{Y})$ is a normal subgroup of $\mathrm{MT}^{\mathrm{der}}(\mathcal{Y})$, if $\mathcal{Z}$ is a connected complex algebraic manifold (see [9], Theorem 1.4). Since $\mathrm{MT}^{\mathrm{der}}(\mathcal{Y}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{Y})$ (follows from [10], Corollary 1.3.19) and $\mathrm{Sp}(H^3(X,\mathbb{Q}), Q)$ is simple, one concludes:

Proposition 3.6. *If* $\mathcal{V}_{\mathbb{Z}}$ *has an infinite monodromy group,* $\mathcal{Z}$ *is a connected complex algebraic manifold,* $\mathcal{Y}_z \cong X$ *and*

$$\mathrm{Hg}(\mathcal{Y}) = \mathrm{Sp}(H^3(X,\mathbb{Q}), Q),$$

one has also

$$\mathrm{Mon}^0(\mathcal{Y}) = \mathrm{Sp}(H^3(X,\mathbb{Q}), Q).$$

Consider the Kuranishi family $\mathcal{X} \to B$ of X and the period map

$$p : B \to \mathrm{Grass}(H^3(X, \mathbb{C}), b_3(X)/2)$$

associating to each $b \in B$ the subspace

$$F^2(H^3(\mathcal{X}_b, \mathbb{C})) \subset H^3(\mathcal{X}_b, \mathbb{C}) \cong H^3(\mathcal{X}_B, \mathbb{C}) \cong H^3(X, \mathbb{C})$$

as described in [14], Chapter 10. We say that $F^2(\mathcal{H}^3)_B$ is constant, if the period map $p : B \to \mathrm{Grass}(H^3(X, \mathbb{C}), b_3(X)/2)$ is constant. Moreover recall that $\mathcal{Y} \to \mathcal{Z}$ is a maximal family of Calabi-Yau 3-manifolds, if $\mathcal{Z}$ can be covered by open subsets U such that each $\mathcal{Y}_U$ is isomorphic to a Kuranishi family.

THEOREM 3.7. *Assume that $\mathcal{Z}$ is a connected complex algebraic manifold and $f : \mathcal{Y} \to \mathcal{Z}$ is a maximal family of Calabi-Yau 3-manifolds with $\mathcal{Y}_z \cong X$ and an infinite monodromy group. Then the following statements are equivalent:*

1. *One has that $F^2(\mathcal{H}^3)_B$ is constant.*
2. *The monodromy representation ρ of $R^3 f_* \mathbb{Q}$ satisfies*

$$\rho(\gamma)(F^2(H^3(X, \mathbb{C}))) = F^2(H^3(X, \mathbb{C})) \quad (\forall \gamma \in \pi_1(\mathcal{Z}, z)).$$

3. *One has*

$$\mathrm{Hg}(\mathcal{Y})_{\mathbb{R}} = C(h_G(i)).$$

Proof. In [11], Section 2, we have seen that (1) implies (2).
In the case of (2) we assume that

$$\rho(\gamma)(F^2(H^3(X, \mathbb{C}))) = F^2(H^3(X, \mathbb{C})) \text{ and } \rho(\gamma)(H^3(X, \mathbb{R})) = H^3(X, \mathbb{R}) \quad (\forall \gamma \in \pi_1(\mathcal{Z}, z)).$$

Hence one has also that

$$\rho(\gamma)(\overline{F^2(H^3(X, \mathbb{C}))}) = \overline{F^2(H^3(X, \mathbb{C}))} \quad (\forall \gamma \in \pi_1(\mathcal{Z}, z)).$$

Thus one concludes that $h_G(S^1)$ commutes with $\mathrm{Mon}^0(\mathcal{Y})$. Hence $\mathrm{Mon}^0(\mathcal{Y})_{\mathbb{R}}$ is a semisimple group contained in the simple group $C^{\mathrm{der}}(h_G(i)) \cong \mathrm{SU}(1,1)$. This implies that $C^{\mathrm{der}}(h_G(i)) = \mathrm{Mon}^0(\mathcal{Y})_{\mathbb{R}}$. Since $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{Y}) = \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})$ is simple by Theorem 2.1, we conclude

$$C^{\mathrm{der}}(h_G(i)) = \mathrm{Mon}^0(\mathcal{Y})_{\mathbb{R}} = \mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$$

from the fact that $\mathrm{Mon}^0(\mathcal{Y})_{\mathbb{R}}$ is a normal subgroup of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$. Due to the fact that $h(S^1)$ is not contained in $C^{\mathrm{der}}(h_G(i))$, the reductive group $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ has a nontrivial center. Thus from Corollary 3.4, we conclude (3).

Now assume that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$. In this case $h_G(i)$ commutes with the elements of $h_b(S^1)(\mathbb{R})$ for each $b \in B$. Hence $h_G(S^1)$ is contained in $C(h_b(S^1))$. Due to the fact that $C(h_b(S^1))$ contains only the complex structures $\pm h_W(i)_b$ and $\pm h_G(i)_b$ (see Remark 1.5), one concludes $h_G(i) = h_G(i)_b$ from the fact that the VHS is continuous. In other terms $F^2(\mathcal{H}^3)_B$ is constant. ∎

EXAMPLE 3.8. We consider an example, which ocurs in [10], 11.3.11. Let $\mathcal{E} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ denote the family of elliptic curves

$$\mathbb{P}^2 \supset V(y^2 z - x(x-z)(x-\lambda z)) \to \lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

with involution $\iota_{\mathcal{E}}$ given by $y \to -y$ over $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Moreover there is a $K3$ surface S with involution ι_S such that

$$\iota_S|_{H^{1,1}(S)} = \mathrm{id} \quad \text{and} \quad \iota_S|_{H^{2,0}(S)\oplus H^{0,2}(S)} = -\mathrm{id}.$$

By blowing up the singular sections of the family $\mathcal{E} \times S/\langle(\iota_{\mathcal{E}}, \iota_S)\rangle$ over $\mathbb{P}^1 \setminus \{0,1,\infty\}$, one obtains a family $\mathcal{Y}$ of Calabi-Yau 3-manifolds. The Hodge numbers are given by $h^{1,1} = 61$ and $h^{2,1} = 1$.

It is a well-known fact that the family $\mathcal{E}$ has a locally injective period map to the upper half plane. By [10], Example 1.6.9,

$$F^3(H^3(\mathcal{Y}_\lambda, \mathbb{C})) = H^{2,0}(S) \otimes H^{1,0}(\mathcal{E}_\lambda) \quad \text{and} \quad F^2(H^3(\mathcal{Y}_\lambda, \mathbb{C})) = H^{2,0}(S) \otimes H^1(\mathcal{E}_\lambda, \mathbb{C}).$$

Thus the F^2-bundle in the VHS of $\mathcal{Y}$ is constant and one concludes that $\mathcal{Y}$ a maximal family from the fact that the period map associated with the F^3-bundle is locally injective. By Theorem 3.7, one concludes $\mathrm{Hg}(\mathcal{Y})_{\mathbb{R}} = C(h_G(i))$.

REMARK 3.9. For the proof that (3) $\Rightarrow$ (1) in Theorem 3.7 one does not need the assumption that the base is algebraic. It is sufficient to consider the local universal deformation. Thus from [11], Section 2 one concludes that X cannot occur as a fiber of a family with maximally unipotent monodromy, if $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$.

EXAMPLE 3.10. In [11] one finds an example of a Calabi-Yau 3-manifold X with Hodge numbers $h^{2,1}(X) = 1$ and $h^{1,1}(X) = 73$. The manifold X has an automorphism α of degree 3, which extends to an automorphism of $\mathcal{X}$ over B and acts by a primitive cubic root of unity on $F^2(H^3(X, \mathbb{C}))$. Since α yields an isometry of the Hodge structure of each fiber, the generic Hodge group is contained in the centralizer $C(\alpha)$ of α in $\mathrm{Sp}(H^3(X, \mathbb{Q}), Q)$. By [11], Lemma 3.4, one has a description of $C(\alpha)_{\mathbb{R}}$ coinciding with the description of $C(h_G(i))$ in Proposition 1.6. Hence $C(\alpha)_{\mathbb{R}} = C(h_G(i))$. Due to the fact that $C^{\mathrm{der}}(h_G(i))$ does not contain any proper simple subgroup and $\mathrm{Hg}^{\mathrm{der}}(\mathcal{X})_{\mathbb{R}}$ is a nontrivial simple subgroup of $C^{\mathrm{der}}(h_G(i))$, one concludes $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$.

4. The third case. Recall that K denotes a maximal compact subgroup of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ and that

$$D = \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})(\mathbb{R})/\mathrm{ad}(K(\mathbb{R}))$$

is a Hermitian symmetric domain (see Proposition 2.7). For $D = \mathbb{B}_1$ we have seen that $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \cong C(h_G(i))$, if and only if $\mathrm{Hg}(\mathcal{X})$ has a nondiscrete center (see Corollary 3.4). In Section 2 we have seen that

$$\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X}) = \mathrm{Sp}^{\mathrm{ad}}(H^3(X, \mathbb{Q}), Q) \quad \text{or} \quad \mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} = \mathrm{PU}(1,1).$$

It remains to consider the third case that $\mathrm{Hg}(\mathcal{X})$ has a discrete center and $D \cong \mathbb{B}_1$. Thus assume that $\mathrm{Hg}(\mathcal{X})$ is simple and has dimension 3. We will study $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ by computing its Lie algebra in this case. Let us start with the following observation:

Recall that $\mathrm{GSp}(H^3(X, \mathbb{R}), Q)$ is given by the matrices $M \in H^3(X, \mathbb{R})$ with

$$M^t Q M = rQ \quad \text{for some} \quad r \in \mathbb{R}.$$

Moreover recall that each representation of $\mathbb{S}$ on a real vector space V is a Hodge structure by the decomposition of $V_{\mathbb{C}}$ into the eigenspaces with respect to the characters $z^p \bar{z}^q$ for $p, q \in \mathbb{Z}$ (see [4], 1.1.1). The conjugation by each diagonal matrix

$\mathrm{diag}(a,a,a,a) \in h(\mathbb{S})(\mathbb{R})$ fixes each element of $\mathrm{GSp}(H^3(X,\mathbb{R}),Q)$. Thus the weight homomorphism

$$\mathrm{ad}_{\mathrm{GSp}(H^3(X,\mathbb{R}),Q)} \circ h \circ w$$

is given by $\mathbb{G}_{m,\mathbb{R}} \to \{e\}$ and the Hodge structure $\mathrm{ad}_{\mathrm{GSp}(H^3(X,\mathbb{R}),Q)} \circ h$ is of weight zero. Therefore the algebra $\mathrm{Lie}(\mathrm{GSp}(H^3(X,\mathbb{R}),Q))_{\mathbb{C}}$ decomposes into eigenspaces with respect to the characters $(z/\bar{z})^k$ for $k \in \mathbb{Z}$.

4.1. Now we compute the eigenspace decomposition of $\mathrm{Lie}(\mathrm{Sp}(H^3(X,\mathbb{R}),Q))$ with respect to the representation $(\mathrm{ad}_{\mathrm{Sp}(H^3(X,\mathbb{R}),Q)} \circ h_X)$ of S^1. This description is obtained from the following facts: Each of the following 3-dimensional subgroups of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$ given with respect to the basis $\{v_{3,0},v_{2,1},v_{1,2},v_{0,3}\}$ contains an 1-dimensional subgroup on which $h(S^1)$ acts trivially by conjugation. Moreover the kernel of the respective restricted adjoint representation on the respective Lie algebra can be obtained from the description of the conjugation by elements of $h(S^1)$ in Remark 1.4. This allows us to determine the characters of the respective restricted adjoint representation, since we have only characters of the type $(z/\bar{z})^k$ for $k \in \mathbb{Z}$ as we have seen above. Since

$$10 = \dim \mathrm{Sp}(H^3(X,\mathbb{R}),Q),$$

one checks easily that one can find a basis of eigenvectors by the computations below:

- The centralizer $C(h(S^1))$ is a twodimensional torus (see Remark 1.5), which yields a corresponding twodimensional eigenspace with character 1.
- The group $C^{\mathrm{der}}(h_W(i))$ is given by the matrices

$$M = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \bar{\alpha} & 0 & -\beta \\ -\bar{\beta} & 0 & \bar{\alpha} & 0 \\ 0 & \bar{\beta} & 0 & \alpha \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1$$

  (this follows from Proposition 1.7 and Remark 1.8). The complexified Lie algebra of $C^{\mathrm{der}}(h_W(i))$ has an eigenspace with character $(\bar{z}/z)^2$ and an eigenspace with character $(z/\bar{z})^2$.
- The group $C^{\mathrm{der}}(h_G(i))$ is given by the matrices

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \bar{\beta} & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & \beta & \alpha \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1$$

  (this follows from Proposition 1.6 and Remark 1.8). The complexified Lie algebra of $C^{\mathrm{der}}(h_G(i))$ has an eigenspace with character $\bar{z}/z$ and an eigenspace with character $z/\bar{z}$.
- By explicit computations using the definition of Q (see (1)), one can easily check that the group CG given by the matrices

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \bar{\beta} & \bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \det(M) = 1$$

is a subgroup of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$. The complexified Lie algebra of the group CG has an eigenspace with character $\bar{z}/z$ and an eigenspace with character $z/\bar{z}$.

- By explicit computations using the definition of Q (see (1)), one can easily check that the group given by the matrices

$$M = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{\beta} & 0 & 0 & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad \det(M) = 1$$

is a subgroup of $\mathrm{Sp}(H^3(X,\mathbb{R}),Q)$. The complexified Lie algebra of this group has an eigenspace with character $(\bar{z}/z)^3$ and an eigenspace with character $(z/\bar{z})^3$.

From now on we make computations with respect to the basis $\{v_{3,0}, v_{2,1}, v_{1,2}, v_{0,3}\}$. The Lie algebra of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ contains clearly the vector space

$$\mathrm{Lie}(h_X(S^1)) = \mathrm{Span}_{\mathbb{R}}(\mathrm{diag}(3i, i, -i, -3i)).$$

Recall that the representation $\mathrm{ad} \circ h_X$ of S^1 on $\mathrm{Lie}(\mathrm{Hg}(\mathcal{X}))$ is a weight zero Hodge structure of type $(1,-1),(0,0),(-1,1)$ (follows from Proposition 3.5) and the maximal torus of the 3-dimensional simple group $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ has dimension 1. The direct sum of the eigenspaces with the characters 1, $z/\bar{z}$ and $\bar{z}/z$ coincides with

$$\mathrm{Lie}(C^{\mathrm{der}}(h_G(i)))_{\mathbb{C}} \oplus \mathrm{Lie}(CG)_{\mathbb{C}}$$

as one concludes from 4.1. Hence

$$\mathrm{Lie}(\mathrm{Hg}(\mathcal{X})) \subset \mathrm{Lie}(C^{\mathrm{der}}(h_G(i))) \oplus \mathrm{Lie}(CG).$$

Moreover recall that $\mathrm{Lie}(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}) \cong \mathfrak{su}(1,1)$, where

$$\mathfrak{su}(1,1) = \mathrm{Span}_{\mathbb{R}}(H,X,Y) \text{ for } H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

(compare Remark 1.9). One computes easily that

$$[H,X] = 2Y, \quad [Y,H] = 2X, \quad [Y,X] = 2H.$$

Moreover H generates the Lie subalgebra of a maximal torus of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ with respect to the identification above. Thus $\mathrm{Span}(H) = \mathrm{Lie}(h_X(S^1))$. Since

$$[H, X - iY] = 2Y + 2iX = 2i(X - iY) \text{ and } [H, X + iY] = 2Y - 2iX = -2i(X - iY),$$

the vector space $\mathrm{Span}_{\mathbb{C}}(X,Y)$ has a basis of eigenvectors with respect to $\mathrm{ad}(H)$. Therefore each $M \in \mathrm{Span}_{\mathbb{R}}(X,Y) \subset \mathrm{Lie}(\mathrm{Hg}(\mathcal{X}))$ has the form

$$M = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix} \in \mathrm{Lie}(C^{\mathrm{der}}(h_G(i))) + \mathrm{Lie}(CG),$$

where CG was introduced in 4.1. The explicit descriptions of $C^{\mathrm{der}}(h_G(i))$ and CG in 4.1 and the explicit description of $\mathrm{SU}(1,1)$ in Remark 1.8, yield natural isomorphisms

$$C^{\mathrm{der}}(h_G(i)) \cong CG \cong \mathrm{SU}(1,1).$$

Thus from the explicit description of $\mathfrak{su}(1,1)$ in Remark 1.9, we conclude

$$M = \begin{pmatrix} 0 & x & 0 & 0 \\ \bar{x} & 0 & y & 0 \\ 0 & \bar{y} & 0 & x \\ 0 & 0 & \bar{x} & 0 \end{pmatrix}$$

for some $x, y \in \mathbb{C}$. One has an $M \in \mathrm{Lie}(\mathrm{Hg}(\mathcal{X}))$ with $x \neq 0$. Otherwise one would have

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Lie}(\mathrm{Hg}(\mathcal{X})),$$

since $\dim \mathrm{Span}_{\mathbb{R}}(X, Y) = 2$. This implies

$$[N_1, N_2] = \mathrm{diag}(0, -2i, 2i, 0) \neq 0.$$

But this cannot hold true, since $\mathrm{Span}_{\mathbb{R}}(\mathrm{diag}(3i, i, -i, -3i))$ is the subvector space of diagonal matrices in $\mathrm{Lie}(\mathrm{Hg}(\mathcal{X})_{\mathbb{R}})$. Moreover one has

$$\left[\begin{pmatrix} 0 & x & 0 & 0 \\ \bar{x} & 0 & y & 0 \\ 0 & \bar{y} & 0 & x \\ 0 & 0 & \bar{x} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & \bar{z} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & xz & 0 \\ 0 & y\bar{z} - z\bar{y} & 0 & -zx \\ -\bar{x}\bar{z} & 0 & \bar{y}z - \bar{z}y & 0 \\ 0 & \bar{x}\bar{z} & 0 & 0 \end{pmatrix} \notin \mathrm{Lie}(\mathrm{Hg}(\mathcal{X}))$$

for $x, z \neq 0$. Hence we conclude:

Proposition 4.2. *Assume that* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{PU}(1,1)$ *and* $\mathrm{Hg}(\mathcal{X})$ *has a discrete center. Then for some* $x, y \in \mathbb{C}$ *we have*

$$\mathrm{Lie}(\mathrm{Hg}(\mathcal{X})) = \mathrm{Span}_{\mathbb{R}}(\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & x & 0 \\ 0 & \bar{x} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & y & 0 \\ 0 & \bar{y} & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}).$$

Now we determine the possible choices of $x, y \in \mathbb{C}$:

$$\left[\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & y & 0 \\ 0 & \bar{y} & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & x & 0 \\ 0 & \bar{x} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right] \tag{6}$$
$$= \begin{pmatrix} 2i & 0 & ix - y & 0 \\ 0 & -2i + \bar{x}y - x\bar{y} & 0 & y - ix \\ \bar{y} + i\bar{x} & 0 & 2i + x\bar{y} - \bar{x}y & 0 \\ 0 & -i\bar{x} - \bar{y} & 0 & -2i \end{pmatrix}$$

Hence one obtains

$$ix - y = 0 \Leftrightarrow ix = y \Leftrightarrow \Im(y) = \Re(x), \quad \Re(y) = -\Im(x).$$

Thus the matrix on the right hand side of (6) is contained in $\mathrm{Span}(\mathrm{diag}(3i, i, -i, -3i))$ and for the second entry in the second column we obtain

$$-2i + \bar{x}y - x\bar{y} = \frac{2}{3}i \Rightarrow \bar{x}y - x\bar{y} = \frac{8}{3}i.$$

We have independent of the choice of x and y that

$$\Re(\bar{x}y - x\bar{y}) = \Re(\bar{x}y - \overline{\bar{x}y}) = 0$$

The previous equations imply:

$$\begin{aligned}\frac{8}{3} = \Im(\bar{x}y - x\bar{y}) &= -\Im(x)\Re(y) + \Re(x)\Im(y) + \Re(x)\Im(y) - \Im(x)\Re(y)\\ &= 2\Re(x)^2 + 2\Im(x)^2 = 2|x|^2.\end{aligned}$$

By using $ix = y$, we compute

$$\left[\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & y & 0 \\ 0 & \bar{y} & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}\right] = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2x & 0 \\ 0 & 2\bar{x} & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\text{and} \quad \left[\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & x & 0 \\ 0 & \bar{x} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 2y & 0 \\ 0 & 2\bar{y} & 0 & 2i \\ 0 & 0 & -2i & 0 \end{pmatrix}$$

By the same arguments as in the proof of Lemma 1.2, we conclude that a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ isomorphic to $\mathrm{SU}(1,1)(\mathbb{R})$ is the group of $\mathbb{R}$-valued points of a group isomorphic to $\mathrm{SU}(1,1)$. Note that that the center of $\mathrm{SU}(1,1)$ consists of $\{\pm 1\}$. Since for each $x \in \mathbb{C}$ with $|x| = \frac{2}{\sqrt{3}}$ there is a Lie algebra isomorphic to $\mathfrak{su}(1,1)$, which has an associated subgroup of $\mathrm{GL}(H^3(X,\mathbb{R}))$ with center consisting of $\{\pm 1\}$, we conclude:

Proposition 4.3. *For each $x \in \mathbb{C}$ with $|x| = \frac{2}{\sqrt{3}}$ there is a simple $\mathbb{R}$-algebraic subgroup*

$$G_x \subset \mathrm{Sp}(H^3(X,\mathbb{R}), Q)$$

isomorphic to $\mathrm{SL}_{\mathbb{R}}(2)$ such that $h(S^1) \subset G_x$.

Corollary 4.4. *In the third case we have*

$$\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} \cong \mathrm{SL}_{\mathbb{R}}(2)$$

and all representations of generic Hodge groups of third type on $H^3(X,\mathbb{R})$ are isomorphic.

We will see that in the third case the representation of $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}}$ on $H^3(X,\mathbb{R})$ is isomorphic to the natural representation of $\mathrm{SL}_{\mathbb{R}}(2)$ on $\mathrm{Sym}^3(\mathbb{R}^2)$ later.

Lemma 4.5. *Each unipotent matrix in G_x has a Jordan block of length ≥ 3.*

Proof. A unipotent matrix in G_x, whose Jordan blocks have the maximal length 2, would correspond to a matrix $M \in \mathrm{Lie}(G_x)$, whose square is zero. One has that

$$(m_{i,j}) = M^2 = \begin{pmatrix} a3i & c+bi & 0 & 0 \\ c-bi & ai & cx+by & 0 \\ 0 & c\bar{x}+b\bar{y} & -ai & c+bi \\ 0 & 0 & c-bi & -3ai \end{pmatrix}^2 = 0$$

with $a, b, c \in \mathbb{R}$ is satisfied, only if

$$m_{1,2} = 4ai(c+bi) = 0.$$

Hence $a = 0$ or $c + bi = 0$. The reader checks easily that M^2 cannot be zero in either case with the exception given by $M = 0$. □

EXAMPLE 4.6. In [5] there is a list of explicitly computed examples of variations of Hodge structures of families $\mathcal{Y} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ of Calabi-Yau 3-manifolds with 1-dimensional complex moduli. Note that each of these variations has a monodromy group containing a unipotent matrix, which has only Jordan blocks of length ≤ 2. Due to the fact that $\mathrm{Mon}^0(\mathcal{Y}) \subseteq \mathrm{Hg}(\mathcal{Y})$, we conclude from Lemma 4.5 that there is no x with $|x| = \frac{2}{\sqrt{3}}$ such that $\mathrm{Hg}(\mathcal{Y})_{\mathbb{R}} \cong G_x$. Moreover each example in [5] has maximally unipotent monodromy. Thus we are not in the case $\mathrm{Hg}(\mathcal{Y})_{\mathbb{R}} = C(h_G(i))$ for these examples. Therefore the examples of [5] have a generic Hodge group given by $\mathrm{Sp}(H^3(Y, \mathbb{Q}), Q)$, where Y denotes an arbitrary fiber of the respective family $\mathcal{Y}$.

It would be very nice to find an example for the third case $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = G_x$. At present there is no example of a family of Calabi-Yau manifolds with 1-dimensional complex moduli known to the author, which satisfies the third case. Nevertheless one finds a Calabi-Yau like variation of Hodge structures of third case, which arises in a natural way over a curve as we will see now(for the definition see 4.10. For this example one uses the construction of C. Borcea [1]:

CONSTRUCTION 4.7. *Let E_1, E_2, E_3 be elliptic curves with involutions $\iota_1, \iota_2, \iota_3$ such that $E_j/\iota_j \cong \mathbb{P}^1$. The singular variety*

$$E_1 \times E_2 \times E_3 / \langle (\iota_1, \iota_2), (\iota_2, \iota_3) \rangle$$

yields a Calabi-Yau 3-manifold C by blowing up the singularities. The isomorphism class of C depends on the choice of the sequence of blowing ups. Nevertheless the Hodge structure on $H^3(C, \mathbb{Z})$ does not depend on the choice of this sequence and is given by the tensor product

$$H^3(C, \mathbb{C}) = H^1(E_1, \mathbb{C}) \otimes H^1(E_2, \mathbb{C}) \otimes H^1(E_3, \mathbb{C})$$

of the respective Hodge structures.

Let $f_1 : \mathcal{E} \to \mathbb{A}^1 \setminus \{0, 1\}$ denote the family of elliptic curves given by

$$\mathbb{P}^2 \supset V(y^2 z = x(x-z)(x-\lambda z)) \to \lambda \in \mathbb{A}^1 \setminus \{0, 1\}.$$

By using the involution of $\mathcal{E}$ over $\mathbb{A}^1 \setminus \{0, 1\}$ and three copies of $\mathcal{E} \to \mathbb{A}^1 \setminus \{0, 1\}$, one can give a relative version of the previous construction. Let $f_3 : \mathcal{C} \to (\mathbb{A}^1 \setminus \{0, 1\})^3$ denote a family obtained by this relative version of C. Borcea's construction.

Recall that a Calabi-Yau 3-manifold X has complex multiplication (CM), if the Hodge group $\mathrm{Hg}(H^3(X,Q),h)$ is a torus. For $\mathrm{Hg}(\mathcal{X})_{\mathbb{R}} = C(h_G(i))$ the pair is a Shimura datum (see Proposition 3.5). Thus we have a dense set of CM fibers.[4] But in this case one cannot have maximally unipotent monodromy (see Remark 3.9). Moreover for $\mathrm{Hg}(\mathcal{X}) = \mathrm{Sp}(H^3(X,\mathbb{Q}),Q)$ the associated Hermitian symmetric domain has a dimension larger than the dimension of the basis. For this case one conjectures that only finitely many CM fibers occur. Hence for families of Calabi-Yau 3-manifolds with onedimensional complex moduli it is feasible to concjecture that the existence of infinitely many nonisomorphic CM fibers and maximally unipotent monodromy exclude each other. This does not hold true for Calabi-Yau 3-manifolds with higher dimensional complex moduli, since the family $f_3 : \mathcal{C} \to (\mathbb{A}^1 \setminus \{0,1\})^3$ has maximally unipotent monodromy and a dense set of CM fibers:

Remark 4.8. Let Δ^* denote the punctured disc. One finds a neighbourhood U of the point $(0,0,0) \in \mathbb{A}^3$ such that $\mathcal{C}$ is locally defined over $(\Delta^*)^3 \subset U$. Let D_1, D_2, D_3 denote the irreducible components of the complement of $(\Delta^*)^3 \subset U$ and γ_i denote a closed path given by a loop around D_i. The family $f_1 : \mathcal{E} \to \mathbb{A}^1 \setminus \{0,1\}$ of elliptic curves has unipotent monodromy around 0 with

$$\rho(\gamma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

with respect to a basis $\{a,b\}$ (follows from the computations in [10], Section 3.3). Thus one computes easily that

$$N_{r,s,t} = r\log\rho(\gamma_1) + s\log\rho(\gamma_2) + t\log\rho(\gamma_3) = \begin{pmatrix} 0 & 2t & 2s & 0 & 2r & 0 & 0 & 0 \\ 0 & 0 & 0 & 2s & 0 & 2r & 0 & 0 \\ 0 & 0 & 0 & 2t & 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2r \\ 0 & 0 & 0 & 0 & 0 & 2t & 2s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2s \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis

$$\mathcal{B} = \{a_1 \otimes a_2 \otimes a_3,\quad a_1 \otimes a_2 \otimes b_3,\quad a_1 \otimes b_2 \otimes a_3,\quad a_1 \otimes b_2 \otimes b_3,$$
$$b_1 \otimes a_2 \otimes a_3,\quad b_1 \otimes a_2 \otimes b_3,\quad b_1 \otimes b_2 \otimes a_3,\quad b_1 \otimes b_2 \otimes b_3\}.$$

By analogue computations, one gets the same result for all maximal-depth normal crossing points of $(\mathbb{A}^1 \setminus \{0,1\})^3$. Thus the family $\mathcal{C} \to (\mathbb{A}^1 \setminus \{0,1\})^3$ has maximally unipotent monodromy around each maximal-depth normal crossing point (for the definition of maximally unipotent monodromy see [8]). Moreover C has CM, if and only if E_1, E_2, E_3 have CM as complex tori (see [1], Proposition 3.1). Since it is a well-known fact that $\mathcal{E}$ has a dense set of fibers $\mathcal{E}_\lambda$ such that $\mathcal{E}_\lambda$ has CM, one concludes that $\mathcal{C}$ has a dense set of CM fibers.

Now we come to the Calabi-Yau like VHS of third type. Let $\Delta \subset (\mathbb{A}^1 \setminus \{0,1\})^3$ be the diagonal obtained from the closed embedding

$$\mathbb{A}^1 \setminus \{0,1\} \hookrightarrow (\mathbb{A}^1 \setminus \{0,1\})^3 \text{ via } x \to (x,x,x).$$

[4]The proof uses arguments, which occur already in [11], Section 4. One has only to replace $C(\alpha)$ by $\mathrm{Hg}(\mathcal{X})$ and use the same arguments, which occur after the proof of [11], Lemma 4.

As we will see the rational VHS of the restricted family $\mathcal{C}_\Delta \to \Delta$ contains a sub-VHS of third type. Let

$$\mathcal{H}^1 = R^1(f_1)_*\mathbb{Q} \otimes \mathcal{O}_\Delta \quad \text{and} \quad \mathcal{H}^3 = R^3(f_3|_{\mathcal{C}_\Delta})_*\mathbb{Q} \otimes \mathcal{O}_\Delta.$$

4.9. One has that $\mathcal{H}^3 = (\mathcal{H}^1)^{\otimes 3}$ (see also [13], Remark 7.4) and $F^3(\mathcal{H}^3)$ is contained in the symmetric product $\mathrm{Sym}^3(\mathcal{H}^1)$. Hence

$$H^{3,0}(\mathcal{C}_{(\lambda,\lambda,\lambda)}), H^{0,3}(\mathcal{C}_{(\lambda,\lambda,\lambda)}) \subset \mathrm{Sym}^3(H^1(\mathcal{E}_\lambda, \mathbb{C}))$$

for each $(\lambda, \lambda, \lambda) \in \Delta$. Since $F^3(\mathcal{H}^3) \subset \mathrm{Sym}^3(\mathcal{H}^1)$, one obtains $\nabla_t\omega(b) \in \mathrm{Sym}(H^1(\mathcal{E}_\lambda, \mathbb{Q}))$ for each section $\omega \in F^3(\mathcal{H}^3_\Delta)(U)$ and $t \in T_b\Delta$. By Bryant-Griffiths [2], one has that $F^2(\mathcal{H}^3)$ is generated by the sections of $F^3(\mathcal{H}^3)$ and their differentials. Therefore one concludes that $F^2(\mathcal{H}^3)\cap\mathrm{Sym}^3(\mathcal{H}^1)$ is of rank 2 and we have a polarized rational variation $\mathcal{V}$ of Hodge structures of type

$$(3,0), \quad (2,1), \quad (1,2), \quad (0,3)$$

with the underlying local system $\mathrm{Sym}^3(R^1(f_1)_*\mathbb{Q})$ of rank 4. This VHS satisfies that $F^2(\mathcal{V})$ is generated by the sections of $F^3(\mathcal{V})$ and their differentials along Δ, and that $F^1(\mathcal{V}) = F^3(\mathcal{V})^\perp$ with respect to the polarization. By [2], these two properties characterize the VHS of a family of Calabi-Yau 3-manifolds. In this sense $\mathcal{V}$ is a Calabi-Yau like sub-VHS of the rational VHS of $\mathcal{C}_\Delta$.

4.10. Let M be connected complex manifold and $\mathcal{W} \to M$ be a Calabi-Yau like VHS with

$$h^{3,0}(\mathcal{W}_m) = h^{2,1}(\mathcal{W}_m) = h^{1,2}(\mathcal{W}_m) = h^{0,3}(\mathcal{W}_m) = 1$$

for each $m \in M$. We say that $\mathcal{W}$ is of third type, if the center of its generic Hodge group is discrete and the associated Hermitian symmetric domain is $\mathbb{B}_1$. Note that all previous arguments are also valid for a Calabi-Yau like VHS in the sense of [2], which is not necessarily the VHS of a family of Calabi-Yau 3-manifolds. Thus there is an $x \in \mathbb{C}$ with $|x| = \frac{2}{\sqrt{3}}$ such that $\mathrm{Hg}(\mathcal{W})_\mathbb{R} = G_x$ for a Calabi-Yau like VHS of third type.

Let E be an elliptic curve and $M \in \mathrm{GL}(H^1(E, \mathbb{Q}))$ be given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(H^1(E, \mathbb{Q}))$$

with respect to a basis $\{e_1, e_2\}$ of $H^1(E, \mathbb{Q})$. Moreover let

$$Kr^3(M) = M \otimes M \otimes M$$

denote the third Kronecker power of M. One can easily check that

$$Kr^3(M)(\mathrm{Sym}^3(H^1(E, \mathbb{Q}))) = \mathrm{Sym}^3(H^1(E, \mathbb{Q}))$$

for each $M \in \mathrm{GL}(H^1(E, \mathbb{Q}))$. Moreover one can easily compute that $Kr(M)$ acts on $\mathrm{Sym}^3(H^1(E, \mathbb{Q}))$ by the matrix

$$r(M) = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & 2abd + b^2c & b^2d \\ ac^2 & acd + bc^2 & ad^2 + 2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix} \tag{7}$$

with respect to the basis

$$\{e_1 \otimes e_1 \otimes e_1, \quad e_1 \otimes e_1 \otimes e_2 \quad + \quad e_1 \otimes e_2 \otimes e_1 \quad + \quad e_2 \otimes e_1 \otimes e_1,$$
$$e_1 \otimes e_2 \otimes e_2 \quad + \quad e_2 \otimes e_1 \otimes e_2 \quad + \quad e_1 \otimes e_2 \otimes e_2, \quad e_2 \otimes e_2 \otimes e_2\}.$$

Lemma 4.11. *One has the homomorphisms*

$$r : \mathrm{GL}(H^1(E,\mathbb{Q})) \to \mathrm{GL}(\mathrm{Sym}^3(H^1(E,\mathbb{Q})))$$

and

$$r|_{\mathrm{SL}(H^1(E,\mathbb{Q}))} : \mathrm{SL}(H^1(E,\mathbb{Q})) \to \mathrm{SL}(\mathrm{Sym}^3(H^1(E,\mathbb{Q})))$$

of $\mathbb{Q}$-algebraic groups.

Proof. From (7) one concludes that r is an regular map. Note that the determinant of $r(M)$ is given by $\det^6(M)$ for each $M \in \mathrm{GL}(H^1(E,\mathbb{Q}))$. This follows by computing $\det(r(J_M))$, where J_M denotes the associated Jordan form of M. Since one can easily check that Kr^3 respects the matrix multiplication, one concludes that the same holds true for r. Thus we obtain the homomorphisms of $\mathbb{Q}$-algebraic groups as claimed. ∎

Let G denote the Zariski closure of $r(\mathrm{SL}(H^1(E,\mathbb{Q})))$ in $\mathrm{GL}(\mathrm{Sym}^3(H^1(E,\mathbb{Q})))$. It is a well-known fact that G is an algebraic group.

Lemma 4.12. *The group G has at most dimension 3.*

Proof. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(H^1(E,\mathbb{Q})).$$

For $(m_{i,j}) = r(M)$ one has that

$$\begin{aligned} m_{2,2}^3 &= (a(ad+2bc))^3 \\ &= a^3(a^3d^3 + 6a^2bcd^2 + 12ab^2c^2d + 8b^3c^3) \\ &= m_{1,1}(m_{1,1}m_{4,4} + \frac{2}{3}m_{1,2}m_{4,3} + \frac{4}{3}m_{1,3}m_{4,2} + 8m_{1,4}m_{4,1}) \end{aligned}$$

(follows from (7)). In an analogue way one can express $m_{2,3}^3, m_{3,2}^3, m_{3,3}^3$ by equations with entries $m_{i,j}$ such that $\{i,j\} \cap \{1,4\} \neq \emptyset$. Note that for all other entries $m_{i,j}$ of $r(M)$ such that $\{i,j\} \cap \{1,4\} \neq \emptyset$ the power $m_{i,j}^3$ satisfies some equation in terms of

$$m_{1,1} = a^3, m_{1,8} = b^3, m_{8,1} = c^3, m_{8,8} = d^3$$

(compare (7)). Due to these facts, one finds enough equations such that the Zariski closure $\overline{r(\mathrm{GL}(H^1(E,\mathbb{Q}))}$ of the group $r(\mathrm{GL}(H^1(E,\mathbb{Q}))(\mathbb{Q})$ has at most dimension 4. Since $\det(r(M)) = \det^6(M)$, the set on the right hand site of the inequality

$$G^0 \subseteq (\overline{r(\mathrm{GL}(H^1(E,\mathbb{Q}))} \cap \mathrm{SL}(\mathrm{Sym}^3 H^1(E,\mathbb{Q})))^0$$

is a proper Zariski closed subset of $\overline{r(\mathrm{GL}(H^1(E,\mathbb{Q}))}^0$. Thus one concludes that

$$\dim G \leq 3. \qquad ∎$$

Note that the Hodge structure of $\mathcal{C}_{(\lambda,\lambda,\lambda)}$ is given by the tensor product $H^1(\mathcal{E}_\lambda,\mathbb{Q})^{\otimes 3}$. Thus the associated representation of S^1 is given by $Kr^3 \circ h_\lambda$, where h_λ denotes the Hodge structure of $\mathcal{E}_\lambda$. Therefore the sub-Hodge structure on $\mathrm{Sym}^3(H^1(\mathcal{E}_\lambda,\mathbb{Q}))$ is given by

$$h' = r \circ h_\lambda.$$

One concludes $h'(S^1) \subset G_{\mathbb{R}}$, since $h_\lambda(S^1) \subset \mathrm{SL}(H^1(E,\mathbb{R}))$ and r yields a homomorphism $\mathrm{SL}(H^1(E,\mathbb{R})) \to G_{\mathbb{R}}$.

PROPOSITION 4.13. *The variation $\mathcal{V}$ of Hodge structures is of third type.*

Proof. Since $h'(S^1) \subset G_{\mathbb{R}}$, the conjugation by $h'(i)$ yields a Cartan involution of $G_{\mathbb{R}}$. Thus G is reductive. Since $\dim G \leq 3$, this group is not only reductive, but simple. This follows from the fact that the smallest simple Lie algebras have dimension 3 and G is clearly not commutative. Therefore the center of $G_{\mathbb{R}}$ is discrete the associated hermitian symmetric domain is $\mathbb{B}_1$. Hence $\mathcal{V}$ is of third type. ☐

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