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NEW THOUGHTS ON WEINBERGER'S FIRST AND SECOND INTEGRAL BOUNDS FOR GREEN'S FUNCTIONS*

JIE XIAO†

Abstract. New thoughts about the first and second integral bounds of Hans F. Weinberger for Green's functions of uniformly elliptic equations are presented by extending the bounds to two optimal monotone principles, but also further explored via: (i) discovering two new sharp Green-function-involved isoperimetric inequalities; (ii) verifying the lower dimensional Pólya conjecture for the lowest eigenvalue of the Laplacian; (iii) sharpening an eccentricity-based lower bound for the Mahler volumes of the origin-symmetric convex bodies.

Key words. Integral bounds, Green's functions, iso-volume-like inequalities, Faber-Krahn type estimates.

AMS subject classifications. 35J, 49K, 53C.

1. Introduction.

1.1. Weinberger's 1st & 2nd integral bounds for Green's functions.

From now on, let (a_{ij}) be an $n\times n$ symmetric matrix on $\mathbb{R}^n$, $n\ge 2$, but also let

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]$$

be self-adjoint, and uniformly elliptic according to that there exists a constant $\lambda>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \forall \quad \big(x,\xi=(\xi_1,...,\xi_n)\big)\in\mathbb{R}^n\times\mathbb{R}^n \tag{1.1}$$

holds. The model of this operator is the Laplacian $\Delta := \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$. Given a bounded domain $D\subset\mathbb{R}^n$ with boundary ∂D and two functions f in D and h on ∂D respectively, the solution (whenever it exists) to the following boundary value problem:

$$\begin{cases} Lu = f & \text{in} \quad D \\ u = h & \text{on} \quad \partial D \end{cases}$$

can be written as

$$u(o) = -\int_D fG(o,\cdot)\,dV(\cdot) + \int_{\partial D} h\frac{\partial G(o,\cdot)}{\partial\nu}\,dS(\cdot) \quad \text{for} \quad o\in D. \tag{1.2}$$

Here and henceforth, $G(o,x) := G_{L,D}(o,x)$ denotes the Green function of D with singularity at any given point $o\in D$ associated to the operator L, i.e., the non-negative solution to

$$\begin{cases} -LG(o,\cdot) = \delta_o(\cdot) & \text{in} \quad D \\ G(o,\cdot) = 0 & \text{on} \quad \partial D \end{cases}$$

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†Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada (jxiao@mun.ca).

for which $\delta_o(\cdot)$ is the Dirac measure giving unit mass to the point o; dS and dV are the surface and volume elements;

$$\frac{\partial G(o,x)}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{G(o,x)}{\partial x_j}\nu_i \tag{1.3}$$

is the directional derivative of $G(o,\cdot)$ along the outward unit normal vector $\nu = (\nu_1, ..., \nu_n)$. For the later use, recall that if $L = \Delta$ then

$$G(o,x) = \begin{cases} \frac{\ln \frac{R_o}{|o-x|}}{2\pi} + H(o,x) & \text{for} \quad n = 2 \\ \frac{|o-x|^{2-n} - R_o^{2-n}}{n(n-2)\sigma_n} + H(o,x) & \text{for} \quad n > 2, \end{cases} \tag{1.4}$$

where

$$LH(o,\cdot) = \Delta H(o,\cdot) = 0 \quad \& \quad H(o,o) = 0,$$

σ_n is the volume of the unit n-ball and R_o is called the conformal respectively harmonic radius of D with respect to o for $n = 2$ respectively $n > 2$; see also [2, p.58-59] and [4]. When D is a Euclidean ball $B_r(o)$ with center o and radius r, $G(o,x)$ can be calculated below:

$$G(o,x) = \begin{cases} (2\pi)^{-1} \ln\left(\frac{\left|\frac{rx}{|x|} - \frac{|x|o}{r}\right|}{|x-o|}\right) & \text{for} \quad n = 2 \\ \big[n(n-2)\sigma_n\big]^{-1}\Big(|x-o|^{2-n} - \big|\frac{rx}{|x|} - \frac{|x|o}{r}\big|^{2-n}\Big) & \text{for} \quad n > 2. \end{cases} \tag{1.5}$$

To improve G. Stampachhia's results in [18], in his 1962 paper [21] (see also MathSciNet: MR0145191(26#2726) and its citations), Hans F. Weinberger obtained two pointwise estimates on the solution (1.2) under the condition $h = 0$. The first is:

$$|u(o)| \le \lambda^{-1} K_{p,n} V(D)^{\frac{2}{n}-\frac{1}{p}} \left(\int_D |f|^p dV\right)^{\frac{1}{p}} \quad \text{for} \quad o \in D, \tag{1.6}$$

where p is any number greater than $\frac{n}{2} > 1$, $V(D)$ is the volume of D, and

$$K_{p,n} = (n-2)^{\frac{1}{p}-2} n^{-\frac{1}{p}} \sigma_n^{-\frac{2}{n}} \left[B\Big(\frac{2p-1}{p-1}, \frac{2}{n-2} - \frac{1}{p-1}\Big)\right]^{1-\frac{1}{p}}$$

is the best possible constant with $B(\cdot,\cdot)$ being the classical Beta function. The second is that if $f = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i} = \text{div} g$, i.e., the divergence of vector-valued function $g = (g_1, ..., g_n)$, then

$$|u(o)| \le \lambda^{-1} \bar{K}_{p,n} V(D)^{\frac{1}{n}-\frac{1}{p}} \left[\int_D \Big(\sum_{i=1}^{n} g_i^2\Big)^{\frac{p}{2}} dV\right]^{\frac{1}{p}} \quad \text{for} \quad o \in D \ \& \ p > n, \tag{1.7}$$

where

$$\bar{K}_{p,n} = \sigma_n^{-\frac{1}{n}} n^{-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}}$$

is the best possible constant.

Weinberger's proofs for both (1.6) and (1.7) use the Hölder inequality, the representation of the solution

$$u(o) = -\int_D G(o,\cdot)f(\cdot)\,dV(\cdot)$$

which also equals

$$\int_D \langle g, \nabla G(o,\cdot)\rangle\,dV(\cdot) \quad \text{whenever} \quad f = \mathrm{div} g,$$

limit arguments, and most importantly, two optimal iso-volume estimates for $G_{L,D}(o,\cdot)$ (when L and D are sufficiently smooth) as follows:

The first integral bound of Green's function is: Under $0 \le q < \frac{n}{n-2}$ with $n \ge 3$,

$$\mathrm{I}(o,D,q,\lambda) := \int_D G(o,\cdot)^q\,dV(\cdot) \le \left(\frac{\left(\frac{n}{n-2}\right)B\left(q+1, \frac{n}{n-2}-q\right)}{[\lambda n(n-2)\sigma_n^{\frac{2}{n}}]^q}\right) V(D)^{1-\frac{q(n-2)}{n}} \tag{1.8}$$

with equality if $L = \Delta$ and $D = B_r(o)$. This has been extended by C. Bandle (cf. [2, p.61, (2.21)] and [3]) to $n = 2$ via replacing the coefficient before $V(D)$ with $\Gamma(1+q)(4\lambda\pi)^{-q}$ where $\Gamma(\cdot)$ is the classical Gamma function.

The second integral bound of Green's function is: Under $0 \le q < \frac{n}{n-1}$,

$$\mathrm{II}(o,D,q,\lambda) := \int_D |\nabla G(o,\cdot)|^q\,dV(\cdot) \le \left[\frac{n(\lambda n\sigma_n^{\frac{1}{n}})^{-q}}{n-q(n-1)}\right] V(D)^{1-\frac{q(n-1)}{n}} \tag{1.9}$$

with equality if $L = \Delta$ and $D = B_r(o)$.

1.2. A monotonicity look at the 1st & 2nd integral bounds and beyond. By normalization, we define

$$\mathbf{I}(o,D,q,\lambda) := \begin{cases} \frac{\mathrm{I}(o,D,q,\lambda)}{\Gamma(1+q)(4\lambda\pi)^{-q}} & \text{for} \quad n = 2 \\ \left(\frac{\mathrm{I}(o,D,q,\lambda)}{\left(\frac{n}{n-2}\right)B\left(q+1,\frac{n}{n-2}-q\right)[\lambda n(n-2)\sigma_n^{\frac{2}{n}}]^{-q}}\right)^{\frac{n}{n-q(n-2)}} & \text{for} \quad n > 2 \end{cases}$$

and

$$\mathbf{II}(o,D,q,\lambda) := \left(\frac{\mathrm{II}(o,D,q,\lambda)}{n(\lambda n\sigma_n^{\frac{1}{n}})^{-q}[n-q(n-1)]^{-1}}\right)^{\frac{n}{n-q(n-1)}}.$$

Then (1.8) and (1.9) can be rewritten as

$$\mathbf{I}(o,D,q,\lambda) \le \mathbf{I}(o,D,0,\lambda) \quad \forall \quad q \in [0, \frac{n}{n-2})$$

and

$$\mathbf{II}(o,D,q,\lambda) \le \mathbf{II}(o,D,0,\lambda) \quad \forall \quad q \in [0, \frac{n}{n-1}).$$

Such a new observation suggests an investigation of the monotonicity properties of $\mathbf{I}(o,D,q,\lambda)$ and $\mathbf{II}(o,D,q,\lambda)$ with respect to q. In the forthcoming two sections, we will prove respectively that $\mathbf{I}(o,D,q,\lambda)$ and $\mathbf{II}(o,D,q,\lambda)$ are strictly decreasing with

q being strictly increasing in two appropriate intervals except $L = \Delta$ and $D = B_r(o)$, and thereby evaluating

$$\liminf_{q\to\frac{n}{n-2}} \mathbf{I}(o, D, q, \lambda) \quad \& \quad \liminf_{q\to\frac{n}{n-1}} \mathbf{II}(o, D, q, \lambda)$$

in terms of two analogues $R_{o,\mathrm{I},\lambda}$ and $R_{o,\mathrm{II},\lambda}$ of the (conformal or harmonic) radius R_o. Here, it is perhaps appropriate to point out that our arguments for the monotonicity properties of $\mathbf{I}(o, D, q, \lambda)$ and $\mathbf{II}(o, D, q, \lambda)$ cannot be obtained from Weinberger's ones for (1.8)-(1.9) which depends on the well-known Pólya-Szegö symmetrization. The key for us is to use the layer cake formula to reduce the desired monotonicity properties to one-dimensional calculus inequalities with sharp constants. Section 4 describes some applications of the ideas developed in Sections 2-3 through:

- discovering two new sharp isoperimetric inequalities via $G_{L,D}(o, \cdot)$;
- establishing a new Faber-Krahn type inequality for L (with strongly uniform ellipticity condition) that particularly confirms Pólya's conjecture for the lowest Laplacian eigenvalue in dimensions $2, 3, 4$;
- using the optimal Faber-Krahn inequality for Laplacian to sharpen an eccentricity-based lower bound for the Mahler volumes of the origin-symmetric convex bodies.

2. The first monotonicity principle.

2.1. The fundamental setting. To reach the monotonicity of $\mathbf{I}(o, D, q, \lambda)$ with respect to q, we need a one-dimensional result which seems to be useful for other sharp inequality problems such as in [14] and [16].

LEMMA 2.1. *For* $0 \le q < \frac{n}{n-2}$, $n \ge 2$ *and* $0 \le t < \infty$ *let* $\Phi_q(t) = -\int_t^\infty s^q \, d\Phi(s)$ *and*

$$\Psi_q(t) = \begin{cases} \frac{c^q \Phi_q(t)}{\Gamma(1+q)} & \text{when} \quad n = 2 \\ \left[\frac{c^q \Phi_q(t)}{(\frac{n}{n-2}) B\left(\frac{n}{n-2} - q, 1+q\right)}\right]^{\frac{n}{n-q(n-2)}} & \text{when} \quad n > 2 \end{cases}$$

with Φ *and* c *being respectively a differentiable self-map of* $[0, \infty)$ *and a positive constant such that*

$$0 \ge \begin{cases} \frac{d}{dt}\left[e^{ct}\Phi(t)\right] & \text{when} \quad n = 2 \\ \frac{d}{dt}\left[\Phi(t)^{\frac{2-n}{n}} - ct\right]^{\frac{n}{2-n}} & \text{when} \quad n > 2. \end{cases}$$

(i) *If* $0 \le q_2 < q_1 < \frac{n}{n-2}$ *then* $\Psi_{q_1}(0) \le \Psi_{q_2}(0)$ *with equality if and only if*

$$\Phi(t) = \begin{cases} \Phi(0)e^{-ct} & \text{when} \quad n = 2 \\ \left[\Phi(0)^{\frac{2-n}{n}} + ct\right]^{\frac{n}{2-n}} & \text{when} \quad n > 2 \end{cases}$$

holds for all $t \in (0, \infty)$.

(ii)

$$\lim_{q\to\frac{n}{n-2}} \Psi_q(0) = \lim_{t\to\infty} \begin{cases} \Phi(t)e^{ct} & \text{when} \quad n = 2 \\ \left[\Phi(t)^{\frac{2-n}{n}} - ct\right]^{\frac{n}{2-n}} & \text{when} \quad n > 2. \end{cases}$$

Proof. (i) We will verify this part according to two cases $n = 2$ and $n > 2$.

Case 1: $n = 2$. With no loss of generality we may assume $\Psi_{q_2}(0) < \infty$. If $q_2 = 0$ then $\Phi_{q_2}(t) = \Phi_0(t) = \Phi(t)$ follows from $d(e^{ct}\Phi(t))/dt \le 0$ which ensures $\Phi(\infty) := \lim_{t\to\infty} \Phi(t) = 0$. Consequently,

$$-\frac{d\Phi_0(t)}{\Phi_0(t)} \ge cdt = \frac{e^{-ct}dt}{\int_t^\infty e^{-cr}dr} \quad \forall \quad t \in [0,\infty).$$

If $q_2 > 0$, then both $d(e^{ct}\Phi(t))/dt \le 0$ and integration-by-part imply that for any $t \in [0,\infty)$,

$$\begin{aligned}\Phi_{q_2}(t) &= t^{q_2}\Phi(t) + q_2\int_t^\infty r^{q_2-1}\Phi(r)dr\\ &\le \Phi(t)\left(t^{q_2} + q_2 e^{ct}\int_t^\infty r^{q_2-1}e^{-cr}dr\right)\\ &= c\Phi(t)e^{ct}\int_t^\infty r^{q_2}e^{-cr}dr.\end{aligned}$$

As a result, we read off

$$-\frac{d\Phi_{q_2}(t)}{\Phi_{q_2}(t)} \ge \frac{ct^{q_2}\Phi(t)dt}{\Phi_{q_2}(t)} \ge \frac{t^{q_2}e^{-ct}dt}{\int_t^\infty r^{q_2}e^{-cr}dr} \quad \forall \quad t \in [0,\infty).$$

Integrating this inequality from 0 to t, we obtain

$$\Phi_{q_2}(t) \le \frac{c^{q_2+1}\Phi_{q_2}(0)}{\Gamma(q_2+1)}\int_t^\infty r^{q_2}e^{-cr}dr \quad \forall \quad t \in [0,\infty).$$

With the help of the last estimate we have that if

$$0 \le q_2 < q_1 < \frac{n}{n-2} = \frac{2}{2-2} = \infty$$

then

$$\begin{aligned}\Phi_{q_1}(0) &= (q_1-q_2)\int_0^\infty t^{q_1-q_2-1}\Phi_{q_2}(t)dt\\ &\le \frac{c^{q_2+1}(q_1-q_2)\Phi_{q_2}(0)}{\Gamma(q_2+1)}\int_0^\infty t^{q_1-q_2-1}\Big(\int_t^\infty r^{q_2}e^{-cr}dr\Big)dt \qquad (2.1)\\ &= c^{q_2-q_1}\left(\frac{\Gamma(q_1+1)}{\Gamma(q_2+1)}\right)\Phi_{q_2}(0),\end{aligned}$$

thereby getting the desired assertion.

Regarding the equality case, we consider two aspects. On the one hand, if

$$\Phi(t) = \Phi(0)e^{-ct} \quad \forall \quad t \in (0,\infty),$$

then

$$\Phi_q(0) = c^{-q}\Gamma(q+1)\Phi(0) \quad \forall \quad q \in [0,\infty),$$

and accordingly the desired equality holds. On the other hand, assume $\Psi_{q_1}(0) = \Psi_{q_2}(0)$ is valid. If the statement "$\Phi(t) = e^{-ct}\Phi(0)\ \forall\ t > 0$" were false, then there would be two positive numbers r_0 and t_0 such that $r_0 > t_0$ and $\Phi(r_0) < e^{-c(r_0-t_0)}\Phi(t_0)$ hold, and hence the continuity of $\Phi(\cdot)$ produces such a constant $\delta > 0$ that $\Phi(r_0) < e^{-c(r_0-t)}\Phi(t)$ when $t \in (t_0-\delta, t_0]$. Therefore $d\big(e^{ct}\Phi(t)\big)/dt \le 0$ is applied to derive that $\Phi(r) < e^{-c(r-t)}\Phi(t)$ as $t \in (t_0-\delta, t_0]$ and $r \ge r_0$. Consequently, we obtain

$$\Phi_{q_2}(t) < c\Phi(t)e^{ct}\int_t^\infty r^{q_2}e^{-cr}dr \quad \forall \quad t \in (t_0-\delta, t_0],$$

whence finding

$$\Phi_{q_2}(t) < \frac{c^{q_2+1}\Phi_{q_2}(0)}{\Gamma(q_2+1)}\int_t^\infty r^{q_2}e^{-cr}dr \quad \forall \quad t \in (t_0-\delta, t_0].$$

This, along with (2.1), yields

$$\Phi_{q_1}(0) = (q_1-q_2)\int_0^\infty t^{q_1-q_2-1}\Phi_{q_2}(t)dt < c^{q_2-q_1}\left(\frac{\Gamma(q_1+1)}{\Gamma(q_2+1)}\right)\Phi_{q_2}(0),$$

contradicting the previous equality assumption.

Case 2: $n > 2$. Since

$$0 \ge \frac{d}{dt}\big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}} \quad \forall \quad t \in [0,\infty),$$

it follows that

$$\big[\Phi(t_2)^{\frac{2-n}{n}} - ct_2\big]^{\frac{n}{2-n}} \le \big[\Phi(t_1)^{\frac{2-n}{n}} - ct_1\big]^{\frac{n}{2-n}} \quad \forall \quad 0 \le t_1 < t_2 < \infty. \tag{2.2}$$

If $q_2 = 0$, then using integration-by-parts, (2.2) and a simple substitution we get

$$\begin{aligned}
\Phi_{q_1}(0) &= -\int_0^\infty s^{q_1}\, d\Phi(s) \\
&= q_1\int_0^\infty \Phi(s)s^{q_1-1}\, ds \\
&\le\ q_1\int_0^\infty \big[\Phi(0)^{\frac{2-n}{n}} + cs\big]^{\frac{n}{2-n}} s^{q_1-1}\, ds \\
&= \Phi(0)q_1c^{-q_1}\int_0^\infty \big[1+\Phi(0)^{\frac{n-2}{n}}s\big]^{\frac{n}{2-n}} s^{q_1-1}\, ds \\
&= \Phi_0(0)^{\frac{n-q_1(n-2)}{n}} q_1c^{-q_1}B\big(\frac{n}{n-2}-q_1, q_1\big),
\end{aligned}$$

whence reaching $\Psi_{q_1}(0) \le \Psi_{q_2}(0)$.

If $q_2 > 0$, then the situation is more complex than $q_2 = 0$. Given $r \in [0,\infty)$ and $q \in (q_2, \frac{n}{n-2})$, an integration-by-parts, the inequality (2.2) and a change of variable yield

$$\begin{aligned}
\Phi_q(r) &= r^q\Phi(r) + q\int_r^\infty \Phi(t)t^{q-1}\,dt \\
&\le r^q\Phi(r) + q\int_r^\infty \big[\Phi(r)^{\frac{2-n}{n}} + c(t-r)\big]^{\frac{n}{2-n}} t^{q-1}\,dt \\
&= \frac{cn}{n-2}\int_r^\infty \big[\Phi(r)^{\frac{2-n}{n}} + c(t-r)\big]^{\frac{2(n-1)}{2-n}} t^q\,dt \\
&= \frac{cn}{n-2}\int_r^\infty \big[\Phi(r)^{\frac{2-n}{n}} - cr + ct\big]^{\frac{2(n-1)}{2-n}} t^q\,dt \\
&= c^{-q}n(n-2)^{-1}\big[\Phi(r)^{\frac{2-n}{n}} - cr\big]^{\frac{n-(n-2)q}{2-n}} \int_{\frac{cr}{\Phi(r)^{\frac{2-n}{n}}-cr}}^\infty \frac{t^q\,dt}{(1+t)^{\frac{2(n-1)}{n-2}}},
\end{aligned}$$

and consequently,

$$\left[\frac{c^q\Phi_q(r)}{qB\big(\frac{n}{n-2}-q,q\big)}\right]^{\frac{n}{n-q(n-2)}} \le \big[\Phi(r)^{\frac{2-n}{n}} - cr\big]^{\frac{n}{2-n}}. \tag{2.3}$$

Observe that

$$\frac{d\Phi_q(t)}{dt} = t^q\frac{d\Phi(t)}{dt} \le \Big(\frac{cn}{2-n}\Big)t^q\Phi(t)^{\frac{2(n-1)}{n}} \quad \forall \quad t\in[0,\infty). \tag{2.4}$$

Now, (2.3) and (2.4) are used to deduce the following differential inequality

$$t^q\left[\big(a\Phi_q(t)\big)^{\frac{2-n}{n-q(n-2)}} + ct\right]^{\frac{2(n-1)}{2-n}} \le \frac{\frac{d}{dt}\Phi_q(t)}{\Big(\frac{cn}{2-n}\Big)} \quad \text{where} \quad a = \frac{c^q}{qB\big(\frac{n}{n-2}-q,q\big)}. \tag{2.5}$$

The estimate $\Phi_q(t)\le\Phi_q(0)$ and the differential inequality (2.5) derive

$$t^q\Big[a^{\frac{2-n}{n-q(n-2)}} + ct\Phi_q(0)^{\frac{n-2}{n-q(n-2)}}\Big]^{\frac{2(n-1)}{2-n}} \le \Big(\frac{2-n}{cn}\Big)\Phi_q(t)^{\frac{2(n-1)}{q(n-2)-n}}\Big(\frac{d\Phi_q(t)}{dt}\Big).$$

Integrating this last inequality over $[0,s]$, we obtain

$$\begin{aligned}
\Phi_q(s) &\le \left[\frac{\int_0^s\big[a^{\frac{2-n}{n-q(n-2)}} + cr\Phi_q(0)^{\frac{n-2}{n-q(n-2)}}\big]^{\frac{2(n-1)}{2-n}}r^q dr}{\frac{n-q(n-2)}{cn(q+1)}} + \Phi_q(0)^{\frac{(2-n)(q+1)}{n-q(n-2)}}\right]^{\frac{n-q(n-2)}{(2-n)(q+1)}} \\
&= \Phi_q(0)\left[1 + \frac{an(q+1)}{c^q(n-q(n-2))}\int_0^{cs[a\Phi_q(0)]^{\frac{n-2}{n-q(n-2)}}} (1+r)^{\frac{2(n-1)}{2-n}}r^q\,dr\right]^{\frac{n-q(n-2)}{(2-n)(q+1)}}.
\end{aligned}$$

Using the above inequality, setting $b = c\big[a\Phi_{q_2}(0)\big]^{\frac{n-2}{n-q(n-2)}}$, and integrating-by-parts, we further get

$$
\begin{aligned}
\Phi_{q_1}(0) &= (q_1-q_2)\int_0^\infty \Phi_{q_2}(s)s^{q_1-q_2-1}\,ds\\
&\le \Phi_{q_2}(0)\int_0^\infty \left[1+\frac{\int_0^{bs}(1+r)^{\frac{2(n-1)}{2-n}}r^{q_2}\,dr}{\frac{c^{q_2}\left(n-q_2(n-2)\right)}{an(q+1)}}\right]^{\frac{n-q(n-2)}{(2-n)(q_2+1)}} ds^{q_1-q_2}\\
&= -\Phi_{q_2}(0)\int_0^\infty s^{q_1-q_2}\frac{d}{ds}\left[1+\frac{\int_0^{bs}(1+r)^{\frac{2(n-1)}{2-n}}r^{q_2}\,dr}{\frac{c^{q_2}\left(n-q_2(n-2)\right)}{an(q_2+1)}}\right]^{\frac{n-q_2(n-2)}{(2-n)(q_2+1)}} ds\\
&= \frac{\left(a\Phi_{q_2}(0)\right)^{\frac{n-q_1(n-2)}{n-q_2(n-2)}}}{\left(\frac{n-2}{n}\right)c^{q_1}}\int_0^\infty \frac{t^{q_1}}{(1+t)^{\frac{2(n-1)}{n-2}}}\left[1+\frac{\int_0^t v^{q_2}(1+v)^{\frac{2(n-1)}{2-n}}\,dv}{\frac{c^{q_2}\left(n-(n-2)q_2\right)}{an(q_2+1)}}\right]^{\frac{2(n-1)}{(2-n)(q_2+1)}} dt\\
&\le \frac{\left(a\Phi_{q_2}(0)\right)^{\frac{n-q_1(n-2)}{n-q_2(n-2)}}}{\left(\frac{n-2}{n}\right)c^{q_1}}\int_0^\infty t^{q_1}(1+t)^{\frac{2(n-1)}{2-n}}\,dt\\
&= \left[\frac{q_1B\left(\frac{n}{n-2}-q_1,q_1\right)}{c^{q_1}}\right]\left[\left(\frac{c^{q_2}}{q_2B\left(\frac{n}{n-2}-q_2,q_2\right)}\right)\Phi_{q_2}(0)\right]^{\frac{n-q_1(n-2)}{n-q_2(n-2)}},
\end{aligned}
$$

whose last inequality becomes equality when $\Phi_{q_2}(0)=0$. Simplifying the just-obtained estimates and using the definition of Ψ_q we immediately find $\Psi_{q_1}(0)\le\Psi_{q_2}(0)$.

Next, let us consider the equality. The 'if' part can be seen from a direct computation. As a matter of fact, if

$$\Phi(t)=[\Phi(0)^{\frac{2-n}{n}}+ct]^{\frac{n}{2-n}}\quad\forall\quad t\in(0,\infty), \tag{2.6}$$

then a simple calculation yields

$$\Phi_q(0)=\int_0^\infty \Phi(t)\,dt^q=c^{-q}\Big(\frac{n}{n-2}\Big)B\Big(\frac{n}{n-2}-q,1+q\Big)\Phi(0)^{\frac{n-q(n-2)}{n}},$$

whence giving $\Psi_{q_1}(0)=\Psi_{q_2}(0)$. On the other hand, if (2.6) is not valid, by (2.2) there is a $t_0\in(0,\infty)$ and $\epsilon>0$ such that

$$\Phi(t)<[\Phi(0)^{\frac{2-n}{n}}+ct]^{\frac{n}{2-n}}\quad\forall\quad t\in(t_0,t_0+\epsilon). \tag{2.7}$$

Applying (2.7) to the beginning estimates in the treatment of either $q_2=0$ or $q_2>0$, we find that (2.3) becomes a strict inequality for $r\in(t_0,t_0+\epsilon)$, and so that (2.5) is actually a strict inequality when $t\in(t_0,t_0+\epsilon)$. With the help of this strictness, from the concluding group of estimates in the treatment of either $q_2=0$ or $q_2>0$ we see either

$$\Phi_{q_1}(0)<\Phi_0(0)^{\frac{n-q_1(n-2)}{n}}\Big(\frac{n}{n-2}\Big)c^{-q_1}B\Big(\frac{n}{n-2}-q_1,1+q_1\Big)$$

or

$$
\begin{aligned}
\Phi_{q_1}(0) &= (q_1 - q_2)\left[\int_0^{t_0} + \int_{t_0}^{t_0+\epsilon} + \int_{t_0+\epsilon}^{\infty}\right]\Phi_{q_2}(s)s^{q_1-q_2-1}\,ds \\
&< \Phi_{q_2}(0)\int_0^{\infty}\left[1 + \frac{\int_0^{bs}(1+r)^{\frac{2(n-1)}{2-n}}r^{q_2}\,dr}{\frac{c^{q_2}\left(n-q_2(n-2)\right)}{an(q+1)}}\right]^{\frac{n-q(n-2)}{(2-n)(q_2+1)}} ds^{q_1-q_2} \\
&\le \Big[\frac{q_1 B\big(\frac{n}{n-2} - q_1, q_1\big)}{c^{q_1}}\Big]\left[\Big(\frac{c^{q_2}}{q_2 B\big(\frac{n}{n-2} - q_2, q_2\big)}\Big)\Phi_{q_2}(0)\right]^{\frac{n-q_1(n-2)}{n-q_2(n-2)}}.
\end{aligned}
$$

Needless to say, we end up with the strict inequality $\Psi_{q_1}(0) < \Psi_{q_2}(0)$, whence completing the argument for the 'only if' part.

(ii) We demonstrate this part in accordance with two cases $n = 2$ and $n > 2$.

Case 1: $n = 2$. From the argument for (i) we may assume that $\Psi_q(0) < \infty$ is valid for all $q \ge q_0$ with some $q_0 \in (0, \infty)$ and so that via integration-by-parts and $d\big(e^{ct}\Phi(t)\big)/dt \le 0$,

$$
\begin{aligned}
\Phi_q(0) &= q\int_0^{\infty} r^{q-1}\Phi(r)dr \\
&= q\int_0^{\infty} e^{cr}\Phi(r)r^{q-1}e^{-cr}dr \\
&= q\Big(e^{ct}\Phi(t)\int_0^t r^{q-1}e^{-cr}\,dr\Big)\Big|_0^{\infty} - q\int_0^{\infty}\Big(\int_0^t r^{q-1}e^{-cr}\,dr\Big)\,d(e^{ct}\Phi(t)) \\
&= c^{-q}\Gamma(q+1)\Big(\lim_{t\to\infty} e^{ct}\Phi(t)\Big) - q\int_0^{\infty}\Big(\int_0^t r^{q-1}e^{-cr}\,dr\Big)\,d(e^{ct}\Phi(t)).
\end{aligned}
$$

Therefore, the desired limit formula follows from showing

$$
0 \ge \mathsf{J}(q, c) := \frac{qc^q}{\Gamma(q+1)}\int_0^{\infty}\Big(\int_0^t r^{q-1}e^{-cr}\,dr\Big)\,d(e^{ct}\Phi(t)) \to 0 \quad \text{as} \quad q \to \infty.
$$

Notice that the condition $d\big(e^{ct}\Phi(t)\big)/dt \le 0$ deduces that for any $\epsilon > 0$ there exists a $t_0 > 0$ such that $-\frac{\epsilon}{2} < \int_{t_0}^{\infty} d(e^{ct}\Phi(t)) \le 0$. So

$$
\begin{aligned}
\mathsf{J}_1(q, c) &:= \frac{qc^q}{\Gamma(q+1)}\int_{t_0}^{\infty}\Big(\int_0^t r^{q-1}e^{-cr}\,dr\Big)\,d\big(e^{ct}\Phi(t)\big) \\
&\ge \int_{t_0}^{\infty} d\big(e^{ct}\Phi(t)\big) > -\frac{\epsilon}{2}.
\end{aligned}
$$

Meanwhile, integrating by parts plus $d\Phi(t)/dt \le 0$ derives

$$\begin{aligned}
\mathsf{J}_2(q,c) &:= \frac{qc^q}{\Gamma(q+1)}\int_0^{t_0}\Big(\int_0^t r^{q-1}e^{-cr}\,dr\Big)\,d\big(e^{ct}\Phi(t)\big)\\
&\ge \frac{c^q}{\Gamma(q+1)}\int_0^{t_0} t^q\,d\big(e^{ct}\Phi(t)\big)\\
&\ge \frac{c^q}{\Gamma(q+1)}\int_0^{t_0} t^q e^{ct}\,d\Phi(t)\\
&\ge \frac{c^q e^{ct_0} t_0^{q-q_0}}{\Gamma(q+1)}\int_0^{t_0} t^{q_0}\,d\Phi(t)\\
&\ge -\frac{c^q e^{ct_0} t_0^{q-q_0}\Phi_{q_0}(0)}{\Gamma(q+1)} \to 0 \quad \text{as} \quad q\to\infty.
\end{aligned}$$

The estimates on $\mathsf{J}_1(q,c)$ and $\mathsf{J}_2(q,c)$, along with $d\big(e^{ct}\Phi(t)\big)/dt \le 0$, imply that

$$0 \ge \mathsf{J}(q,c) = \mathsf{J}_1(q,c) + \mathsf{J}_2(q,c) > -\epsilon$$

holds for sufficiently large q. Thus, $\lim_{q\to\infty}\mathsf{J}(q,c) = 0$, as required.

Case 2: $n > 2$. From (2.3) it turns out that for a given $r \in [0,\infty)$,

$$\begin{aligned}
\Phi_q(0) &= \int_0^r \Phi(t)\,dt^q + \Phi_q(r)\\
&\le \int_0^r \Phi(t)\,dt^q + \big[\Phi(r)^{\frac{2-n}{n}} - cr\big]^{\frac{n-q(n-2)}{2-n}} c^{-q}\Big(\frac{n}{n-2}\Big)B\Big(\frac{n}{n-2}-q, 1+q\Big).
\end{aligned}$$

Using the Adams inequality [1, (17)]:

$$(\alpha+\beta)^\gamma \le \alpha^\gamma + \gamma 2^{\gamma-1}(\beta^\gamma + \beta\alpha^{\beta-1}) \quad \text{for} \quad 0 \le \alpha, \beta, \gamma - 1 < \infty,$$

as well as the asymptotic behavior of $B(\cdot,\cdot)$, we get

$$\lim_{q\to\frac{n}{n-2}} \Psi_q(0) \le \big[\Phi(r)^{\frac{2-n}{n}} - cr\big]^{\frac{n}{2-n}},$$

thereby obtaining

$$\lim_{q\to\frac{n}{n-2}} \Psi_q(0) \le \lim_{t\to\infty}\big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}}. \tag{2.8}$$

For the reversed one of (2.8), noting that $\big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}}$ decreases with t increasing, and so using (2.2), we obtain

$$\phi := \lim_{t\to\infty}\big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}} \le \big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}} \le \Phi(0).$$

Clearly, it follows from (2.8) that ϕ is nonnegative. But, if $\phi = 0$ then (2.8) gives $\lim_{q\to\frac{n}{n-2}}\Psi_q(0) = 0$ and hence the limit formula in (ii) (under $n > 2$) is true. So, it remains to deal with the case $\phi > 0$. Using this condition, we get

$$\Phi_q(0) \ge \int_0^\infty \big[\phi^{\frac{2-n}{n}} + ct\big]^{\frac{n}{2-n}}\,dt^q = \phi^{\frac{n-q(n-2)}{n}}\Big(\frac{n}{n-2}\Big)c^{-q}B\Big(\frac{n}{n-2}-q, 1+q\Big).$$

Naturally, this last estimate yields

$$\lim_{q\to\frac{n}{n-2}} \Psi_q(0) \ge \lim_{t\to\infty}\big[\Phi(t)^{\frac{2-n}{n}} - ct\big]^{\frac{n}{2-n}}. \tag{2.9}$$

A combination of (2.8) and (2.9) gives the desired limit formula. □

2.2. A monotone integration for Green's functions. Using the preceding lemma, we get the following monotonicity for Green's functions.

THEOREM 2.2. *Let the uniformly elliptic operator L and the bounded domain D be so smooth that $G(o,\cdot) = G_{L,D}(o,\cdot)$ exists.*

(i) *If $0 \le q_2 < q_1 < \frac{n}{n-2}$ then*

$$\mathbf{I}(o,D,q_1,\lambda) \le \mathbf{I}(o,D,q_2,\lambda) \tag{2.10}$$

where inequality in (2.10) becomes equality when $L = \Delta$ and $D = B_r(o)$.

(ii) *If $0 \le q < \frac{n}{n-2}$, $t \in [0,\infty)$ and $D_t = \{x \in D : \ G(o,x) > t\}$ then*

$$R_{o,\mathrm{I},\lambda} := \left[\sigma_n^{-1} \liminf_{q \to \frac{n}{n-2}} \mathbf{I}(o,D,q,\lambda)\right]^{\frac{1}{n}}$$

defines the type I *radius of D with respect to $o \in D$ which can be evaluated by*

$$\lim_{t\to\infty} \begin{cases} \left[\pi^{-1} V(D_t) e^{\kappa_n t}\right]^{\frac{1}{n}} & \text{when } n = 2 \\ \sigma_n^{-\frac{1}{n}} \left[V(D_t)^{\frac{2-n}{n}} - \kappa_n t\right]^{\frac{1}{2-n}} & \text{when } n > 2, \end{cases}$$

where

$$\kappa_n := \begin{cases} 4\pi\lambda & \text{when } n = 2 \\ n(n-2)\sigma_n^{\frac{2}{n}}\lambda & \text{when } n > 2. \end{cases}$$

Consequently

$$\sigma_n R^n_{o,\mathrm{I},\lambda} \le \mathbf{I}(o,D,q,\lambda) \le V(D) \tag{2.11}$$

where equalities in (2.11) occur and so $R_{o,\mathrm{I},\lambda} = R_o$ whenever $L = \Delta$ and $D = B_r(o)$. Moreover

$$1 = \lim_{t\to\infty} \begin{cases} \frac{V(D_t)}{\pi(e^{-2\pi t} R_o)^2} & \text{when } n = 2 \\ \frac{V(D_t)}{\sigma_n [n(n-2)\sigma_n t + R_o^{2-n}]^{\frac{n}{2-n}}} & \text{when } n > 2 \end{cases}$$

is valid for $L = \Delta$.

Proof. (i) For $t \ge 0$ consider the level set D_t and put

$$\mathrm{I}(o,D_t,q,\lambda) = \int_{D_t} G(o,\cdot)^q \, dV(\cdot).$$

According to the well-known co-area formula (cf. [2, p.53, Lemma 2.5]) and Sard's theorem (cf. [17, Theorem 10.4]), we may assume $|\nabla G(o,x)| > 0$ exist for all $x \in \partial D_t$, and thus have

$$-\frac{d}{dt}\mathrm{I}(o,D_t,q,\lambda) = t^q \int_{\partial D_t} |\nabla G(o,x)|^{-1} \, dS(x).$$

Note that

$$\frac{\partial G(o,x)}{\partial x_i} = -|\nabla G(o,x)|\nu_i \quad \text{when} \quad x \in \partial D_t,$$

and from the definition of Green's function we read

$$\int_{\partial D_t} \frac{\partial G(o,x)}{\partial \nu} dS(x) = -1, \tag{2.12}$$

thereby finding via (2.12), (1.3) and (1.1)

$$\begin{aligned} 1 &= -\int_{\partial D_t} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial G(o,x)}{\partial x_j} \nu_i \, dS(x) \\ &= \int_{\partial D_t} |\nabla G(o,x)| \sum_{i,j=1}^{n} a_{ij}(x) \nu_i \nu_j \, dS(x) \\ &\geq \lambda \int_{\partial D_t} |\nabla G(o,x)| \, dS(x). \end{aligned} \tag{2.13}$$

Now that the isoperimetric inequality is valid for D_t and its boundary ∂D_t, i.e.,

$$V(D_t)^{\frac{n-1}{n}} \leq (n\sigma_n^{\frac{1}{n}})^{-1} S(\partial D_t). \tag{2.14}$$

So, using the Cauchy-Schwarz inequality and (2.12)-(2.13)-(2.14) we get

$$\frac{d}{dt} V(D_t) \leq -\lambda (n\sigma_n^{\frac{1}{n}})^2 V(D_t)^{\frac{2(n-1)}{n}}. \tag{2.15}$$

Upon letting $\Phi(t) = V(D_t)$ and using the layer-cake-formula we find

$$\Phi_q(t) := \int_{D_t} G(o,x)^q \, dV(x) = -\int_t^{\infty} s^q \, d\Phi(s).$$

From (2.15) we know that the above-defined Φ obeys the differential inequality required in Lemma 2.1 with $c = \kappa_n$, and consequently use Lemma 2.1 (i) to achieve (2.10). The equality of (2.10) follows from a direct computation with the precise formula (1.5) of Green's function of $B_r(o)$ associated to Δ.

(ii) This follows from Lemma 2.1 (ii), the just-checked (i), and (1.4) which determines the radius R_o under $L = \Delta$:

$$R_o = \lim_{x \to o} \begin{cases} |o-x| \exp[2\pi G(o,x)] & \text{when} \quad n = 2 \\ \left[|o-x|^{2-n} - n(n-2)\sigma_n G(o,x)\right]^{\frac{1}{2-n}} & \text{when} \quad n > 2. \end{cases}$$

□

3. The second monotonicity principle.

3.1. A monotone integration for the gradients of Green's functions. Despite being still reduced to a one-dimensional sharp estimate, the monotonicity of $\mathbf{II}(o, D, q, \lambda)$ will be derived without introducing any additional assertion similar to Lemma 2.1.

THEOREM 3.1. *Let the uniformly elliptic operator L and the bounded domain D be so smooth that $G(o,\cdot) = G_{L,D}(o,\cdot)$ exists.*

(i) *If $0 \leq q_2 < q_1 \leq 1 < \frac{n}{n-1}$ then*

$$\mathbf{II}(o, D, q_1, \lambda) \leq \mathbf{II}(o, D, q_2, \lambda) \tag{3.1}$$

where inequality in (3.1) becomes equality when $L = \Delta$ and $D = B_r(o)$.

(ii) *If $0 \leq q < \frac{n}{n-1}$, $t \in [0, \infty)$ and $D_t = \{x \in D : \ G(o, x) > t\}$ then*

$$R_{o,\mathrm{II},\lambda} := \left[\sigma_n^{-1} \liminf_{q \to \frac{n}{n-1}} \mathbf{II}(o, D, q, \lambda)\right]^{\frac{1}{n}}$$

defines the type II *radius of D with respect to $o \in D$. Consequently*

$$\sigma_n R^n_{o,\mathrm{II},\lambda} \leq \mathbf{II}(o, D, q, \lambda) \leq V(D) \tag{3.2}$$

where equalities in (3.2) occur and so $R_{o,\mathrm{II},\lambda} = R_o$ whenever $L = \Delta$ and $D = B_r(o)$. Moreover

$$1 = \lim_{t \to \infty} \begin{cases} \dfrac{\int_{\partial D_t} |\nabla G(o, \cdot)|^{q-1} \, dS(\cdot)}{(2\pi e^{-2\pi t} R_o)^{2-q}} & \text{when} \quad n = 2 \\ \dfrac{\int_{\partial D_t} |\nabla G(o, \cdot)|^{q-1} \, dS(\cdot)}{\left(n\sigma_n [n(n-2)\sigma_n t + R_o^{2-n}]^{\frac{n-1}{2-n}}\right)^{2-q}} & \text{when} \quad n > 2 \end{cases}$$

is valid for $L = \Delta$.

Proof. (i) In the sequel, let $0 \leq q < \frac{n}{n-1}$, $t \in [0, \infty)$ and

$$\Lambda_q(t) = \int_{D_t} |\nabla G(o, \cdot)|^q \, dV(\cdot).$$

By the co-area formula, we get

$$\frac{d}{dt}\Lambda_q(t) = -\int_{\partial D_t} |\nabla G(o, x)|^{q-1} \, dS(x).$$

So,

$$\mathrm{II}(o, D, q, \lambda) = -\int_0^\infty \frac{d}{dt}\Lambda_q(t) \, dt.$$

By (2.14), Cauchy-Schwarz's inequality and (2.13) we obtain

$$\begin{aligned} &\Lambda_0(t)^{\frac{n-1}{n}} \leq (n\sigma_n^{\frac{1}{n}})^{-1} S(\partial D_t) \\ &\leq (n\sigma_n^{\frac{1}{n}})^{-1} \left[\int_{\partial D_t} |\nabla G(o, x)|^{-1} dS(x)\right]^{\frac{1}{2}} \left[\int_{\partial D_t} |\nabla G(o, x)| dS(x)\right]^{\frac{1}{2}} \\ &\leq \frac{\sqrt{-\frac{d}{dt}\Lambda_0(t)}}{(n\sigma_n^{\frac{1}{n}})\sqrt{\lambda}}. \end{aligned} \tag{3.3}$$

Meanwhile, we employ Hölder's inequality and (2.13) again to obtain

$$-\frac{d}{dt}\Lambda_q(t) \leq \lambda^{-\frac{q}{2}} \Big[-\frac{d}{dt}\Lambda_0(t)\Big]^{1-\frac{q}{2}}. \tag{3.4}$$

To continue, we apply (3.4) and (3.3) to get

$$\lambda\Lambda_q(s) = \int_s^\infty \Big[-\lambda\frac{d}{dt}\Lambda_q(t)\Big]\,dt \tag{3.5}$$
$$\le \lambda^{1-\frac{q}{2}}\int_s^\infty \Big[-\frac{d}{dt}\Lambda_0(t)\Big]^{1-\frac{q}{2}}\,dt$$
$$\le \lambda(\lambda n\sigma_n^{\frac{1}{n}})^{-q}\Big[\frac{n}{n-q(n-1)}\Big]\Lambda_0(s)^{\frac{n-q(n-1)}{n}}.$$

Both (3.5) and (3.3) produce

$$-\frac{d}{dt}\Lambda_0(t) \ge \gamma_{q,n}\Lambda_q(t)^{\frac{2(n-1)}{n-q(n-1)}}. \tag{3.6}$$

In the above and below,

$$\gamma_{q,n} := \lambda^{\frac{n+q(n-1)}{n-q(n-1)}}(n\sigma_n^{\frac{1}{n}})^{\frac{2n}{n-q(n-1)}}\Big[1-\frac{q(n-1)}{n}\Big]^{\frac{2(n-1)}{n-q(n-1)}}.$$

An application of (2.13) and Hölder's inequality derives that if $0 \le q_2 < q_1 < \frac{n}{n-1}$ then

$$-\frac{d}{dt}\Lambda_{q_1}(t) \le \lambda^{\frac{q_2-q_1}{2-q_2}}\Big[\int_{\partial D_t}|\nabla G(o,\cdot)|^{q_2-1}\,dS(\cdot)\Big]^{\frac{2-q_1}{2-q_2}}$$

and hence

$$\Big[-\lambda\frac{d}{dt}\Lambda_{q_1}(t)\Big]^{\frac{1}{2-q_1}} \le \Big[-\lambda\frac{d}{dt}\Lambda_{q_2}(t)\Big]^{\frac{1}{2-q_2}}. \tag{3.7}$$

Using (3.7) with $q_2 = q < 1 = q_1$, (2.14) and (3.5) we find

$$-\frac{d}{dt}\Lambda_q(t) \ge \delta_{q,n}\Lambda_q(t)^{\frac{(n-1)(2-q)}{n-q(n-1)}} \quad \forall \quad q\in[0,1], \tag{3.8}$$

where

$$\delta_{q,n} := \lambda^{-1}(\lambda n\sigma_n^{\frac{1}{n}})^{\frac{n(2-q)}{n-q(n-1)}}\Big[1-\frac{q(n-1)}{n}\Big]^{\frac{(n-1)(2-q)}{n-q(n-1)}}.$$

As a consequence of (3.7) and (3.8), we further obtain that if $0 \le q_2 < q_1 \le 1 < \frac{n}{n-1}$ then

$$\Lambda_{q_1}(0) = -\int_0^\infty \frac{d}{dt}\big[\Lambda_{q_1}(t)\big]\,dt$$
$$\le -\int_0^\infty \Big[-\lambda\frac{d}{dt}\Lambda_{q_2}(t)\Big]^{\frac{q_2-q_1}{2-q_2}}\,d\Lambda_{q_2}(t)$$
$$\le -\lambda^{\frac{q_2-q_1}{2-q_2}}\int_0^\infty \Big[\delta_{q_2,n}\big(\Lambda_{q_2}(t)\big)^{\frac{(n-1)(2-q_2)}{n-q_2(n-1)}}\Big]^{\frac{q_2-q_1}{2-q_2}}\,d\Lambda_{q_2}(t)$$
$$= \Big(\lambda\gamma_{q_2,n}\Big)^{\frac{q_2-q_1}{2}}\Big[\frac{n-q_2(n-1)}{n-q_1(n-1)}\Big]\big[\Lambda_{q_2}(0)\big]^{\frac{n-q_1(n-1)}{n-q_2(n-1)}}.$$

A simplification of the above estimates gives the desired inequality. In addition to this, the equality case can be checked through a direct computation with the explicit formula (1.5) of Green's function of $B_r(o)$ attached to Δ.

(ii) Clearly, $R_{o,\mathrm{II},\lambda}$ makes sense, enjoys (3.2), and equals R_o whenever $L=\Delta$ and $D=B_r(o)$, thereby assuring $\mathbf{II}(o,D,q,\lambda)=V(D)$.

Next, suppose $L=\Delta$. Then $\lambda=1$. Two cases are considered in what follows.

Case 1: $n=2$. Under this condition, we employ (1.4) to obtain

$$G(o,x)=(2\pi)^{-1}\ln\frac{R_o}{|o-x|}+H(o,x),$$

whence finding

$$|\nabla G(o,x)|=(2\pi|o-x|)^{-1}+o(1)\quad\text{as}\quad |o-x|\to 0.$$

Furthermore, if $G(o,x)=t$, then

$$R_o=|o-x|e^{2\pi t}+o(1)\quad\text{as}\quad |o-x|\to 0,$$

and hence

$$\int_{\partial D_t}|\nabla G(o,x)|^{q-1}dS(x)=(2\pi R_o e^{-2\pi t})^{2-q}+o(1)\quad\text{as}\quad t\to\infty.$$

This verifies the desired limit formula for $n=2$.

Case 2: $n>2$. Under this assumption, we read from (1.4) that

$$G(o,x)=\frac{|o-x|^{2-n}-R_o^{2-n}}{n(n-2)\sigma_n}+H(o,x),$$

and so that

$$|\nabla G(o,x)|=(n\sigma_n)^{-1}|o-x|^{1-n}+o(1)\quad\text{as}\quad |o-x|\to 0.$$

When $G(o,x)=t$, we also have

$$R_o=\big[|o-x|^{2-n}-n(n-2)\sigma_n t\big]^{\frac{1}{2-n}}+o(1)\quad\text{as}\quad |o-x|\to 0,$$

thereby getting

$$\int_{\partial D_t}|\nabla G(o,x)|^{q-1}dS(x)=\Big(n\sigma_n\big[n(n-2)\sigma_n t+R_o^{2-n}\big]^{\frac{n-1}{2-n}}\Big)^{2-q}+o(1)\text{ as }t\to\infty.$$

Obviously, this last estimate yields the desired limit formula for $n>2$. □

3.2. Two sharp Sobolev-like inequalities. Totally motivated by Theorems 2.2 & 3.1 and their arguments, we figure out two interesting Sobolev-like inequalities with sharp constants.

COROLLARY 3.2. *Let the uniformly elliptic operator L and the bounded domain D be so smooth that $G(o,\cdot)=G_{L,D}(o\cdot)$ exists. For $0\le q\le 1<\frac{n}{n-1}$ and $0\le p<\frac{n}{n-2}-\frac{q(n-1)}{n-2}$ set*

$$\eta_{p,q,n}:=\begin{cases}\delta_{q,2}^{-p}\Gamma(p+1) & \textit{when}\quad n=2\\ \delta_{q,n}^{-p}\Big[\frac{n-q(n-1)}{n-2}\Big]^{p+1}B\Big(\frac{(n-1)(2-q)}{n-2}-p-1,p+1\Big) & \textit{when}\quad n>2.\end{cases}$$

Then

(i)

$$\mathbf{I}(o,D,p,\lambda)\le \left(n\sigma_n^{\frac{1}{n}}\right)^{\frac{n}{1-n}}\lambda^{\frac{(q-1)n}{(2-q)(n-1)}}\left(\int_{\partial D}|\nabla G(o,\cdot)|^{q-1}\,dS(\cdot)\right)^{\frac{n}{(n-1)(2-q)}}$$

with equality if $L=\Delta$ *and* $D=B_r(o)$.

(ii)

$$\int_D G(o,\cdot)^p|\nabla G(o,\cdot)|^q\,dV(\cdot)\le \frac{\eta_{p,q,n}}{\left(\frac{n\left(\lambda n\sigma_n^{\frac{1}{n}}\right)^q}{n-q(n-1)}\right)^{\frac{p(n-2)}{n-q(n-1)}-1}}\big[\mathbf{II}(o,D,q,\lambda)\big]^{n-q(n-1)-p(n-2)}$$

with equality if $L=\Delta$ *and* $D=B_r(o)$.

Proof. (i) This follows immediately from (3.7) and

$$\mathbf{I}(o,D,p,\lambda)\le V(D)\le\left[\frac{S(\partial D)}{n\sigma_n^{\frac{1}{n}}}\right]^{\frac{n}{n-1}}.$$

(ii) Keeping the notation $\Lambda_q(\cdot)$, we integrate (3.8) with respect to dt to get the following inequality for $t>r\ge 0$:

$$\Lambda_q(t)\le\begin{cases}\Lambda_q(r)\exp[-\delta_{q,2}(t-r)] & \text{when}\quad n=2\\ \left[\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}+\frac{(n-2)\delta_{q,n}}{n-q(n-1)}(t-r)\right]^{\frac{n-q(n-1)}{2-n}} & \text{when}\quad n>2.\end{cases}\tag{3.9}$$

So, if $d\mu_q:=|\nabla G(o,\cdot)|^q dV(\cdot)$ then by substitution and integration-by-parts we have

$$\begin{aligned}&\int_{D_r}G(o,\cdot)^p|\nabla G(o,\cdot)|^q\,dV(\cdot)=\int_r^\infty\mu_q(D_t)\,dt^p\\&=-\int_r^\infty t^p\,d\mu_q(D_t)\\&=-\int_r^\infty t^p\,d\Lambda_q(t)\\&=r^p\Lambda_q(r)+p\int_r^\infty\Lambda_q(t)t^{p-1}\,dt.\end{aligned}$$

Case 1: $n=2$. Regarding this, we get from and the above upper bound estimate (3.9) for $\Lambda_q(t)$ and integration-by-parts,

$$\begin{aligned}&p\int_r^\infty\Lambda_q(t)t^{p-1}\,dt\le e^{\delta_{q,2}r}\Lambda_q(r)\Big(-r^pe^{-\delta_{q,2}r}+\delta_{q,2}\int_r^\infty t^pe^{-\delta_{q,2}t}\,dt\Big)\\&=-r^p\Lambda_q(r)+\delta_{q,2}^{-p}e^{\delta_{q,2}r}\Lambda_q(r)\int_{\delta_{q,2}r}^\infty e^{-t}t^p\,dt\\&\le -r^p\Lambda_q(r)+\delta_{q,2}^{-p}e^{\delta_{q,2}r}\Lambda_q(r)\Gamma(p+1).\end{aligned}$$

Case 2: $n>2$. Concerning this, let $\tau_{q,n}:=\frac{(n-2)\delta_{q,n}}{n-q(n-1)}$. Similarly, we get from (3.9) and an integration-by-parts,

$$
\begin{aligned}
p\int_r^\infty \Lambda_q(t)t^{p-1}\,dt &\le p\int_r^\infty \left[\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}+\tau_{q,n}(t-r)\right]^{\frac{n-q(n-1)}{2-n}} t^{p-1}\,dt\\
&= -r^p\Lambda_q(r)+\delta_{q,n}\int_r^\infty t^p\left[\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}+\tau_{q,n}(t-r)\right]^{\frac{n-q(n-1)}{2-n}-1}dt\\
&= -r^p\Lambda_q(r)+\delta_{q,n}\left[\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}-\tau_{q,n}r\right]^{\frac{n-q(n-1)}{2-n}-1}\times\\
&\times\int_r^\infty t^p\left[1+\frac{\tau_{q,n}t}{\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}-\tau_{q,n}r}\right]^{\frac{(2-q)(n-1)}{2-n}}dt\\
&\le -r^p\Lambda_q(r)+\frac{\delta_{q,n}^{-p}}{\left(\frac{n-2}{n-q(n-1)}\right)^{p+1}}\left[\Lambda_q(r)^{\frac{2-n}{n-q(n-1)}}-\tau_{q,n}r\right]^{p+\frac{n-q(n-1)}{2-n}}\times\\
&\times B\left(\frac{(n-1)(2-q)}{n-2}-p-1,p+1\right).
\end{aligned}
$$

A combination of the above two cases with $r=0$ gives the desired inequality. Moreover, if $L=\Delta$ and $D=B_{r_0}(o)$ (for some $r_0>0$) then the inequalities (under $r=0$) stated in the above argument become equalities, and hence the equality in Corollary 3.2 (ii) holds in this case. □

4. Applications.

4.1. Two new optimal isoperimetric inequalities via Green's functions.

A consideration of the cases $q<0$ of $\mathbf{I}(o,D,q,\lambda)$ and $\mathbf{II}(o,D,q,\lambda)$, whenever L and D are so smooth that $G(o,\cdot)=G_{L,D}(o,\cdot)$ exists, reveals (by Hölder's inequality) the following inequalities

$$
(4.1)\qquad \left(\int_D\frac{V(D)^{-1}dV(\cdot)}{G(o,\cdot)^p}\right)^{\frac1p}\le\left(\int_D\frac{V(D)^{-1}dV(\cdot)}{G(o,\cdot)^q}\right)^{\frac1q}\quad\forall\ 0<p<q<\infty
$$

and

$$
(4.2)\qquad \left(\int_D\frac{V(D)^{-1}dV(\cdot)}{|\nabla G(o,\cdot)|^p}\right)^{\frac1p}\le\left(\int_D\frac{V(D)^{-1}dV(\cdot)}{|\nabla G(o,\cdot)|^q}\right)^{\frac1q}\quad\forall\ 0<p<q<\infty
$$

with equalities in (4.1) and (4.2) respectively if and only if $G(o,\cdot)$ and $|\nabla G(o,\cdot)|$ are constants on D respectively. But, (1.3) clearly shows that the equality cases cannot happen at all. Namely, (4.1) and (4.2) are actually strict. A similar argument plus (2.13) derives

$$
(4.3)\qquad S(\partial D_t)=\int_{\partial D_t}dS\le\lambda^{-\alpha}\left(\int_{\partial D_t}|\nabla G(o,\cdot)|^{\frac{\alpha}{\alpha-1}}\,dS(\cdot)\right)^{1-\alpha}\quad\forall\ \alpha\in(0,1)
$$

with equality if and only if $|\nabla G(o,\cdot)|=[\lambda S(\partial D_t)]^{-1}$ on ∂D_t.

An application of Hölder's inequality along with (4.3) yields the following monotonicity estimate

$$
(4.4)\qquad \left(\int_{\partial D_t}\frac{\lambda\,dS(\cdot)}{|\nabla G(o,\cdot)|^p}\right)^{\frac{1}{1+p}}\le\left(\int_{\partial D_t}\frac{\lambda\,dS(\cdot)}{|\nabla G(o,\cdot)|^q}\right)^{\frac{1}{1+q}}\quad\forall\ 0<p<q<\infty
$$

with the equality in (4.4) if and only if $|\nabla G(o,\cdot)| = [\lambda S(\partial D_t)]^{-1}$ on ∂D_t – this can certainly happen, for example, when $L = \Delta$ and $D = B_r(o)$.

Furthermore, we have the following new sharp isoperimetric inequalities.

PROPOSITION 4.1. *Let the uniformly elliptic operator L and the bounded domain D are so smooth that $G(o,\cdot) = G_{L,D}(o,\cdot)$ exists.*

(i) *If $0 \le q < \frac{n}{n-2}$ and $0 < \alpha < 1$ then*

$$\int_D G(o,\cdot)^q \, dV(\cdot) \le \lambda^{\frac{-\alpha}{2+\alpha}} \int_0^\infty t^q \left(\int_{\partial D_t} |\nabla G(o,\cdot)|^{-\alpha-1} \, dS(\cdot) \right)^{\frac{2}{2+\alpha}} dt$$

with equality when $L = \Delta$ and $D = B_r(o)$.

(ii) *If $0 \le q < \frac{n}{n-1}$ then*

$$\int_{\partial D} |\nabla G(o,\cdot)|^{-1} \, dS(\cdot) \ge \gamma_{q,n} \left(\int_D |\nabla G|^q \, dV \right)^{\frac{2(n-1)}{n-q(n-1)}}$$

with equality when $L = \Delta$ and $D = B_r(o)$.

Proof. (i) An immediate application of (4.4) yields

$$\begin{aligned} -\frac{d}{dt} \int_{D_t} G(o,\cdot)^q \, dV(\cdot) &= t^q \int_{\partial D_t} |\nabla G(o,\cdot)|^{-1} \, dS(\cdot) \\ &\le t^q \left(\lambda^{-\frac{\alpha}{2}} \int_{\partial D_t} |\nabla G(o,\cdot)|^{-\alpha-1} \, dS(\cdot) \right)^{\frac{2}{2+\alpha}} . \end{aligned}$$

An integration with respect to $t \in [0,\infty)$ derives the desired inequality whose equality case is obvious.

(ii) This follows from the special case $t = 0$ of (3.6). □

As the endpoint $q = 0$ of (i) and (ii), the following sharp isoperimetric inequalities are very natural (cf. [6, p.53]):

$$V(D) \le \lambda^{-\frac{\alpha}{2+\alpha}} \int_0^\infty \left(\int_{\partial D_t} |\nabla G(o,\cdot)|^{-\alpha-1} \, dS(\cdot) \right)^{\frac{2}{2+\alpha}} dt \quad \forall \quad \alpha \in (0,1)$$

and

$$V(D) \le \left(\sqrt{\lambda} n \sigma_n^{\frac{1}{n}} \right)^{-\frac{n}{n-1}} \left(\int_{\partial D} |\nabla G(o,\cdot)|^{-1} \, dS(\cdot) \right)^{\frac{n}{2(n-1)}}$$

which can be also established via (2.14) and (4.3) (with $t = 0$).

4.2. The lowest eigenvalue of an elliptic operator & Pólya's conjecture. According to [2, p.110], if there exists another constant $\Lambda \ge \lambda$ such that the following strongly uniform ellipticity condition

$$\Lambda|\xi|^2 \ge \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \forall \; \big(x, \xi = (\xi_1, ..., \xi_n)\big) \in \mathbb{R}^n \times \mathbb{R}^n \tag{4.5}$$

holds, then under some suitable regularity conditions (say, C^∞) on this elliptic operator L and the bounded domain D, the solution pair (u, λ) to

$$-Lu = \lambda u \quad \text{in} \quad D \quad \text{subject to} \quad u = 0 \quad \text{on} \quad \partial D \tag{4.6}$$

is decided by the extreme function of the following minimizing problem

$$\lambda_1(L,D) := \inf_{v\in H_0^1(D)} \left(\int_D v^2\, dV\right)^{-1} \int_D \sum_{i,j=1}^n a_{ij}(\cdot)\Big(\frac{\partial v}{\partial x_i}\Big)\Big(\frac{\partial v}{\partial x_j}\Big)\, dV(\cdot), \tag{4.7}$$

where $H_0^1(D)$ is the Sobolev space defined as the closure of all C^∞ smooth functions with compact support in D that are square integrable with square integrable derivatives.

PROPOSITION 4.2. *With (4.5), (4.6) and (4.7), one has*

$$2n\lambda\sigma_n^{\frac{2}{n}} \le \lambda_1(L,D)V(D)^{\frac{2}{n}}.$$

In particular, the following Pólya's conjecture (cf. [10, p.305] and [9])

$$(2\pi)^2\sigma_n^{-\frac{2}{n}} \le \lambda_1(\Delta,D)V(D)^{\frac{2}{n}}$$

is true for the lower dimensions $n = 2, 3, 4$.

Proof. Assume that $u \in H_0^1(D)$ enjoys $-Lu = \lambda_1(L,D)u$ in D and $u|_{\partial D} = 0$. Via a limit argument, we may assume that L and D are so smooth that $G(o,\cdot) = G_{L,D}(o,\cdot)$ exists. Then, an application of (1.2) and Theorem 2.2 (i) derives

$$\begin{aligned} u(o) &= \lambda_1(L,D)\int_D uG(o,\cdot)\, dV(\cdot) \\ &\le \lambda_1(L,D)\big[\sup_{x\in D} u(x)\big]\int_D G(o,\cdot)\, dV(\cdot) \\ &\le \lambda_1(L,D)\big[\sup_{x\in D} u(x)\big]\left[\frac{V(D)^{\frac{2}{n}}}{2n\lambda\sigma_n^{\frac{2}{n}}}\right], \end{aligned}$$

and so

$$1 \le \lambda_1(L,D)\left[\frac{V(D)^{\frac{2}{n}}}{2n\lambda\sigma_n^{\frac{2}{n}}}\right]$$

which gives the desired inequality.

Since

$$2n\sigma_n^{\frac{2}{n}} \ge (2\pi)^2\sigma_n^{-\frac{2}{n}}$$

holds for $n = 2, 3, 4$, Pólya's conjecture is true for those lower dimensions. □

However, for the higher dimensions $n \ge 5$ the above Pölya's conjecture is still open; see also [10, p.305]. Interestingly, an argument similar to the above can be found in the paper [20] by G.-J. Tian and X.-J. Wang.

4.3. A sharp eccentricity-based lower bound for the Mahler volumes. Due to Proposition 4.2 and its proof, we naturally recall the following Faber-Krahn inequality under (4.5) (cf. [7] or [2, p.111, Theorem 3.3])

$$\lambda_1(L,D) \ge \lambda\Big[\frac{\sigma_n}{V(D)}\Big]^{\frac{2}{n}} \mathrm{j}^2_{\frac{n-2}{2}} \tag{4.8}$$

with equality if and only if $D = B_r(o)$ and (a_{ij}) is λ times the identity matrix (δ_{ij}), where $\mathsf{j}_{\frac{n-2}{2}}$ is the first zero of the Bessel function of order $\frac{n-2}{2}$. Surprisingly, this review produces a way to sharpen an eccentricity-based lower bound for the Mahler volumes of the origin-symmetric convex bodies.

PROPOSITION 4.3. *Suppose that D is a convex body (open bounded convex set) and symmetric with respect to the origin. For the unit ball B of $\mathbb{R}^n$ define the circumradius $R(D)$ and the inradius $r(D)$ of D to be the best quantities such that*

$$r(D)B := \{x \in \mathbb{R}^n : |x| < r(D)\} \subseteq D \subseteq \{x \in \mathbb{R}^n : |x| < R(D)\} =: R(D)B$$

and write $e(D) := R(D)/r(D)$ for the eccentricity of D. If

$$D^\circ := \{y \in \mathbb{R}^n : |\langle x, y\rangle| < 1 \quad \forall \quad x \in D\}$$

is the polar body of D, then the Mahler volume

$$M(D) := V(D)V(D^\circ)$$

is not less than $e(D)^{-n}\sigma_n^2$ (cf. [19]). Moreover, $M(D)$ equals $e(D)^{-n}\sigma_n^2$ if and only if D is an origin-centered ball.

Proof. Although the first part of the conclusion is known, in order to verify the second part of the conclusion, we use (4.8) with $L = \Delta$ to give an alternative proof for $M(D) \ge e(D)^{-n}\sigma_n^2$. In fact, (4.8) tells us

$$\lambda_1(\Delta, D^\circ)V(D^\circ)^{\frac{2}{n}} \ge \mathsf{j}^2_{\frac{n-2}{2}}\sigma_n^{\frac{2}{n}} \tag{4.9}$$

with equality if and only if D° is an origin-centered ball. Without loss of generality, we may assume

$$e(D)^{-\frac{1}{2}}B \subseteq D \subseteq e(D)^{\frac{1}{2}}B.$$

Then

$$e(D)^{-\frac{1}{2}}B \subseteq D^\circ \subseteq e(D)^{\frac{1}{2}}B.$$

Also because of

$$\lambda_1(\Delta, \rho B) = \left(\rho^{-1}\mathsf{j}_{\frac{n-2}{2}}\right)^2 \quad \forall \quad \rho > 0,$$

we have by the monotonicity of $\lambda_1(\Delta, \cdot)$,

$$\left(\mathsf{j}_{\frac{n-2}{2}}e(D)^{-\frac{1}{2}}\right)^2 \le \lambda_1(\Delta, D^\circ) \le \left(\mathsf{j}_{\frac{n-2}{2}}e(D)^{\frac{1}{2}}\right)^2 \tag{4.10}$$

thereby getting via (4.9),

$$V(D^\circ) \ge \sigma_n e(D)^{-\frac{n}{2}}.$$

This yields

$$M(D) \ge V(D^\circ)\sigma_n e(D)^{-\frac{n}{2}} \ge \sigma_n^2 e(D)^{-n} \tag{4.11}$$

as desired.

The proof of the second part is completed via the following argument. If $M(D) = e(D)^{-n}\sigma_n^2$, then (4.11) gives

$$V(D^\circ) = \sigma_n e(D)^{-\frac{n}{2}}.$$

This, along with (4.9) and the most right inequality of (4.10), deduces

$$\mathsf{j}^2_{\frac{n-2}{2}}\sigma_n^{\frac{2}{n}} \le \lambda_1(\Delta, D^\circ)V(D^\circ)^{\frac{2}{n}} \le \mathsf{j}^2_{\frac{n-2}{2}}\sigma_n^{\frac{2}{n}}$$

and so

$$V(D^\circ)^{\frac{2}{n}}\lambda_1(\Delta, D^\circ) = \sigma_n^{\frac{2}{n}}\mathsf{j}^2_{\frac{n-2}{2}}.$$

As a result of the equality situation of (4.9), we see that $D^\circ = rB$ for some $r > 0$, and so is D. □

Here, it should be pointed out that the Santaló inequality $M(D) \le M(B)$ is always valid for any origin-symmetric convex body D (cf. [15]). And, it would be very interesting to find out a pass from $\lambda_1(\Delta, D^\circ)$ or $\lambda_1(\Delta, D)$ to the Mahler conjecture:

$$M(D) \ge M(Q) = \frac{4^n}{\Gamma(n+1)} \quad \forall \quad \text{origin-symmetric convex body} \quad D,$$

where $Q \subset \mathbb{R}^n$ stands for the unit cube centered at the origin. Though the Mahler conjecture is still open in general, several important steps: [12]; [13]; [8]; [5]; [11], have approached toward this conjecture.

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REFERENCES

[1] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. Math., 128 (1988), pp. 385–398.

[2] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman Monographs and Studies in Mathematics, 1980.

[3] C. Bandle, A. Brillard and M. Flucher, *Green's function, harmonic transplantation, and best Sobolev constant in spaces of constant curvature*, Trans. Amer. Math. Soc., 350 (1998), pp. 1103–1128.

[4] C. Bandle and M. Flucher, *Harmonic radius and concentration of energy, hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{\frac{n+2}{n-2}}$, SIAM Review, 38 (1996), pp. 191–238.

[5] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Invent. Math., 88 (1987), pp. 319–340.

[6] I. Chavel, *Isoperimetric inequalities. Differential geometric and analytic perspectives*, Cambridge Tracts in Mathematics, 145, Cambridge University Press, Cambridge, 2001.

[7] E. Krahn, *Ueber Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat), A9 (1926), pp. 1–144.

[8] G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals.* Geom. Funct. Anal. **18** (2008), 870–892.

[9] P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys., 88 (1983), pp. 309–318.

[10] E. H. Lieb and M. Loss, *Analysis, second edition*, GSM, vol. 14, Amer. Math. Soc. Providence, Rhode Island, 2001.

[11] E. Lutwak, D. Yang and G. Zhang, *A volume inequality for polar bodies*, J. Differential Geom., 84 (2010), pp. 163–178.

[12] F. Nazarov, *The Hörmander proof of the Bounrgain-Milman theorem*, Preprint, (2010), 8 pages.

[13] F. Nazarov, F. Petrov, D. Ryabogin and A. Zvavitch, *A remark on the Mahler conjecture: local minimality of the unit cube*, Duke Math. J., 154 (2010), pp. 419–430.

[14] M. Ritoré and C. Sinestrari, *Mean Curvature Flow and Isoperimetric Inequalities*, Advanced Courses in Math CRM Barcelona, Birkhäuser Verlag, 2010.

[15] L. Santaló, *Un invariante afin para los cuerpos convexos del espacio de n dimensiones,* Portugal Math., 8 (1949), pp. 155–161.

[16] F. Schulze, *Nonlinear evolution by mean curbature and isoperimetric inequalities*, J. Differential Geom., 79 (2008), pp. 197–241.

[17] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Mat. Analysis, vol. 3, Australia National Univ., Centre for Math. Analysis, Canberra, 1983.

[18] G. Stampacchia, *Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per equazioni del secondo ordine ellitiche*, Ann. Scuola Norm. Super. Pisa (s.3), 12 (1958), pp. 223–244.

[19] T. Tao, *Santalo's inequality*, http://terrytao.wordpress.com/2007/03/08/open-problem-the-mahler-conjecture-on-convex-bodies/.

[20] G.-J. Tian and X.-J. Wang, *A class of Sobolev type inequalities*, Methods Appl. Anal., 15 (2008), pp. 263–276.

[21] H. F. Weinberger, *Symmetrization in uniformly elliptic problems*, 1962 Studies in mathematical analysis and related topics pp. 424–428 Stanford Univ. Press, Stanford, Calif.