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FANO THREEFOLDS OF GENUS 6*

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Abstract. Ideas and methods of *Clemens C. H., Griffiths Ph. The intermediate Jacobian of a cubic threefold* are applied to a Fano threefold X of genus 6 — intersection of $G(2,5) \subset P^9$ with P^7 and a quadric. Main results:

1. The Fano surface $F(X)$ of X is smooth and irreducible. Hodge numbers and some other invariants of $F(X)$ are calculated.

2. Tangent bundle theorem for X is proved, and its geometric interpretation is given. It is shown that $F(X)$ defines X uniquely.

3. The Abel - Jacobi map Φ : Alb $F(X) \to J^3(X)$ is an isogeny.

4. As a necessary step of calculation of $h^{1,0}(F(X))$ we describe a special intersection of 3 quadrics in P^6 (having 1 double point) whose Hesse curve is a smooth plane curve of degree 6.

5. im $\Phi(F(X)) \subset J^3(X)$ is algebraically equivalent to $\frac{2\Theta^8}{8!}$ where $\Theta \subset J^3(X)$ is a Poincaré divisor (a sketch of the proof).

Key words. Fano threefolds, Fano surfaces, middle Jacobian, tangent bundle theorem, global Torelli theorem.

AMS subject classifications. Primary 14J30, 14J45; Secondary 14J25, 14C30.

0. Introduction — statement of results. The main object of the present paper is a Fano threefold X of genus 6 of the principal series which is an intersection of the Grassmannian variety $G(2,5) \subset P^9$ with P^7 and a quadric. The surface that parametrizes conics on X is called its Fano surface and is denoted by $F(X)$. The middle Jacobian of X is denoted by $J^3(X)$. We investigate the Abel - Jacobi map $\Phi : F(X) \to J^3(X)$. This investigation is the main component of solution of problems of rationality and of global Torelly theorem for X.

Sections 1 and 2 are a part of introduction. Section 1 contains a definition and the main properties of intermediate Jacobian of a variety, and of an Abel - Jacobi map. Section 2 contains a definition of Fano threefolds, and the main steps of proofs of non-rationality of a cubic threefold and of a two-sheeted covering of P^3 ramified at a quartic. This justifies the subject of the present paper.

Other sections contain new results. In Sections 3 and 4 we find some simple properties of $F(X)$, calculate some of its invariants and prove the tangent bundle theorem. Namely, let V be a vector space of dimension 5, $G = G(2,V)$ a Grassmann variety, it is included into $P(\lambda^2(V))$ via Plücker embedding. Let H_1, H_2 be hyperplanes in $P(\lambda^2(V))$ and Ω a quadric in $P(\lambda^2(V))$ such that $X = G \cap H_1 \cap H_2 \cap \Omega$ is a smooth threefold. X is a Fano threefold of the principal series that will be studied in the present paper. We consider only those X that satisfy some conditions (3.6), (3.7) (which are fulfilled for a generic X), although most likely all results of the paper are true for all X. Fano surface $F_c = F_c(X)$ is a set of conics on X.

Main results of Section 3 are the following:

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Proposition 0.1. F_c is a smooth surface containing a distinguished point c_Ω and a distinguished straight line l_2 ($c_\Omega \notin l_2$)). Let $r_F : F \to F_c$ be the blowing up of F_c at c_Ω and $l_1 = r_F^{-1}(c_\Omega)$. There exists an involution i_F on F without stable points such that $i_F|_{l_1} : l_1 \to l_2$ is an isomorphism. We denote a two-sheeted covering that corresponds to i_F by $p_F : F \to F_0$, and we denote $l = p_F(l_1)$. l_1 and l_2 are exceptional lines on F, i.e. there are blowings down maps $r : F \to F_m$ (resp. $r_0 : F_0 \to F_{0m}$) that send lines $l_1 \cup l_2$ (resp. l) to points $c_\Omega \cup c'_\Omega$ (resp. $c_{0\Omega}$), as well as an involution map $i_{F_m} : F_m \to F_m$ and the corresponding two-sheeted covering $p_{F_m} : F_m \to F_{0m}$, so we have a commutative diagram:

$$\begin{tikzcd}
 & F \arrow[rr, "i_F"] \arrow[dl, "r_F"'] \arrow[dd, "r"] \arrow[dr, "p_F"] & & F \arrow[dl, "p_F"'] \arrow[dd, "r"] \\
F_c \arrow[dr] & & F_0 \arrow[dd, "r_0"] & \\
 & F_m \arrow[rr, "i_{F_m}"] \arrow[dr, "p_{F_m}"'] & & F_m \arrow[dl, "p_{F_m}"] \\
 & & F_{0m} &
\end{tikzcd}$$

We denote $Z = P_{P(V^*)}(O \oplus O(1))$, there are projections $\eta : Z \to P(V^*)$ and $r_Z : Z \to P^5$ where r_Z is the blowing up map of a point $t \in P^5$.

Proposition 0.2. There are canonical inclusions $\bar\phi : F_0 \hookrightarrow Z$, $\bar\phi_m : F_{0m} \hookrightarrow P^5$ making the following diagram commutative (the vertical maps are r_0, r_Z):

$$\begin{array}{ccc}
F_0 & \overset{\bar\phi}{\to} & Z \\
\downarrow & & \downarrow \\
F_{0m} & \overset{\bar\phi_m}{\to} & P^5
\end{array}$$

such that $\bar\phi_m(c_{0\Omega}) = t$. Further, P^5 can be identified with a space which is dual to the space of quadrics in P^7 which contain X. The point $t \in P^5$ corresponds to a hyperplane of quadrics which contain $G \cap H_1 \cap H_2$.

Proposition 0.3. There exists a locally free sheaf M of rank 4 on $P(V^*)$ and a bundle of quadrics $O_\eta(-1) \to S^2(\eta^*(M^*))$ on Z such that $F_0 \hookrightarrow Z$ is its rank 2 determinantal variety.

Proposition 0.4. Let $\sigma_m \in \operatorname{Pic} F_{0m}$ and $\sigma = r^*(\sigma_m) \in \operatorname{Pic} F_0$ are sheaves of order 2 that correspond to two-sheeted coverings p_{F_m}, p_F respectively. There is an explicit formula on F_0 for σ (see (3.8)). The canonical sheaf of F_{0m} (resp. F_m) is $\omega_{F_{0m}} = \bar\phi_m^*(O_{P^5}(3)) \otimes \sigma_m$ (resp. $\omega_{F_m} = p_{F_m}^*(\bar\phi_m^*(O_{P^5}(3)))$).

Proposition 0.5. $\deg \bar\phi_m(F_{0m}) = 40$, $c_1(\Omega_{F_m})^2 = 720$, $c_2(\Omega_{F_m}) = 384$. Taking into consideration that $h^{0,0}(F) = h^{0,0}(F_0) = 1$ and that $h^{1,0}(F) = 10$, $h^{1,0}(F_0) = 0$ (see Theorem 0.13) we get the following table of Hodge numbers $h^{i,j}(F_m)$:

$$\begin{array}{ccc}
1 & 10 & 101 \\
10 & 220 & 10 \\
101 & 10 & 1
\end{array} \tag{0.6}$$

and their decomposition on i_{F_m}-invariant part $h^{i,j}(F_m)^+ = h^{i,j}(F_{0m})$:

$$\begin{array}{ccc} 1 & 0 & 45 \\ 0 & 100 & 0 \\ 45 & 0 & 1 \end{array} \tag{0.7}$$

and on i_{F_m}-antiinvariant part $h^{i,j}(F_m)^- = h^i(F_{0m}, \lambda^j\Omega_{F_{0m}} \otimes \sigma_m)$:

$$\begin{array}{ccc} 0 & 10 & 56 \\ 10 & 120 & 10 \\ 56 & 10 & 0 \end{array} \tag{0.8}$$

Proposition 0.9. A conic c on X and a conic $i_F(c)$ which is involutory to c, meet at 2 (possibly coinciding) points $\gamma_1(c)$, $\gamma_2(c)$. We define a surface $W = \cup_{c\in F}(\gamma_1(c) \cup \gamma_2(c))$. Then for a generic point $x \in W$ there exists only one (up to i_F) conic c such that $x \in \{\gamma_1(c), \gamma_2(c)\}$.

Proposition 0.10. $O_X(W) = O_{P^7}(21)|_X$. There are 39 conics passing through a fixed generic point on X.

Remark 0.11. 0.4 – 0.8 remain true for a Fano surface $F(X')$ of X' — a two-sheeted covering of P^3 ramified at a quartic W, while 0.9 and 0.10 are not true for X'. The end of Section 3 contains a description of a special Fano threefold of genus 6 defined in [1] — a two-sheeted covering of a section of G by 3 hyperplanes ramified at an intersection with a quadric — whose properties are similar to the ones of X'.

Section 4 of the present paper contains a proof of the tangent bundle theorem for $F(X)$. Let us give the necessary definitions. There exists a locally free sheaf $\tau_{2,M}$ of rank 2 on F such that for $c \in F$ $P(\tau_{2,M}(c))$ is the linear envelope of $\gamma_1(c), \gamma_2(c)$. Let $i : l_1 \cup l_2 \to F$ be the natural inclusion.

Theorem 0.12. (Tangent bundle theorem). There exists an exact sequence of sheaves on F:

$$0 \to \tau^*_{2,M} \to r^*(\Omega_{F_m}) \to i_*(2O_{l_1\cup l_2}) \to 0.$$

The main resilt of Section 6 is a the following

Theorem 0.13. $h^{1,0}(F) = 10$, $h^{1,0}(F_0) = 0$, $h^{0,0}(F) = 1$, $h^{0,0}(F_0) = 1$.

The method of the proof is to consider a degeneration of X in a family of Fano threefolds $f_X : X_4 \to \Delta$ whose base Δ is a complex analytic neighbourhood of 0. A generic fibre of f_X is a smooth Fano threefold, a special fibre $X_0 = f_X^{-1}(0)$ has one double point. X_0 is birationally equivalent to a special intersection of 3 quadrics in P^6 denoted by X_0'.

Let us consider main objects related to a generic intersection of 3 quadrics in P^{n-1} (denoted by X_8^{n-4}) for an odd n ([28], [47] for any n, [21], [50] for $n = 7$). Namely, attached to X_8^{n-4} is a Hesse curve H_n which is a plane curve of degree n, and its two-sheeted non-ramified covering $p_{H_n} : \hat{H}_n \to H_n$. We define $S = S(p_{H_n})$ as a connected component of the set of effective divisors of degree n on $\hat{H}_n$ such that $d \in S \iff O_{H_n}(p_{H_n*}(d)) = O_{P^2}(1)|_{H_n}$. We define a Fano surface $F(X_8^{n-4})$ as a set of $\frac{n-5}{2}$-dimensional quadrics on X_8^{n-4}. Then:

(1) There exists an isomorphism Pr $\hat{H}_n/H_n \to J^{n-4}(X_8^{n-4})$ ([21]), where Pr is the Prym variety;

(2) There exists an isogeny Alb $S \to$ Pr $\hat{H}_n/H_n$ ([50]);

(3) There exists an isomorphism $F(X_8^{n-4}) \to S$ ([47]).

Section 5 contains a proof of the following

THEOREM 0.14. An analog of a Hesse curve for X_0' is a smooth plane curve H_6 of degree 6. Sing $(F(X_0))$ is a double curve which is isomorphic to H_6. Analogs of the above $\widetilde{(1) - (3)}$ are true for X_0' after replacement of $\underbrace{F(X_0) \text{ and } X_0'}$ by their desingularizations $\widetilde{F(X_0)}$ and $\tilde{X}_0'$ respectively. Particularly, $h^{1,0}(\widetilde{F(X_0)}) = \dim \text{Pr } (\hat{H}^6/H^6) = 9$.

To finish a proof of Theorem 0.13, we use a Clemens - Schmidt exact sequence ([40]) for a family $f_F : F_3 \to \Delta$ whose fibres are Fano surfaces of fibres of f_X.
The next result of Section 6 is

THEOREM 0.15. Abel - Jacobi map Φ : Alb $F(X) \to J^3(X)$ is an isogeny.

Finally, a sketch of a proof of the following theorem is given:

THEOREM 0.16. $\Phi(F) \subset J^3(X)$ is algebraically equivalent to $\frac{2\Theta^8}{8!}$ where $\Theta \subset J^3(X)$ is a Poincaré divisor.

The idea of the proof is the following. A result of Beauville ([28]) implies that an analog of this result is true for the image of $S(p_{H_6})$ in Pr $\hat{H}^6/H^6$ (i.e. that it is equivalent to $\frac{2\Theta^7}{7!}$). Investigation of topology of degeneration of abelian varieties given in [25], [31], permits to deduce (0.16) from this property.

Section 7 contains proofs of 2 theorems:

THEOREM 0.17. (Geometric interpretation of the tangent bundle theorem). Let us denote $V_{10} = H^0(F, \Omega_F)^*$, and let $B_{10} : F_m \to G(2, V_{10})$ be a map defined by Ω_{F_m}. Let $B_8 : F \to G(2, V_8)$ be a map defined as follows: $B_8(f) =< \gamma_1(f), \gamma_2(f) >$ (here $V_8 = H_1 \cap H_2 \subset \lambda^2(V)$). Then there exists a natural projection $p_{10,8} : V_{10} \to V_8$ such that

$$B_8 = G(2, p_{10,8}) \circ B_{10} \circ r$$

(recall that $r : F \to F_m$ is a blowing down map). If B_{10} is regular at $c_{0\Omega} \in F_m$ then $p_{10,8}$ is a projection from a plane $B_{10}(c_{0\Omega}) \subset V_{10}$.

THEOREM 0.18. Let X satisfies the property: B_{10} is regular at $c_\Omega \in F_m$. Then, if for another Fano threefold X' of the same type we have $F_c(X) = F_c(X')$ then $X = X'$.

The proof of this theorem is based on the above geometric interpretation of the tangent bundle theorem that permits to recover uniquely a map $B_8 : F \to G(2, V_8)$ by a given $F_c(X)$, and then to recover uniquely X by B_8.

1. Introduction — Intermediate Jacobian of algebraic varieties and Abel - Jacobi map. Let Y be a smooth n-dimensional algebraic variety over $\mathbb{C}$. We consider the Hodge decomposition of its Betti cohomology of an odd dimension k:

$$H^k(Y, \mathbb{C}) = \sum_{i+j=k} H^{i,j}(Y, \mathbb{C})$$

and we denote

$$H^+ = H^k(Y,\mathbb{C})^+ = \sum_{i+j=k, i>j} H^{i,j}(Y,\mathbb{C})$$

$$H^- = H^k(Y,\mathbb{C})^- = \sum_{i+j=k, i<j} H^{i,j}(Y,\mathbb{C})$$

so we have

$$H^+ = \overline{H^-};\ \ H^+ \cap H^- = 0. \tag{1.1}$$

(1.1) implies that the image of $H^k(Y,\mathbb{Z})$ in H^- under the projection along H^+ is a lattice. A complex torus $H^k(Y,\mathbb{C})^-/\mathrm{im}\ H^k(Y,\mathbb{Z})$ is called the k-th intermediate Jacobian of Y, it is denoted by $J^k(Y)$. For $k=n$ it is called also the middle Jacobian.

To define a structure of a polarized abelian variety on a complex torus V/R, it is necessary to define an hermitian form H on V satisfying conditions:

(i) H is positively defined;

(ii) im H restricted on R takes integer values.

There exists a Hodge real bilinear form Q on $H^k(Y,\mathbb{C})$ (see, for example, [39]) which is defined (for $k \le n$) as follows:

$$Q(\xi,\eta) = \xi \wedge \eta \wedge L^{n-k} \in H^{2n}(Y,\mathbb{C}).$$

Here $H^{2n}(Y,\mathbb{C})$ is identified with $\mathbb{C}$, $L \in H^{1,1}(Y,\mathbb{C}) \cap H^2(Y,\mathbb{Z})$ is the fundamental class of a hyperplane section of Y or, which is the same, the cohomology class that corresponds to a Kähler form on Y. Now we define H on H^- as follows:

$$H(\xi,\eta) = iQ(\xi,\bar{\eta}).$$

H is hermitian, because Q is skew symmetric on odd-dimensional cohomology. A map

$$\wedge L^{n-k} : H^k(Y,\mathbb{Z}) \to H^{2n-k}(Y,\mathbb{Z})$$

is an inclusion, and a pairing $H^k(Y,\mathbb{Z}) \otimes H^{2n-k}(Y,\mathbb{Z}) \to \mathbb{Z}$ is unimodular, according a Poincaré theorem. This means that (ii) is satisfied for H, and moreover if $\wedge L^{n-k}$ is an isomorphism then $B(H) := \mathrm{im}\ H$ is unimodular. Nevertheless, (i) is not satisfied in general. We denote $k = 2l+1$. Hodge - Riemann bilinear relations show that a sufficient condition for $J^k(Y)$ to be an abelian variety is the following: $H^- = H^{l,l+1}(Y,\mathbb{C})$ and $H^{l-1,l}(Y,\mathbb{C}) = 0$ (i.e. all elements of $H^{l,l+1}(Y,\mathbb{C})$ are primitive). These conditions are always true for $k=1$, and $J^1(Y) = \mathrm{Pic}\ ^0(Y)$.

For $k=n$ $\wedge L^{n-k}$ is an identical operator, hence $B(H)$ is unimodular. Complex torus V/R having an hermitian form H on V satisfying (ii) such that $B(H)$ is unimodular, is called a principal torus. So, a middle Jacobian $J^n(X)$ is a principal torus (Griffiths, [38]).

For a principal torus T (like for principally polarized abelian varieties) there exists a notion of a dual torus which is isomorphic to T. For $J^n(Y) = H^-/\mathrm{im}\ H^n(Y,\mathbb{Z})$ the dual torus is $(H^+)^*/\mathrm{im}\ H_n(Y,\mathbb{Z})$. This is a convenient way to define an Abel - Jacobi map (it is possible to define it for all $J^k(Y)$ without consideration of the dual torus, see for example [42], but we do not need it).

Let F be a family of l-dimensional algebraic cycles on Y. Abel - Jacobi map $\Phi : F \to J^n(Y)$ (or $\Phi : \mathrm{Alb}\ F \to J^n(Y)$) is a generalization for $n \ge 1$ of a map $Y \to \mathrm{Alb}\ Y$ for $n=1$. A family of cycles F is defined by its graph $\Gamma_F \overset{i}{\hookrightarrow} F \times Y$ such that for $f \in F$ $(p_F \circ i)^{-1}(f)$ is a cycle that corresponds to f, here p_F is a projection from $F \times Y$ to F. Then there are cylinder maps

$$\Phi_{\mathbb{Z}} : H_1(F, \mathbb{Z}) \to H_n(Y, \mathbb{Z})$$

$$\Phi_{\mathbb{C}} : \begin{array}{ccc} H^1(F,\mathbb{C})^* & \to & H^n(Y,\mathbb{C})^* \\ \| & & \| \\ H_1(F,\mathbb{C}) & \to & H_n(Y,\mathbb{C}) \end{array}.$$

Since F is an algebraic family, $\Phi_{\mathbb{C}}$ behaves good with respect to the Hodge decomposition:

$$\Phi_{\mathbb{C}}(H^{i,j}(F,\mathbb{C})^*) \subset H^{i+l,j+l}(Y,\mathbb{C})^*.$$

For $(i,j) = (1,0)$ we have $\Phi_{\mathbb{C}}(H^{1,0}(F,\mathbb{C})^*) \subset H^{l+1,l}(Y,\mathbb{C})^*$ and hence $\Phi_{\mathbb{C}}$ and $\Phi_{\mathbb{Z}}$ define a map $\Phi : \mathrm{Alb}\ F \to J^n(Y)$ which is called the Abel - Jacobi map.

Further we shall consider only the middle Jacobian of threefolds. To have a possibility to distinguish rational and non-rational threefolds we should know what is the behaviour of the Jacobian under the simplest rational maps.

Proposition 1.2. ([32]; see also [20]). Let Y be a threefold and $\tilde{Y}$ be the blowing up of a point (resp. of a curve C) on Y. Then $J^3(\tilde{Y}) = J^3(Y)$ (resp. $J^3(\tilde{Y}) = J^3(Y) \oplus J(C)$).

According Hironaka ([41]) any birational map of algebraic varieties over $\mathbb{C}$ is a composition of blowings up and down. This implies a general theorem which is used to apply the method of middle Jacobian to proofs of non-rationality of threefolds:

Theorem 1.3. ([32]). Middle Jacobian of a rational threefold is an abelian variety which is isomorphic to the direct sum of Jacobians of curves.

To apply this theorem we need to know properties of Jacobians of curves. Here is one of them:

Theorem 1.4. (Andreotti - Mayer criterion, [24]). Let (X,Θ) be a principally polarized abelian variety. If (X,Θ) is the Jacobian of a curve C then $\mathrm{Codim}\ _X\mathrm{Sing}\ \Theta = 4$ (resp. 3) if C is not a hyperelliptic curve (resp. if C is a hyperelliptic curve).

2. Introduction — Fano threefolds. A threefold X is called a Fano threefold if its anticanonical sheaf K_X^{-1} is ample on X. A survey on Fano threefolds can be found in [2], [3], [16]. According a Kodaira criterion, a Fano threefold X has an important property $H^i(X, O_X) = 0$ for $i > 0$, i.e. X satisfies an obvious necessary condition of rationality. $J^3(X)$ is an abelian variety whose dimension is $h^{2,1}(X)$.

Let us consider the main steps of the proof of Clemens - Griffiths ([32]) of non-rationality and global Torelli theorem of a cubic threefold. Firstly we give some general definitions.

Let $\phi : Y \to A$ be a map of a k-dimensional variety Y to an abelian variety A. For any $y \in Y$ a map of tangent spaces is defined: $T_\phi(y) : T_Y(y) \to T_A(\phi(y))$. Let ϕ satisfy a property that T_ϕ is an inclusion at a generic point. Then there exists a Gauss map which is a rational map $G(\phi) : Y \to G(k, T_A(0))$ where $G(k, T_A(0))$ is the Grassmann variety. The Gauss map is defined as follows. For $y \in Y$ $T_\phi(y)(T_Y(y))$ is an element of $G(k, T_A(\phi(y)))$. The group law on an abelian variety defines a canonical isomorphism of tangent spaces at any its points, and hence an isomorphism $G(k, T_A(\phi(y))) \to$

$G(k, T_A(0))$. An image of the above element under this isomorphism is the desired element $G(\phi)(y)$.

Let $\Phi : Y \to \text{Alb } Y$ be the Albanese map. If its Gauss map exists then it coincides with the map $Y \to G(k, H^0(Y, \Omega_Y)^*)$ associated with the sheaf Ω_Y (since $T_{\text{Alb } Y}(0)$ is canonically isomorphic to $H^0(Y, \Omega_Y)$).

Let now V be a vector space of dimension 5 and $X \subset P(V)$ a cubic threefold. By definition, its Fano surface $F = F(X)$ is a set of straight lines on X. It is a subvariety of $G(2, V)$, we denote the corresponding inclusion $F \to G(2, V)$ by i_F.

THEOREM 2.1. dim Alb $F = h^{1,0}(F) = 5$.

A proof of this theorem given in [32] uses a technique of degeneration of X to a threefold X_0 having one double point x_0. A set of straight lines on X_0 which contain x_0 is a smooth curve D_0 on $F_0 = F(X_0)$; it is easy to calculate that its genus is 4 and that Sing $F_0 = D_0$. The tangent cone to X_0 at x_0 has 2 rulings. This implies: if $r : \tilde{F}_0 \to F_0$ is a desingularization of F_0 then $r^{-1}(D_0) = D_1 \cup D_2$ is a pair of curves which are isomorphic to D_0. Further, we have: $\tilde{F}_0 = S^2(D_0)$, $h^{1,0}(\tilde{F}_0) = h^{1,0}(S^2(D_0)) = g(D_0) = 4$. Finally, degeneration theory for X and F implies that $h^{1,0}(F) = h^{1,0}(\tilde{F}_0) + 1 = 5$. The situation for the Fano threefolds considered in the present paper is the same (see Proposition 5.12).

Theorem 2.1 implies the following

COROLLARY 2.2. Abel - Jacobi map $\Phi : \text{Alb } F \to J^3(X)$ is an isogeny.

Really, Φ is an isomorphism, but we do not need it.

The next step is a proof of the tangent bundle theorem for F. It establishes a connection between Ω_F (or T_F) and a sheaf defined on X "geometrically". For the case of a cubic threefold its statement is very simple:

THEOREM 2.3. There is an isomorphism of sheaves on F: $\Omega_F = i_F^*(\tau_{2,V}^*)$.

A proof of this theorem for the Fano surface of a cubic threefold is also very easy (comparatively to proofs of analogous theorems for other Fano threefolds). The main idea of the proof is a calculation of T_F by means of deformation theory.

Theorems 2.1 and 2.3 imply

COROLLARY 2.4. (Geometric interpretation of the tangent bundle theorem). There exists an isomorphism $H^0(F, \Omega_F)^* \to V$ such that the canonical map $F \to G(2, H^0(F, \Omega_F)^*)$ coincides with the inclusion $i_F : F \to G(2, V)$.

Corollaries 2.2 and 2.4 imply that the Gauss map of Φ (i.e. a map $G(\Phi) : F \to T_{J^3(X)}(0)$) coincides with i_F.

THEOREM 2.5. (Analog of a Riemann theorem). A set of points of type $\Phi(f_1) - \Phi(f_2)$ where $f_1, f_2 \in F$ is (up to a shift) a Poincaré divisor $\Theta \overset{i_\Theta}{\hookrightarrow} J^3(X)$.

Theorem 2.5 and Corollary 2.4 are used for a description of the Gauss map of i_Θ, i.e. a map $G(i_\Theta) : \Theta \to G(4, V) = P(V^*)$. It is clear that if $f_1, f_2 \in F$ and the corresponding straight lines in X do not meet one another, then (2.4) implies that $G(i_\Theta)(\Phi(f_1) - \Phi(f_2))$ is a space $P^3 \in P(V^*)$ spanned on f_1, f_2 considered as straight lines in $P(V)$. It is possible to prove that if t is an intersection point of f_1, f_2 considered as straight lines in $P(V)$ then $G(i_\Theta)(\Phi(f_1) - \Phi(f_2))$ is the tangent space of X at t.

This means that $G(i_\Theta) : \Theta \to G(4, V) = P(V^*)$ is a n-fold covering where $n = \#(G(i_\Theta)^*(P^3))$ for a $P^3 \in P(V^*)$ is equal to the number of pairs of non-meeting straight lines on a smooth cubic surface $P^3 \cap X$. We have $n = 216$. This is sufficient to make a conclusion that $J^3(X)$ is not a Jacobian of a curve C of genus 5. Really, for $J(C)$ the degree of covering of the Gauss map $G(i_\Theta) : \Theta \to G(4, V)$ is clearly $\binom{8}{4} = 70$. Another way to see that $J^3(X) \neq J(C)$ is to investigate the ramification divisor of $G(i_\Theta)$. For $J^3(X)$ (resp. $J(C)$) it is the set of P^3 which are tangent to X (resp. to C). Finally, the above recovering of X by ramification divisor of $G(i_\Theta)$ means that the global Torelli theorem is proved for it:

Theorem 2.6. The polarized abelian variety $(J^3(X), \Theta)$ defines X uniquely.

Analogs of results of [32] for cubic threefolds are obtained for the following Fano threefolds:

1. Two-sheeted covering of P^3 with ramification in a quartic ([11], [12], [14], [49], [51]). Since many properties of X^3_{10} — a Fano threefold of genus 6 (the main object of the present paper) are analogous to the ones of a two-sheeted covering of P^3 with ramification in a quartic, a brief survey of them will be given below.
2. X^3_8 — an intersection of 3 quadrics in P^6 ([21], [28], [50], [51]). Main theorems concerning X^3_8 are given in Section 5, because they are necessary for a calculation of irregularity of X^3_{10}. Here there is no complete analogy: proofs of non-rationality and of global Toreeli theorem for X^3_8 use a fact that $J^3(X^3_8)$ is a Prym variety. Tangent bundle theorem for X^3_8 is unknown.
3. "Double cone Veronese" X^3_1 ([13]). Here the situation is more complicated than for other investigated types of Fano varieties. Namely, $F(X^3_1)$ is not reduced, and X^3_1 is the only known Fano threefold for which $h^{1,0}(F(X^3_1)) \neq h^{2,1}(X^3_1)$ (here we have $h^{1,0}(F(X^3_1)) = 2h^{2,1}(X^3_1)$).

We give here main steps and results of a proof of non-rationality of a two-sheeted covering of $P^3 \subset P(V_4)$ with ramification in a quartic $W \subset P^3$, because they are analogs of the results of the present paper. Let $v : P^3 \to P^9 = P(S^2(V_4))$ be a Veronese map (i.e. defined by the complete linear series $O_{P^3}(2)$), t a point in a $P^{10} - P^9$, K a cone over $v(P^3)$ and a vertex t, $p : K \to v(P^3)$ a projection from t, Ω a quadric in P^{10} such that $t \notin \Omega$. Then "a double P^3" $X = K \cap \Omega$. Without loss of generality we can suppose that P^9 is the polar hyperplane for Ω with respect to t. Then the ramification quartic $W = v^{-1}(\Omega \cap v(P^3))$.

Smoothness of X implies smoothness of W. We consider below only the case when W does not contain straight lines (this restriction is rejected in [15]). It is easy to see that a condition that W does not contain straight lines is analogous to a condition of the lemma 3.7 of the present paper for X^3_{10}.

If c is a conic on X then $p(c)$ is a conic on $v(P^3)$ and $v^{-1}(p(c))$ is a straight line in $P(V_4)$ which is tangential to W at 2 points (a bitangent of W). We define a Fano surface $F = F(X)$ of X as a set of conics on X, and we denote a set of bitangents of W in $P(V_4)$ by F_0. A map $c \mapsto v^{-1}(p(c))$ is a two-sheeted non-ramified covering $p_F : F \to F_0$, we denote the corresponding involution on F by i_F. All conics on X are smooth. Involutory conics c and $i_F(c)$ meet at 2 points $\gamma_1(c), \gamma_2(c) \in v(W)$, and $v^{-1}(\gamma_i(c))$ are points of tangency of $p_F(c)$ of W. There are 12 conics passing through a fixed generic point of X and 6 pairs of involutory conics passing through a fixed generic point of $v(W)$.

By definition, F_0 is included in $G(2, V_4) \subset P^5 = P(\lambda^2(V_4))$. It was indicated above (in (0.11)) that (0.4) and (0.5) are true for this inclusion and hence Hodge numbers of F and F_0 are given in (0.6) - (0.8).

Main theorems for a "double P^3" are the following:

THEOREM 2.7. (Tangent bundle theorem). There exists a locally free sheaf τ_2 on F of rank 2 such that for $c \in F$ $\quad P(\tau_2(c))$ is a linear envelope of $\gamma_1(c)$, $\gamma_2(c)$. Then $\Omega_F = \tau_2^*$.

THEOREM 2.8. $h^{1,0}(F) = 10$. Moreover, the Abel - Jacobi map $\Phi : \mathrm{Alb}\, F \to J^3(X)$ is an isomorphism.

COROLLARY 2.9. (Geometric interpretation of the tangent bundle theorem). Let us consider a map $\rho_1 : F \to G(2, H^0(F, \Omega_F)^*)$ given by Ω_F, and a map $\rho_2 : F \to G(2, S^2(V_4))$ which sends $c \in F$ to a linear envelope of $\gamma_1(c)$, $\gamma_2(c) \subset P(S^2(V_4))$. Then there exists a natural isomorphism $\iota : H^0(F, \Omega_F)^* \to S^2(V_4)$ such that $\iota \circ \rho_1 = \rho_2$.

THEOREM 2.10. $F(X)$ defines X uniquely.

THEOREM 2.11.(Analog of a Riemann theorem). Let us consider a map $\Phi_5 : \prod_{i=1}^5 F \to J^3(X)$ defined as follows: $\Phi_5(f_1, \dots, f_5) = \sum_{i=1}^5 \Phi(f_i) - \Phi(i_F(f_i))$. Then its ramification divisor is a Poincaré divisor Θ on $J^3(X)$.

THEOREM 2.12. Θ is irreducible, and codim $_{J^3(X)}\mathrm{Sing}\, \Theta = 2$.

So, according Andreotti - Meier criterion, $J^3(X)$ is not the Jacobian of a curve, and hence X is not rational.

THEOREM 2.13. (Global Torelli theorem). $(J^3(X), \Theta)$ defines X uniquely.

3. Fano surface as a determinantal variety. Here we introduce some notations that will be used throughout the paper.

A sheaf of type $O_\alpha(a) \otimes O_\beta(b) \otimes \dots$ will be denoted by $O(_\alpha a,_\beta b \dots)$. The inverse image of a sheaf will be denoted often by the same symbol as the sheaf itself, if it is clear what map we have in mind (the map whose inverse image is considered). The blowing up map of a variety Y along $X \subset Y$ will be denoted by r with some subscript, and for $Z \subset Y$ the notation $r^{-1}(Z)$ will mean the total inverse image of Z, if $Z \subset X$, and the proper inverse image if $Z \not\subset X$. $V_\bullet$ means a vector space, and the subscript is its dimension. If E is a locally free sheaf on a variety Y, then the fibre of the corresponding vector bundle at a point $y \in Y$ will be denoted by $E(y)$. $P_Y(E) \to Y$ and $G_Y(k, E) \to Y$ mean the geometric projectivization and grassmannization respectively, i.e. $P_Y(E) = \mathrm{Proj}(\sum_k S^*(E^*))$. Members of tautological exact sequences on $G_Y(k, E)$ are denoted as follows:

$$0 \to \tau_{k,E} \to E \to \tau^*_{n-k,E} \to 0$$
$$0 \to \tau_{n-k,E^*} \to E^* \to \tau^*_{k,E} \to 0$$

(here $n = \dim E$). For $t \in G_Y(k, V_n)$ $\quad t_V$ means a k-dimensional subspace in V_n that corresponds to t. In some cases a vector space and its projectivization will be denoted by the same symbol.

Here we give without proofs some results about $G(2, 5)$ and its hyperplane sections (see, for example, [48]).

We identify $G(2,V_n)$ and its image in $P(\lambda^2(V_n))$ under the Plücker embedding. For $n=4$ $G(2,4)$ is a quadric hypersurface in $P(\lambda^2(V_4))$, we shall denote the corresponding element in $P(S^2(\lambda^2(V_4^*)))$ by $Pl(V_4)$. Further we shall consider a space $V=V_5$ and a Grassmannian $G=G(2,V_5)$.

For a conic line c we denote by $\pi(c)$ the plane spanned on c. Let c be a conic line on G. If $\pi(c)\not\subset G$ or if $\pi(c)\subset G$ as an α-plane (i.e. as a Schubert cycle $\Omega(1,4)$), then there exists the only $V_4\subset V$ such that $c\subset G(2,V_4)$. If $\pi(c)\subset G$ as a β-plane (i.e. $P(V_3)$ for some $V_3\subset V$), then all spaces V_4 which contain V_3 satisfy the condition $c\subset G(2,V_4)$.

Let H_1, H_2 be hypersurfaces in $\lambda^2(V)$, i.e. elements of $P(\lambda^2(V^*))$, and E_2 a plane that they generate in $\lambda^2(V^*)$. We denote $V_8=H_1\cap H_2\subset\lambda^2(V)$ and $G_4=G(2,V)\cap H_1\cap H_2\subset P(V_8)$. We can associate to an element $H\in\lambda^2(V^*)$ a skew-symmetric map $S_H:V\to V^*$, and we can associate to an inclusion $E_2\hookrightarrow\lambda^2(V^*)$ a map $S:E_2\otimes V\to V^*$ such that for $H\in E_2$, $v\in V$ we have $S(H\otimes v)=S_H(v)$.

For generic H_1, H_2 G_4 is smooth. Condition of smoothness of G_4 is equivalent to a condition that $\forall H\in P(E_2)$ rank $S_H=4$. Further, taking into consideration (3.5), we shall suppose that the pair H_1, H_2 satisfies this condition.

We shall often consider the space $P(V^*)$, and we shall consider an element $V_4\in P(V^*)$ as a hyperplane in V. The tautological exact sequence on $P(V^*)$ is the following:

$$0\to\tau_4\overset{i_V}{\hookrightarrow}V\otimes O\to O(1)\to 0.$$

For $V_4\in P(V^*)$ we denote

$$M(V_4)=\lambda^2(V_4)\cap H_1\cap H_2\subset V_8,$$

$$Q_G(V_4)=G(2,V_4)\cap P(V_8)=G(2,V_4)\cap P(M(V_4)).$$

It is clear that $Q_G(V_4)$ is a quadric hypersurface in $P(M(V_4))$.

Lemma 3.1. $\forall\ V_4\in P(V^*)$ we have: dim $M(V_4)=4$.

Proof. dim $\lambda^2(V_4)=6$. If dim $M(V_4)>4$, then there exists an element $H\in P(E_2)$ such that $H\supset\lambda^2(V_4)$. This means that the composition map

$$V_4\hookrightarrow V\overset{S_H}{\to}V^*\to V_4^*$$

is 0, hence $S_H(V_4)\subset(V/V_4)^*$, i.e. rank $(S_H)=2$ — a contradiction. □

Let us define a sheaf M on $P(V^*)$ as the kernel of the composition

$$\lambda^2(\tau_4)\overset{\lambda^2(i_V)}{\hookrightarrow}\lambda^2(V)\otimes O\to E_2^*\otimes O$$

where an epimorphism $\lambda^2(V)\to E_2^*$ is dual to the inclusion $E_2\to\lambda^2(V^*)$. It is clear that the fibre of M at $V_4\in P(V^*)$ is $M(V_4)$. Lemma 3.1 implies that M is a locally free sheaf of rank 4.

Let a locally free sheaf E on a variety Y be given. A bundle of quadrics Q on (Y,E) is a map that associates to any $y\in Y$ a quadric (a fibre of Q in y) $Q(y)\subset P(E(y))$ such that they form an algebraic family. This means that there exists an invertible sheaf $L(Q)$ on Y and a map of sheaves $i(Q):L(Q)\to S^2(E^*)$ such that

$$\forall y\in Y\quad i(Q)(y)=Q(y)\in P(S^2(E(y))^*).$$

So, there exists a bundle of quadrics Q_G on $(P(V^*), M)$ whose fibre at $V_4 \in P(V^*)$ is $Q_G(V_4)$.

LEMMA 3.2. Let E be a locally free sheaf of rank 4 on Y. Then there exists a bundle of quadrics Pl (E) on $(Y, \lambda^2(E)$ having Pl $(E)(y) =$ Pl $(E(y))$. Then $L(\text{Pl }(E)) = (\det E)^{-1}$.

Proof. Let $U_1, U_2 \subset Y$ be affine open subsets such that $E|_{U_i}$ is a free O_{U_i}-module. Let $e_i = (e_{i1}, \ldots, e_{i4})$ $(i = 1, 2)$ be its basis, and $e^*_{i1}, \ldots, e^*_{i4}$ a basis of $E^*|_{U_i}$ which is dual to (e_i). Further, let $e^*_{ikl} = e^*_{ik} \wedge e^*_{il}$. We have:

$$\text{Pl }(e_i) = e^*_{i12} \circ e^*_{i34} - e^*_{i13} \circ e^*_{i24} + e^*_{i14} \circ e^*_{i23} \in S^2(\lambda^2(E^*))|_{U_i}$$

is an equation of the Plücker quadric. Let us consider a map of sheaves $O \to S^2(\lambda^2(E^*))$ on U_i defined by the element $Pl(e_i)$. Let on $U_1 \cap U_2$ we have $e_1 = A(e_2)$ for an $A \in GL(E|_{U_1 \cap U_2})$. Then it is easy to check that Pl $(e_1) = (\det A)^{-1} Pl(e_2)$ on $U_1 \cap U_2$. This implies the desired. □

Let us apply this lemma to τ_4 on $P(V^*)$. Since det $(\tau_4) = O(-1)$, we have: the bundle Pl (τ_4) can be given by a map

$$i(\text{Pl }(\tau_4)) : O(1) \to S^2(\lambda^2(\tau_4^*)).$$

The restriction of Pl (τ_4) to $M \subset \lambda^2(\tau_4)$ gives us the bundle Q_G. This means that $L(Q_G) = O(1)$.

Let us describe now the set of planes on G_4. We define an inclusion $\psi_1 : P(E_2) \to P(V)$ as follows: $\psi_1(H) = \text{Ker } S_H$. Then im ψ_1 is a conic line, we denote it by c_u and its plane $\pi(c_u)$ by $U = P(U_3)$, where U_3 is a subspace of V. We have: $U^* = P(U_3^*) \subset G_4$ is the only β-plane on G_4. A straight line in $P(V^*)$ which is dual to U will be denoted by l, i.e. $l = P((V/U_3)^*) \subset P(V^*)$, and a point of l is a subspace V_4 such that $U_3 \subset V_4 \subset V$. There exists an isomorphism $\psi : c_u \to l$ such that $\psi(v) = S(E_2 \otimes v) \in P(V^*)$, and we denote by $\alpha(v)$ a Schubert cycle $\Omega(v, \psi(v))$. Then $\forall v \in c_u \quad \alpha(v) \subset G_4$, and all α-planes on G_4 are $\alpha(v)$ for some $v \in c_u$. Further, we have $\alpha(v) \cap U^* \subset G_4$ is a straight line, we denote it by $l_g(v)$. It is dual to the point v with respect to the plane U. All lines $l_g(v)$ are tangent to the conic line $c^*_u \subset U^*$ (the dual to c_u). It is clear that $\forall\ V_4 \in l$ we have $Q_G(V_4) = U^* \cup \alpha(\psi^{-1}(V_4))$, i.e. is a pair of planes.

(3.3). The converse is also true: if for $V_4 \in P(V^*) \quad Q_G(V_4)$ is a pair of planes, then it follows easily from the description of the set of planes on G_4 that $V_4 \in l$.

(3.4). There exists a map $b : G_4 \to P^4$ having the following properties: it is a birational isomorphism; it is a restriction to G_4 of the linear projection p_{85} whose center is U^*, from $P(V_8)$ to $P^4 = P(V_8/U_3^*)$. There exist: (a) the unique normcubic $c_3 \subset P^4$ whose linear envelope is a $P^3 \subset P^4$, (b) an isomorphism $\psi_2 : c_u \to c_3$ and (c) an isomorphism $\tilde{b} : (\tilde{G}_4)_{U^*} \to (\tilde{P}^4)_{c_3}$ - a desingularization of b such that the following diagram is commutative:

$$\begin{array}{ccc} (\tilde{G}_4)_{U^*} & \xrightarrow{\tilde{b}} & (\tilde{P}^4)_{c_3} \\ \downarrow & & \downarrow \\ G_4 & \xrightarrow{b} & P^4 \end{array}$$

where left and right vertical maps (denoted by r_G and r_P respectively) are blowings up of G_4 and P^4 respectively. Further, for $v \in c_u$ we have: $r_G^*(\alpha(v))$ is a plane, and

$$\tilde{b}(r_G^*(\alpha(v))) = r_P^*(\psi_2(v)).$$

For $t \in U^* \subset G$ we denote $\{v_1, v_2\} = t_V \cap c_u$. Then $< \psi_2(v_1), \psi_2(v_2) >$ is a bisecante of c_3 in P^3, and

$$\tilde{b}(r_G^*(t)) = r_P^*(< \psi_2(v_1), \psi_2(v_2) >).$$

For $V_4 \in P(V^*) - l$ the map $b|_{Q_G(V_4)} : Q_G(V_4) \to P^2$ can be described as follows. $Q_G(V_4) \cap U^* = \{t\}$ (one point); it is clear that $t_V = V_4 \cap U_3$. Then $b|_{Q_G(V_4)}$ is the blowing up of t to a bisecante of c_3 and blowing down of t_1, t_2 to intersection points of this bisecante with c_3, where t_1, t_2 are straight lines on $Q_G(V_4)$, which contain t.

Let us describe now conic lines on X. Let $X = G_4 \cap \Omega$, where Ω is a quadric hypersurface on $P(V_8)$.

Lemma 3.5. G_4 is smooth.

Proof. If G_4 is not smooth, then $\exists H \in P(E_2)$ such that rank $S_H = 2$. Then Sing $(G \cap H) = G(2, \text{Ker } S_H) = P^2$, hence Sing $X \supset P^2 \cap H_1 \cap \Omega$. This means that Sing X is not empty - a contradiction. □

We denote by $F_c = F_c(X)$ the set of conics on X; it is an algebraic variety which is called the Fano surface of X (later we shall see that F_c is really a surface). Taking into consideration that for a generic conic $c \subset G \quad \exists! V_4 \in P(V^*)$ such that $c \subset G(2, V_4)$, we define a surface $F = F(X)$ (which we shall call the Fano surface as well) as a set of pairs $(c \in F_c, V_4 \in P(V^*))$ such that $c \subset G(2, V_4)$. There are natural projections

$$r_F : F \to F_c, \quad \tilde{\phi} : F \to P(V^*).$$

There exists a bundle of quadrics Q_Ω on $(P(V^*), M)$ whose fibre at $V_4 \in P(V^*)$ is $Q_\Omega(V_4) = P(M(V_4)) \cap \Omega$. It is clear that

(a) $L(Q_\Omega) = O$;

(b) $X \cap G(2, V_4) = Q_G(V_4) \cap Q_\Omega(V_4)$.

Further, we have $\forall\ V_4 \in P(V^*) \quad \dim Q_G(V_4) \cap Q_\Omega(V_4) = 1$, because smoothness of X implies that Pic (X) is generated by $O_{P(V_8)}(1)|_X$, so the degree of any surface on X is a multiple of 10 (= deg X). But if $Q_G(V_4) \cap Q_\Omega(V_4)$ contains a surface, then its degree is 1 or 2.

It is clear that $(c, V_4) \in F \iff c \subset Q_G(V_4) \cap Q_\Omega(V_4)$. Since $Q_G(V_4) \cap Q_\Omega(V_4)$ is a curve of degree (2,2) on a quadric surface, and c is a curve of degree (1,1), then $Q_G(V_4) \cap Q_\Omega(V_4) - c$ is also a conic line. We denote it by c'. So, there exists an involution i_F on F defined as follows: $i_F(c, V_4) = (c', V_4)$. Let F_0 be the quotient surface F/i_F and $p_F : F \to F_0$ the corresponding two-sheeted covering. A point $f_0 \in F_0$ will be denoted as follows: $f_0 = (c, c', V_4)$, where $f_0 = p_F(c, V_4) = p_F(c', V_4)$. It is clear that c and c' have 2 (possible coinciding) intersection points which we denote by $\gamma_1(f_0)$ and $\gamma_2(f_0)$ (or $\gamma_1(c)$ and $\gamma_2(c)$). Their linear envelope (i.e. a line $< \gamma_1(c), \gamma_2(c) >$) will be denoted by $l_8(c)$ (or $l_8(f_0)$).

(3.6). There exist smooth threefolds X which contain a pair of involutory conics c and c' such that both of them is a pair of straight lines such that $l_8(c)$ is their common component. Considerations analogous to ones used in the proof of Lemma

3.7 show that the set of X having this property has codimension 1 in the set of all possible X. These X will not be considered in the present paper, although most likely main results which are obtained in the present paper for a "generic" X, are true for them as well (for example, smoothness of F).

It is clear that there exists a map $\phi : F_0 \to P(V^*)$ satisfying a condition $\phi \circ p_F = \tilde{\phi}$. Further, it is clear that

$$(c, V_4) \in F \iff \pi(c) \cup \pi(c') \in < Q_G(V_4), Q_\Omega(V_4) >_{P(S^2(M(V_4))^*)},$$

i.e. some linear combination of $Q_G(V_4)$, $Q_\Omega(V_4)$ is a pair of planes. So, let us define Z as a variety of pairs (V_4, Q) where $V_4 \in P(V^*)$, $Q \in < Q_G(V_4), Q_\Omega(V_4) >$ together with a natural projection $\eta : Z \to P(V^*)$. Since $L(Q_G) = O(1)$, $L(Q_\Omega) = O$, we have

$$Z = P_{P(V^*}(O + O(1)).$$

There is a tautological exact sequence on Z:

$$0 \to O_\eta(-1) \xrightarrow{i_\eta} O + O(1) \to O(1,_\eta 1) \to 0$$

as well as the dual sequence. Further, Z is a blowing up of $P^5 = P(V_6)$ at a point $t \in P^5$; we denote the corresponding map by $r_Z : Z \to P^5$. It is clear that $V_6/t = V^*$ and $r_Z^*(O_{P^5}(1)) = O(1,_\eta 1)$. There are divisors D_G, D_Ω on Z which are defined as follows: a point of Z can be considered as a pair (V_4, Q); we have:

$$(V_4, Q) \in D_G \iff Q = Q_G(V_4);$$

$$(V_4, Q) \in D_\Omega \iff Q = Q_\Omega(V_4).$$

It is clear that $D_G = r_Z^*(t)$ is the exceptional divisor on Z, and $r_Z(D_\Omega)$ is a hypersurface in P^5.

Let $(V_4, Q) \in Z$; we associate it the quadric $Q \subset P(M(V_4))$. This gives us a bundle of quadrics (denoted by T) on (Z, M). It is clear that $L(T) = O_\eta(-1)$ and

$$i(T) = (i(Q_G) \oplus i(Q_\Omega)) \circ i_\eta : O_\eta(-1) \to S^2(M^*).$$

There exists an inclusion $\bar{\phi} : F_0 \to Z$ defined as follows: $\bar{\phi}(c, c', V_4) = (V_4, \pi(c) \cup \pi(c'))$. It is clear that $z \in \operatorname{im} \bar{\phi} \iff T(z)$ is a pair of planes, i.e. a rank 2 quadric surface. This means that $F_0 \hookrightarrow Z$ is the second determinantal of T.

Let us remember the definition and main properties of determinantal varieties. Let E be a locally free sheaf of rank n on an algebraic variety Y, Q a bundle of quadrics on (Y, E) and $i(Q) : L^* \to S^2(E^*)$ the corresponding map. For any $y \in Y$ we can consider the fibre $i(Q)(y)$ as a symmetric map from $E(y)$ to $E(y)^*$. The determinantal $D_k = D_k(Q)$ of Q is defined as a set of points $y \in Y$ such that $\dim \operatorname{im} i(Q)(y) \le k$. We define also a variety $\tilde{D}_k = \tilde{D}_k(Q)$ as a set of pairs (y, V_k) such that $y \in D_k$ and $\operatorname{im} i(Q)(y) \subset V_k \subset E(y)^*$. The natural projection $\pi_k : \tilde{D}_k \to D_k$ is an isomorphism outside of $\pi_k^{-1}(D_{k-1})$. There is also a tautological inclusion $\tilde{D}_k \to G_y(k, E^*)$, and we have a commutative diagram:

$$\begin{array}{ccc} \tilde{D}_k & \hookrightarrow & G_Y(k, E^*) \\ \downarrow & & \downarrow \\ D_k & \hookrightarrow & Y \end{array} .$$

There is a sheaf $C = C(k) = \text{Coker } (S^2(\tau_{k,E^*}) \to S^2(E^*))$ on $G_Y(k, E^*)$. A sheaf $L \otimes C$ has a distinguished section coming from a map

$$O \overset{i(Q)\otimes L}{\to} L \otimes S^2(E^*) \to L \otimes C.$$

It is clear that $\tilde{D}_k$ is the set of zeros of this section. Q is called k-regular, if for $i = k$ and $i = k - 1$

$$\text{Codim }_{G_Y(i,E^*)} \tilde{D}_i = \dim L \otimes C(i)$$

(if $D_i \neq \emptyset$). If so, we have Codim $_Y D_k = \frac{(n-k)(n-k+1)}{2}$. Further, for a k-regular bundle Q we have $\bar{D}_{k-1} = \pi_k^{-1}(D_{k-1})$ is a divisor in $\tilde{D}_k$, which is the determinantal of rank $k-1$ of a bundle of quadrics $L^* \to S^2(\tau_k)$, which is defined naturally on $\tilde{D}_k$. Porteus formula ([7]) implies that

$$O_{\tilde{D}_k}(\bar{D}_{k-1}) = L^k \otimes O_G(-2).$$

Particularly, if $D_{k-1} = \emptyset$, then $\bar{D}_{k-1} = \emptyset$ as well, and $L^k = O_G(2)$ in Pic (D_k).

Let us consider the case of even k. For a quadric Q of rank k the set of projective spaces of the maximal possible dimension which are contained in Q, has 2 connected components. We denote this set of connected components by $\Gamma(Q)$, and we define a two-sheeted covering $p_{D_k} : \hat{\tilde{D}}_k \to \tilde{D}_k$ as follows: $p_{D_k}^{-1}(y) = \Gamma(Q(y))$ for any $y \in \tilde{D}_k - \bar{D}_{k-1}$. We have: $\bar{D}_{k-1}$ is the ramification divisor of p_{D_k}, and a sheaf that corresponds to p_{D_k} is $L^{\frac{k}{2}} \otimes O_G(-1)$.

Let us consider the case $k = 2$. For $y \in D_2$ $\quad Q(y)$ is a pair of hyperplanes in $P(E(y))$, and a choice of a point in $p_{D_2}^{-1}(y)$ is a choice of one of these hyperplanes. So, there exists an inclusion $\hat{\tilde{D}}_2 \to P_Y(E)$, and there exist sheaves $\tau_{n-1,E}$ and $O_\pi(1)$ on $\hat{\tilde{D}}_2$ (here $\pi : P_Y(E) \to Y$ is a projection). It is clear that $\hat{\tilde{D}}_2$ is the set of zeros of a section of $L \otimes S^2(\tau^*_{n-1,E})$ (here $L \otimes S^2(\tau^*_{n-1,E})$ is a sheaf on $P_Y(E^*)$).

Let us return to study of the Fano surface. As we have seen, $F_0 = D_2(T)$.

Lemma 3.7. For a generic X we have $D_1(T) = \emptyset$.

Proof. It is clear that

$$D_1(T) \neq \emptyset \iff \exists V_4 \in P(V^*),\ \exists V_3 \subset M(V_4)$$

such that the double plane $2P(V_3) \in < Q_G(V_4), Q_\Omega(V_4) >$. Let us fix G_4. For given $V_4 \in P(V^*)$, $V_3 \subset M(V_4)$ $\quad Q_G(V_4)$ is defined uniquely, and the set of quadrics $Q_\Omega(V_4)$ such that $2P(V_3) \in < Q_G(V_4), Q_\Omega(V_4) >$ has codimension 8 in the space of all quadrics in $P(V_8)$. This means that the set of quadrics Ω such that $2P(V_3) \in < Q_G(V_4), Q_\Omega(V_4) >$ has codimension 8 in the space of all quadrics in $P(V_8)$. But the set of pairs $\{V_4 \in P(V^*),\ V_3 \subset M(V_4)\}$ is 7-dimensional, hence the set of quadrics Ω such that for $X = G_4 \cap \Omega$ $\quad D_1(T) \neq \emptyset$ is a union of a 7-dimensional set of subvarieties of codimension 8, hence has codimension ≥ 1. $\square$

(4.2) (see below) implies that dim $F = 2$. This means that a condition $D_1(T) = \emptyset$ for X is equivalent to a condition of 2-regularity of the bundle T. Further we shall consider only X that satisfy this condition.

Properties of determinantal varieties given above imply the following

PROPOSITION 3.8. There exist inclusions

$$F_0 \to G_Z(2, M^*), \quad F \to P_Z(M^*)$$

hence tautological sheaves on $G_Z(2, M^*)$ and $P_Z(M^*)$ can be restricted to F_0 and F respectively. Further, F_0 is the set of zeros of a section of $O_\eta(1) \otimes C(2)$ on $G_Z(2, M^*)$, and F is the set of zeros of a section of $O_\eta(1) \otimes S^2(\tau^*_{3,M})$ on $P_Z(M^*)$. We have equalities of sheaves $O_G(2) = O_\eta(2)$ (respectively $O_G(1) = O_\eta(1)$) on F_0 (respectively on F). The covering $p_F : F \to F_0$ is non-ramified, and the corresponding sheaf is $\sigma = O({}_\eta 1,{}_G -1)$. For $f_0 \in F_0 \quad l_8(f_0) = P(\tau_{2,M}(f_0))$. ∎

If for $V_4 \in P(V^*) \quad Q_G(V_4)$ or $Q_\Omega(V_4)$ is already a pair of planes then it is clear that $V_4 \in \phi(F_0)$. We denote the set of these V_4 (and the set of their ϕ-inverse images on F_0) by l_G and l_Ω respectively. They are curves on F_0, and it is clear that $l_G = \bar{\phi}^*(D_G)$, $l_\Omega = \bar{\phi}^*(D_\Omega)$. l_Ω is not defined by X uniquely, it depends on a choice of Ω such that $X = G_4 \cap \Omega$. (3.3) implies that $l_G = l$. It is clear that $p_F^{-1}(l)$ is a pair of straight lines, we denote them by l_1 and l_2 by such a way that

$$(c, V_4) \in l_1 \iff \{c = U^* \cap \Omega,\ V_4 \in l\}$$

$$(c, V_4) \in l_2 \iff \{c = \alpha(\psi^{-1}(V_4)) \cap \Omega,\ V_4 \in l\}.$$

It is clear that $p_F|_{l_i} : l_i \to l$ are isomorphisms $(i = 1, 2)$ and that $r_F(l_1)$ is a point $c_\Omega \in F_c$; the corresponding conic line $U^* \cap \Omega$ on X is also c_Ω. Using properties of conic lines on G we see that for $c \in F_c \quad r_F^{-1}(c)$ is either a point or a straight line on F. If $r_F^{-1}(c)$ is a straight line, then $\pi(c)$ is a β-plane on G. In this case $\pi(c) \cap H_1 \cap H_2 \supset c$, hence $\pi(c) \subset G_4$ and $\pi(c) = U^*$, $c = c_\Omega$. So, c_Ω is the only point of F_c such that $r_F^{-1}(c_\Omega)$ is not a point. This means that $r_F : F \to F_c$ is the blowing up of c_Ω.

PROPOSITION 3.9. $O_{F_0}(l) = O_\eta(1)$, $O_{F_0}(l_\Omega) = O(1,{}_\eta 1)$.

Proof. This important proposition (it implies smoothness of F_c at c_Ω, see below) can be proved by different ways. Here is one of them. Let us consider maps of sheaves on $G_Z(2, M^*)$:

$$0 \to O_\eta(-1) \overset{i_\eta}{\hookrightarrow} O \oplus O(1) \overset{i_1}{\twoheadrightarrow} O(1,{}_\eta 1) \to 0 \tag{3.10}$$

$$O(1) \overset{i_2}{\hookrightarrow} O \oplus O(1) \overset{i_3}{\to} S^2(M^*) \overset{i_4}{\twoheadrightarrow} C(2) \tag{3.11}$$

where $i_3 = i(Q_\Omega) \oplus i(Q_G)$. By definition, for $f_0 \in F_0 \quad i_4 \circ i_3 \circ i_\eta(f_0) = 0$, and exactness of (3.10) implies that there exists an inclusion $i_5 : O(1,{}_\eta 1) \to C(2)$ on F_0 satisfying a condition $i_5 \circ i_1 = i_4 \circ i_3$. It is clear that

$$f_0 \in l \iff \text{im } (i_3 \circ i_2(f_0)) = \text{im } (i_3 \circ i_\eta(f_0)).$$

This implies that for $f_0 \in F_0 \quad f_0 \in l \iff i_1 \circ i_2(f_0) = 0$, i.e. l is the set of zeros of $i_1 \circ i_2$ on F_0. This means that $O_{F_0}(l) = O_\eta(1)$. Replacing in (3.11) $O(1)$ by O, we get analogously that $O_{F_0}(l_\Omega) = O(1,{}_\eta 1)$. ∎

We denote the projection $P_Z(M^*) \to Z$ by π, and its inversible sheaf by $O_\pi(1)$. Further, we denote $c_1(O(1)) = H$, $c_1(O_\eta(1)) = E$, $c_1(O_\pi(1)) = P$. Chow ring

$A(P_Z(M^*))$ is generated by H, E, P satisfying relations

$$H^5 = 0,\ E^2 = -EH,\ P^4 = -3P^3H - 5P^2H^2 - 5PH^3.$$

Proposition 3.13. Intersection indices on F_0 are: $EH = 1$, $E^2 = -1$.

Proof. Equality $EH = 1$ is obvious, because cl $(l) = E$, l is a straight line in $P(V^*)$, and H is a class of a hyperplane section in $P(V^*)$. Equality $E^2 = -EH$ is true even on Z and moreover on F_0. □

Corollary 3.14. c_Ω is a non-singular point on F_c.

Proof. $< E^2 >_F = -2$. i_F interchanges l_1 and l_2 and hence $< l_1^2 >_F = < l_2^2 >_F$. Further, $l_1 \cap l_2 = \emptyset$, hence $< l_1^2 >_F = < l_2^2 >_F = -1$. This implies that c_Ω is a non-singular point on F_c. □

Maps of blowing down of $l_1 \cup l_2$ on F and of l on F_0 are denoted by $r : F \to F_m$ and $r_0 : F_0 \to F_{0m}$ respectively. There exists an involution $i_{F_m} : F_m \to F_m$ and a projection $p_{F_m} : F_m \to F_{0m}$ making the following diagram commutative:

$$\begin{tikzcd}
 & F \arrow[rr, "i_F"] \arrow[dl, "r_F"'] \arrow[dd, "r"] \arrow[dr, "p_F"] & & F \arrow[dl, "p_F"'] \arrow[dd, "r"] \\
F_c \arrow[dr] & & F_0 \arrow[dd, "r_0"] & \\
 & F_m \arrow[rr, "i_{F_m}"] \arrow[dr, "p_{F_m}"'] & & F_m \arrow[dl, "p_{F_m}"] \\
 & & F_{0m} &
\end{tikzcd}$$

Let us consider a map $r_Z \circ \bar{\phi} : F_0 \to P^5$. Since $l \subset D_\Omega = r_Z^*(t)$, we have: $r_Z \circ \bar{\phi}$ factors through $\bar{\phi}_m : F_{0m} \to P^5$, hence we have a commutative diagram

$$\begin{array}{ccc}
F_0 & \overset{\bar{\phi}}{\to} & Z \\
\downarrow & & \downarrow \\
F_{0m} & \overset{\bar{\phi}_m}{\to} & P^5
\end{array}$$

where vertical maps are r_0 and r_Z respectively. Further, we have $l = F_0 \underset{Z}{\times} D_G$. This implies

Proposition 3.15. $\bar{\phi}_m$ is regular at $c_{0\Omega}$, and $\bar{\phi}_m(c_{0\Omega}) = t$. □

Since $l_\Omega = \bar{\phi}^*(D_\Omega)$ and $r_Z(D_\Omega)$ is a hyperplane in P^5, we have that $\bar{\phi}_m(l_\Omega)$ is a hyperplane section of $\bar{\phi}_m(F_{0m})$. We denote by $P^{5'}$ the set of quadrics in $P(V_8)$ that contain X; it is the linear envelope of Ω and $q(P(V))$ (the definition of $q(P(V))$ is given in the beginning of Section 5). To each $\Omega' \in P^{5'} - q(P(V))$ we can associate its own straight line $l_{\Omega'}$ and hyperplane $r_Z(D_{\Omega'})$ in P^5. It is easy to see that the map $\Omega' \to r_Z(D_{\Omega'})$ gives us a natural isomorphism $P^{5'} \to (P^5)^*$ such that quadrics in $q(P(V))$ correspond to hyperplanes that contain t.

LEMMA 3.16. deg $\phi(l_\Omega) = 40$.

Proof. We have $l_\Omega = D_Z(Q_\Omega)$ on $P(V^*)$. According the Porteus formula ([7]), we have: deg $\phi(l_\Omega) = 4(c_1(M^*)c_2(M^*) - c_3(M^*))$. Further, $M = 8O - 5O(1) + O(2)$ in $K_0(P(V^*))$, hence $c_t(M^*) = 1 + 3H + 5H^2 + 5H^3$, and deg $\phi(l_\Omega) = 40$. □

Since cl $_{F_0}(l_\Omega) = H + E$, we have deg $l_\Omega =< H + E, H >_{F_0}$, and $< H^2 >_{F_0}=$ deg $\phi(F_0) = 39$. It is easy to see that deg $_{P^5}(\bar{\phi}_m(F_{0m})) = 40$ and cl $_Z(\bar{\phi}(F_0)) = 40H^3 + 39H^2E$ (this formula can be got by application of the Porteus formula for T, but calculations will be longer than in Lemma 3.16).

REMARK. $< H^2 >_{F_0}$ is equal to the number of conics passing through a fixed generic point on X. Really, let $x \in X$, x_V the corresponding straight line in $P(V)$ and $\tilde{x}_V = P(V/x_V)^* \subset P(V^*)$ the dual plane in $P(V^*)$, i.e. the set of P^3 that contain x_V. Let $f = (c, V_4) \in F$ and $x \in c$. Then $x_V \subset P(V_4)$, i.e. $V_4 \in \tilde{x}_V$. But $V_4 \in \phi(F_0)$, i.e. if $x \in c$, then $\tilde{\phi}(f) \in \tilde{x}_V \cap \phi(F_0)$. Clearly the converse is also true, so the number of conics passing through x is equal to the number of intersection points of $\phi(V_0)$ and $\tilde{x}_V$. So, there are 39 conics passing through a generic point of X.

We have an equality in $K_0(P_Z(M^*))$:

$$\Omega_F = \Omega_{P_Z(M^*)} - O_\eta(-1) \otimes S^2(\tau_{3,M}). \tag{3.17}$$

We deduce easily from this formula that $\omega_F = O(3,_\eta 4)$. Further, we have an exact sequence (where i is an inclusion $l_1 \cup l_2 \to F$)

$$0 \to r^*(\Omega_{F_m}) \to \Omega_F \to i_*(\Omega_{l_1 \cup l_2}) \to 0. \tag{3.18}$$

It implies that $r^*(\omega_{F_m}) = O(3,_\eta 3)$, i.e. $\omega_{F_m} = p^*_{F_m}(\bar{\phi}^*_m(O_{P^5}(3)))$. Let σ_m be a sheaf of order 2 that corresponds to a two-sheeted covering p_{F_m}; it is clear that $r_0^*(\sigma_m) = \sigma = O(_G{-1},_\eta 1)$. This means that $\omega_{F_0} = O(3,_G 1,_\eta 3)$ and $\omega_{F_{0m}} = \bar{\phi}^*_m(O_{P^5}(3)) \otimes \sigma_m$. Since deg $\bar{\phi}_m(F_{0m}) = 40$, we get that $c_1(\Omega_{F_{0m}})^2 = 360$ and $c_1(\Omega_{F_m})^2 = 720$.

To calculate $c_2(\Omega_F)$, it is necessary to find cl (F) in $A(P_Z(M^*))$, which is equal $c_6(O_\eta(1) \otimes S^2(\tau^*_{3,M}))$ (see (3.8)). A calculation gives the desired result:

$$\text{cl } (F) = 80P^3H^3 + 240P^2H^4 + 78P^3H^2E + 235P^2H^3E.$$

Further, we get (taking into consideration (3.12)) intersection indices of divisors on F:

$$H^2 = 78;\ P^2 = -13;\ PH = 1;\ EH = 2;\ EP = -1.$$

We deduce from (3.17) after some calculations:

$$c_2(\Omega_F) = 4H^2 - 4P^2 + 13EH + 4EP = 386$$

and using (3.18) we get

$$c_2(\Omega_{F_m}) = 384.$$

We define a surface $W \subset X$ as a set of intersection points of involutory conics, i.e.

$$W = \bigcup_{f_0 \in F_0} (\gamma_1(f_0) \cup \gamma_2(f_0)).$$

Further, we define a surface $\bar{W}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \bar{W} & \hookrightarrow & P_{F_0}(\tau_{2,M}) \\ & \underset{p_W}{\searrow} & \downarrow \\ & & F_0 \end{array}$$

where the vertical map is p, and p_W is a two-sheeted covering of F_0 such that for $f_0 \in F_0$ we have

$$p_W^{-1}(f_0) = \{\gamma_1(f_0), \gamma_2(f_0)\} \subset p^{-1}(f_0) = l_8(f_0).$$

It is clear that there exists a map $\xi : P_{F_0}(\tau_{2,M}) \to P(V_8)$, that corresponds to an inclusion of sheaves $\tau_{2,M} \hookrightarrow M \hookrightarrow V_8 \otimes O$ on F_0. We have: $\xi|_W : \bar{W} \to W$ is an epimorphism and $\xi^*(O_{P(V_8)}(1)) = O_P(1)$.

Lemma 3.19. Let $f_a = (c_a, c'_a, V_{4a}) \in F_0 - l$, $f_b = (c_b, c'_b, V_{4b}) \in F_0$ and $\gamma_1(f_a) = \gamma_1(f_b)$. Then $j = j(f_a, f_b) = P(M(V_{4a}) \cap M(V_{4b}))$ is a straight line in G_4 which is tangent to Ω.

Proof. Since $P(V_{4a} \cap V_{4b})$ is a plane, we have $P(\lambda^2(V_{4a} \cap V_{4b})) \subset G$ and $j = P(\lambda^2(V_{4a} \cap V_{4b})) \cap H_1 \cap H_2$ is a linear subspace in G_4. Since $f_a \notin l$, j is not a plane. If j is a point, then $j = \gamma_1(f_a) = \gamma_1(f_b)$ and $T_{P(M(V_{4a}))}(j)$, $T_{P(M(V_{4b}))}(j)$ are linearly independent in $T_{P(V_8)}(j)$. But dim $T_X(j) \cap T_{P(M(V_{4a}))}(j) = 2$, because this space coincide with $T_{Q_G(V_{4a})}(j)$ or $T_{Q_\Omega(V_{4a})}(j)$. Analogously, dim $T_X(j) \cap T_{P(M(V_{4b}))}(j) = 2$, and linear independence of these spaces implies that dim $T_X(j) \geq 4$ that contradicts to non-singularity. This means that j is a straight line. It is easy to see that it is tangential to Ω at $\gamma_1(f_a)$. □

Lemma 3.20. Let γ be a straight line on G_4 which is tangential to Ω at a point t_0. Let $\pi(j) = \cup_{t \in j} t_V$ be a plane in $P(V)$ and $\widetilde{\pi(j)}$ the dual straight line in $P(V^*)$, i.e. the set of subspaces $P(V_4) \in P(V^*)$ that contain $\pi(j)$. If $(c, c', V_4) \in F_0$ and $V_4 \in \widetilde{\pi(j)}$, then $t_0 \in \{\gamma_1(c), \gamma_2(c)\}$ or $j \subset X$.

Proof. The above conditions imply that $j \subset Q_G(V_4)$. It is clear that $j \cap \Omega = \{j \cap c, j \cap c'\}$. Since j is tangential to Ω at a point t_0, then:

If $j \not\subset X$, then $t_0 = j \cap c = j \cap c'$,

and hence $t_0 \in c \cap c'$. □

Proposition 3.21. For a generic point $t \in W$ $\quad (\xi|_{\bar{W}})^{-1}(t)$ is one point.

Proof. Let for a point $t \in W$ $\quad (\xi|_{\bar{W}})^{-1}(t)$ is more than one point. f_a, $f_b \in p_W(\xi|_{\bar{W}})^{-1}(t)$ and $j = j(f_a, f_b)$. Then a straight line $\widetilde{\pi(j)}$ meets $\phi(F_0)$ at points $\phi(f_a), \phi(f_b)$. But a variety $B \subset G(2, V^*)$ of bisecants of $\phi(F_0)$ is 4-dimensional, while a variety B' of straight lines on G_4 which are tangential to Ω is 3-dimensional. Let j be such a line. We can associate it a straight line $\widetilde{\pi(j)}$ in $P(V^*)$. These lines form a threefold $B' \subset G(2, V^*)$. Intersection of B and B' in $G(2, V^*)$ is a curve, hence the set of points $t \in W$ such that $(\xi|_{\bar{W}})^{-1}(t)$ is more than one point is 1-dimensional. □

Proposition 3.22. $O_X(W) = O_X(21)$.

Proof. $\forall f_0 \in F_0 \quad \{\gamma_1(f_0), \gamma_2(f_0)\}$ is a quadric in $P(\tau_{2,M}(f_0))$, and these quadrics form a bundle Γ on $(F_0, \tau_{2,M})$. Let us find $L(\Gamma)$. There are maps of sheaves on F_0:

$$O_\eta(-1) \xrightarrow{i_\eta} O \oplus O(1) \to S^2(M^*) \to S^2(\tau^*_{2,M}).$$

Their composition is 0, hence there exists a map

$$O(1,_\eta 1) = \text{Coker } i_\eta \to S^2(M^*) \to S^2(\tau^*_{2,M}).$$

It is easy to see that this map is $i(\Gamma)$ and $L(\Gamma) = O(1,_\eta 1)$. This implies that

$$O_{P(\tau_{2,M})}(\bar{W}) = O(_P 2, -1,_\eta -1). \tag{3.23}$$

We denote $c_1(O_{P(V_8)}(1))$ by R. We have

$$\deg W =< W, R, R >_{P(V_8)} =< \bar{W}, R, R >_{P(\tau_{2,M})} .$$

In $A(P_{F_0}(\tau_{2,M}))$ we have an equality $R^2 = -c_1 R - c_2$, where $c_1 = c_1(\tau_{2,M}) = -3H - E$, $c_2 = c_2(\tau_{2,M})$. (3.23) implies that cl $\bar{W} = 2R - H - E$. This means that

$$< \bar{W}, R, R >_{P(\tau_{2,M})} = (2R^3 - HR^2 - ER^2)_{P(\tau_{2,M})}$$

$$= ((2c_1^2 - 2c_2 + Hc_1 + Ec_1)R + 2c_1 c_2 + Hc_2 + Ec_2)_{P(\tau_{2,M})}$$

$$= (2c_1^2 - 2c_2 + Hc_1 + Ec_1)_{F_0} = 210.$$

Since $\deg X = 10$, we have $O_X(W) = O_X(21)$. □

Let us compare properties of a threefold X under consideration and another threefold X' which is a two-sheeted covering of P^3 with ramification at a quartic surface W (see [11], [12], [49]). $c_1(\Omega)^2$ and $c_2(\Omega)$ of their Fano surfaces $F(X')$ and $F_m(X)$ coincide. Since irregularities coincide as well:

$$h^{1,0}(F_m(X)) = h^{1,0}(F(X')) = 10, \; h^{1,0}(F_{0m}(X)) = h^{1,0}(F_0(X')) = 0$$

we have that their Hodge numbers as well as their decomposition on symmetric and antisymmetric part with respect to $i_{F_m(X)}$, $i_{F(X')}$ also coincide. Moreover, we have the following properties of inclusions $F_m(X) \hookrightarrow P^5$, $F(X') \hookrightarrow P^5$: $\deg F_m(X) = \deg F(X') = 40$, $\omega_{F_{0m}(X)} = \omega_{F_0(X')} = O_{p^5}(3) \otimes \sigma$. Nevertheless, some properties of $F_m(X)$ and $F(X')$ are different. For example, we have for X':

(a) There are 12 conics passing through a generic point $x \in X'$;

(b) $\xi|_{\bar{W}} : \bar{W} \to W$ is a six-sheeted covering;

(c) image of $F(X')$ in P^5 is contained in the Plücker quadric (apparently this is not true for the image of $F_m(X)$ in P^5);

(d) There are no exceptional points and lines on $F(X')$.

[16] contains a conjecture that X and X' can be birationally isomorphic. Differences of properties of their Fano surfaces given above show that this is few likely. Nevertheless, there exists a Fano threefold X'' ([1]) whose properties are in some sense "intermediate" between properties of X and X'.

Namely, let G_3 be a $G(2,5) \cap P^3$. X'' is a two-sheeted covering of G_3 with ramification in $W = G_3 \cap \Omega$, where Ω is a quartic surface in $P(\lambda^2(V))$. There exists a variation in a smooth family of X in X''. Let K be a cone over G_4 with vertex t and $X_4 = K \cap \Omega$. Then a section of X_4 by a hyperplane which does not contain t is isomorphic to a X and a section of X_4 by a hyperplane which contains t is isomorphic to a X''. From another side, some properties of X'' are analogous to the ones of X'. For example, $F_0(X'')$ (an analog of $F_0(X')$) is isomorphic to the set of conics on G_3 which are bitangent to W.

4. Tangent bundle theorem. Firstly we shall prove this theorem "locally". Remember that there exist maps $F \to F_0 \to G_Z(2, M^*)$ and $F \to P_Z(M^*)$ so the corresponding tautological sheaves $\tau_{2,m}$ and $\tau_{3,m}$ are defined on F. We denote $l = l_1 \cup l_2 \hookrightarrow F$.

PROPOSITION 4.1. Let $c \in F - l$. Then there exists a natural isomorphism $\Omega_F(c) \to \tau^*_{2,m}(c)$.

Proof of this proposition is a logically necessary step of the proof of the Tangent Bundle Theorem for F, because it implies the following

COROLLARY 4.2. F is smooth, dim $F = 2$, T is a regular bundle. □

Steps of the proof of Proposition 4.1 correspond to ones of the proof of Theorem 4.14 below. The proof of (4.14) is more complicated only because of necessity to consider exceptional lines l_1, l_2.

Proof of 4.1. We introduce the following notation. Let $\alpha : P_{P(V^*)}(M) \to P(V^*)$ is the projectivization of M and $O_\alpha(1)$ is the tautological sheaf on $P_{P(V^*)}(M)$. There exists an inclusion of sheaves $M \to V_8 \otimes O$ on $P(V^*)$ and hence maps

$$\begin{array}{ccccccc} P(M(V_4)) & & & X & \hookrightarrow & G_4 & \hookrightarrow & G \\ \downarrow & & & & & \downarrow & & \downarrow \\ P_{P(V^*)}(M) & \hookrightarrow & P_{P(V^*)}(V_8 \otimes O) = P(V^*) \cdot P(V_8) & & \overset{\mathrm{pr}}{\to} & P(V_8) & \hookrightarrow & P(\lambda^2(V)) \end{array}$$

where vertical maps are inclusions and $P_{P(V^*)}(V_8 \otimes O) = P(V^*) \times P(V_8)$.

We denote by $O_\alpha(1)$ the sheaf $O_{P(\lambda^2(V))}(1)$ and its inverse images in all obgects of this diagram, as well as in subvarieties of $P(M(V_4))$ (for a generic V_4).

Let $c = (c, V_4) \in F - l$. We consider diagrams

$$\begin{array}{ccc} c & \to & \pi(c) \\ \downarrow & & \downarrow \\ Q_G(V_4) & \to & P(M(V_4)) \end{array} \tag{4.3}$$

$$\begin{array}{ccc} c & \to & X \\ \downarrow & & \downarrow \\ Q_G(V_4) & \to & G_4 \end{array}. \tag{4.4}$$

According deformation theory, $T_F(c) = H^0(c, N(c, X))$, if

$$H^1(c, N(c, X)) = 0. \tag{4.5}$$

Later it will be shown that condition (4.5) is satisfied, hence we denote $H^0(c, N(c, X))$ by $T_F(c)$. Analogous notations will be used in other cases where we shall use deformation theory. According Serre duality for c we have:

$$H^0(c, N(c, X)) = H^1(c, N(c, X)^* \otimes \omega_c)^* = H^1(c, N(c, X)^* \otimes \det N(c, X)^* \otimes \omega_c)^*$$

$$= H^1(c, N(c, X) \otimes \omega_X)^*$$

and analogously $H^1(c, N(c, X)) = H^0(c, N(c, X) \otimes \omega_X)^*$, i.e.

$$\Omega_F(c) = H^1(c, N(c, X) \otimes \omega_X)^* \tag{4.6}$$

if $H^0(c, N(c, X) \otimes \omega_X) = 0$.

Since $c \notin l$ we have that (4.3) is a fibred product, and hence

$$N(c, Q_G(V_4)) = N(\pi(c), P(M(V_4))) = O_\alpha(1).$$

Exact sequences of normal sheaves for both pairs of inclusions in (4.4) are the following:

$$0 \to N(c, X) \to N(c, G_4) \to N(X, G_4)|_c \to 0 \tag{4.7}$$

$$0 \to N(c, Q_G(V_4)) \to N(c, G_4) \to N(Q_G(V_4), G_4)|_c \to 0. \tag{4.8}$$

Let us consider the composite map

$$O_\alpha(1) = N(c, Q_G(V_4)) \to N(c, G_4) \to N(X, G_4)|_c = O_\alpha(2). \tag{4.9}$$

Further, let us consider a diagram

$$\begin{array}{ccccc} c & \hookrightarrow & c \cup c' & \to & X \\ & & \downarrow & & \downarrow \\ & & Q_G(V_4) & \to & G_4 \end{array}$$

where $c' = i_f(c)$. We have $c \cup c' = X \underset{G_4}{\times} Q_G(V_4)$, so $\tilde{N}(c \cup c', Q_G(V_4))|_c \to N(X, G_4)|_c$ is an isomorphism (here and below $\tilde{N}(Y_1, Y_2)$ means — for an inclusion of varieties $Y_1 \subset Y_2$ — a locally free sheaf on Y_2 such that Y_1 is the set of zeros of a section of it).

(4.10). There is a map $N(c, Q_\Omega(V_4)) \to \tilde{N}(c \cup c', Q_G(V_4))|_c$ which is a multiplication by a section of a sheaf $O_c(c \cap c') = O_\alpha(1)$ which corresponds to the divisor $c \cap c'$ on c.

It is clear that $N(Q_G(V_4), G_4) = \tau^*_{2,V}|_{Q_G(V_4)}$. Multiplying (4.7) and (4.8) by $\omega_X = O_\alpha(-1)$ we get

$$0 \to N(c, X) \otimes \omega_X \to N(c, G_4) \otimes O_\alpha(-1) \to O_\alpha(1) \to 0 \tag{4.11}$$

$$0 \to O \to N(c, G_4) \otimes O_\alpha(-1) \to \tau_{2,V}|_c \to 0. \tag{4.12}$$

According (4.10), the composite map

$$\phi_c : O \to N(c, G_4) \otimes O_\alpha(-1) \to O_\alpha(1)$$

corresponds to the divisor $c \cap c'$ on c. Let us consider the long exact cohomology sequences for (4.11) and (4.12). Since $c \notin l$, we have $\tau_{2,V}|_c = 2O_c(-1)$ and $H^i(c, \tau_{2,V}|_c) = 0$. This means that the long exact cohomology sequence for (4.11) is the following:

$$0 \to H^0(N(c, X) \otimes \omega_X) \to H^0(O_c) \overset{H^0(\phi_c)}{\to} H^0(O_\alpha(1)) \to \Omega_F(c) \to 0. \tag{4.13}$$

We have $H^0(c, O_\alpha(1)) = \tau^*_{3,M}(c)$ (see (4.43) below) and $H^0(\phi_c)$ is dual to an epimorhpism

$$\tau_{3,M}(c) \to \tau_{3,M}(c)/\tau_{2,M}(c)$$

This means that $H^0(\phi_c)$ is an inclusion, $H^0(N(c, X) \otimes \omega_X) = 0$ and $\Omega_F(c) = \tau^*_{2,M}(c)$. □

Theorem 4.14. (The Tangent Bundle Theorem). There exists an exact sequence of sheaves on F:

$$0 \to \tau^*_{2,M} \to r^*(\Omega_{F_m}) \to i_*(O_l \oplus O_l) \to 0$$

where i is an inclusion $l \to F$.

Proof. Let us consider a Fano family $D_F(c) \hookrightarrow F \times X$ defined as follows: $(c,x) \in D_F(c) \iff x \in c$, and let

$$\begin{array}{ccc} D_F(c) & \hookrightarrow & f \times X \\ & \overset{\beta}{\searrow} & \downarrow \\ & & F \end{array}$$

be a diagram of natural projections on F, the vertical map is α. According the deformation theory, $r^*_F(T_{F_c}) = \beta_*(N(D_F(c), F \times X))$, if $\beta_{*1}(N(D_F(c), F \times X)) = 0$. Like in (4.6), the relative Serre duality for β implies that

$$r^*_F(\Omega_{F_c}) = \beta_{*1}(N(D_F(c), F \times X) \otimes O_\alpha(-1))$$

if $\beta_*(N(D_F(c), F \times X) \otimes O_\alpha(-1)) = 0$.

Let us define the following subvarieties of $P_F(M)$ (here (f,x) is an element of $P_F(M) \subset F \times P(\lambda^2(V))$ and $f = (c, V_4)$):

$$D_F(G) = \{(f,x) | x \in Q_G(V_4)\}$$

$$D_F(P^2) = P_F(\tau_{3,M}) = \{(f,x) | x \in \pi(c)\}$$

$$D_l(P^2) = P_l(\tau_{3,M}) = \{(f,x) | f \in l \ \ and \ \ x \in \pi(c)\}.$$

Recall that for $f = (c, V_4) \in F$ we have $Q_G(V_4) \cap \pi(c) = c$ if $f \notin l$ and $Q_G(V_4) \cap \pi(c) = \pi(c)$ if $f \in l$. Hence there is a diagram which is a fibred product:

$$\begin{array}{ccc} D_l(P^2) \cup D_F(c) & \to & D_F(P^2) \\ \downarrow & & \downarrow \\ D_F(G) & \to & P_F(M) \end{array} . \tag{4.15}$$

Normal sheaves of inclusions of this diagram are the following. Since $N(l,F) = O_\eta(1)$, we have

$$N(D_l(P^2), D_F(P^2)) = O_\eta(1) \tag{4.16}$$

and since $M/\tau_{3,M} = O_\eta(1)$ (recall that the projectivization map $P_Z(M^*) \to Z$ is denoted by π) we have

$$N(D_F(P^2), P_F(M)) = O({}_\alpha 1, {}_\pi 1). \tag{4.17}$$

Let us consider a diagram

$$\begin{array}{ccc} D_F(c) & \to & F \times X \\ \downarrow & & \downarrow \\ D_F(G) & \to & F \times G_4 \end{array} . \tag{4.18}$$

We have

$$N(F \times X, F \times G_4) = O_\alpha(2) \tag{4.19}$$

$$\tilde{N}(D_F(G), F\times G_4) = O(1)\otimes \tau^*_{2,V} \tag{4.20}$$

where (4.20) is obtained by restriction to $F\times G_4$ of the corresponding equality for a general flag variety $D_{P(V_n^*)}(G)\subset P(V_n^*)\times G(k,V_n)$ defined as follows:

$$D_{P(V_n^*)}(G) = \{(V_{n-1},g)|V_{n-1}\in P(V_n^*), g\in G(k,V_n), g_{V_n}\subset V_{n-1}\}.$$

We have $N(D_{P(V_n^*)}(G), P(V_n^*)\times G(k,V_n)) = O_{P(V_n^*)}(1)\otimes \tau^*_{k,V_n}$.

LEMMA 4.21. Sing $D_F(G) = S$, where $S\subset F\times Q_G(V_4)$ is defined as follows:

$$S = \{(f,x)|f\in l, x\in\{\gamma_1(f),\gamma_2(f)\}\}$$

($f\in F,\quad x\in Q_G(V_4)$). S is a double curve on $D_F(G)$.

Proof. Explicit calculations in coordinates of the equation of the tangent space of points of $D_F(G)$ and of the equation of the tangent cone of points of S. We omit these calculations. ∎

Let us consider the blowing up of (4.15) along S (recall that the blowing up along S of a variety $Y\supset S$ is denoted by $\tilde{Y}\overset{r}{\to} Y$):

$$\begin{array}{ccc} \widetilde{D_l(P^2)}\cup\widetilde{D_F(c)} & \to & \widetilde{D_F(P^2)} \\ \downarrow & & \downarrow \\ \widetilde{D_F(G)} & \to & \widetilde{P_F(M)} \end{array}. \tag{4.22}$$

(4.22) is also a fibred product. (4.16), (4.17) imply that normal sheaves for inclusions are the following:

$$N(\widetilde{D_l(P^2)}, \widetilde{D_F(P^2)}) = O({}_\eta 1, {}_r 1) \tag{4.23}$$

$$N(\widetilde{D_F(P^2)}, \widetilde{P_F(M)}) = O({}_\alpha 1, {}_\pi 1, {}_r 1) \tag{4.24}$$

where $O_r(1)$ is the tautological sheaf of the map r. (4.23), (4.24) imply

$$N(\widetilde{D_F(c)}, \widetilde{D_F(G)}) = O({}_\alpha 1, {}_\pi 1, {}_\eta -1). \tag{4.25}$$

LEMMA 4.26. Let us denote $N = N(\widetilde{D_F(G)}, \widetilde{F\times G_4})$. Then N appears in the exact sequence (4.31) below.

Proof. Let us consider a diagram of restictions of blowings up on S

$$\begin{array}{ccccccc} \tilde{S}_1 & \hookrightarrow & \tilde{S}_2 & \hookrightarrow & \tilde{S}_3 & \hookrightarrow & \tilde{S}_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & = & S & = & S & = & S \end{array}$$

where $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{S}_3$, $\tilde{S}_4$ are inverse images of S in $\widetilde{D_F(c)}$, $\widetilde{D_F(G)}$, $\widetilde{P_F(M)}$, $\widetilde{F\times G_4}$ respectively; projections (vertical arrows) are denoted by r_1, r_2, r_3, r_4 respectively. For $s\in S\quad r_1^{-1}(s) = P^1$, $r_2^{-1}(s)$ is a 2-dimensional quadric, $r_3^{-1}(s) = P^3$ which is the linear envelope of $r_2^{-1}(s)$, and $r_4^{-1}(s) = P^4$.

(4.20) and properties of blowings up imply that

$$\tilde{N}(\tilde{S}_3\cup\widetilde{D_F(G)}, \widetilde{F\times G_4}) = O(1,{}_r 1)\otimes\tau^*_{2,V}. \tag{4.27}$$

If we have a diagram of inclusions

$$
\begin{array}{ccc}
Y'=Y_1\cap Y_2 & \xrightarrow{i} & Y_1\\
\downarrow & & \downarrow\\
Y_2 & \to & Y
\end{array}
\tag{4.28}
$$

where all objects are smooth varieties, codim ${}_Y Y_i = k$, codim ${}_{Y_i} Y' = 1$ $(i = 1, 2)$, then there exists an exact sequence of sheaves on Y_1

$$0 \to N(Y_1, Y) \to \tilde{N}(Y_1 \cup Y_2, Y)|_{Y_1} \to i_*(N(Y', Y_1) \otimes N(Y', Y_2)) \to 0. \tag{4.29}$$

We let in (4.28): $Y = \widetilde{F \times G}_4$, $Y_1 = \widetilde{D_F(G)}$, $Y_2 = \tilde{S}_3$, $Y' = \tilde{S}_2$. In this case (4.29) is the following:

$$0 \to N \to O(1,_r 1) \otimes \tau^*_{2,V} \to i_*(N(\tilde{S}_2, \tilde{S}_3) \otimes N(\tilde{S}_2, \widetilde{D_F(G)})) \to 0. \tag{4.30}$$

We have $N(\tilde{S}_2, \widetilde{D_F(G)}) = O_r(-1)$, and since for $s \in S$ $\quad r_2^{-1}(s)$ is a quadric in $r_3^{-1}(s) = P^3$, we have $N(\tilde{S}_2, \tilde{S}_3) = O_r(2) \otimes r_2^*(L)$ for some sheaf L on S. It is possible to show that $L = \tilde{N}(D_F(G), P_F(M))|_S = O(-1,_\alpha 2)$, but we do not need it. So, (4.30) can be written as follows:

$$0 \to N \to O(1,_r 1) \otimes \tau^*_{2,V} \to i_*(O_r(1) \otimes r_2^*(L)) \to 0. \tag{4.31}$$

This is an exact sequence of sheaves on $\widetilde{D_F(G)}$ that we need. $\square$

Now we need an exact sequence of normal sheaves for a pair of inclusions $\widetilde{D_F(c)} \hookrightarrow \widetilde{D_F(G)} \hookrightarrow \widetilde{F \times G}_4$:

$$0 \to O({}_\alpha 1,_\pi 1,_\eta -1) \to r^* N(D_F(c), F \times G_4) \otimes O_r(1) \to N|_{\widetilde{D_F(c)}} \to 0. \tag{4.32}$$

Let us restrict (4.31) to $\widetilde{D_F(c)}$. Since $\tilde{S}_2$ and $\widetilde{D_F(c)}$ are divisors in $\widetilde{D_F(G)}$, $\tilde{S}_2 \cap \widetilde{D_F(c)} = \tilde{S}_1$ and codim ${}_{\widetilde{D_F(G)}} \tilde{S}_1 = 2$, we have: Tor ${}_1^{\widetilde{D_F(g)}}(i_* O_{\tilde{S}_2}, i_* O_{\widetilde{D_F(c)}}) = 0$, and the restriction of (4.31) is the following:

$$0 \to N|_{\widetilde{D_F(c)}} \to O(1,_r 1) \otimes \tau^*_{2,V} \to i_* O_r(1) \otimes L \to 0 \tag{4.33}$$

where i is an inclusion $\tilde{S}_1 \to \widetilde{D_F(c)}$.

Multiplying (4.32) and (4.33) by $O_r(-1)$ we get

$$0 \to O({}_\alpha 1,_\pi 1,_\eta -1,_r -1) \to r^* N(D_F(c), F \times G_4) \to N \otimes O_r(-1)|_{\widetilde{D_F(c)}} \to 0 \tag{4.34}$$

$$0 \to N \otimes O_r(-1)|_{\widetilde{D_F(c)}} \to O(1) \otimes \tau^*_{2,V} \to i_* L \to 0. \tag{4.35}$$

Now we consider direct images of the above sequences to $D_F(c)$. To do it, we consider an exact sequence of sheaves on $\widetilde{D_F(c)}$:

$$0 \to O \to O_r(-1) \to i_* O_r(-1)|_{\tilde{S}_1} \to 0.$$

We get that $r_*O_r(-1) = O$, $r_{*i}O_r(-1) = 0$ for $i > 0$. Long exact r_*-direct images sequences of (4.34), (4.35) are the following (here we denote $N' = r_*(N \otimes O_r(-1)|_{\widetilde{D_F(c)}})$):

$$0 \to O({}_\alpha 1,{}_\pi 1,{}_\eta -1) \to N(D_F(c), F \times G_4) \to N' \to 0. \tag{4.36}$$

We get: $r_{*1}(N \otimes O_r(-1)|_{\widetilde{D_F(c)}}) = 0$, and

$$0 \to N' \to O(1) \otimes \tau^*_{2,V} \to i_*L \to 0. \tag{4.37}$$

An exact sequence of normal sheaves for a pair of inclusions $D_F(c) \hookrightarrow F \times X \hookrightarrow F \times G_4$ is the following:

$$0 \to N(D_F(c), F \times X) \to N(D_F(c), F \times G_4) \to O_\alpha(2) \to 0. \tag{4.38}$$

Multiplying (4.36) — (4.38) by $O_\alpha(-1)$ and taking the long exact β_*-direct image sequence we get (here we denote $B = N(D_F(c), F \times G_4) \otimes O_\alpha(-1)$):

$$0 \to O({}_\pi 1,{}_\eta -1) \to \beta_*(B) \to \beta_*(N' \otimes O_\alpha(-1)) \to 0 \tag{4.39}$$

$$0 \to \beta_{*1}(B) \to \beta_{*1}(N' \otimes O_\alpha(-1)) \to 0 \tag{4.40}$$

$$0 \to \beta_*(N' \otimes O_\alpha(-1)) \to O(1) \otimes \beta_*\tau_{2,V} \to \beta_*i_*(L \otimes O_\alpha(-1)) \to \beta_{*1}(N' \otimes O_\alpha(-1)) \to$$

$$O(1) \otimes \beta_{*1}\tau_{2,V} \to \beta_{*1}i_*(L \otimes O_\alpha(-1)) \to 0 \tag{4.41}$$

$$0 \to \beta_*(N(D_F(c), F \times X) \otimes O_\alpha(-1)) \to \beta_*(B) \to \beta_*O_\alpha(1) \to r_F^*(\Omega_{F_c}) \to$$

$$\beta_{*1}(B) \to \beta_{*1}O_\alpha(1) \to 0. \tag{4.42}$$

It is easy to see that the restriction of $\tau_{2,V}$ on a conic line $c \subset G$ is equal to $2O_c(-1)$ (resp. $O_c \oplus O_c(-2)$) if c is not contained in a α-plane (resp. if c is contained in a α-plane). There exists the following property of the map $\beta : D_F(c) \to F$:

For $f \in F$ we have: $\beta^{-1}(f)$ is a conic line on a α-plane $\iff f \in l_2$.

Hence, only for $f \in l_2$ cohomology of a sheaf $\tau_{2,V}|_{\beta^{-1}(f)}$ are non-zero. This means that $\beta_*\tau_{2,V} = 0$ and $\beta_{*1}\tau_{2,V}$ is a sheaf whose support is l_2, and it is invertible on l_2. (4.41) implies that $\beta_*(N' \otimes O_\alpha(-1)) = 0$, and (4.39) implies that $\beta_*(B)$ is isomorphic to $O({}_\pi 1,{}_\eta -1)$.

Since we have a commutative diagram

$$\begin{array}{ccc} S & \overset{i}{\hookrightarrow} & D_F(c) \\ \downarrow & & \downarrow \\ l & \overset{i}{\hookrightarrow} & F \end{array}$$

where vertical maps are β_S, β respectively, we see that $\beta_{*k}i_*(L \otimes O_\alpha(-1)) = i_*\beta_{S*k}(L \otimes O_\alpha(-1))$. Since β_S is a two-sheeted covering, we have $\beta_{*1}i_*(L \otimes O_\alpha(-1)) = 0$ and $\beta_*i_*(L \otimes O_\alpha(-1))$ is a locally free sheaf of rank 2 on l. Let us consider a diagram

$$\begin{array}{ccccc} D_F(c) & \overset{i_c}{\hookrightarrow} & D_F(P^2) & = & P_F(\tau_{3,M}) \\ & \overset{\beta}{\searrow} & \downarrow & & \\ & & F & & \end{array}$$

where the vertical map is α. Since for any $f \in F$ we have an equality of sheaves on $\alpha^{-1}(f)$: $O_{\alpha^{-1}(f)}(\beta^{-1}(f)) = O_\alpha(2)$, we have globally on $D_F(P^2)$: $O_{D_F(P^2)}(D_F(c)) = \alpha^* L_F \otimes O_\alpha(2)$, where L_F is some invertible sheaf on F. There is an exact sequence of sheaves on $D_F(P^2)$:

$$0 \to L_F^{-1} \otimes O_\alpha(-1) \to O_\alpha(1) \to i_{c*}(O_\alpha(1)|_{D_F(c)}) \to 0.$$

Considering its long exact α_*-direct image sequence we see that

$$\beta_* O_\alpha(1) = \tau^*_{3,M} \tag{4.43}$$

and that $\beta_{*1}(O_\alpha(1)) = 0$.

Now let us consider a map $i_B : \beta_*(B) \to \beta_* O_\alpha(1)$ which is isomorphic to $O(_\pi 1,_\eta -1) \to \tau^*_{3,M}$. Its fibre at a point $c = (c, V_4) \in F - l$ is $i_B(c) = H^0(\phi_c)$ because of (4.13), i.e. $i_B(c)$ is an inclusion $(\tau_{3,M}(c)/\tau_{2,M}(c))^* \to \tau^*_{3,M}(c)$. Tautological exact sequences on $P_Z(M^*)$ and on $G_Z(2, M^*)$ and an equality of sheaves on F: $O_\eta(1) \otimes \det \tau^*_{2,M} = O$ imply an equality $(\tau_{3,M}/\tau_{2,M})^* = O(_\pi 1,_\eta -1)$. So, there are 2 maps of sheaves

$$(\tau_{3,M}/\tau_{2,M})^* = O(_\pi 1,_\eta -1) \to \tau^*_{3,M}$$

namely i_B and the natural inclusion $(\tau_{3,M}/\tau_{2,M})^* \hookrightarrow \tau^*_{3,M}$ whose cokernel is $\tau^*_{2,M}$. For all $f \in F - l$ their fibres at f coincide, hence the maps coincide as well. So, the exact sequence (4.42) (resp. (4.41), taking into consideration (4.40)) can be rewritten as follows:

$$0 \to \tau^*_{2,M} \to r^*(\Omega_{F_c}) \to \beta_{*1}(B) \to 0 \tag{4.44}$$

respectively

$$0 \to \beta_* i_*(L \otimes O_\alpha(-1)) \to \beta_{*1}(B) \to O(1) \otimes \beta_{*1}\tau_{2,V} \to 0 \tag{4.45}$$

where $\beta_* i_*(L \otimes O_\alpha(-1))$ is a locally free sheaf of rank 2 on l and $O(1) \otimes \beta_{*1}\tau_{2,V}$ is an invertible sheaf on l_2.

Let us prove that $O(1) \otimes \beta_{*1}\tau_{2,V} = \Omega_{l_2}$ and a composition map

$$r^*(\Omega_{F_c}) \to \beta_{*1}(B) \to O(1) \otimes \beta_{*1}\tau_{2,V}$$

coincides with the natural epimorphism $r^*(\Omega_{F_c}) \to i_{2*}(\Omega_{l_2})$ associated to an inclusion $i_2 : l_2 \to F_c$.

We define respectively $D_{l_2}(c)$, $D_{l_2}(P^2)$, $D_{l_2}(G) = D_{l_2}(P^2) \cup l_2 \times U^*$ as restrictions on l_2 of $D_F(c)$, $D_F(P^2)$, $D_F(G)$ respectively. We find Ω_{l_2} by using deformation theory and a fact that l_2 is a variety of α-planes on G_4. Let

$$\begin{array}{ccc} D_{l_2}(P^2) & \hookrightarrow & l_2 \times G_4 \\ & \searrow & \downarrow \\ & & l_2 \end{array}$$

be the corresponding family (both maps to l_2 are α); then we have:

$T_{l_2} = \alpha_*(N(D_{l_2}(P^2), l_2 \times G_4))$, if

$$\alpha_{*1}(N(D_{l_2}(P^2), l_2 \times G_4)) = 0. \tag{4.46}$$

Serre duality implies that $\Omega_{l_2} = \alpha_{*2}(N(D_{l_2}(P^2), l_2 \times G_4) \otimes \omega_{G_4})$. Now we apply (4.29) to a diagram

$$\begin{array}{ccc} P_{l_2}(\tau_{2,m}) & \overset{i}{\to} & D_{l_2}(P^2) \\ \downarrow & & \downarrow \\ l_2 \times U^* & \to & l_2 \times G_4 \end{array}.$$

We get an exact sequence of sheaves on $D_{l_2}(P^2)$:

$$0 \to N(D_{l_2}(P^2), l_2 \times G_4) \to O(1) \otimes \tau^*_{2,V} \to i_*(N_1 \otimes N_2) \to 0$$

where $N_1 = N(P_{l_2}(\tau_{2,M}), D_{l_2}(P^2))$, $N_2 = N(P_{l_2}(\tau_{2,M}), l_2 \times U^*)$. We multiply it by $\omega_{G_4} = O_\alpha(-3)$ and consider a long exact α_*-direct image sequence. Since $\forall x \in l_2$ we have $\tau^*_{2,V}|_{\alpha^{-1}(x)} = O \oplus O_\alpha(1)$, then $\alpha_{*i}(O(1,_\alpha -3) \otimes \tau^*_{2,V})$ is 0 for $i = 0, 1$ and one-dimensional for $i = 2$. Restriction of both sheaves N_1 and N_2 on $\tau_{2,M}(x)$ (i.e. the fibre of the projection $P_{l_2}(\tau_{2,M}) \to l_2$ at x) is $O_\alpha(1)$, so $N_1 \otimes N_2 \otimes O_\alpha(-3)$ at this fibre is equal to $O_\alpha(-1)$, and its cohomology is 0. This implies that the condition (4.46) is satisfied and

$$\Omega_{l_2} \to \alpha_{*2}(O(1,_\alpha -3) \otimes \tau^*_{2,V}) \tag{4.47}$$

is an isomorphism. It is clear that a divisor $D_{l_2}(c) \overset{i}{\hookrightarrow} D_{l_2}(P^2)$ corresponds to a sheaf $O_\alpha(2)$, so we consider an exact sequence on $D_{l_2}(P^2)$:

$$0 \to O_\alpha(-2) \to O \to i_*(O_{D_{l_2}(c)}) \to 0.$$

We multiply it by $O(1) \otimes \tau_{2,V} = O(1,_\alpha -1) \otimes \tau^*_{2,V}$ and consider a long exact α_*-direct image sequence:

$$0 \to \alpha_{*1}(O(1) \otimes \tau_{2,V}|_{D_{l_2}(c)}) \to \alpha_{*2}(O(1,_\alpha -3) \otimes \tau^*_{2,V}) \to 0. \tag{4.48}$$

(4.47) and (4.48) imply an isomorphism that we need. It is clear that it is compatible with the map $r^*_F(\Omega_{F_c}) \to i_{2*}(\Omega_{l_2})$. The kernel of this map is $r^*(\Omega_{F_m})$, where $r : F \to F_m$, and the exact sequences(4.44), (4.45) can be rewritten as follows:

$$0 \to \tau^*_{2,M} \to r^*(\Omega_{F_m}) \to \beta_* i_*(L \otimes O_\alpha(-1)) \to 0. \tag{4.49}$$

It is clear that $r^*(\Omega_{F_m})|_l = 2O_l$, and hence the restriction of (4.49) on l gives us an exact sequence

$$2O_l \to \beta_* i_*(L \otimes O_\alpha(-1))|_l \to 0. \tag{4.50}$$

Since $\beta_* i_*(L \otimes O_\alpha(-1))|_l$ is a locally free sheaf of rank 2, (4.50) imply that it is isomorphic to $2O_l$ (equality $2O_l = \beta_* i_*(L \otimes O_\alpha(-1))$ can be proved also by means of explicit calculations). So, (4.49) can be rewritten as follows:

$$0 \to \tau^*_{2,M} \to r^*(\Omega_{F_m}) \to 2O_l \to 0. \tag{4.51}$$ □

It is clear that (4.51) is isomorphic the the exact sequence associated to the divisor $l \subset F$

$$0 \to O_\eta(-1) \to O \to i_*(O_l) \to 0$$

multiplied by $r^*(\Omega_{F_m})$. So, $\tau^*_{2,M} = r^*(\Omega_{F_m}) \otimes O_\eta(-1)$.

Remark. (4.51) can be used for a calculation of Chern classes of Ω_F, because calculation of the class of $\tau^*_{2,M}$ in $K_0(F)$ is much easier than the one of Ω_F.

5. Special intersection of 3 quadrics in P^6. In this section X will mean a special intersection of 3 quadrics in P^6 (see below for its definition).

Firstly we continue to describe some properties of $G(2,5)$. For a variety $Y \subset P^n = P(V_{n+1})$ we denote by $TP_Y(t) \subset P^n$ (resp. $TV_Y(t) \subset V_{n+1}$) the projective (resp. vector) tangent space to Y at a point $t \in Y$, i.e. $TP_Y(t) = P(TV_Y(t))$.

For any n $G(2,V_n)$ is an intersection of quadrics in $P(\lambda^2 V_n)$, and the set of these quadrics is isomorphic to $G(n-4,V_n)$. Namely, for $V_{n-4} \subset V_n$ the corresponding quadric is Pl $(V_n/V_{n-4}) \in P(S^2\lambda^2 V_n/V_{n-4})^*$. For $n=5$ we denote the corresponding inclusion by $q : P(V) \to P(S^2\lambda^2 V^*)$; it is linear. All quadrics $q(P(V))$ are cones whose vertex has dimension 3.

Let $t \in G$, $v \in P(V)$. Then $t \in \text{Sing } q(v) \iff v \in t_V$.

5.1. Let $TV_G(t) = V_7 \subset \lambda^2(V)$. Then V_7 is a member of exact sequences

$$0 \to \lambda^2 t_V \to V_7 \to t_V \otimes V/t_V \to 0$$

$$0 \to V_7 \to \lambda^2 V \to \lambda^2(V/t_V) \to 0.$$

$\forall v \in t_V$ we have $\text{Sing } q(v) \subset TP_G(t) \subset q(v)$. The same inclusions are valid for projections from t.

5.2. Let $c \subset G$ be a conic line such that $\pi(c) \not\subset G$. Then there exists the only $V_4 \in P(V^*)$ such that $v \in P(V_4) \iff \pi(c) \subset q(v)$. It is easy to see that $c \subset G(2,V_4)$.

Let $t \in G$; for any variety Y we denote by Y_t the projection of Y from t. Since $\deg G = 5$, we have: $\deg G_t = 4$, for $v \in t_V$ $(q(v))_t$ is a quadric containing G_t. This means that G_t is a complete intersection of 2 quadrics. They generate a straight line P^1 which is isomorphic to t_V, all quadrics belonging to this line are cones whose vertex is 2-dimensional, and they contain $(TP_G(t))_t$.

Let $t \in G_4$ and $t_V \cap U = \emptyset$; then $(G_4)_t$ is a complete intersection of two quadrics in P^6, they generate a straight line P^1 which is isomorphic to t_V, all quadrics belonging to this line are cones whose vertex is a point, they contain $P^3_T = (TP_G(t) \cap H_1 \cap H_2)_t$, and their vertices are contained in P^3_T and form a normcubic $\bar{c}_3$ in it. The set of straight lines on G_4 passing through t is canonically isomorphic to $\bar{c}_3$; a projection from t sends any such a line to the corresponding point of $\bar{c}_3$.

Let us recall now properties of X^3_8 - a generic intersection of 3 quadrics in $P_6 = P(V_7)$ (see [21], [28], [47], [50]). Let P^2 be a linear envelope in $P(S^2 V^*_7)$ of these 3 quadrics, and for $h \in P^2$ let $Q(h)$ be the corresponding quadric. The Hesse curve $H_7 = H_7(X^3_8)$ is defined as follows: $h \in H_7 \iff Q(h)$ is a cone. The set of planes on a non-singular 4-dimensional quadric Q has 2 connected components, we denote this set of components by $\Gamma(Q)$. Planes belonging to one component either coincide or their intersection is one point; intersection of planes belonging to different components is either empty or a straight line, and analogously for P^3 lying on a cone over such a quadric. We denote a two-sheeted covering $p = p_{H_7} : \hat{H}_7 \to H_7$ as follows: $p^{-1}(h) = \Gamma(Q(h))$.

For any plane curve H of degree n and its two-sheeted covering $p : \hat{H} \to H$ whose involution is denoted by $i_H : \hat{H} \to \hat{H}$, we define a set $\bar{S} = \bar{S}(p)$ of effective divisors on $\hat{H}$ as follows:

$d \in \bar{S} \iff O_H(p_*(d)) = O_{P^2}(1)|_H \iff p_*(d)$ is a sum of n points $H \cap l$ where l is a straight line in P^2.

$\bar{S}$ is non-connected, $\bar{S} = S \cup S'$, and if $d \in \bar{S}$, $d = \sum_{i=1}^{n} \hat{h}_i$ for $\hat{h}_i \in \hat{H}$, then d and $d - \hat{h}_1 + i_H(\hat{h}_1)$ belong to different components of $\bar{S}$. Particularly, if n is even then S has an involution i_S: $i_S(\sum \hat{h}_i) = \sum i_H(\hat{h}_i)$. S is irreducible ([50]).

There exists an isomorphism $\phi_7 : F(X_8^3) \to S = S(p_{H_7})$, where $F(X_8^3)$ is a set of conics in X_8^3. ϕ_7 is defined as follows. For a conic $c \subset X_8^3$ we define firstly a straight line $l(c) \subset P^2$ as follows: $h \in l(c) \iff Q(h) \supset \pi(c)$. For $h \in l(c) \cap H_7 \quad \pi(c)$ defines an element $\gamma(c,h) \in \Gamma(Q(h)) = p^{-1}(h) \subset \hat{H}_7$. We let

$$\phi_7(c) = \sum_{h \in l(c) \cap H_7} \gamma(c,h).$$

A fact that ϕ_7 is an isomorphism follows easily from the following proposition ([47]):

5.3. Let $l \subset P(S^2V_7^*)$ be a straight line in a space of quadrics in P^6, $h_i \in l$ $(i = 1, \ldots, 7)$ are such that $Q(h_i)$ are cones. Then $\cap_{h \in l} Q(h)$ contains 2^6 planes. Let π be such a plane. It defines a component in $\Gamma(Q(h_i))$ denoted by $\gamma(\pi, h_i)$. Let us consider $\gamma(\pi, h_i)$ as a function in i, it maps a set $\{1, \ldots, 7\}$ to $\cup_{i=1}^{7} \Gamma(Q(h_i))$. Then these functions are different for different π, and values of 2 such functions are different at even numbers of elements of $\{1, \ldots, 7\}$.

A composition map $S \to \mathrm{Div}\ \hat{H}_7 \to \mathrm{Pic}\ \hat{H}_7$ (defined up to a shift) factors through a map $\mathrm{Pr}\ (\hat{H}_7/H_7) \hookrightarrow \mathrm{Pic}\ \hat{H}_7$.

Theorem 5.4. ([50]) The corresponding map $\mathrm{Alb}\ (S) \to \mathrm{Pr}\ (\hat{H}_7/H_7)$ is an isogeny. □

Theorem 5.5. ([21]) There exists an isomorphism $\mathrm{Pr}\ (\hat{H}_7/H_7) \to J^3(X_8^3)$. □

We denote the Abel - Jacobi map for conics by Φ. We get a diagram where the left vertical map is 2Φ:

$$\begin{array}{ccc} \mathrm{Alb}\ F & \overset{\mathrm{Alb}\ (\phi_7)}{\to} & \mathrm{Alb}\ S \\ \downarrow & & \downarrow \\ J^3(X_8^3) & \leftarrow & \mathrm{Pr}\ (\hat{H}_7/H_7) \end{array}.$$

Theorem 5.6. ([28]) This diagram is commutative. □

Let us consider now properties of a threefold $G_4 \cap \Omega$ which is smooth everythere except one point t which is an ordinary double point. Recall that in this section it will be denoted by X. The set of conics on X will be denoted by $F = F(X)$. It is well-known that the projection of X from t is an intersection of 3 quadrics in P^6; this fact is used for a proof of non-rationality of a "generic" $G(2,5) \cap H_1 \cap H_2 \cap \Omega$ ([26]).

Lemma 5.7. There exists a cone $\Omega' \in P(S^2V_8^*)$ such that t is its vertex and such that $X = G_4 \cap \Omega'$.

Proof. We deprojectivize a map $q : P(V) \to P(S^2\lambda^2V^*)$ and we restrict it to V_8. We get a map $q : V \to S^2V_8$. Let $T : V \to V_8^*$ be a map defined as follows: $T(v)$ is an equation of the polar line of $q(v)$ with respect to t. Then $\mathrm{im}\ T \subset (V_8/TV_{G_4}(t))^*$ and $\mathrm{Ker}\ T$ is a set of points $v \in V$ such that $t \in \mathrm{Sing}\ q(v)$, i.e. $\mathrm{Ker}\ t = t_V$, hence T is an epimorphism on $(V_8/TV_{G_4}(t))^*$. Since t is a singular point in $G_4 \cap \Omega$, we have $TV_\Omega(t) \in (V_8/TV_{G_4}(t))^*$ and $\exists v \in V$ such that $T(v) = TV_\Omega(t)$. This means that the equation of the tangent hyperplane at t of the quadric $\Omega - q(v)$ is 0, i.e. $\Omega' = \Omega - q(v)$ is a desired cone. $\square$

Further we shall suppose that Ω is such a cone. We see that X_t is an intersection of 3 quadrics: Ω_t and $(q(t_V))_t$. As earlier, let P^2 be the plane that they generate, and we denote by t_P the straight line that corresponds to quadrics $(q(t_V))_t$, or, in formal notations, $h \in t_P \iff Q(h) \in (q(t_V)|_{H_1 \cap H_2})_t$.

t_P is a component of $H_7(X_t)$, another component is a curve of degree 6 denoted by $H_6 = H_6(X)$. It is clear that for a generic X it is non-singular, later we shall consider only these X. Let $t_P \cap H_6 = \{R_1, \ldots, R_6\}$. $Q(R_i)$ are ordinary cones, like all other quadrics corresponding to points of t_P and H_6. Hence there are a two-sheeted non-ramified covering $p = p_{H_6} : \hat{H}_6 \to H_6$ and a corresponding surface $S = S(p_{H_6})$ together with an involution i_S. Vertices of $Q(R_i)$ are projections of 6 straight lines on X passing through t, and they are the only singular points on X_t. These vertices are intersection points of $\bar{c}_3$ and Ω_t as well.

Proposition 5.8. There exists a birational isomorphism $\phi_6 : F(X) \to S(p_{H_6})$.

Proof. Let $c = (c, V_4) \in F$. If $t \notin \pi(c)$ then the projection $c_t \subset X_t$ is a conic. We denote by χ a rational map $(c, V_4) \mapsto c_t$ from $F(X)$ to $F(X_t)$. c_t defines a straight line $l(c) = l(c_t) \subset P^2$. We denote $l(c) \cap t_P = h_0(c)$, $\quad l(c) \cap H_6 = \{h_1(c), \ldots, h_6(c)\}$. If $c \notin l_1 \cup l_2$, then $l(c) \neq t_P$ and (5.2) implies that $Q(h_0(c)) = q(t_V \cap V_4)$.

Lemma 5.9. Sing $Q(h_0(c)) = P(M(V_4))_t \cap P_T^3$.

Proof. This statement is formulated on $G(2, V)$ as follows. For some given $t \in G$, $V_4 \subset V$ we set $s = t_V \cap V_4$. Then the linear envelope of t and $P(\lambda^2(V_4)) \cap TP_G(t)$ coincide with Sing $q(s)$. Really, let s_0, t_1 (resp. s_0, v_1, v_2, v_3) be a basis of t_V (resp. V_4). Then (considering vector spaces) we have

a basis of $t \subset \lambda^2 V$ is $s_0 \wedge t_1$

a basis of $TV_G(t)$ is $s_0 \wedge t_1$, $s_0 \wedge v_i$, $t_1 \wedge v_i$

a basis of $\lambda^2 V_4$ is $s_0 \wedge v_i$, $v_i \wedge v_j$

and a basis of Sing $q(s)$ is $s_0 \wedge t_1$, $s_0 \wedge v_i$ (see (5.1); $i, j = 1, 2, 3$). This implies the proposition for G. Intersecting it with H_1, H_2 and projecting from t we get the desired. $\square$

$\forall h \in t_P$ the quadric $Q(h)$ contains P_T^3, we denote the corresponding element of $\Gamma(Q(h))$ by $\beta(h)$.

Lemma 5.10. $\beta(h_0(c)) \neq \gamma(c_t, h_0(c))$.

Proof. Since $\pi(c) \subset P(M(V_4))$, we have: $\pi(c_t)$ and $(P_T^3)_{\text{Sing } Q(h_0(c))}$ have an empty intersection after the projection from Sing $Q(h_0(c))$. $\square$

We define $\phi_6(c) = \sum_{i=1}^6 \gamma(c_t, h_i(c))$. (5.10) and (5.3) imply that at a generic point ϕ_6 is an inclusion. Since S is irreducible and dim $F =$ dim S we have that ϕ_6 is a birational isomorphism. $\square$

Lemma 5.11. ϕ_6 commutes with involutions i_F, i_S.

Proof. Let (c, V_4) and $(c', V_4) \in F$ be 2 involutory conics. Then $\pi(c) \cap \pi(c')$ is a straight line, hence (if $t \notin P(M(V_4))$, i.e. if $V_4 \not\supset t_V$) $\quad \pi(c_t) \cap \pi(c'_t)$ is also a straight line. For $h \in l(c) \quad Q(h)$ contains c'_t and $\pi(c_t) \cap \pi(c'_t)$, i.e. a curve of degree 3 on

$\pi(c'_t)$ and hence the whole $\pi(c'_t)$, i.e. $l(c) = l(c')$, $h_i(c) = h_i(c')$. The intersection of projections of $\pi(c_t)$ and $\pi(c'_t)$ from Sing $Q(h_i(c))$ $(i = 1, \ldots, 6)$ is a straight line, and if

$$\text{Sing } Q(h_i(c)) \notin P(M(V_4))_t$$

then $\gamma(c_t, h_i(c)) \neq \gamma(c'_t, h_i(c'))$. But else

$$P(M(V_4))_t \subset Q(h_i(c)) \Rightarrow \text{Sing } Q(h_0(c)) \in Q(h_i(c)).$$

If $h_i(c) \notin \{R_1, \ldots, R_6\}$ then $h_i(c) \notin t_P$ and Sing $Q(h_0(c))$ belongs to 3 basis quadrics of P^2 and hence to X_t, i.e. $h_0(c) \in \{R_1, \ldots, R_6\}$. This is not true at an open set, hence at this set c and c' are mapped to involutory elements of S, hence the last is true at all F. □

The above description of ϕ_6 does not indicate images of points $(c, V_4) \in F$ for which $V_4 \supset t_V$, as well as images of straight lines $l_1, l_2 \subset F$. The set of spaces V_4 which contain t_V is isomorphic to $P^2_T = P((V/t_V)^*) \hookrightarrow P(V^*)$. For this V_4 $Q_\Omega(V_4)$ is a cone in $P(M(V_4))$ whose vertex is t, and $Q_G(V_4))$ is a generic quadric containing t. Let $(c, V_4) \in F$, $V_4 \in P^2_T$ and

$$Q = T(\bar{\phi}(c)) = \pi(c) \cup \pi(c') \in < Q_\Omega(V_4), Q_G(V_4) >$$

is a pair of planes. If $Q \notin Q_\Omega(V_4)$, then $Q_\Omega(V_4) \cap Q_G(V_4) = c \cup c' = Q \cap Q_\Omega(V_4)$. Let $t \in \pi(c)$, then $Q_\Omega(V_4) \cap \pi(c)$ is a pair of straight lines passing through t, and $Q_\Omega(V_4) \cap \pi(c')$ is a generic conic. There are 6 straight lines on X passing through t, i.e. there are only $\binom{6}{2} = 15$ such spaces V_4 while there are 30 conics. It is possible to prove (although we do not need it) that ϕ_6 maps these conics (as well as lines $l_1, l_2 \subset F$) to 32 points $S \cap ((p_{H_6})^{-1}(\sum_i R_i))$.

Let now $Q = Q_\Omega(V_4)$. We denote the set of these $V_4 \in P^2_T$ by K_0, i.e. $K_0 = l_\Omega \cap P^2_T$. We multiply an inclusion of vector spaces $(t) \to V_8$ by $O_{P^2_T}$; it is clear that the obtained map $t \otimes O_{P^2_T} \to V_8 \otimes O_{P^2_T}$ factors through an inclusion $M|_{P^2_T} \to V_8 \otimes O_{P^2_T}$, and we denote Coker $(t \otimes O_{P^2_T} \to M|_{P^2_T})$ by M_t. Since Ω is a cone whose vertex is t, we have: a bundle of quadrics Q_Ω restricted to $P^2_T \subset P(V^*)$ induces a bundle of quadrics Q_{Ω_t} on (P^2_T, M_t) whose fibre at $V_4 \in P^2_T$ is a projection $Q_\Omega(V_4)$ from its vertex. It is clear that $K_0 = D_2(Q_{\Omega_t})$, hence K_0 is a curve on P^2_T, and according the Porteus formula ([7]) we have deg $K_0 = 2c_1(M^*_t) = 6$.

Since $K_0 \subset F_0(X)$, there exists a two-sheeted non-ramified covering $p_K : K \to K_0$ whose involution is denoted by $i_K : K \to K$, and $p_K = p_F|_K$, $i_K = i_F|_K$. Involutory conics corresponding to points of K have 2 points of intersection, one of each is t. For a generic cone Ω whose vertex is t, we have: K_0 is non-singular and K is connected.

Let $d : \tilde{F} \to F$ be a desingularization map and $\tilde{K} = d^{-1}(K)$. There exists an involution $i_{\tilde{F}}$ on $\tilde{F}$ such that $i_F \circ d = d \circ i_{\tilde{F}}$. There exist rational maps $\tilde{\phi}_6 : \tilde{F} \to S$ and $\tilde{\chi} : \tilde{F} \to F(X_t)$.

Proposition 5.12. $\tilde{K}$ is non-connected, $\tilde{K} = K_1 \cup K_2$ and $d_i = d|_{K_i} : K_i \to K$ are isomorphisms $(i = 1, 2)$. For $f \in \tilde{K}$ we have: $d(f)$ is a conic on X passing through t, $\tilde{\chi}(f)$ is a pair of straight lines on X_t, one of them is contained in a quadric $Q_2 = P^3_T \cap \Omega_t$.

Proof. Since S is smooth and $\tilde{K}$ is not a union of straight lines, we have that $\tilde{\phi}_6(\tilde{K})$ is a curve on S. Let us find which points $s \in S$ do not belong to $\phi_6(F - K)$.

Let $s = \sum_{i=1}^{6} \hat{h}_i$, $\hat{h}_i \in \hat{H}_6$, $p_{h_6}(\hat{h}_i) = h_i \in H_6$, and l a straight line in P^2 such that $\{h_i\} = l \cap H_6$. We can suppose that $l \neq t_P$ (this is true for all except a finite number points $s \in S$). We let $h_0 = l \cap t_P$ and $\hat{h}_0 \in \Gamma(Q(h_0))$, $\hat{h}_0 \notin \beta(h_0)$. According (5.3), there exists a plane $\pi = \pi(s)$ on $\cap_{h\in l} Q(h)$ that corresponds to the set $\{\hat{h}_i\}$ $(i = 0, \dots, 6)$. Really, their images under the projection from Sing $Q(h_0)$ must intersect namely by this way. Let $\pi \cap P^3_T = \emptyset$; then $c = \pi \cap X_t$ is a conic. Let us consider the 2-dimensional cone containing c and having a vertex $t \in P(V_8) \supset X$. Since X is an intersection of quadrics, any straight line either intersects X at most at 2 points, or is contained in X. Since $c \subset X_t$, any ruling line of the above cone intersects X at one point (the other intersection point is t). This point does not coincide with t because $c \cap P^3_T = \emptyset$. So, the set of these intersection points is a conic, and $s \in S$ is its image under the map ϕ_6. (If we choose a plane π on $\cap_{h\in l} Q(h)$ that corresponds to a set $\beta(h_0), \hat{h}_1, \dots, \hat{h}_6$, then $\pi \cap P^3_T$ is one point, and instead of a conic we should get a normcubic on X passing throught t).

So, if $s = \tilde{\phi}_6(c)$ for $c \in \tilde{K}$, then $\pi(s) \cap P^3_T = P^1$. It is clear that $\pi(s) = \pi(\tilde{\chi}(c))$ and $\forall h \in P^2 - l(c)$ we have $\tilde{\chi}(c) = \pi(s) \cap Q(h)$. Choosing $h \in t_P$ we get that $\tilde{\chi}(c) \supset \pi(s) \cap P^3_T = P^1$. This means that $\tilde{\chi}(c)$ is a pair of straight lines, one of which lies on a quadric $Q_2 = \Omega_t \cap P^3_T$, and other intersects it.

Conversely, let l_X be any straight line on X_t that intersects Q_2. There are 2 ruling lines u_1, u_2 of Q_2 that pass through a point $l_X \cap Q_2$, hence we get 2 conics $c_i(l_X) = l_X \cup u_i$ $(i = 1, 2)$. We can associate them:

(a) 2 different straight lines $l(c_i(l_X))) \subset P^2$;

(b) points $h_{ij} = h_j(c_i(l_X)) \in H_6$;

(c) elements $\gamma(c_i(l_X), h_{ij}) \in \hat{H}_6$

such that the element

$$s_i(l_X) = \sum_{j=1}^{6} \gamma(c_i(l_X), h_{ij}) \in S$$

and $\pi(s_i(l_X)) \cap P^3_T = u_i$. Let us find $\tilde{\phi}_6^{-1}(s_i(l_X))$. A cone over $c_i(l_X)$ having a vertex t in $P(V_8)$ is a plane $< l_X, t >$, and each straight line in this plane passing through t meets X exactly at one point (except t). Further, one straight line in this plane (it is the inverse image of $l_X \cap Q_2$ under the projection from t) is tangential to X at t. So, $< l_X, t > \cap X$ is a conic $c_X(l_X) \in K$, and $\tilde{\phi}_6^{-1}(s_i(l_X)) \in d^{-1}(c_X(l_X))$. This means that $\forall c \in K$ $\quad d^{-1}(c)$ are 2 points, i.e. $\tilde{K} \to K$ is a non-ramified two-sheeted covering, and for $c \in \tilde{K}$ we have $\tilde{\chi}(c) = d(c)_t \cup u(c)$, where $u(c)$ is a ruling line of Q_2. Since there are 2 systems of rulings on Q_2, we have: $\tilde{K} = K_1 \cup K_2$ is non-connected, namely $c \in K_i$ iff $u(c)$ is a ruling line of type i. It is clear that restrictions $d_i = d|_{K_i} : K_i \to K$ are isomorphisms $(i = 1, 2)$.

Remark. Let $\tilde{r} : \tilde{F} \to \tilde{F}_m$ be a map of blowing down of $l_1 \cup l_2 \subset \tilde{F}$ to points. There exists an isomorphism $i : \tilde{F}_m \to S$ such that $\tilde{\phi}_6 = i \circ \tilde{r}$.

Let us denote an isomorphism $d_2^{-1} \circ d_1 : K_1 \to K_2$ by α, and let $i_{K_1} = d_1^{-1} \circ i_K \circ d_1$ be an involution of K_1.

Lemma 5.13. For $c \in K_1$ we have $i_{\tilde{F}}(c) = \alpha(i_{K_1}(c))$.

Proof. $i_{\tilde{F}}(c)$ is equal to $i_{K_1}(c)$ or $\alpha(i_{K_1}(c))$. We have: the intersection of $\tilde{\chi}(c)$

and $\tilde{\chi}(i_{K_1}(c))$ is 1 point, while $\forall f \in \tilde{F}$ the intersection of $\tilde{\chi}(f)$ and $\tilde{\chi}(i_{\tilde{F}}(f))$ is 2 points. $\square$

We denote by $\tilde{X}$ the desingularization of X. Any point of $\tilde{F}$ defines a conic on $\tilde{X}$ by an obvious way (for $c \in \tilde{F}$ the corresponding conic is the inverse image of $\tilde{\chi}(c)$ under the desingularization map $\tilde{X} \to X$). So, there exists the Abel - Jacobi map $\Phi : \tilde{F} \to J^3(\tilde{X})$.

The next 3 propositions are analogs for X_t of the theorems (5.4) - (5.6) for a generic X_8^3, and their proofs follow the proofs of these theorems.

Proposition 5.14. A map Alb $S(p_{H_6}) \to$ Pr $\hat{H}_6/H_6$ is an isogeny.

Proof. [50], Proposition 1.3.3 implies that this map is an epimorphism, hence it is sufficient to prove that $h^{1,0}(S) \leq 9$. The proof of ([50], Proposition 3.4.4) implies that:

(1) There exist plane curves C_i of degree 6 ($i = 0, \dots, 6$), their non-ramified two-sheeted coverings $p_i : \hat{C}_i \to C_i$ and points $P_1, \dots, P_6 \in P^2$ in a general position such that:

(a) $p_0 = p_{H_6}$;

(b) Sing $C_i = \{P_1, \dots, P_i\}$ and these points are ordinary double points on C_i;

(2) $\forall i = 0, \dots, 5$ there exists a commutative diagram

$$\begin{array}{ccc} \hat{D}_i & \to & D_i \\ & \searrow & \downarrow \\ & & T_i \end{array}$$

where T_i is an open subset in P^1, D_i is a smooth surface, the fibre of the covering $\hat{D}_i \to D_i$ at a point $t \in T_i$ is a non-ramified two-sheeted covering of smooth curves $(\hat{D}_i)_t \to (D_i)_t$ for all t except $t = 0$, where $(D_i)_t$ has one double point and $(\hat{D}_i)_t$ has 2 double points which are its inverse images. Further, for $t = t_0$ the above fibre is isomorphic to the normalization of the covering p_i denoted by $\tilde{p}_i : \tilde{\hat{C}}_i \to \tilde{C}_i$, and for $t = 0$ it is isomorphic to the blowing up of the covering p_{i+1} at $P_1 \cup \dots \cup P_i$ for C_{i+1} and their p_{i+1}-inverse images for $\hat{C}_{i+1}$.

(3) There exist maps $D_i \to T_i \times P^2$ over T_i whose fibres at points t_0 and $0 \in T_i$ are ordinary maps $\tilde{C}_i \twoheadrightarrow C_i \hookrightarrow P^2$ and $(\tilde{C}_{i+1})_{P_1 \cup \dots \cup P_i} \twoheadrightarrow C_{i+1} \hookrightarrow P^2$.

Some properties of C_i and coverings $D_i \to T_i$ of purely technical nature are omitted, their complete list is given in [50], 2.2.1.

These conditions and ([50], 2.2.7) imply that

$$\forall i \quad h^{1,0}(S(\tilde{p}_i)) \leq h^{1,0}(S(\tilde{p}_{i+1})) + 1.$$

If points $P_1, \dots, P_6$ do not belong to one conic then the linear system of a sheaf $O_{P^2}(1)|_{\tilde{C}_6}$ is non-special. ([50], 1.3.8) implies that $h^{1,0}(S(\tilde{p}_6)) = g(\tilde{C}_6) - 1 = 3$ and hence $h^{1,0}(S(p_{H_6})) \leq 9$. $\square$

Proposition 5.15. There exists an isomorphism $I :$ Pr $\hat{H}_6/H_6 \to J^3(\tilde{X})$.

Proof. Let $\hat{h} \in \hat{H}_6$ and $p(\hat{h}) = h$. Let us consider a cone $Q(h)$. The set of spaces P^3 of type $\hat{h}$ on $Q(h)$ is isomorphic to P^2, so for all these P^3 curves $P^3 \cap X$ are birationally equivalent. We denote such a curve by $C_4(\hat{h})$, they can be lifted on $\tilde{X}$ and these lifts define a cylinder map $\Phi : H_1(\hat{H}_6, \mathbb{Z}) \to H_3(\tilde{X}, \mathbb{Z})$ and an Abel - Jacobi map $\hat{I} : J(\hat{H}_6) \to J^3(\tilde{X})$.

Let us choose a generic straight line $l \subset X$; $\quad \forall \hat{h} \in \hat{H}_6$ there exists the only P^3 of type $\hat{h}$ such that $P^3 \supset l$. We have

$$P^3 \cap X = C_4(\hat{h}) = l \cup C_3(\hat{h}, l)$$

where $C_3(\hat{h}, l)$ is a normcubic in P^3 meeting l at 2 points. A system of curves C_3 defines the same cylinder map $\Phi : H_1(\hat{H}_6, \mathbb{Z}) \to H_3(\tilde{X}, \mathbb{Z})$. Let us apply a formula ([20], p. 44) to Φ:

$$\forall \gamma_1, \gamma_2 \in H_1(\hat{H}_6, \mathbb{Z}) \quad < \Phi(\gamma_1), \Phi(\gamma_2) >= k < \gamma_1, \gamma_2 > + < i(\gamma_1), \gamma_2 >$$

where $k \in \mathbb{Z}$ and $i \in \text{End } H_1(\hat{H}_6, \mathbb{Z})$ corresponds to a correspondence C on $\hat{H}_6 \times \hat{H}_6$ defined as follows: $(\hat{h}_1, \hat{h}_2) \in C \iff C_3(\hat{h}_1, l) \cap C_3(\hat{h}_2, l) \neq \emptyset$. Further, [20], p. 96 implies that $i = H_1(i_H)$.

From another side, we have $g(H_6) = 10$, $h^{2,1}(\tilde{X}) = 9$ (because $h^{2,1}(X) = 10$ and because $\tilde{X}$ is a blowing up of a degenerated fibre of a Lefschetz pencil). This implies that dim Ker $\Phi \geq h_1(\hat{H}_6) - h_3(\tilde{X}) = 20$, i.e. for a 20-dimensional space of cycles γ we have $i(\gamma) + k\gamma = 0$. This can be only for $k = -1$, hence Ker $\Phi = J(H_6)$ and Pr $\hat{H}_6/H_6 \to J^3(\tilde{X})$ is an isomorphism. $\square$

Proposition 5.16. A diagram

$$\begin{array}{ccc} \text{Alb } (\tilde{F}) & \overset{\text{Alb } (\tilde{\phi}_6)}{\rightarrow} & \text{Alb } (S) \\ \downarrow & & \downarrow \\ J^3(\tilde{X}) & \overset{I}{\leftarrow} & \text{Pr } \hat{H}_6/H_6 \end{array}$$

where the left vertical map is 2Φ, is commutative.

Proof. Let $c \in \tilde{F}$, $\quad l(c) \cap t_P = h_0$, $\quad l(c) \cap H_6 = \{h_1, \ldots, h_6\}$ and $\gamma(c_t, h_i) = \hat{h}_i$. Then $\phi_6(c) = \sum_{i=1}^{6} \hat{h}_i$ and it is sufficient to prove that codimension 2 cycles $2c - \sum_{i=1}^{6} C_4(\hat{h}_i)$ on $\tilde{X}$ are linearly equivalent for all $c \in \tilde{F}$. Let $X_4 = \cap_{h \in l(c)} Q(h)$ be an intersection of 2 quadrics and P_i^3 the only P^3 of type $\hat{h}_i$ containing $\pi(c_t)$. Then $X_4 \cap P_i^3$ is a quadric containing $\pi(c_t)$, i.e. $X_4 \cap P_i^3$ is a pair of planes $\pi(c_t) \cup L_i$. According ([47], Section 3, Lemma 6) there is a linear equivalence ($\equiv$) on X_4 of 2 cycles:

$$\sum_{i=0}^{6} L_i + 5\pi(c_t) \equiv 3(P^4 \cap X_4).$$

Intersecting this equivalence with X we get:

$$\sum_{i=0}^{6} L_i \cap X + 6c \equiv 2c - (c + L_0 \cap X) + \text{const} \; .$$

But $c + L_0 \cap X = const$, because $c \cup (L_0 \cap X) = P_0^3 \cap X$ and P_0^3 varies along $t_P = P^1$. Equality $C_4(\hat{h}_i) = (L_i \cap X) \cup c)$ implies that $\sum_{i=1}^{6} C_4(\hat{h}_i) - 2c \equiv const.$ $\square$

6. Abel - Jacobi map of $F(X)$ is an isogeny. Here X will mean again a Fano threefold of genus 6. Let us prove that $h^{1,0}(F(X)) = \dim J^3(X) = 10$. We consider $X_4' = G \cap H_1 \cap \Omega$ and a Lefschetz pencil $f' : X_4' \to P^1$, we restrict it to a neighbourhood of zero $\Delta \subset P^1$, so the only singular fibre of ${f'}^{-1}(t)$, $t \in \Delta$, is a threefold $X_0 = {f'}^{-1}(0)$ having the only double point. We denote $F_3' = \cup_{t\in\Delta} F({f'}^{-1}(t))$, and $f_F' : F_3' \to \Delta$ is the projection.

LEMMA 6.1. For a generic $f' : X_4' \to P^1$ F_3' is non-singular.

Proof. Non-singularity of F_3' is not obvious only at points of $K = F(X_0) \subset F_3'$. For a given X_0 it is possible to choose $X_4' \subset P^8 = P(H_1)$ such that $X_0 = X_4' \cap P^7$, where $P^7 \subset TP_{X_4'}(t)$ and such that all points of K are non-singular on $F(X_4')$ — the variety of conics on X_4'. A generic $P^6 \subset P^7$ defines a Lefschetz pencil $f' : X_4' \to P^1$; we denote by $U(P^6)$ a set of conics $c \subset P^8$ such that $\dim (\pi(c) \cap P^6) \geq 1$. We have $F_3' = U(P^6) \cap F(X_4')$. Let $c \in K$. Let us look which conditions must be imposed on $P^6 \subset P^7$ in order to achieve non-singularity of $F_3' = F_3'(P^6)$ at points of c. For any variety $Y \ni c$ we denote $TV_Y(c)$ by $\bar{Y}$, and we denote by U_7 (resp. U_8) varieties of conics in P^7 (resp. P^8). We get a diagram of tangent spaces at c:

$$\begin{array}{ccccc} \overline{F(X_0)} & & \hookrightarrow & & \overline{F(X_4')} \\ \downarrow & & & & \downarrow \\ \bar{U}_7 & \hookrightarrow & \overline{U(P^6)} & \hookrightarrow & \bar{U}_8 \end{array}$$

(vertical maps are inclusions). Here we have: $\overline{F(X_0)} = \overline{F(X_4')} \cap \bar{U}_7$, $\operatorname{codim}_{\bar{U}_8} \bar{U}_7 = 3$, $\dim \overline{F(X_4')} = 5$, $\dim \overline{F(X_0)} = 3$, because c is a smooth point on $F(X_0)$. Condition of smoothness of $F_3'(P^6)$ at c is equivalent to a condition that $\dim \overline{U(P^6)} \cap \overline{F(X_4')} = 3$, i.e. that $\overline{U(P^6)} \not\subset < \overline{F(X_4')}, \bar{U}_7 >$. But $\overline{U(P^6)} \in P(\bar{U}_8/\bar{U}_7) = P^2$, and a map $P^6 \mapsto P(\overline{U(P^6)}/\bar{U}_7) \in P^2$ is a linear projection from P^{7*} to P^2 whose center $P^4 \subset P^{7*}$ is dual to $\pi(c) \subset P^7$. The inverse image of $< \overline{F(X_4')}, \bar{U}_7 >$ under this projection is a hyperplane in P^{7*} containing P^4, i.e. the set of $P^6 \subset P^7$ which contain a fixed point in $\pi(c)$. It is easy to see that this point is t. So, for a non-singularity of F_3' it is sufficient to choose a generic P^6 which does not contain t. □

So, it is clear that for any X we can choose $f' : X_4' \to P^1$ such that:

1. $X = {f'}^{-1}(t_0')$ for $t_0' \in \Delta$;
2. The special fibre satisfies properties of general position of Section 5 (smoothness of H_6, K_0);
3. F_3' is non-singular.

6.2. To transform f', f_F' to a desired degeneration we use a standard method (see, for example, [30]). Let us consider blowings up of X_4' and of F_3' along singular points of the fibre over 0, we denote the corresponding maps by $\tilde{f}' : \tilde{X}_4' \to \Delta$, $\tilde{f}_F' : \tilde{F}_3' \to \Delta$. Their fibres at 0 are $\tilde{X}_0 \cup P^3$, $\widetilde{F(X_0)} \cup R$ respectively, where P^3, R are exceptional divisors of blowing up of X_4' and F_3', and R is a fibration over $K = K(F(X_0))$ whose fibres are isomorphic to P^1. It is clear that $\tilde{X}_0 \cap P^3 = Q$ is a non-singular quadric in P^3 and $\widetilde{F(X_0)} \cap R = K_1 \cup K_2$. Components of the fibre at 0 of P^3 and R are double, that's why we consider a map $\phi_2 : \Delta \to \Delta$: $t \mapsto t^2$ and a diagram which is a fibred product:

$$\begin{array}{ccc} X_4 & \to & \tilde{X}'_4 \\ \downarrow & & \downarrow \\ \Delta & \stackrel{\phi_2}{\to} & \Delta \end{array}$$

where the right vertical map is $\tilde{f}'$. We denote the left vertical map by f. The fibre of f at $t_0 = (t'_0)^{\frac{1}{2}}$ (resp. at $t = 0$) is X (resp. $X(0) = \tilde{X}_0 \cup \bar{P}_3$), where $\bar{P}_3$ is a two-sheeted covering of P^3 ramified at Q, so $\tilde{X}_0 \cap \bar{P}_3 = Q$, and both components have multiplicity 1. Analogously we define a map $f_F : F_3 \to \Delta$ having a singular fibre $F(0) = F \cup \bar{R}$, where $F = \widetilde{F(X_0)}$, and $\bar{R} \to R$ is a two-sheeted covering ramified at $K_1 \cup K_2$, so $F \cap \bar{R} = K_1 \cup K_2$, and composition maps $K_i \hookrightarrow \bar{R} \to R \to K$ coincide with d_i $(i = 1, 2)$.

Clemens - Schmidt exact sequence ([8], [40]) for f_F is the following:

$$0 \to H^1(F(0)) \stackrel{\nu}{\to} H^1(F(X)) \stackrel{N}{\to} H^1(F(X)). \tag{6.3}$$

There exists a mixed Hodge structure on $H^1(F(0))$, $H^1(F(X))$, and ν and N are strict morphisms with respect to these structures of degrees (0,0) and (-1, -1) respectively. We shall need only weight filtrations; their definition in our case is the following ([8]): for $H^1(F(X))$ we have

$$0 \subset W_0 \subset W_1 \subset W_2 = H^1(F(X))$$

where $W_0 = \text{im } N$, $W_1 = \text{Ker } N$. For $H^0(F(0))$ we have

$$0 \subset W_0 \subset W_1 = H^1(F(0))$$

where $W_0 = H^1(\Pi(F(0)))$ and $\Pi(F(0))$ is a CW-complex whose vertices correspond to components of $F(0)$ and edges correspond to irreducible components of intersections of these components. We have

$$W_1/W_0 = \text{Ker } ((H^1(F) \oplus H^1(\bar{R})) \to H^1(F \cap \bar{R}))$$

for a natural restriction map. In our case $\Pi(F(0)) = S^1$ (2 vertices, 2 edges), i.e. $\dim W_0(H^1(F(0))) = 1$. Exactness of (6.3) implies that $\dim W_2/W_1 = \dim W_0(H^1(F(X))) = \dim W_0(H^1(F(0))) = 1$ and W_1/W_0 for $H^1(F(X))$ and $H^1(F(0))$ coincide. So, an equality $h^1(F(X)) = 20$ is a corollary of

$$\dim \text{Ker } ((H^1(F) \oplus H^1(\bar{R})) \to H^1(F \cap \bar{R})) = 18. \tag{6.4}$$

So, we have to prove (6.4).

Lemma 6.5. Let us identify K_1 and K_2 via α. Then (6.4) follows from the following statement: the image of a restriction map

$$H^1(F) \to H^1(K_1) \oplus H^1(K_2) \tag{6.6}$$

is contained in the diagonal.

Proof. Since $\bar{R} \to K$ is a fibration with a fibre P^1, we have: $H^1(K) \to H^1(\bar{R})$ is an isomorphism. This means that the restriction map $H^1(\bar{R}) \to H^1(K_1) \oplus H^1(K_2)$ is an inclusion, and its image is the diagonal. According (5.14), $h^1(F) = 18$, this implies the lemma. ∎

LEMMA 6.7. (6.6) is equivalent to the following condition

$$\forall h \in H^1(F) \quad h|_K = i_K^*((i_F^*(h))|_K) \tag{6.8}$$

(here and below $K = K_1$).

Proof. Follows from commutativity of the diagram

$$\begin{array}{ccccc} K_1 & \overset{i_K}{\to} & K_1 & \overset{\alpha}{\to} & K_2 \\ \downarrow & & & & \downarrow \\ F & & \overset{i_F}{\to} & & F \end{array}$$

(vertical maps are natural inclusions). ∎

There exists a composite map defined up to a shift

$$f : F \overset{\tilde{\phi}_6}{\to} S \to \mathrm{Pr}\ \hat{H}^6/H^6.$$

LEMMA 6.9. $f \circ i_F = -f$ (up to a shift).

Proof. For $c \in F$ let $f(c) = \sum_{i=1}^6 \hat{h}_i$ for $\hat{h}^i \in \hat{H}_6$; then $f(i_F(c)) = \sum_{i=1}^6 i_H(\hat{h}_i)$. But $O_{\hat{H}_6}(\sum_{i=1}^6(\hat{h}_i + i_H(\hat{h}_i))) = O_{P^2}(1)|_{\hat{H}_6}$, i.e. $\sum \hat{h}_i \equiv -\sum i_H(\hat{h}_i) + \text{const}$. ∎

This means that $H^1(i_F)$ is a multiplication by -1. This is equvalent to the following: $H^1(F_0) = 0$, where $F_0 = \widetilde{F_0(X_0)}$.

Let $E_i \hookrightarrow H^1(K)$ $(i = \pm 1)$ be eigenspaces of $H^1(i_K)$ with eigenvalues i, then $E_1 = H^1(K)$, $E_{-1} = H^1(\mathrm{Pr}\ K/K_0)$, and (6.8) is equivalent to the following:

The image of the restriction map $H^1(F) \to H^1(K)$ is contained in E_{-1}, or, which is the same, the composition map Alb $K_0 \to$ Alb $K \to$ Alb F is 0.

Taking into consideration (5.15) and (5.16), it is sufficient to prove that the map Alb $K_0 \to J^3(\tilde{X}_0)$ is 0. Let $k_{10}, k_{20} \in K_0$ and k_1, k_2 their representatives in K. Let us find a cycle D on $\tilde{X}_0$ which is the image of $k_{10} - k_{20}$:

$$D = \tilde{\chi}(k_1) + \tilde{\chi}(i_K(k_1)) - \tilde{\chi}(k_2) - \tilde{\chi}(i_K(k_2)).$$

We have $\forall f \in F \quad \Phi(\tilde{\chi}(f)) + \Phi(\tilde{\chi}(i_F(f))) = \text{const}$ and $i_F(i_K(k_1)) = \alpha(k_1)$, hence

$$D = \tilde{\chi}(k_1) - \tilde{\chi}(\alpha(k_1)) - \tilde{\chi}(k_2) + \tilde{\chi}(\alpha(k_2)) = U(k_1) - U(\alpha(k_1)) + U(\alpha(k_2)) - U(k_2)$$

i.e. $D \equiv 0$. So, we have proved that $h^{1,0}(F(X)) = 10$.

REMARK. The above considerations imply that there exists a map Pr $K/K_0 \to J^3(\tilde{X}_0) \to$ Pr $\hat{H}_6/H_6$. For a generic covering p_{H_6} existence of this map implies that $p_K = p_{H_6}$ which implies in its turn that $p_K(X_0) = p_{H_6}(X_0)$ is true for any X_0.

Calculation of $h^{1,0}(F_0(X))$, as well as a proof of connectedness of $F(X)$ and of $F_0(X)$ is made by the same method, but easier. Analogously to a fibration $f_F : F_3 \to \Delta$ we construct a fibration $f_{0F} : F_{03} \to \Delta$ having a generic fibre $f_{0F}^{-1}(t_0) = F_0(X)$ and a special fibre $f_{0F}^{-1}(0) = F_0(0) = F_0 \cup \bar{R}_0$, where $F_0 = \widetilde{F_0(X_0)}$ and $\bar{R}_0$ is a two-sheeted covering of R_0 (R_0 is a fibration over K_0 whose fibre is P^1) ramified at $K \subset R_0$, and $F_0 \cap \bar{R}_0 = K$. There exists a weight filtration on $H^1(F_0(0))$:

$$0 \subset W_0 \subset W_1 = H^1(F_0(0))$$

where $W_0 = H^1(\Pi(F_0(0)))$, $W_1/W_0 = \text{Ker }((H^1(F_0) \oplus H^1(R_0)) \to H^1(K))$. Since $\Pi(F_0(0))$ is a segment, then $W_0 = 0$. As we have seen (Lemma 6.9), $H^1(F_0) = 0$. Since the cohomology map $H^1(K_0) \to H^1(\bar{R}_0)$ of the fibration $\bar{R}_0 \to K_0$ is an isomorphism, we have:

$$\text{Ker }(H^1(\bar{R}_0) \to H^1(K)) = \text{Ker }(H^1(K_0) \to H^1(K)) = 0.$$

So, $H^1(F_0(0)) = 0$. It follows immediately from the Clemens - Schmidt exact sequence of the fibration f_{0F} that $H^1(F_0(X)) = 0$ as well.

To calculate $H^0(F(X))$ we consider the even Clemens - Schmidt exact sequence of the fibration f_F:

$$0 \to H^0(F(0)) \to H^0(F(X)) \xrightarrow{N} H^0(F(X)).$$

Connectedness of F implies that $h^0(F(0)) = 1$ and N is 0. This implies connectedness of $F(X)$ and hence of $F_0(X)$ as well.

Proposition 6.10. The Abel - Jacobi map $\Phi : \text{Alb } F \to J^3(X)$ is an epimorphism (here $F = F(X)$).

Proof. Let F_1 be the family of straight lines on X, $D \subset F_1 \times F_1$ the incidence divisor, i.e. $(t_1, t_2) \in D \iff t_1 \neq t_2$ and $t_1 \cap t_2 \neq \emptyset$, $p_j : D \to F_1$ a projection to the j-th component ($j = 1, 2$), $N = \deg p_j$, i.e. the number of straight lines meeting a given straight line, $i : D \to D$ the natural involution (permutation of t_1, t_2) and $p_D : D \to D_0$ a factorization of D by i. Then $D_0 \overset{d}{\hookrightarrow} F$ is a set of reducible conics. Let us consider cylinder maps in complex homology

(a) for conics: $\Phi_F : H_1(F, \mathbb{C}) \to H_3(X, \mathbb{C})$;
(b) for straight lines: $\Phi_{F_1} : H_1(F_1, \mathbb{C}) \to H_3(X, \mathbb{C})$.

It is clear that for $\gamma \in H^1(D)$ we have

$$\Phi_F \circ d_* \circ p_{D*}(\gamma) = \Phi_{F_1}(p_{1*}(\gamma)) + \Phi_{F_1}(p_{2*}(\gamma)) \overset{\text{def}}{=} B(\gamma).$$

If $\gamma = p_1^*(\gamma')$, where $\gamma' = H_1(F_1)$, then

$$B(\gamma) = \Phi_{F_1}(p_{1*}(p_1^*(\gamma'))) + \Phi_{F_1}(p_{2*}(p_1^*(\gamma'))) = N \cdot \Phi_{F_1}(\gamma') + \Phi_{F_1}(p_{1*}(i^*(p_1^*(\gamma')))).$$

According ([20], Section 4.3, Lemma 6) we have

$$\Phi_{F_1}(p_{1*}(i^*(p_1^*(\gamma')))) = (1-n)\Phi_{F_1}(\gamma')$$

where n is a number of straight lines passing through a generic point X_4', so $B(\gamma) = (N+1-n)\Phi_{F_1}(\gamma')$. For X we have $N = 11$, $n = 6$. According to a lemma of Clemens ([20], Section 4.3) Φ_{F_1} is an epimorphism, hence $\gamma' \mapsto B(\gamma) = \Phi_F \circ d_* \circ p_D(\gamma)$ is an epimorhism as well. This implies that Φ_F is also an epimorphism, which is equivalent to (6.10). □

Since Alb F and $J^3(X)$ have the same dimension, we get

Theorem 6.11. Abel - Jacobi map $\Phi : \text{Alb } F \to J^3(X)$ is an isogeny. □

Now we give a sketch of a proof of the following theorem:

Theorem 6.12. The class of $\Phi(F)$ in $A(J^3(X))$ is $\frac{2\Theta^8}{8!}$ where Θ is the class of a Poincaré divisor.

Sketch of proof. For a degeneration $f : X_4 \to \Delta$ (see 6.2) we denote $f^{-1}(t) = X_t$ and $F(X_t) = F_t$ for $t \neq 0$. There exists a complex analytic fibration $j : J_{11} \to \Delta$ whose fibre at $t \in \Delta^* = \Delta - \{0\}$ is $j^{-1}(t) = J^3(X_t)$ ([52], Section 2). Its fibre $j^{-1}(0) = J(0)$ at 0 is a fibration over $J^3(\tilde{X}_0)$ whose fibre is $\mathbb{C}^*$ ([52], 4.56). There exists a compactification of $J(0)$. It is a fibration $\mu : \tilde{J}_0 \to J^3(\tilde{X}_0)$ whose fibre is P^1. It has 2 sections α_0, $\alpha_\infty : J^3(\tilde{X}_0) \to \tilde{J}_0$ which map any point $t \in J^3(\tilde{X}_0)$ to points (0), $(\infty) \in P^1 = \mu^{-1}(t)$ such that $J(0) = \tilde{J}_0 - \text{im}\ \alpha_0 - \text{im}\ \alpha_\infty$. There exists also a compactification of J_{11} which is a fibration $j_c : J_{11,c} \to \Delta$ whose non-singular fibre is the same as the one of j and whose singular fibre is $j_c^{-1}(0) = J_0$. Further, there is a map $\nu : \tilde{J}_0 \to J_0$ which is a normalization of J_0. ν identifies (with a shift) im α_0 and im α_∞, outside these subsets ν is an inclusion.

Let us consider now the Abel - Jacobi map. We fix a section $s_0 : \Delta \to F_3$ such that $s_0(0) \in F(X_0)-K$, and for any section $s : \Delta \to F_3$ such that $s(0) \in F(X_0)-K$ we consider an element $\Phi_s(t) = \Phi(s(t) - s_0(t)) \in J^3(X_t)$. They form a section $\Delta^* \to J_{11}$ which can be expanded to a section $\Phi_s : \Delta \to J_{11}$. We have: $\Phi_s(0)$ depends only on $s(0)$ but not on s and $\mu(\Phi_s(0)) = \Phi_{\tilde{X}_0}(s(0)-s_0(0))$ ([52], 4.58) where $\Phi_{\tilde{X}_0}$ is the Abel - Jacobi map of $\tilde{X}_0$. We denote by $\cup_s \Phi_s(t) = F_\Phi(t) \subset j^{-1}(t)$ the image of F_t under the Abel - Jacobi map shifted by such a way that $s_0(t)$ goes to 0. For $t = 0$ we have an inclusion $F(X_0) - K \hookrightarrow J(0)$. There exists a commutative diagram where the above inclusion is the left vertical map:

$$\begin{array}{ccccccc} F(X_0) - K & \hookrightarrow & \widetilde{F(X_0)} & \twoheadrightarrow & F(X_0) & = & F_\Phi(0) \\ \downarrow & & \downarrow & & \downarrow & & \\ J(0) & \hookrightarrow & \tilde{J}_0 & \overset{\nu}{\twoheadrightarrow} & J_0 & & \end{array}$$

(vertical maps are inclusions) and we get a family of surfaces $(F_\Phi \to \Delta) \subset (j_c : J_{11,c} \to \Delta)$ whose fibre at $t \in \Delta$ is $F_\Phi(t) = F(X_t)$. Since

$$\mu(\widetilde{F(X_0)}) = \Phi_{\tilde{X}_0}(\widetilde{F(X_0)}) \subset J^3(\tilde{X}_0), \quad J^3(\tilde{X}_0) = \text{Pr}\ (\hat{H}_6/H_6) \text{ and } \widetilde{F(X_0)} = S(p_{H_6})$$

then according ([28]) the homology class of $\Phi_{\tilde{X}_0}(\widetilde{F(X_0)})$ in $H_4(J^3(\tilde{X}_0), \mathbb{Z})$ is

$$\frac{2\Theta^7}{7!} = 2\sum_{i<j} \gamma_i \times \delta_i \times \gamma_j \times \delta_j \tag{6.13}$$

where Θ is the class of a Poincaré divisor and (γ_i, δ_i) is a simplectic basis of $H_1(J^3(\tilde{X}_0), \mathbb{Z})$ with respect to the principal polarization associated with $\Theta \subset J^3(\tilde{X}_0)$.

There exists a map of topological spaces $(0,0) : J^3(X_t) \to J_0$ ([25]) such that $(0,0)(F_\Phi(t))$ is homologically equivalent to $F_\Phi(0) \hookrightarrow J_0$. By analogy with proofs of ([25], Lemma 2.6) and ([31], Theorem 8.27) it is possible to show — using (6.13) — that the class of $F_\Phi(t) \subset J^3(X_t)$ in $H_4(J^3(X_t), \mathbb{Z})$ is

$$2\sum_{i<j} \gamma_i \times \delta_i \times \gamma_j \times \delta_j = \frac{2\Theta^8}{8!}. \qquad \square$$

7. Geometric interpretation of the tangent bundle theorem and recovering of X by its Fano surface. We denote $V_{10} = H^0(F, \Omega_F)^*$. Here we impose the following condition on F:

7.1. The natural map $V_{10}^* = H^0(F_m, \Omega_{F_m}) \to \Omega_{F_m}(c_\Omega)$ is an epimorphism.

Apparently this condition is true for any X or at least for a generic X. Theorem about recovering of X by its Fano surface $F(X)$ will be proved only for X that satisfy (7.1).

Let us define a map $B_8 : F \to G(2, V_8)$ as follows: $B_8(f) =< \gamma_1(f), \gamma_2(f) >$. We denote by $B_{10} : F_m \to G(2, V_{10})$ a map associated to Ω_{F_m}. (7.1) means that B_{10} is regular at c_Ω. It is clear that B_8 factors through $p_F : F \to F_0$. Since $H^0(F_0, \Omega_{F_0}) = 0$, we have: B_{10} factors through $p_{F_m} : F_m \to F_{0m}$, and the corresponding map $F_{0m} \to G(2, V_{10})$ is associated to a sheaf $\Omega_{F_{0m}} \otimes \sigma_m$.

Theorem 7.2. (Geometric interpretation of the tangent bundle theorem). A map $B_{10} \circ r : F \to G(2, V_{10})$ is regular. There exists the only epimorphism $p_{10,8} : V_{10} \to V_8$ such that

$$B_8 = G(2, p_{10,8}) \circ B_{10} \circ r. \tag{7.3}$$

If (7.1) is satisfied then $p_{10,8}$ is a projection from the straight line $B_{10}(c_\Omega)$.

Proof. We denote $H = H^0(F, \tau_{2,M}^*)^*$ and let $B_H : F \to G(2, H)$ be the map difined by $\tau_{2,M}^*$. Let us consider the long exact cohomology sequence for (4.51):

$$0 \to H^* \xrightarrow{i_1} V_{10}^* \to H^0(2O_{l_1 \cup l_2}).$$

It is clear that (7.1) is equivalent to a condition $H = V_8$. Since the linear envelope of straight lines $< \gamma_1(f), \gamma_2(f) >= l_8(f) \subset P(V_8)$ is the whole $P(V_8)$, there exists an inclusion $i_2 : V_8^* = H^0(G(2, V_8), \tau_{2,V_8}^*) \hookrightarrow H^* = H^0(F, \tau_{2,M}^*)$, and we have

$$B_8 = G(2, i_2^*) \circ B_H. \tag{7.4}$$

This implies that $\forall f \in F \quad B_H$ is regular at f. For $f \in F - l$ the restriction of (4.51) to f is

$$0 \to \tau_{2,M}^*|_f \to r^*\Omega_{F_m}|_f \to 0$$

and there exists a commutative diagram

$$\begin{array}{ccccccc} & & H^* & \overset{i_1}{\hookrightarrow} & V_{10}^* & & \\ & & \downarrow & & \downarrow & & \\ H^0(\tau_{2,M}^*|_f) & = & \tau_{2,M}^*(f) & \to & H^0(r^*\Omega_{F_m}|_f) & = & r^*\Omega_{F_m}(f) \end{array}. \tag{7.5}$$

Since $H^* \to \tau_{2,M}^*(f)$ is an epimorphism, we have that $V_{10}^* \to r^*\Omega_{F_m}(f)$ is also an epimorphism, i.e. B_{10} is regular at $F - l$. If so, there exists a continuation of B_{10} to l, i.e. B_{10} is regular on F. Commutativity of the diagram dual to (7.5) shows that

$$B_H(f) = G(2, i_1^*) \circ B_{10}(f). \tag{7.6}$$

We denote now $p_{10,8} = (i_1 \circ i_2)^*$. (7.4) and (7.6) imply that on $F - l$ we have $B_8 = G(2, p_{10,8}) \circ B_{10} \circ r$, hence this is true on the whole F.

If (7.1) is true then for $f \in l$ we have $B_{10} \circ r(f) = B_{10}(c_\Omega)$ which does not depend on f. But $B_8(f)$ for $f \in l$ are different straight lines having no commun points. This observation shows that (7.3) can be satisfied only if $p_{10,8}$ is the projection from the straight line $B_{10}(c_\Omega)$. $\square$

Theorem 7.7. Here we suppose that (7.1) is satisfied for F_c. Then X satisfying the property $F_c = F_c(X)$ is defined uniquely up to isomorphism.

Proof. Let us consider the map $F_c \to G(2, V_{10})$ associated to Ω_{F_c}. It factors through F_{0m} and hence it permits us to recover uniquely the point $c_\Omega \in F_c$, the straight line $l_2 \subset F_c$, the involution i_F on F and the straight line $B_{10}(c_\Omega) \subset P(V_{10})$. Since (7.1) implies that $p_{10,8}$ is the projection from $B_{10}(c_\Omega)$, we can recover as well the map $B_8 : F_0 \to G(2, V_8)$. Since the straight line $l \subset F_0$ can be recovered by F_c and the straight lines $l_0(c)$ for $c \in l$ are contained in U^* and are tangent to the conic c_u^*, we have that U^*, c_u^* and the isomorphism $l \to c_u^*$ are recovered by F_c.

PROPOSITION 7.8. $c_\Omega \subset U^*$ can be recovered uniquely by F_c.

Proof. We prove

(a) If $c \in F_0 - l$ and $l_8(c) \cap U^* \neq \emptyset$ then $l_8(c) \cap U^* \in c_\Omega$.

(b) The quantity of these points $c \in F_0 - l$ is finite and > 4.

So, having F_c it is possible to construct more than 4 points on the conic line c_Ω, i.e. to recover c_Ω itself.

(a) can be proved easily. For $c \in F_0 - l$ we have: $X \cap l_8(c) = G_4 \cap l_8(c)$. Since $U^* \subset G_4$, we have $l_8(c) \cap U^* \in G_4$, and hence $l_8(c) \cap U^* \in X$. But $c_\Omega = X \cap U^*$.

Let us define now a curve S as a set of pairs (t, c) where $t \in c_\Omega$, $c \in c_u \cap t_V$. We define a straight line $j(t, c)$ as a line lying in α-plane $\alpha(c)$ (see Section 3) and which is the tangent of $\alpha(c) \cap \Omega$ at t. Lemma 3.20 (whose notations are used here) implies that either $j(t, c) \subset X$ or $\{$if for $f_0 \in F_0 - l$ holds $\tilde{\phi}(f_0) \in \pi(\widetilde{j(t, c)})\}$ $t \in l_8(f_0)$. We define a surface

$$R = \bigcup_{(t,c) \in S} \pi(\widetilde{j(t, c)}) \subset P(V^*).$$

It is clear that $R \supset l$. Let $V_4 \in (R - l) \cap (\phi(F_0 - l))$, $V_4 = \phi(f_0)$, $V_4 \in \pi(\widetilde{j(t, c)})$. Then either

(i) $l_8(f_0) \cap c_\Omega = \{t\}$ or

(ii) There exists a straight line on X passing through t.

Let us find for how many $V_4 \in (R - l) \cap (\phi(F_0 - l))$ we have the case (ii). Since the union of all straight lines on X is an intersection of X and a hypersurface of degree 10, there are 20 points on c_Ω which belong to a straight line on X. Let t and $j(t)$ are one of these points and lines. We define $c = \cap_{t' \in j(t)} t'_V$; it is clear that $c \in c_u \cap t_V$ and $j(t) = j(t, c)$. Since $j(t)$ meets 11 straight lines on X, we have: a straight line $\pi(\widetilde{j(t, c)}) \subset R$ meets $\phi(F_0)$ at 11 points. But one of them belongs to l. We get that for $20 \cdot (11 - 1) = 200$ intersection points of $R - l$ and $\phi(F_0 - l)$ there exist corresponding lines on X.

Let us find now $\#(R - l) \cap (\phi(F_0 - l))$. Let $r : \widetilde{P(V^*)} \to P(V^*)$ be the blowing up along l, $\tilde{R}$ and $\tilde{F}_0$ the proper inverse images of R and $\phi(F_0)$ respectively. Then $\#(R - l) \cap (\phi(F_0 - l)) = < \tilde{R}, \tilde{F}_0 >_{\widetilde{P(V^*)}}$.

LEMMA 7.9. $< \tilde{R}, \tilde{F}_0 >_{\widetilde{P(V^*)}} = 36d + 8$ where $d = \deg R$.

Proof. $\widetilde{P(V^*)}$ is a variety of pairs (V_2, V_4) where $V_2 \subset V_4 \subset V$ and $V_2 \subset U_3$; $r(V_2, V_4) = V_4$. There exists an isomorphism

$$\widetilde{P(V^*)} = P_{U^*}(O \oplus O \oplus O(1)) \xrightarrow{\mu} U^*$$

and $\mu(V_2,V_4)=V_2\in U^*$. The ring $A(\widetilde{P(V^*)})$ has generators $D=\mu^*(c_1(O_{U^*}(1)))$, $\tilde H=c_1(O_\mu(1))$ and relations $D^3=0$, $\tilde H^3=D\tilde H^2$. Further, we have $r^{-1}(l)=\tilde H-D$, $r^{-1}(H)=\tilde H$, $r_*(\tilde H^2)=r_*(\tilde HD)=r_*(D^2)=H^2$ where $H=c_1(O(1))$. Let us find the class of R in $A(\widetilde{P(V^*)})$. By definition,

$$\{(V_2,V_4)\in\tilde R\iff P(V_2)=t_V \text{ for some } t\in c_\Omega \text{ and } P(V_4)\supset\pi(\widetilde{j(t,c)})\}$$

where $c\in t_V\cap c_u$. So, $\mu(\tilde R)=c_\Omega$, and for $t\in c_\Omega$ $\quad\mu^{-1}(t)\cap\tilde R$ is a pair of straight lines $\pi(\widetilde{j(t,c_i)})$ $(i=1,2)$, where $\{c_1,c_2\}=t_V\cap c_u$. This means that $\tilde R$ is a divisor in $\mu^{-1}(c_\Omega)$ and $O_{\mu^{-1}(c_\Omega)}(\tilde R)=L\otimes O_\mu(2)$ where L is an invertible sheaf on c_Ω. Since $\mu^{-1}(c_\Omega)$ is a divisor on $\widetilde{P(V^*)}$ of degree $2D$, we have: cl $(\tilde R)=4D\tilde H+nD^2$ where n is defined by the condition $r_*(\tilde R)=dH^2=(4+n)H^2$, i.e. $n=d-4$.

Let us find now the class of $\tilde F_0$ in $A(\widetilde{P(V^*)})$. Let us assume that there exists a locally free sheaf E of rank 2 on $P(V^*)$ such that $\phi(F_0)$ is the set of zeros of a section of E. Then $\tilde F_0$ is the set of zeros of a section of $r^*(E)\otimes O_r(1)$ on $\widetilde{P(V^*)}$ and

$$\text{cl }(\tilde F_0)=c_2(r^*(E)\otimes O_r(1))=r^*c_2(E)+r^*c_1(E)\cdot c_1(O_r(1))+c_1(O_r(1))^2.$$

We have $c_2(E)=\deg\phi(F_0)=39H^2$, $c_1(O_r(1))=D-\tilde H$, and $c_1(E)$ can be easily found by consideration of the exact sequence of normal sheaves for inclusions $l\hookrightarrow\phi(F_0)\hookrightarrow P(V^*)$:

$$0\to O_l(-1)\to 3O_l(1)\to E|_l\to 0.$$

Namely, det $E|_l=O_l(4)$, i.e. $c_1(E)=4H$ and

$$\text{cl }\tilde F_0=39\tilde H^2+4\tilde H(D-\tilde H)+(D-\tilde H)^2=36\tilde H^2+2\tilde HD+D^2.$$

Really, E does not exist, but it is clear that the above formalism is correct. So, $<\tilde R,\tilde F_0>_{\widetilde{P(V^*)}}=(4\tilde HD+(d-4)D^2)(36\tilde H^2+2\tilde HD+D^2)=36d+8.$ $\square$

This means that there are $36d+8-200$ straight lines of type $l_8(f_0)$ which meet c_Ω. Since this number is ≥ 0, we get that $d\geq 6$ and this number is ≥ 24. So, we can recover c_Ω uniquely by F_c. $\square$

Let us consider now the birational isomorphism $b:G_4\to P^4$ (see (3.4)) which is the restriction to G_4 of the projection $p_{85}:P(V_8)\to P^4=P(V_8/U_3^*)$.

Proposition 7.10. It is possible to recover the normcubic $c_3\subset P^3\subset P^4$ and the isomorphism $\psi_2:c_u\to c_3$ by F_c.

Proof. For $f_0\in F_0$ we let $l_5(f_0)=G(2,p_{85})(l_8(f_0))$, i.e. l_5 is a map from F_0 to $G(2,V_8/U_3^*)$. Since l is a divisor in F_0, then $l_5(f_0)$ is defined also for a generic point $f_0\in l$. We shall consider $l_5(f_0)$ also as a straight line in P^4. A natural map of straight lines $p_{85}|_{l_8(f_0)}:l_8(f_0)\to l_5(f_0)$ is defined also for a generic point $f_0\in l$. Analogously, the map $\xi_5=p_{85}\circ\xi:P_{F_0}(\tau_{2,M})\to P^4$ is defined at a generic point of $P_l(\tau_{2,M})$.

It is clear that in $P_{F_0}(\tau_{2,M})$ we have: $S=\bar W\cap P_l(\tau_{2,M})$ (here we identify c_u and l via ψ). Let us consider a commutative diagram

$$\begin{array}{ccccccccc}
S & \hookrightarrow & \bar W & \overset{\xi|_W}{\to} & W & \hookrightarrow & X & \hookrightarrow & G_4\\
\downarrow & & \downarrow & & & & & & \downarrow\\
P_l(\tau_{2,M}) & \hookrightarrow & P_{F_0}(\tau_{2,M}) & & & \overset{\xi_5}{\to} & & & P^4
\end{array}$$

where the left and middle vertical maps are inclusions, the right vertical map is b. Let $t_1 \in c_u$, then $l_8(\psi(t_1)) \subset U^*$, and let $\{t', t''\} = l_8(\psi(t_1)) \cap c_\Omega$. We denote $\{t_1, t_2\} = t'_V \cap c_u$ and $\{t_1, t_3\} = t''_V \cap c_u$. Then $(t', t_1) \in S$, $(t', t_2) \in S$. The image of these elements in G_4 is t', hence their images in P^4 belong to a bisecant $< \psi_2(t_1), \psi_2(t_2) >$ of c_3. Since c_Ω can be recovered by F_c, we have: for $t_1 \in c_u$ we can find images of points (t', t_1) and (t', t_2) in P^4 going along the left-lower way of the above diagram, and hence we can find the bisecant $< \psi_2(t_1), \psi_2(t_2) >$. Analogously we can find the bisecant $< \psi_2(t_1), \psi_2(t_3) >$; their intersection point indicates us the point $\psi_2(t_1) \in P^4$. □

Let us consider an isomorphism $I : P(V_8) \to P(V'_8)$ and a map $b^{-1} : P^4 \to I(G_4)$ such that $b^{-1} \circ b = I|_{G_4}$. Since we can recover uniquely $c_3 \subset P^4$, we can also recover $b^{-1} : P^4 \to I(G_4)$ and $P(V'_8)$ as the linear envelope of $I(G_4)$.

Proposition 7.11. Isomorphism $I : P(V_8) \to P(V'_8)$ can be recovered by F_c.

Proof. Let $\phi(f_0) = V_4$ where $f_0 \in F_0 - l$, and let $\{t\} = Q_G(V_4) \cap U^*$. Then (see 3.4) $b(Q_G(V_4)) = P^2$ and since points $b(\gamma_1(f_0))$, $b(\gamma_2(f_0))$ belong to P^2, we have: $l_5(f_0) \subset P^2$. This implies that $b^{-1}(l_5(f_0))$ is a conic on $I(Q_G(V_4))$ passing through $I(t)$, $I(\gamma_1(f_0))$, $I(\gamma_2(f_0))$. Since b is the restriction of $p_{85} : P(V_8) \to P^4$ to G_4, we have that for $a \in l_8(f_0)$ points $I(t)$, $I(a)$ and $b^{-1}(p_{85}(a))$ are collinear.

We can recover an isomorphism $S^2(c_3) \to I(U^*)$ and hence $I|_{U^*} : U^* \to I(U^*)$ as well. For f_0 we can recover a point $l_5(f_0) \cap P^3$. There exists only one bisecant of c_3 passing through this point, and the linear envelope of this bisecant and $l_5(f_0)$ is equal clearly $P^2 = b(Q_G(V_4))$. So, we can recover this P^2.

Let now $f_1, f_2 \in F_0 - l$ be such that $l_5(f_1) \cap l_5(f_2) = \{m\}$, where $m = m(f_1, f_2) \in P^4$, and $m_1 \in l_8(f_1)$, $m_2 \in l_8(f_2)$ are inverse images of m: $p_{85}(m_1) = p_{85}(m_2) = m$. Let, further, $t_1 = Q_G(\phi(f_1)) \cap U^*$, $t_2 = Q_G(\phi(f_2)) \cap U^*$. Then the point $b^{-1}(m)$ belongs to both lines $< I(t_i), I(m_i) >$ $(i = 1, 2)$, hence $I^{-1}(b^{-1}(m)) =< t_1, m_1 > \cap < t_2, m_2 >$, because these lines are different. If f_1, f_2 are given, then we can construct points m, m_1, m_2, t_1, t_2, $b^{-1}(m)$, and hence $I^{-1}(b^{-1}(m))$. So, there are many points in $P(V_8)$ whose I-image can be constructed. To prove that they permit to recover uniquely the linear map I it is sufficient to prove that all points of type $b^{-1}(m(f_1, f_2))$ do not belong to a hyperplane containing $I(U^*)$, or — the same — to prove that all points of type $m(f_1, f_2)$ do not belong to a hyperplane in P^4.

Lemma 7.12. Let Y be a surface on $G(2, V)$, $d = c_1(\tau_{2,V}|_Y)^2 - c_2(\tau_{2,V}|_Y)$. Then for a generic point $y \in Y$ we have: the straight line y_V meets $d - 3$ straight lines of type y'_V where $y' \in Y - y$.

Proof. Follows easily from a calculation of intersection index on $\tilde{G}_y$. □

Chern classes of $l_5^*(\tau_2)$ (this sheaf corresponds to the inclusion $l_5 \times F_0 \to G(2, (V_8/V_3)^*)$) can be easily found by using exact sequences (4.51) and

$$0 \to \tau_2^*|_{F_0} \to \tau_{2,M}^*|_{F_0} \to i_*(\tau_{2,M}^*|_l) \to 0$$

where $i : l \to F_0$. We get: $d = 168$. This means that straight lines of type $l_5(f_0)$ for $f_0 \in F$ do not belong to one hypersurface and a generic straight line of this type meets 165 others. This implies that all points of type $m(f_1, f_2)$ do not belong to a hyperplane in P^4. □

Now we can recover $G_4 \subset P(V_8)$ by F_c, because $G_4 = I^{-1}(b^{-1}(P^4))$. It is clear that for $f_0 \in F_0 - l$

$$\{\gamma_1(f_0), \gamma_2(f_0)\} = G_4 \cap l_8(f_0)$$

hence we can recover W by F_c. Since $O_X(W) = O_X(21)$ and X is an intersection of quadrics in $P(V_8)$, X is an intersection of all quadrics in $P(V_8)$ which contain W. So, we can recover uniquely X by F_c. $\square$

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