ASIAN J. MATH.
Vol. 16, No. 2, pp. 271–278, June 2012

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ON THE CONJECTURE OF KOSNIOWSKI*

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Abstract. The aim of this paper is to address some results closely related to the conjecture of Kosniowski about the number of fixed points on a unitary S^1-manifold with only isolated fixed points. More precisely, if certain S^1-equivariant Chern characteristic number of a unitary S^1-manifold M is non-zero, we give a sharp (in certan cases) lower bound on the number of isolated fixed points in terms of certain integer powers in the S^1-equivariant Chern number. In addition, we also deal with the case of oriented unitary T^n-manifolds.

Key words. Unitary G-manifolds, ABBV localization theorem, isolated fixed points, Kosniowski's conjecture.

AMS subject classifications. 57S25, 55N91.

1. Introduction and main results. A smooth manifold equipped with a complex vector bundle structure on the stable tangent bundle is called a *unitary manifold* or *stable complex manifold*, while a smooth manifold equipped with a complex vector bundle structure on the tangent bundle is called an *almost complex manifold*. If a Lie group G acts smoothly on a unitary manifold (resp., almost complex manifold) and if the differential of each element of G preserves the given complex vector bundle structure, then M is called a *unitary G-manifold* (resp., *almost complex G-manifold*). In particular, a *unitary torus manifold* (or *unitary toric manifold*) is a closed oriented stable complex manifold of real dimension $2n$ admitting an effective T^n-action with a non-empty fixed-point set. In fact, in this case the fixed-point set consists of only isolated points, since the action of T^n is assumed to be effective.

Now let S^1 act on a closed connected manifold whose fixed points are all isolated. Since the tangent space at a fixed point has the complex structure determined by the isotropy representation of S^1, the real dimension of M is always even, although the complex structure is not canonical, in general. However, it is the case if the manifold is a unitary S^1-manifold. Let P be an isolated fixed point. Then the tangent space T_PM has two orientations: one induced from the orientation of M and the other induced from the complex structure. We define the sign of the point P by

$$\varepsilon(P) = \pm 1,$$

depending on whether or not these two orientations agree. Note that if the manifold M is an almost complex S^1-manifold, then we always have $\varepsilon(P) = 1$.

The aim of this paper is to address some results closely related to the following conjecture of Kosniowski ([4], Conjecture A) about the number of fixed points on a unitary S^1-manifold with only isolated fixed points.

CONJECTURE 1.1. *Let M be a connected oriented closed unitary S^1-manifold of dimension $2n$ with only isolated fixed points. If M does not bound a unitary S^1-manifold equivariantly, then the number of isolated fixed points is greater than or equal to a linear function $f(n)$ of n.*

*Received June 13, 2011; accepted for publication June 24, 2011.
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According to the paper [4] of Kosniowski, the linear function $f(n)$ is expected to be $\frac{n}{2}$. This conjecture suggests that the number of fixed points is large if the dimension is large and if the manifold is not a boundary. In view of our results of this paper (e.g., Theorem 1.3 below), however, his conjecture seems to be a little bit rough. Related, but not directly, to this conjecture, note also that recently there are some works ([10], [7], [6]) by Pelayo and Tolman, Li and Liu, and Li (see also [8]).

On the other hand, as in the book ([3], Appendix D, Section 1.5) of Guillemin, Ginzburg, and Karshon concerning the equivariant boundedness of a stable complex T^k-manifold, cobordant oriented stable complex manifolds have the same characteristic numbers, and the converse is also true. For the sake of clarity and later reference, we state this fact as follows.

Theorem 1.2. *For each $1 \le k \le n$, let M be a closed oriented stable complex manifold admitting a T^k-action with isolated fixed points. Then M bounds a unitary T^k-manifold equivariantly if and only if all T^k-equivariant Chern characteristic numbers of M are zero.*

With these understood, our first main result that can be regarded as a more refinement of the conjecture of Kosniowski is

Theorem 1.3. *Let M be a unitary S^1-manifold of dimension $2n$ with only isolated fixed points. Let $i_1, i_2, \cdots, i_n$ be non-negative integers such that $i_1 + 2i_2 + \cdots + ni_n = n$. Suppose that M does not bound a unitary S^1-manifold equivariantly in such a way that*

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_n(M)^{i_n}, [M] \rangle \neq 0.$$

Then the number of isolated fixed points is greater than or equal to

$$\max\{i_1, i_2, \cdots, i_n\} + 1.$$

Here $c_i(M)$ means the i-th (S^1-equivariant) Chern class of M. The ideas that have been successfully used by Tolman-Weitsman, Pelyao-Tolman, and later Li-Liu in their papers ([11], [10], [7]) also play a crucial role in the proof of Theorem 1.3.

Note that S^6 has the standard almost complex structure which can be given by writing it as the quotient $G_2/SU(3)$ of the Lie group G_2 by $SU(3)$, and it can be provided with a suitable S^1-action with two isolated fixed points. Thus, S^6 equipped with the standard almost complex structure does not bound a unitary S^1-manifold equivariantly (see [4], p. 338 for details). Since the number of isolated fixed points of an effective S^1-action on S^6 is equal to the Euler-Poincaré characteristic of S^6, $\langle c_3(S^6), [S^6] \rangle$ is equal to two that is clearly non-zero. Hence, the example of S^6 shows that the lower bound of Theorem 1.3 is very sharp in this special case. In Section 2, we will also provide more concrete and interesting case concerning the conjecture of Kosniowski (see Corollary 2.4).

As an easy consequence of Theorem 1.3, we can reprove a result of Hattori ([2], Corollary 4.3 or see also [7], Corollary 1.5).

Corollary 1.4. *Let M be a closed oriented unitary S^1-manifold of dimension $2n$ with only isolated fixed points. If*

$$\langle c_1(M)^n, [M] \rangle \neq 0,$$

then the number of the isolated fixed points is greater than or equal to $n+1$.

As another special case, the following corollary holds.

COROLLARY 1.5. *Let M be a closed oriented unitary S^1-manifold of dimension $2n$ with non-empty isolated fixed points. Suppose that M does not bound a unitary S^1-manifold equivariantly. Then the number of the isolated fixed points is greater than or equal to* 3, *unless n is equal to* 1 *or* 3.

Proof. Since the number of isolated fixed points of an effective S^1-action is equal to the Euler-Poincaré characteristic of M, $\langle c_n(M), [M] \rangle$ is non-zero. Hence, it follows from Theorem 1.3 that the number of isolated fixed points is greater than or equal to 2. On the other hand, a corollary of Theorem 5 in [4] says that if M is a unitary S^1-manifold with two fixed points, then M bounds S^1-equivariantly or the dimension of M is two or six. Therefore, in our case M cannot have two isolated fixed points. This completes the proof of Corollary 1.5 □

It has been known that a circle action on an even dimensional manifold cannot have only one isolated fixed point (e.g., see [1], Proposition 3.3). Corollary 1.5 and its proof reproves this fact and more for certain unitary S^1-manifolds. Note also that the lower bound in Corollary 1.5 is very sharp, as the complex projective space $\mathbb{C}P^2$ clearly shows. Finally we remark that Corollary 1.5 continues to hold even for smooth S^1-manifolds (refer to [5], p. 31).

Our second main result which is also closely related to the conjecture of Kosniowski (Conjecture 1.1) and can be regarded as an immediate consequence of Theorem 1.3 is the following theorem.

THEOREM 1.6. *Let M be a closed oriented unitary T^n-manifold of dimension $2n$ with only isolated fixed points. Let $i_1, i_2, \cdots, i_n$ be non-negative integers such that $i_1 + 2i_2 + \cdots + ni_n = n$. Suppose that M does not bound a unitary T^n-manifold equivariantly in such a way that*

$$\langle (c_1^{T^n}(M))^{i_1} (c_2^{T^n}(M))^{i_2} \cdots (c_n^{T^n}(M))^{i_n}, [M] \rangle \neq 0.$$

Then the number of the isolated fixed points is greater than or equal to

$$\max\{i_1, i_2, \cdots, i_n\} + 1.$$

Here $c_i^{T^n}(M)$ means the i-th T^n-equivariant Chern class of M (see Section 3 for more details).

We organize this paper as follows. In Section 2, we give a lower bound on the number of fixed points for a unitary S^1-manifold with only isolated fixed points under the non-triviality of certain S^1-equivariant Chern characteristic number. In the same section, we also present a result related to the conjecture of Kosniowski, Conjecture 1.1 (see Corollary 2.4). In Section 3, in a similar vein we give a lower bound on the number of fixed points for a unitary T^n-manifold with only isolated fixed points.

2. Proof of Theorem 1.3: Unitary S^1-manifolds. The goal of this section is to first set up some basic notations and give some elementary materials for the later use. Then we give a proof of Theorem 1.3 which will play an important role in the proof of Theorem 1.6.

Let E be a complex vector bundle of rank m over a smooth manifold M of real dimension $2n$. For the sake of simplicity, let S^1 act on a unitary M whose fixed points are all isolated, and let P_1, P_2, $\cdots$, P_r denote all the fixed points. Now suppose that the S^1-action on M can be lifted to E. Then the fiber E_{P_i} at the point P_i is a complex S^1-module to which we can associate integer weights $a_1^{(i)}, a_2^{(i)}, \cdots, a_m^{(i)}$.

For instance, the tangent bundle TM can be taken to be such a complex vector bundle E, so that $T_{P_i}M$ can be written as $\oplus_{j=1}^n V_j^{(i)}$. Here $V_j^{(i)}$ is isomorphic to $\mathbb{C}$ by an isomorphism under which the representation of S^1 on $V_j^{(i)}$ is given by $t \mapsto t^{k_j^{(i)}}$ with some non-zero integer $k_j^{(i)}$. We may also assume without loss of generality that the integer weights $k_1^{(i)}, k_2^{(i)}, \cdots, k_n^{(i)}$ are chosen in such a way that the orientations on $V_j^{(i)} = \mathbb{C}$ induce the orientation of $T_{P_i}M$.

Let us denote by $\sigma_j(P_i)$ the j-th elementary symmetric function of n variables $k_1^{(i)}, k_2^{(i)}, \cdots, k_n^{(i)}$. Then the well-known ABBV localization theorem of Atiyah, Bott, Berline and Vergne can be stated as follows.

Theorem 2.1. *Let M be a unitary S^1-manifold of dimension $2n$ with only isolated fixed points $P_1, P_2, \cdots, P_r$. Assume that $i_1 + 2i_2 + \cdots + ni_n$ is equal to n for some non-negative integers $i_1, i_2, \cdots, i_n$. Then we have*

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_n(M)^{i_n}, [M]\rangle = \sum_{i=1}^{r} \varepsilon(P_i) \frac{\sigma_1(P_i)^{i_1} \sigma_2(P_i)^{i_2} \cdots \sigma_n(P_i)^{i_n}}{\prod_{j=1}^n k_j^{(i)}} \tag{2.1}$$

In order to prove the main theorem, we need to set up more notations. For each $1 \le j \le n$, let

$$\{\sigma_j(P_i)\}_{i=1}^r = \{\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{l_j}^{(j)}\}.$$

Note that $\tau_1^{(j)}, \tau_2^{(j)}, \cdots, \tau_{l_j}^{(j)}$ are mutually distinct by the very definition. Observe also that l_j is always less than or equal to r. Now we are ready to state and prove the following

Theorem 2.2. *Let M be a unitary S^1-manifold of dimension $2n$ with only isolated fixed points $P_1, P_2, \cdots, P_r$. Let $i_1, i_2, \cdots, i_n$ be non-negative integers such that $i_1 + 2i_2 + \cdots + ni_n = n$. Suppose that*

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_n(M)^{i_n}, [M]\rangle = \sum_{i=1}^{r} \varepsilon(P_i) \frac{\sigma_1(P_i)^{i_1} \sigma_2(P_i)^{i_2} \cdots \sigma_n(P_i)^{i_n}}{\prod_{j=1}^n k_j^{(i)}} \neq 0. \tag{2.2}$$

Then r is greater than or equal to $\max\{i_1, i_2, \cdots, i_n\} + 1$.

Proof. For the sake of simplicity, assume that $i_1, i_2, \cdots, i_k$ are all non-zero integers in the equation (2.2), since the proof of other cases is similar. Then it follows from the assumption (2.2) that

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_k(M)^{i_k}, [M]\rangle = \sum_{i=1}^{r} \varepsilon(P_i) \frac{\sigma_1(P_i)^{i_1} \sigma_2(P_i)^{i_2} \cdots \sigma_k(P_i)^{i_k}}{\prod_{j=1}^n k_j^{(i)}} \neq 0 \tag{2.3}$$

with $i_1 + 2i_2 + \cdots + ki_k = n$ and $i_1, i_2, \cdots, i_k \ge 1$.

Due to the ABBV localization formula and the dimensional reason, the following lemma is obvious, but it plays an important role in the proof.

LEMMA 2.3. *For each $0 \leq t_1 \leq i_1 - 1$, we have*

$$\sum_{i=1}^{r} \varepsilon(P_i) \frac{\sigma_1(P_i)^{t_1} \sigma_2(P_i)^{i_2} \cdots \sigma_k(P_i)^{i_k}}{\prod_{j=1}^{n} k_j^{(i)}} = 0.$$

Now, assume first that i_1 is greater than or equal to l_1. Then we will derive a contradiction. Therefore, we can conclude that i_1 is strictly less than l_1. Since l_1 is always less than or equal to r, we should have $i_1 + 1 \leq r$.

For the proof, let

$$A_s = \sum_{1 \leq i \leq r,\, \sigma_1(P_i) = \tau_s^{(1)}} \varepsilon(P_i) \frac{\sigma_2(P_i)^{i_2} \cdots \sigma_k(P_i)^{i_k}}{\prod_{j=1}^{n} k_j^{(i)}}, \quad 1 \leq s \leq l_1.$$

Then, it follows from Lemma 2.3 that we can obtain a system of equations, as follows.

$$\begin{aligned} & A_1 + A_2 + \cdots + A_{l_1} = 0 \\ & \tau_1^{(1)} A_1 + \tau_2^{(1)} A_2 + \cdots + \tau_{l_1}^{(1)} A_{l_1} = 0 \\ & \quad \cdots \\ & (\tau_1^{(1)})^{i_1 - 1} A_1 + (\tau_2^{(1)})^{i_1 - 1} A_2 + \cdots + (\tau_{l_1}^{(1)})^{i_1 - 1} A_{l_1} = 0. \end{aligned} \tag{2.4}$$

Since i_1 is assumed to be greater than or equal to l_1, $\tau_1^{(1)}, \tau_2^{(1)}, \cdots, \tau_{l_1}^{(1)}$ are mutually distinct, and the coefficient matrix of the first l_1 lines in the system of equations (2.4) is non-singular, we should have

$$A_1 = A_2 = \cdots = A_{l_1} = 0.$$

But this would imply from the equation (2.3) that

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_k(M)^{i_k}, [M] \rangle = \sum_{s=1}^{l_1} (\tau_s^{(1)})^{i_1} A_s = 0,$$

which is clearly a contradiction to the hypothesis of Theorem 2.2.

Now apply the exactly same argument to all other cases of $i_1, i_2, \cdots, i_k$, so that we can conclude that $i_j + 1 \leq l_j \leq r$ for $j = 2, 3, \cdots, k$. To do so, Lemma 2.3 needs to be suitably modified in such a way that we can apply it to the cases of $i_2, i_3, \cdots, i_k$, and clearly it can be made without any difficulty. This implies that r is greater than or equal to

$$\max\{i_1, i_2, \cdots, i_n\} + 1,$$

which completes the proof of Theorem 2.2. □

As an interesting corollary related to the conjecture of Kosniowski above, we have the following

COROLLARY 2.4. *Let M be a closed connected unitary S^1-manifold of dimension $2n$ with only isolated fixed points. Let $i_1, i_2, \cdots, i_k$ be positive integers such that $i_1 + 2i_2 + \cdots + ki_k = n$. If M satisfies*

$$\langle c_1(M)^{i_1} c_2(M)^{i_2} \cdots c_k(M)^{i_k}, [M] \rangle \neq 0,$$

then the number r of isolated fixed points is greater than or equal to

$$f(n) := \left[\frac{2n}{k(k+1)}\right] + 1.$$

In particular, if $k = 2$, then $r \geq f(n) = \left[\frac{n}{3}\right] + 1$.

Proof. By Theorem 2.2, the number r is greater than or equal to $\max\{i_1, i_2, \cdots, i_k\} + 1$. Assume that $\max\{i_1, i_2, \cdots, i_k\}$ is attained at i_l. Since we have

$$n = i_1 + 2i_2 + \cdots + ki_k \leq (1 + 2 + \cdots + k)i_l = \frac{k(k+1)}{2} i_l,$$

i_l is greater than or equal to $\frac{2n}{k(k+1)}$. Since r is an integer, r is greater than or equal to $f(n)$, as required. This completes the proof. □

3. Proof of Theorem 1.6: Unitary T^n-manifolds. The goal of this section is to give a proof of Theorem 1.6 which is related to Conjecture 1.1. To do so, we first need to recall some basic notions of a unitary torus manifold.

Let M be a closed oriented unitary manifold of dimension $2n$. Then there is a closed, connected real codimension 2 submanifold of M fixed by a certain circle subgroup of T^n which contains at least one fixed point. This is called a *characteristic submanifold* of M, and M has only finitely many characteristic submanifolds. Let $M_1, M_2, \cdots, M_m$ denote such characteristic submanifolds. For each $1 \leq i \leq m$, let T_i be the circle subgroup of T^n fixing M_i pointwisely, and let ζ_i denote the corresponding normal bundle of M_i. Then for each fixed point P one can write the tangent space $T_P M$ as a representation of T^n as

$$T_P M = \oplus_{i \in I(P)} \zeta_i|_P,$$

where $I(P) = \{i \,|\, P \in M_i\} \subset \{1, 2, \cdots, m\}$, and $\zeta_i|_P$ is the restriction of ζ_i to the fixed point P. Note that in case of a unitary torus manifold, the order of $I(P)$ is n. The total T^n-equivariant Chern characteristic class $c(M)$ of the tangent bundle TM of M can be written as

$$c^{T^n}(M) = \prod_{i \in \{1,2,\cdots,m\}} (1 + \lambda_i),$$

where λ_i is the element in $H^2_{T^n}(M; \mathbb{Z})$ associated with each characteristic submanifold M_i (see [9] for more details). Thus, the total equivariant Chern class of TM restricted to an isolated fixed point P is given by

$$c^{T^n}(M)|_P = \prod_{i \in I(P)} (1 + \lambda_i|_P) = 1 + \sum_{i=1}^{n} \tilde{\sigma}_i(P),$$

where $\tilde{\sigma}_i(P)$ for $1 \leq i \leq n$ denotes the i-th elementary symmetric function over n variables $\lambda_i|_P$.

For an n-tuple $(i_1, i_2, \cdots, i_n)$ of non-negative integers, the *T^n-equivariant Chern characteristic number* of an oriented unitary T^n-manifold is defined by

$$\langle (c_1^{T^n}(M))^{i_1} (c_2^{T^n}(M))^{i_2} \cdots (c_n^{T^n}(M))^{i_n}, [M] \rangle \in H^*(BT^n; \mathbb{Z}) = \mathbb{Z}[t_1, t_2, \cdots, t_n],$$

where $c_i^{T^n}(M)$ is the i-th equivariant Chern class of M. Note that in case of an oriented unitary T^n-manifold like an oriented unitary S^1-manifold, the T^n-equivariant Chern characteristic number defined as above may not be zero, even though the sum $i_1+2i_2+\cdots+ni_n$ is greater than $\frac{\dim M}{2}=n$. For the sake of simplicity, in this paper as in Section 2 we only deal with the case that $i_1+2i_2+\cdots+ni_n$ is equal to n.

The goal of this section is to prove the following

THEOREM 3.1. *Let M be a closed connected unitary T^n-manifold (or torus manifold) of dimension $2n$ with only isolated fixed points. Let $i_1, i_2, \cdots, i_n$ be non-negative integers such that $i_1+2i_2+\cdots+ni_n=n$. Suppose that M does not bound a unitary T^n-manifold equivariantly in such a way that*

$$\langle (c_1^{T^n}(M))^{i_1}(c_2^{T^n}(M))^{i_2}\cdots(c_n^{T^n}(M))^{i_n}, [M]\rangle \neq 0.$$

Then the number of the isolated fixed points is greater than or equal to

$$\max\{i_1, i_2, \cdots, i_n\}+1.$$

Proof. The proof of this theorem is completely similar to Theorem 2.2. To be more precise, let $P_1, P_2, \cdots, P_r$ denote all the isolated fixed points. Then, for each $1\le j\le r$, let

$$\{\tilde\sigma_j(P_i)\}_{i=1}^r=\{\tilde\tau_1^{(j)}, \tilde\tau_2^{(j)}, \cdots, \tilde\tau_{l_j}^{(j)}\}.$$

Here $\tilde\tau_s^{(j)}$ is not an integer but an element in $H^*(BT^n;\mathbb{Z})=\mathbb{Z}[t_1, t_2, \cdots, t_n]$, contrary to the values of $\tau_s^{(j)}$'s. But it is important to note that at any rate l_j is less than or equal to r.

It is obvious that the ABBV localization formula (Theorem 2.1) in our case can be stated as follows.

$$\begin{aligned}&\langle (c_1^{T^n}(M))^{i_1}(c_2^{T^n}(M))^{i_2}\cdots(c_n^{T^n}(M))^{i_n}, [M]\rangle\\ &=\sum_{i=1}^r\varepsilon(P_i)\frac{\tilde\sigma_1(P_i)^{i_1}\tilde\sigma_2(P_i)^{i_2}\cdots\tilde\sigma_n(P_i)^{i_n}}{\tilde\sigma_n(P_n)}.\end{aligned}\tag{3.1}$$

For the sake of simplicity, as before assume that $i_1, i_2, \cdots, i_k$ are all non-zero integers in the equation (3.1). Then, analogously to Lemma 2.3, the following lemma holds due to the ABBV localization formula and the dimensional reason.

LEMMA 3.2. *For each $0\le t_1\le i_1-1$, we have*

$$\sum_{i=1}^r\varepsilon(P_i)\frac{\tilde\sigma_1(P_i)^{t_1}\tilde\sigma_2(P_i)^{i_2}\cdots\tilde\sigma_k(P_i)^{i_k}}{\tilde\sigma_n(P_i)}=0.$$

Next, let

$$\tilde A_s=\sum_{1\le i\le r, \tilde\sigma_1(P_i)=\tilde\tau_s^{(1)}}\varepsilon(P_i)\frac{\tilde\sigma_2(P_i)^{i_2}\cdots\tilde\sigma_k(P_i)^{i_k}}{\tilde\sigma_n(P_i)},\quad 1\le s\le l_1.$$

With these understood, it is now easy to see that we can apply the exactly same arguments as in the proof of Theorem 2.1 in order to show that $i_1<l_1\le r$. So we leave the details of the rest of the proof to the reader.

Similarly, it is also true that $i_j<l_j\le r$ for all $2\le j\le k$. This completes the proof of Theorem 3.1. □

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0001651, 2011-0001181). We also want to acknowledge the support by the second stage of the Brain Korea 21 Project, The Development Project of Human Resources in Mathematics, KAIST in 2011.

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