ASIAN J. MATH.
Vol. 16, No. 2, pp. 237–270, June 2012

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ON THE NORMAL BUNDLES OF RATIONAL CURVES ON FANO 3-FOLDS*

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Abstract. A component of very free rational curves on a variety is called unbalanced if the normal bundle of a general member is unbalanced. In this paper we show that all components of very free rational curves on a Fano threefold of Picard number one are balanced with the only exception being the space of conics on $\mathbb{P}^3$.

Key words. Rational curves, Fano threefolds.

AMS subject classifications. Primary 14M22; Secondary 14D15.

1. Introduction. In this paper, we work over the field $\mathbb{C}$ of complex numbers. A variety X is Fano if it is smooth projective with the anticanonical class, $-K_X$, being ample. It is known that Fano varieties are rationally connected and hence contain a lot of rational curves, see [4] and [24]. The geometry of the space of rational curves carries a lot of information of the variety itself. Unfortunately, some basic questions concerning moduli space of rational curves on a Fano variety are still open. For example, let X be a smooth Fano variety of Picard number one. Let $M_e = M_e(X)$ be the space of degree e rational curves on X. One can ask the following

QUESTION 1.1. Is M_e irreducible, at least for e sufficiently large?

We see that the answer is "yes" for $X = \mathbb{P}^n$. A positive answer for X being a quadric hypersurface follows from [22] or [35]. The case of cubic threefolds is treated in [33] and cubic hypersurfaces of higher dimensions are treated in [6]. In [15], the authors give a positive answer to the above question for a general hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ with degree $d < \frac{n-1}{2}$.

In this paper, we consider a variation of the above question. Let X be a smooth projective variety. Assume that $\dim X \geq 3$. Let $M \subset \overline{\mathcal{M}}_{0,0}(X,\beta)$ be a component of the Kontsevich moduli space of genus 0 curves on X. Assume that for a general member $[C] \in M$, the corresponding rational curve $\phi : \mathbb{P}^1 \cong C \to X$ is birational onto image and very free. Recall that ϕ being very free means that $\phi^* T_X$ is ample, see [23]. We call M *a component of very free rational curves* on X. Then it follows that $\phi : C \to X$ is a closed immersion for general $[C] \in M$. The normal bundle of a general such curve C in X has splitting type

$$\mathscr{N}_{C/X} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_{n-1}), \quad n = \dim X,$$

with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. The sequence $(a_1, \ldots, a_{n-1})$ is an invariant of the component M.

DEFINITION 1.2. We say that M (and also $\mathscr{N}_{C/X}$) is *balanced* if $a_{n-1} - a_1 \leq 1$ and that M (and also $\mathscr{N}_{C/X}$) is *unbalanced* if $a_{n-1} - a_1 \geq 2$.

QUESTION 1.3. For a Fano variety X of Picard number one, is M_e balanced (for sufficiently large e)?

*Received May 24, 2011; accepted for publication June 3, 2011.
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The deformation of a curve on X is controlled by its normal bundle and Question 1.3 is essentially asking for the normal bundle of a general rational curve on X. If the normal bundle is balanced, then its splitting type is completely determined by its degree. Without the restriction on Picard number, there could be a lot of unbalanced rational curves, for example when $X = \mathbb{P}^1 \times \mathbb{P}^n$. But when X has Picard number one, we do not expect a general rational curve to move much more freely in some direction than in others. In this paper, we carry out this idea in the three dimensional case.

Assume that a smooth projective threefold $X/\mathbb{C}$ has an unbalanced component M of very free rational curves. Then a general member of M has unbalanced normal bundle, so it moves more freely in some direction. By deforming the curve in this direction, we get a surface Σ (the construction is given in section 2 in a more general setting). Actually we also show that the unbalancedness gives a canonical foliation on M which is algebraically integrable with a leaf being all curves lying on a fixed Σ. In section 3, we study those surfaces Σ. There are two different types of surface Σ that we can get. Accordingly, M is either of conic type or fibration type, see Definition 3.16. If M is of conic type, then $C \subset X$ is étale locally equivalent to a conic in $\mathbb{P}^3$; If M is of fibration type and $-K_X$ is nef, then there is a rational component S of rational curves on X with trivial normal bundle, see Theorem 3.15. After that, we focus on the case when X is Fano of Picard number one. In section 5, we show that the Abel-Jacobi mapping defined by S is never trivial as long as X has nonzero intermediate jacobian; this shows that S can not be rational. The main theorem of this paper is the following

Theorem 1.4. *Let X be a Fano threefold of Picard number one. If X has an unbalanced component M of very free rational curves, then $X = \mathbb{P}^3$ and M is the space of conics on X.*

The intermediate jacobian of X is zero only if $X = X_5$ or $X = X_{22}$. The cases $X = X_5$ and $X = X_{22}$ are ruled out by a ramification argument. The case X_5 follows from the paper [11]. The author carries out the construction of the space of conics on X_{22} in the appendix. After this was done, Prof. J. Kollár informed the author of the paper [25], where a similar construction is carried out using a different model of X_{22}. We note that the above theorem gives a new characterization of $\mathbb{P}^3$. Namely, $\mathbb{P}^3$ would be the only Fano threefold of Picard number one which has an unbalanced component of very free rational curves.

Acknowledgement. This work is the main part of my PhD thesis. I would like to thank my advisor, Aise Johan de Jong, for his careful and patient instructions. Several discussions with Jason Starr and Matt DeLand were also very helpful and Professor Friedman assured me of some results about intermediate jacobians. I would like to thank them all for their help.

2. Basic constructions. Notations and assumptions: Let X be a smooth projective variety over the field $\mathbb{C}$ of complex numbers. Assume that $\dim X = n$. Let $M \subset \overline{\mathcal{M}}_{0,0}(X, \beta)$ be an unbalanced component of very free rational curves on X. For a general member $[C] \in M$, we assume that the splitting type of the normal bundle to be

$$\mathscr{N}_{C/X} \cong \mathcal{O}(a)^{\oplus(n-r-1)} \oplus \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_r), \tag{1}$$

where $3 \le a+2 \le b_1 \le \cdots \le b_r$. Let $M^0 \subset M$ be the open subscheme parametrizing smooth rational curves $C \cong \mathbb{P}^1 \subset X$ with normal bundle (1). Let

$$\begin{tikzcd} \mathcal{U} \arrow[r, "u"] \arrow[d] & X \\ M & \end{tikzcd}$$

be the universal family over M. For any $[C] \in M$, let $u_{[C]} : C \to X$ denote the corresponding morphism. We say that C passes through a point $P \in X$ or that P is on C if P is on the image of $u_{[C]}$. Let $P_i \in X$, $i = 1, 2, \ldots, k$, be distinct points on X. We define

$$M^0(P_1, \ldots, P_k) \subset M^0$$

to be the subscheme that consists of $[C] \in M^0$ such that C passes through all the points P_i. We use $\mathcal{U}^0(P_1, \ldots, P_k) \to X$ to denote the universal family over $M^0(P_1, \ldots, P_k)$. Let $[C] \in M^0(P_1, \ldots, P_k)$. The obstruction of deforming C in X passing through $\{P_1, \ldots, P_k\}$ is in $\mathrm{H}^1(C, \mathscr{N}_{C/X}(-P_1 - \cdots - P_k))$. Since $\mathscr{N}_{C/X}$ has splitting type (1) for $[C] \in M^0$, we know that $M^0(P_1, \ldots, P_k)$ is smooth when $k \le a+1$.

Definition 2.1. Notations and assumptions as above, we use

$$\mathrm{Def}_X(C; P_1, \ldots, P_k) \subset M^0(P_1, \ldots, P_k)$$

to denote the union of the irreducible components of $M^0(P_1, \ldots, P_k)$ containing $[C]$. And we get the corresponding universal family.

$$\begin{tikzcd} \mathcal{U}(C; P_1, \ldots, P_k) \arrow[r, "v = v(C; P_1, \ldots, P_k)"] \arrow[d, "\pi"] & X \\ \mathrm{Def}_X(C; P_1, \ldots, P_k) & \end{tikzcd}$$

Let

$$\sigma_i : \mathrm{Def}_X(C; P_1, \ldots, P_k) \to \mathcal{U}(C; P_1, \ldots, P_k)$$

be the section that gets contracted by v to the point $P_i \in X$, where $i = 1, \ldots, k$. We use the notation $\Sigma(C; P_1, \ldots, P_k) \subset X$ to denote the closure of the image of $v(C; P_1, \ldots, P_k)$.

Remark 2.2. If $[C]$ is a smooth point of $M^0(P_1, \ldots, P_k)$, for example when $k \le a+1$, then $\mathrm{Def}_X(C; P_1, \ldots, P_k)$ is irreducible. Actually, it is smooth everywhere if $k \le a+1$.

In our situation, let $[C] \in M^0$ and we take $k = a+1$ and pick $\{P_1, \ldots, P_{a+1}\}$ to be distinct points on C, hence $\mathrm{Def}_X(C; P_1, \ldots, P_{a+1})$ is smooth. Note that the Zariski tangent space T_p of $\mathcal{U}(C; P_1, \ldots, P_k)$ at $p \in \pi^{-1}([C'])$ fits into the following short exact sequence naturally

$$0 \longrightarrow T_{C',p} \longrightarrow T_p \xrightarrow{d\pi} \mathrm{H}^0(C', \mathscr{N}'(-a-1)) \longrightarrow 0$$

for any point $[C'] \in \mathrm{Def}_X(C; P_1, \ldots, P_{a+1})$, where $\mathscr{N}' = \mathscr{N}_{C'/X}$. Consider the differential of v at p, we have the following

$$\begin{tikzcd} 0 \arrow[r] & T_{C',p} \arrow[r] \arrow[d, "\mathrm{id}"'] & T_p \arrow[r, "d\pi"] \arrow[d, "dv"] & \mathrm{H}^0(C', \mathscr{N}'(-a-1)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & T_{C',p} \arrow[r] & T_{X,p} \arrow[r] & T_{X,p}/T_{C',p} = \mathscr{N}'_p \arrow[r] & 0 \end{tikzcd} \tag{2}$$

Here the last column is the evaluation of a section at the point p. It is easy to see that dv has rank $r+1$ if $p \notin \sigma_i$ for all $i = 1, \ldots, a+1$. Let $\Sigma = \Sigma(C; P_1, \ldots, P_{a+1}) \subset X$ and we have $\dim \Sigma = r+1$ since everything is over $\mathbb{C}$. Let $\phi : \Sigma' \to \Sigma$ be the normalization of Σ and $\tilde{\phi} : \tilde{\Sigma} \to \Sigma'$ be a resolution of Σ'. To make further constructions, we need the following

Lemma 2.3. *Let $f : U \to V$ be a morphism between smooth algebraic varieties over an algebraically closed field k of characteristic 0. Assume that df is generically of rank r. Let $\Sigma \subset V$ be the closure of $f(U)$ and Σ' be the normalization of Σ. Then f naturally lifts to $f' : U \to \Sigma'$*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "f'"'] & \Sigma \arrow[r, "i"] & V \\ & \Sigma' \arrow[u, "\rho"'] & \end{tikzcd}$$

and for any closed point $x \in U$ with $df(x)$ having rank r, Σ' is smooth at $f'(x)$ and df' has rank r at x.

Proof. The existence of f' is easy since U is smooth and hence normal. We only need to prove the remaining part. Let $y = f(x) \in \Sigma \subset V$ and $y' = f'(x)$. The problem is local, so we can choose an r dimensional closed subvariety $Z \subset U$ which is smooth at x and $df(x)$ is injective on $T_{Z,x} \otimes k(x)$. We can replace U by Z. So we assume that U has dimension r. We have the following local homomorphisms between local rings.

$$\begin{tikzcd} \mathcal{O}_{V,y} \arrow[r, two heads, "i^*"] & \mathcal{O}_{\Sigma,y} \arrow[r, "f^*"] \arrow[d, "\rho^*"'] & \mathcal{O}_{U,x} \\ & \mathcal{O}_{\Sigma',y'} \arrow[ur, "f'^*"'] & \end{tikzcd}$$

Choose a set of local parameters $\{t_1, \ldots, t_n\}$ of V at y such that $\{t_1, \ldots, t_r\}$ pull back to local parameters of U at x. So we get the following diagram

$$k[t_1, \ldots, t_r] \hookrightarrow \mathcal{O}_{\Sigma',y'} \hookrightarrow \mathcal{O}_{U,x}$$

with $\mathcal{O}_{\Sigma',y'}$ being an intermediate normal domain of an étale ring extension. After taking the completions, we get a splitting

$$k[[t_1, \ldots, t_r]] \hookrightarrow \hat{\mathcal{O}}_{\Sigma',y'} \to k[[t_1, \ldots, t_r]]$$

with the composition being identity. So $\hat{\mathcal{O}}_{\Sigma',y'} = k[[t_1, \ldots, t_r]] \oplus I$, where I is an ideal of $\hat{\mathcal{O}}_{\Sigma',y'}$ and a module over $k[[t_1, \ldots, t_r]]$. By the analytical irreducibility and analytical normality of normal varieties, (see [38]), we know that $\hat{\mathcal{O}}_{\Sigma',y'}$ is an integrally closed integral domain of dimension r. This forces I to be zero. Indeed for any $a \in I$, by dimension reason, a should be algebraic over $k[[t_1, \dots, t_r]]$. Let

$$f_n a^n + f_{n-1} a^{n-1} \cdots + f_0 = 0, \qquad \text{with} \quad f_i \in k[[t_1, \dots, t_r]]$$

be an equation for a with minimal degree. Then f_0 is also in I and hence has to be 0. This means $a = 0$. □

Apply the above lemma to $v : \mathcal{U}' \to X$ and we get

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, "\tilde{\phi}"] & \Sigma' \arrow[r, "\phi"] & \Sigma \\ & \mathcal{U}' \arrow[u, "v'"] \arrow[ur, "v"'] & \end{tikzcd}$$

where $\mathcal{U}' = \mathcal{U}(C; P_1, \dots, P_{a+1})$. Let $\mathcal{U}'_0 := \mathcal{U}' - \cup_{i=1}^{a+1} \sigma_i$, where σ_i is the section that gets contracted to P_i. Then we also know that the image $v'(\mathcal{U}'_0)$ is in the smooth locus of Σ' and $v'|_{\mathcal{U}'_0}$ is a smooth morphism. Pick an arbitrary point $[C'] \in \mathrm{Def}_X(C; P_1, \dots, P_k)$. If we restrict the above maps to $[C']$, we get

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r] & \Sigma' \arrow[r] & \Sigma \\ & C' \arrow[ul, "\tilde{v}_{[C']}"] \arrow[u, "v'_{[C']}"'] \arrow[ur, "v_{[C']} = u_{[C']}"'] & \end{tikzcd}$$

Note that $\mathrm{Def}_X(C; P_1, \dots, P_{a+1})$ remains the same if we replace C by C'. From now on, by abuse of notation, we will use C in stead of C' to denote an arbitrary curve from the family $\mathcal{U}(C; P_1, \dots, P_{a+1})$.

LEMMA 2.4. *For all points $Q \in C - \{P_1, \dots, P_{a+1}\}$, Σ' is smooth at $v'_{[C]}(Q)$. The morphism $\tilde{\Sigma} \to \Sigma'$ is an isomorphism along $C - \{P_1, \dots, P_{a+1}\}$.*

Proof. Indeed, we already see that $v'(\mathcal{U}'_0)$ is in the smooth locus of Σ'. □

PROPOSITION 2.5. *For any $[C] \in M^0$, the variety $\Sigma = \Sigma(C; P_1, \dots, P_{a+1})$ is independent of the choice of $\{P_1, \dots, P_{a+1}\}$.*

Proof. Consider the morphism $v' : \mathcal{U}' = \mathcal{U}(C; P_1, \dots, P_{a+1}) \to \Sigma'$. Let Q be a point of C that is different from the P_i's. Since Σ' is smooth at $Q' = v'_{[C]}(Q)$ and v' is smooth above the point Q', we get $Z = v'^{-1}(Q')$ is smooth. Let Z_0 be the component of Z that contains the point Q above $[C]$. Note that $\mathcal{U}'$ is a smooth irreducible component of $\mathcal{U}^0(P_1, \dots, P_{a+1})$. Hence $\pi(Z_0)$ becomes a component of $M^0(P_1, \dots, P_{a+1}, Q)$ on which $[C]$ is a smooth point. By definition we have $Z_0 \cong \pi(Z_0) \cong \mathrm{Def}_X(C; P_1, \dots, P_{a+1}, Q)$. Consider the universal family

$$w : \mathcal{U}'' = \mathcal{U}(C; P_1, \dots, P_{a+1}, Q) \to X$$

Since $\mathcal{U}'' \subset \mathcal{U}'$ and $w = v|_{\mathcal{U}''}$, we see that the image of w is contained in Σ. Hence we get $w : \mathcal{U}'' \to \Sigma$. Note that Z_0 is smooth and hence $\mathcal{U}''$ is smooth. So we can lift w

to get $w' : \mathcal{U}'' \to \Sigma'$. By dimension count, we have

$$\begin{aligned}\dim Z_0 &= \dim \mathcal{U}' - \dim \Sigma' \\ &= \dim \mathrm{H}^0(C, \mathscr{N}_{C/X}(-a-1)) + 1 - (r+1) \\ &= \Sigma_{i=1}^{r}(b_i - a) - r \\ &= \dim \mathrm{H}^0(C, \mathscr{N}_{C/X}(-a-2))\end{aligned}$$

This means that the deformation of C in X with the points $\{P_1, \dots, P_{a+1}, Q\}$ fixed is actually unobstructed. Then we can use a same argument to show that the rank of dw' is generically r as we did for dv'. This implies $\Sigma = \Sigma(C; P_1, \dots, P_{a+1}, Q)$. Then we have

$$\Sigma(C; P_1, \dots, P_{a+1}) = \Sigma(C; P_1, \dots, P_a, P_{a+1}, P'_{a+1}) = \Sigma(C; P_1, \dots, P_a, P'_{a+1})$$

and hence by induction we can replace $\{P_i\}_{i=1}^{a+1}$ by another set of $a+1$ points. Thus Σ is independent of the choice of $\{P_i\}_{i=1}^{a+1}$. □

Definition 2.6. Let $\varphi : U \to V$ be a morphism between smooth varieties, we define the normal sheaf of the morphism to be

$$\mathscr{N}_{U/V} = \mathscr{N}_\varphi = \mathrm{coker}(d\varphi : T_U \to \varphi^* T_V).$$

We say that φ has *injective tangent map* at a closed point $x \in U$ if $d\varphi(x)$ is injective. Note that $\mathscr{N}_\varphi$ is locally free at points where φ has injective tangent map.

Corollary 2.7. *Assume that $[C] \in M^0$. Then the variety Σ' is smooth along $v'_{[C]}(C) \cong C$ and $\mathscr{N}_{C/\Sigma'} \cong \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_r)$. The normal bundle $\mathscr{N}_{\tilde{\Sigma}/X}$ is locally free along the curve $\tilde{v}_{[C]}(C) \cong C$ and $\mathscr{N}_{\tilde{\Sigma}/X}|_C \cong \mathcal{O}(a)^{\oplus(n-r-1)}$.*

Proof. Let $Q \in C$ be an arbitrary point. Pick a set of $a+1$ distinct points $\{P_1, \dots, P_{a+1}\}$ that are different from Q. Then by Proposition 2.5, we have $\Sigma = \Sigma(C; P_1, \dots, P_{a+1})$. By Lemma 2.4, we know that Σ' is smooth at $v'_{[C]}(Q)$. Consider the deformation of $v_{[C]} : C \to \Sigma'$ with the points $\{P_1, \dots, P_{a+1}\}$ fixed. Such deformations still form a covering family. This means that $\mathscr{N}_{C/\Sigma'}(-a-1)$ is globally generated. Assume that

$$\mathscr{N}_{C/\Sigma'} \cong \mathcal{O}(b'_1) \oplus \cdots \oplus \mathcal{O}(b'_r)$$

then we get $b'_i \geq a+1$ for all $i = 1, \dots, r$. We first show that $\mathscr{N}_{\tilde{\Sigma}/X}$ is locally free along C. For any point $x \in C$, pick a set $\{P_1, \dots, P_{a+1}\}$ of $a+1$ distinct points on C that are different from the given point x. Then we get the morphism

$$v : \mathcal{U}' = \mathcal{U}(C; P_1, \dots, P_{a+1}) \to X$$

which factors through $v' : \mathcal{U}' \to \Sigma'$. Correspondingly we have the induced maps between Zariski tangent spaces at x,

$$T_{\mathcal{U}',x} \xrightarrow{dv'(x)} T_{\Sigma',v'(x)} \xrightarrow{d\phi(v'(x))} T_{X,v(x)}$$

and the composition is exactly $dv(x)$. We already know that $dv(x)$ and $dv'(x)$ have rank $r+1$. Together with the fact that $\dim T_{\Sigma',x} = r+1$, we know that $d\phi(v'(x))$ is injective. So ϕ has injective tangent map along C. Note that $\tilde{\Sigma} \to \Sigma'$ is isomorphism along C. Hence $\mathscr{N}_{\tilde{\Sigma}/X}$ is locally free of rank $n-r-1$ along C. Now we consider the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{C/\tilde{\Sigma}} \xrightarrow{\ \eta\ } \mathscr{N}_{C/X} \xrightarrow{\ \theta\ } \mathscr{N}_{\tilde{\Sigma}/X}|_C \longrightarrow 0$$

Since $b_i' \geq a+1$, the image of η lies in the summand $\sum_{i=1}^r \mathcal{O}(b_i)$. Then we get the following diagram

$$\begin{tikzcd}
 & & \mathcal{O}(a)^{\oplus(n-r-1)} \arrow[r, "\cong"] & \mathcal{O}(a)^{\oplus(n-r-1)} & \\
0 \arrow[r] & \mathscr{N}_{C/\tilde{\Sigma}} \arrow[r, "\eta"] \arrow[d, equal] & \mathscr{N}_{C/X} \arrow[r, "\theta"] \arrow[u] & \mathscr{N}_{\tilde{\Sigma}/X}|_C \arrow[r] \arrow[u, "\theta'"'] & 0 \\
0 \arrow[r] & \mathscr{N}_{C/\tilde{\Sigma}} \arrow[r, "\eta'"] & \sum_{i=1}^r \mathcal{O}(b_i) \arrow[r] \arrow[u] & \mathcal{Q} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the second and third columns are also short exact sequences and $\mathcal{Q}$ is a torsion sheaf. We have already shown that $\mathscr{N}_{\tilde{\Sigma}/X}|_C$ is locally free of rank $n-r-1$. This forces $\mathcal{Q}$ to be 0. Hence η' and θ' are isomorphisms. ◻

DEFINITION 2.8. Given $[C] \in M^0$, let Σ, Σ' and $\tilde{\Sigma}$ be as above. Since the morphism $v_{[C]} : C \to X$ lifts to $\tilde{v}_{[C]} : C \to \tilde{\Sigma}$, we can define β' to be the homology class of C on $\tilde{\Sigma}$. Let $\mathrm{Def}_{\tilde{\Sigma}} \subset \overline{\mathcal{M}}_{0,0}(\tilde{\Sigma}, \beta')$ be the space of curves $C \cong \mathbb{P}^1 \to \tilde{\Sigma}$ such that the composition $C \to \tilde{\Sigma} \to X$ is a point on M^0. Hence we can view $\tilde{v}_{[C]}$ as a point on $\mathrm{Def}_{\tilde{\Sigma}}$. By abuse of notation, we still use $[C]$ to denote this point. Let $\mathrm{Def}_{\tilde{\Sigma}}(C) \subset \mathrm{Def}_{\tilde{\Sigma}}$ be the irreducible component that contains the point $[C]$. From the corollary above, we know that $\mathrm{Def}_{\tilde{\Sigma}}(C)$ is actually a smooth open subscheme of $\overline{\mathcal{M}}_{0,0}(\tilde{\Sigma}, \beta')$. By composing $C \to \tilde{\Sigma}$ with $\tilde{\Sigma} \to X$, we have a morphism between smooth varieties $\alpha : \mathrm{Def}_{\tilde{\Sigma}}(C) \to M^0$.

PROPOSITION 2.9. *The morphism α is a closed immersion. Furthermore, there is a nonempty open subscheme $U \subset M^0$ and a smooth morphism $\psi : U \to B$ such that for any $[C] \in U$ we have $\psi^{-1}(\psi([C])) = \mathrm{Def}_{\tilde{\Sigma}}(C) \cap U$. The quotient B is smooth of dimension $(a+1)(n-r-1)$.*

Proof. For simplicity, we set $D(C) = \mathrm{Def}_{\tilde{\Sigma}}(C)$. First, we show that α separates points. If C_1 and C_2 on $\tilde{\Sigma}$ map to the same C on X, then Σ has two branches along the curve C. But this is impossible since in the definition of Σ, the nearby deformation of C should swipe out a unique branch of Σ. Now we prove that the differential $d\alpha(t) = d\alpha \otimes k(t)$ is injective for all closed points $t \in D(C)$ and that $D(C)$ is closed in M^0. Consider the universal family $\pi^0 : \mathcal{U}^0 \to M^0$. Let $\mathscr{N}$ be the cokernel of $T_{\mathcal{U}^0/M^0} \to (u^0)^* T_X$ where $u^0 : \mathcal{U}^0 \to X$ is the universal morphism. Then by the definition of M^0, the sheaf $\mathscr{N}$ is locally free and splits uniformly along the closed fibers of π^0. Let $\mathscr{V}_\eta \subset \mathscr{N}_\eta$ be the part of Harder-Narasimhan filtration on the generic fiber that corresponds to $\oplus_{i=1}^r \mathcal{O}(b_i)$ on the geometric generic fiber, c.f. [16]. Since the splitting of $\mathscr{N}$ is uniform, $\mathscr{V}_\eta$ extends to a subbundle $\mathscr{V}$ of $\mathscr{N}$. Actually, we can write down $\mathscr{V}$ explicitly as in [3]. Let $\mathscr{D} = (\pi^0)_* \mathscr{V} \subset T_{M^0} = (\pi^0)_* \mathscr{N}$ be the corresponding subbundle. If we write down the the differential $d\alpha([C])$ explicitly, we have

$$d\alpha([C]) : T_{D(C),[C]} = \mathrm{H}^0(C, \mathscr{N}_{C/\tilde{\Sigma}}) \to T_{M^0,[C]} = \mathrm{H}^0(C, \mathscr{N}_{C/X}).$$

Since we already see that $\mathscr{N}_{C/\tilde{\Sigma}}$ maps isomorphically onto $\mathscr{V}|_C$ inside $\mathscr{N}_{C/X} = \mathscr{N}|_C$. This implies that $T_{D(C),[C]} = \mathscr{D} \otimes k([C])$ and in particular, the differential $d\alpha(t)$ is injective for all closed point $t \in D(C)$. Thus $\mathscr{D}$ defines a foliation on M^0 and $D(C) = \mathrm{Def}_{\tilde{\Sigma}}(C)$ defines a leaf of $\mathscr{D}$, c.f. [9]. Let $\bar{D}(C)$ be the Zariski closure of $D(C)$ in M^0. Since $\mathscr{D}$ is a subbundle of T_{M^0}, we conclude that $\bar{D}(C)$ is smooth and still a leaf, c.f. [9] (Lemma 2.3 there). Now we claim that $D(C) = \bar{D}(C)$. Otherwise, let $[C'] \in \bar{D}(C)$ be a point that is not contained in $D(C)$. Then both $D(C')$ and $\bar{D}(C)$ are leaves through $[C']$; they have to agree on an open part. Thus $D(C')$ and $D(C)$ meet each other. This can happen only when $D(C) = D(C')$. This means that $[C'] \in D(C)$, which is a contradiction. Hence we proved that α is a closed immersion. Since all the leaves of the foliation $\mathscr{D}$ are algebraic, hence $\mathscr{D}$ is algebraically integrable. This means that there is a nonempty open $U \subset M^0$ and a morphism $\psi : U \to B$ such that $T_{U/B} = \mathscr{D}|_U$, c.f. [9] (Proposition 2.1 there). The smoothness results are from direct local computations. ☐

3. Three dimensional case. SITUATION 3.1. In this whole section, we fix the following assumptions and notations:

- $X/\mathbb{C}$ is a smooth projective algebraic variety with $\dim X = 3$.
- $M \subset \overline{\mathcal{M}}_{0,0}(X, \beta)$ is an unbalanced component of very free rational curves on X. Let $M^0 \subset M$ be as in the previous section. We always use C to denote a curve on X such that $[C] \in M^0$.
- $\mathscr{N}_{C/X} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $1 \leq a \leq b-2$.
- Let $\Sigma_C = \Sigma(C; P_1, \ldots, P_{a+1})$ be the surface as is constructed in the previous section; Let Σ'_C be the normalization of Σ_C and $\tilde{\Sigma}_C$ be a resolution of Σ'_C. We frequently drop the subscript C when there is no confusion.

DEFINITION 3.2. ([14]) Let $C_i \subset X_i$ be a curve on a variety X_i, $i = 1, 2$. We say that $C_1 \subset X_1$ is equivalent to $C_2 \subset X_2$ and write $(C_1 \subset X_1) \cong (C_2 \subset X_2)$ if there is an open neighborhood V_i of C_i in X_i and an isomorphism $f : V_1 \to V_2$ with $f|_{C_1} : C_1 \to C_2$ being also an isomorphism.

PROPOSITION 3.3. *The pair $C \subset \Sigma' = \Sigma'_C$ is equivalent to one of the following*

(i) $\sigma \subset F_n$, where $F_n = \mathbb{P}(\mathcal{O}(-n) \oplus \mathcal{O}) \to \mathbb{P}^1$ is the Hirzebruch surface and σ is a section;

(ii) a smooth conic on $\mathbb{P}^2$.

Proof. Since we only care about a neighborhood of $C \subset \Sigma'$, we may replace Σ' by $\tilde{\Sigma}$, see Lemma 2.4. Consider the complete linear system $|C|$. Since C is a very free rational curve on $\tilde{\Sigma}$, we know that $\tilde{\Sigma}$ is a smooth rational surface and hence $h^i(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}) = 0$ for $i = 1, 2$. From the long exact sequence associated to the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{\Sigma}} \longrightarrow \mathcal{O}_{\tilde{\Sigma}}(C) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(b) \longrightarrow 0$$

we get $\dim |C| = b + 1$. Since $h^1(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}) = 0$, any nearby deformation of C in $\tilde{\Sigma}$ is in $|C|$. Then the fact that C being very free implies that $|C|$ separates points and tangent vectors along C. Hence $|C|$ defines an immersion $\phi = \phi_{|C|}$ on a neighborhood of C. Let $\bar{\Sigma} \subset \mathbb{P}^{b+1}$ be the closure of the image of ϕ. Then $\deg \bar{\Sigma} = C^2 = b$, this means that $\bar{\Sigma}$ is a surface of minimal degree. The proposition is a direct application of a theorem of Del Pezzo and Bertini (c.f.[8]). ☐

3.1. Case I: Smooth conic on $\mathbb{P}^2$.

SITUATION 3.4. In this subsection, we make the following further assumptions in addition to Situation 3.1.

- $C \subset \Sigma'_C$ is equivalent to a smooth conic on $\mathbb{P}^2$.
- Let $U' = U'_C \subset \Sigma' = \Sigma'_C$ be the largest open neighborhood of C such that $C \subset U'$ is isomorphic to an open neighborhood of a smooth conic on $\mathbb{P}^2$ and $\mathscr{N}_{U'/X}$ is locally free.
- A curve on U' is called a line/conic if it is so when we identify U' with an open subset of $\mathbb{P}^2$.

LEMMA 3.5. *With the above assumptions, we have $a = 2$ and $b = 4$.*

Proof. Since $C \subset \Sigma'$ is equivalent to a smooth conic on $\mathbb{P}^2$, by Lemma 2.7, we have $\mathcal{O}(b) \cong \mathscr{N}_{C/\Sigma'} = \mathcal{O}(4)$. Hence we have $b = 4$. Then a is equal to either 1 or 2. Let $L' \subset U'$ be a line and assume that $\mathscr{N}_{U'/X}|_L \cong \mathcal{O}(c)$. Then $\mathcal{O}(a) \cong \mathscr{N}_{U'/X}|_C \cong \mathcal{O}(2c)$. This implies that $a = 2c = 2$ is the only possibility. ☐

DEFINITION 3.6. Let $L \cong \mathbb{P}^1 \subset X$ be a smooth rational curve. We say that L is a pseudo-line on X if there exists some $[C] \in M^0$ such that L is the image of a line $L' \subset U'_C \subset \mathbb{P}^2$. Let

$$F(X) = \{[L] \in \mathrm{Hilb}(X) \,|\, L \cong \mathbb{P}^1 \subset X \text{ is a pseudo-line}\} \subset \mathrm{Hilb}(X)$$

be the moduli space of pseudo-lines on X. Given point $x \in X$, let

$$F_x(X) = \{[L] \in F(X) \,|\, x \in L\} \subset F(X)$$

be the space of pseudo-lines on X that pass through the point x. We use $P(X)$ and $P_x(X)$ to denote the universal family of pseudo-lines over $F(X)$ and $F_x(X)$ respectively.

REMARK 3.7. We will see from the next proposition that $F(X)$ is actually an irreducible smooth open subscheme of $\mathrm{Hilb}(X)$.

PROPOSITION 3.8. *Let L be a pseudo-line on X. Then the following are true*

(i) The normal bundle $\mathscr{N}_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$;

(ii) The definition of a pseudo-line is independent of the choice of $L' \subset \Sigma'$ in the following sense: If there is another $[C_1] \in M^0$ and $L'_1 \cong \mathbb{P}^1 \subset U'_{C_1}$ is a rational curve whose image is a curve $L_1 \cong \mathbb{P}^1 \subset X$ with $\mathscr{N}_{L_1/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, then L'_1 is a line and hence L_1 is a pseudo-line;

(iii) Any nearby deformation of L in X is still a pseudo-line on X. Namely, if $Y \to T$ is a family of rational curves on X and Y_{t_0} is a pseudo-line, then there is a nonempty open $T^0 \subset T$ such that Y_t is a pseudo-line for all $t \in T^0$;

(iv) The space $F(X)$ is smooth and irreducible;

(v) Let L_1 and L_2 be two intersecting pseudo-lines on X, then they are lying on a unique Σ;

(vi) The space $F_x(X)$ is smooth and irreducible;

(vii) Through a general pair of points on X, there are only finitely many pseudo-lines.

Proof. (i) Let $L' \subset U'$ be the line that maps to L. Since $\mathscr{N}_{U'/X}|_C \cong \mathcal{O}(a) = \mathcal{O}(2)$, we get $\mathscr{N}_{U'/X}|_{L'} \cong \mathcal{O}(1)$. Since $\mathscr{N}_{L'/U'} = \mathcal{O}(1)$, part (i) of the proposition follows from the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{L'/U'} \longrightarrow \mathscr{N}_{L/X} \longrightarrow \mathscr{N}_{U'/X}|_{L'} \longrightarrow 0\,.$$

(ii) Let $L'_1 \subset U'_{C_1} \subset \Sigma'_{C_1}$ be a rational curve that maps to L_1. Then we still have the corresponding short exact sequence as above. Since the left term $\mathscr{N}_{L'_1/U'_{C_1}}$ is ample (note that L'_1 can be viewed as a curve on $\mathbb{P}^2$) and the middle term is still $\mathcal{O}(1) \oplus \mathcal{O}(1)$, we get $\mathscr{N}_{L'_1/U'_{C_1}} \cong \mathcal{O}(1)$. Hence L'_1 is a line.

To prove (iii), let $Y \to T$ be a family of smooth curves on X such that Y_{t_0} is a pseudo-line, where T is a smooth curve. We want to show that Y_t is a pseudo-line on X for general $t \in T$. We may assume that the normal bundle of Y_t in X is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. After shrinking T and replacing T by a finite cover if necessary, we can find a family, $Z \to T$, of pseudo-lines on X such that Z_t and Y_t meet at a single point and both Z_{t_0} and Y_{t_0} lie on the same Σ' associated to some $[C] \in M^0$. Then $Y_{t_0} \vee Z_{t_0}$ is a degeneration of C and hence $[Y_{t_0} \vee Z_{t_0}] \in M$. Deformation theory tells us that the obstruction of deforming $Y_t \vee Z_t$ is in the second hyper-extension group of the cotangent complex, of the morphism $\phi : C'_t = Y_t \vee Z_t \to X$, by the structure sheaf of C'_t. Namely, the obstruction is in $\mathbb{E}\mathrm{xt}^2_{\mathcal{O}_{C'_t}}(L^*_\phi, \mathcal{O}_{C'_t})$, where L^*_ϕ is the cotangent complex of ϕ, see [26]. A long exact sequence associated to the spectral sequence is

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{C'_t}}(\phi^*\Omega^1_{X/k}, \mathcal{O}_{C'_t}) \longrightarrow \mathbb{E}\mathrm{xt}^2_{\mathcal{O}_{C'_t}}(L^*_\phi, \mathcal{O}_{C'_t}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{C'_t}}(\Omega_{C'_t/k}, \mathcal{O}_{C'_t})$$

This shows that $Y_t \vee Z_t$ is unobstructed in X, see [32]. Hence $\{Y_t \vee Z_t : t \in T\}$ corresponds to a curve inside the smooth locus of M. To show that Y_t is a pseudo-line, we pick points $P_1, P_2 \in Y_t$ and $Q \in Z_t$ which are different from the node. The deformation of $Y_t \vee Z_t$ in X passing through P_1, P_2 and Q is still unobstructed and a general deformation gives a smooth rational curve $C_1 \subset X$. This can be seen from the same long exact sequence as above with all the sheaves twisted by $\mathcal{O}_{C'_t}(-P_1 - P_2 - Q)$. Hence Y_t is on the surface Σ_1 that is associated to C_1. Since Y_t is a component of the degeneration of C_1 on Σ_1, we get that Y_t is a pseudo-line on X.

(iv) Now we know that $F(X)$ is an open subscheme of the Hilbert scheme of X. The smoothness follows directly from the unobstructedness of pseudo-lines on X. To show that $F(X)$ is irreducible, one only needs to show that it is connected. According to Proposition 2.9, there is an open subscheme $U \subset M^0$ with a quotient map $\psi : U \to B$. Let $\mathcal{U}^0|_U \to X$ be the universal family over U. Consider the following diagram.

(3)
$$\begin{tikzcd}
\mathcal{U}^0|_U \arrow[rr] \arrow[d, "\pi^0|_U"'] \arrow[dr] & & X \times U \arrow[d, "1\times\psi"] \\
U \arrow[dr, "\psi"'] & \mathcal{V} \arrow[r] \arrow[d] & X \times B \arrow[dl] \\
& B &
\end{tikzcd}$$

Here $\mathcal{V}$ is the closure of the image of $\mathcal{U}^0|_U$ in $X \times B$. Let $\mathcal{V}' \to \mathcal{V}$ be the normalization and $\tilde{\mathcal{V}} \to \mathcal{V}$ be a resolution of singularity. For each $b \in B$, there is a canonically associated Σ_b. Then for general $b \in B$, $\tilde{\mathcal{V}}_b = \tilde{\Sigma}_b$ and $\mathcal{V}'_b = \Sigma'_b$. Let $L_1 \subset \Sigma_{b_1}$ and $L_2 \subset \Sigma_{b_2}$ be two general pseudo-lines on X. By deforming L_1 in $\tilde{\mathcal{V}}$, we can connect L_1 with a pseudo-line $L_3 \subset \Sigma_{b_2}$ by a one dimensional family of pseudo-lines. Note that here we use the fact that B is irreducible which is a consequence of the irreducibility of M. By deforming L_2 inside Σ'_{b_2}, we can connect L_2 with L_3 by another one dimensional family of pseudo-lines. This shows that $F(X)$ is connected.

To prove (v), let L_1 and L_2 be two intersecting pseudo-lines with intersection point $x \in X$. We only need to show that $[L_1 \vee L_2] \in M$. By deforming $L_1 \vee L_2$, we may assume that x is a general point. Fix another pair of general intersecting pseudo-lines $[L_3 \vee L_4] \in M$ with intersection point y. The next step is to construct a one dimensional family of pairs of intersecting pseudo-lines with $L_1 \vee L_2$ and $L_3 \vee L_4$ being two special fibers. Consider the universal family of pseudo-lines on X.

$$\begin{tikzcd} P(X) \arrow[r, "f"] \arrow[d, "p"] & X \\ F(X) & \end{tikzcd} \tag{4}$$

By Bertini theorem, we can find a smooth irreducible curve $\Gamma \subset X$ that passes through x and y such that $f^{-1}(\Gamma)$ is smooth irreducible. Note that the morphism $f^{-1}(\Gamma) \to \Gamma$ is smooth. Let Γ' be the normalization of Γ inside $f^{-1}(\Gamma)$, then $f^{-1}(\Gamma) \to \Gamma'$ has connected fibers. The pseudo-lines L_i determine points $Q_i \in f^{-1}(\Gamma)$. After taking some finite covering $\tilde{\Gamma}$ of Γ' we may assume that there are sections σ_1, σ_2 of $\mathscr{C} = f^{-1}(\Gamma) \times_{\Gamma'} \tilde{\Gamma} \to \tilde{\Gamma}$ such that $Q_1, Q_3 \in \sigma_1(\tilde{\Gamma})$ and $Q_2, Q_4 \in \sigma_2(\tilde{\Gamma})$. By composing with the morphism p in (4), each of the σ_1 and σ_2 defines a family of pseudo-lines. The two families of pseudo-lines defined by σ_i give a family of intersecting pseudo-lines $Y \to \tilde{\Gamma}$ such that $L_1 \vee L_2$ and $L_3 \vee L_4$ are two fibers. Since $[L_3 \vee L_4] \in M$, we get $[L_1 \vee L_2] \in M$. The existence and uniqueness of Σ containing L_1 and L_2 follows from deforming $L_1 \vee L_2$ with three points fixed as before.

(vi) By deformation theory, we see that $F_x(X)$ is smooth, so we only need to show that it is connected. Let L_1 and L_2 be two pseudo-lines passing through x, by (v) we get a unique Σ. By deformation inside Σ, we see that there is a curve connecting $[L_1]$ and $[L_2]$ in $F_x(X)$.

(vii) Given two general points on X, all pseudo-lines passing through the two points form a zero dimensional subvariety of $F(X)$ and hence finite. □

LEMMA 3.9. *Assume that there is a unique pseudo-line through a general pair of points on X, then $C \subset X$ is equivalent to a conic in $\mathbb{P}^3$ for $[C] \in M^0$ and pseudo-lines on X correspond to lines on $\mathbb{P}^3$.*

Proof. Let $U_C \subset \Sigma_C$ be the image of $U'_C \subset \Sigma'_C$. First we claim that under the assumption of the lemma, the surface U_C is smooth. In fact, if U is not smooth then there are two points $P_1, P_2 \in U'_C \subset \Sigma'_C$ that map to the same point $P \in U$. Pick a general point $Q' \in U'$ which maps to $Q \in U$. The two lines connecting Q' with P_1, P_2 will give two pseudo-lines on X connecting P and Q, which is a contradiction. Then we claim that the complete linear system $|\Sigma|$ is three dimensional. Since X is rationally connected, $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. Hence rational equivalence is the same as algebraic equivalence for divisors on X. Let $Z \to T$ be a flat family of divisors on X over a one dimensional smooth base T. Assume that $Z_{t_0} = \Sigma_0$ and let C_0 be a curve that defines Σ_0. Consider the deformation of C_0 in Z. By the first claim we know that C_0 is in the smooth locus of Z. Hence C_0 moves to nearby divisors. Hence $Z_t = \Sigma_t$ is swept out by some C_t for general $t \in T$. This shows that $B \subset |\Sigma| \cong \mathbb{P}^3$ is an open subset, where B is the quotient as in Proposition 2.9 which is three dimensional. Let $\varphi = \varphi_{|\Sigma|} : X \dashrightarrow \mathbb{P}^3$ be the map defined by the linear system $|\Sigma|$. Next we show that φ defines an isomorphism on a neighborhood of C in X and maps C to a smooth conic. But this is clear. Since C is very free on X, one sees that $|\Sigma|$ separates points and also separates tangent vectors in a neighborhood of C. The splitting of $\mathscr{N}_{C/X}$ shows that C maps to a conic on $\mathbb{P}^3$. ☐

Now we return to the general case. Consider the universal family of pseudo-lines passing through a general point $x \in X$. We write $P_x := P_x(X)$ and $F_x := F_x(X)$. Then we have the following diagram

$$\begin{tikzcd} P_x \arrow[r, "f_x"] \arrow[d, "\pi_x"'] & X \\ F_x & \end{tikzcd}$$

where π_x has a section $s_x : F_x \to P_x$, which is contracted by f_x to the point x. Let $P_x^0 = P_x - s_x(F_x)$, then f_x is étale on P_x^0. Let Y be the normalization of X inside the function field of P_x via f_x. Then we have the following diagram.

(5)
$$\begin{tikzcd} & P_x \arrow[r, "p_2"] \arrow[d, "\phi"] & X \\ P_x^0 \arrow[ur, "i"] \arrow[r, "i'"] & Y \arrow[ur, "\pi"] & \Sigma \arrow[u] \arrow[l, dotted] \\ & & \Sigma' \arrow[u] \arrow[ul, dotted, "\sigma"] \end{tikzcd}$$

In the above diagram i and i' are open immersions; Σ is one of the surfaces that pass through x and Σ' its normalization. The existence of σ is due to the fact that for a general point $y \in \Sigma'$, there is a unique line $L \subset \Sigma'$ connecting x and y. Since σ is defined on an open set whose complement has at least codimension 2, we may assume that σ is defined on a neighborhood of $C \subset \Sigma'$ and by choosing Σ general, we may also assume that $\sigma(C)$ is in the smooth locus of Y.

Proposition 3.10. *Under the assumptions of Situation 3.4, there exists a normal variety Y and a finite morphism $\pi : Y \to X$ with the following properties*
(i) There is an open subset $V \subset X$ such that $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is étale and $C \subset V$ for general $[C] \in M$. There is an open immersion $\rho : \pi^{-1}(V) \to \mathbb{P}^3$ such that for general $[C] \in M$ with $C \subset V$, any lifting $C \subset \pi^{-1}(V)$ is equivalent to a conic on $\mathbb{P}^3$ under ρ.
(ii) A general line $L' \subset \pi^{-1}(V)$ maps to a pseudo-line $L \subset X$.
(iii) The degree $d = \deg(\pi)$ of the morphism π is equal to the number of pseudo-lines connecting a general pair of points on X. The inverse image $\pi^{-1}(L)$ of a general pseudo-line $L \subset X$ is a disjoint union of d lines in $\pi^{-1}(V)$.

Proof. Pick a general point $x \in X$ and let Y be the normalization of X in the function field of P_x as in (5). The proof of the proposition will be divided into several steps.

Step 1. For general $[C] \in M^0$, the curve $C \subset X$ can always be lifted to a curve on Y such that π is étale along the lifting.

Proof of step 1. Pick a surface $\Sigma' = \Sigma'_C$ as in (5) with $x \in \Sigma_C$. Then σ gives a lifting of C to Y. Fix such a curve, we show that π is étale along $\sigma(C)$. Pick an arbitrary point $x' \in C$. We can always find a conic C_1 on $U'_C \subset \Sigma'_C$ such that $x, x' \in C_1$ and $[C_1] \in M^0$. Hence we also get that $\Sigma_1 := \Sigma_{C_1}$ is the same as Σ and $\sigma_1 : \Sigma'_1 \dashrightarrow Y$ is the same as σ. It is easy to see that the image of $d\pi(\sigma(x'))$: $T_Y \otimes k(\sigma(x')) \to T_X \otimes k(x')$ contains $\mathrm{Im}(T_{\Sigma'_1} \otimes k(x') \to T_X \otimes k(x'))$. This is true as long as C_1 passes through both x and x'. By deforming C_1 in X passing through the fixed points x and x', we get a family T of Σ_1's passing through x and x'. The Zariski tangent space $T_X \otimes k(x')$ is generated by the images $\mathrm{Im}(T_{\Sigma'_1} \otimes k(x') \to T_X \otimes k(x'))$ as Σ_1 runs through the family T. So $d\pi(\sigma(x'))$ is surjective and hence π is étale at $\sigma(x')$. Since $x' \in C$ is arbitrary, we know that π is étale along $\sigma(C)$. As a result we have $\mathscr{N}_{\sigma(C)/Y} \cong \mathscr{N}_{C/X}$. It follows that the deformation of $\sigma(C)$ in Y covers an open neighborhood of $[C]$ in M^0. Hence for a general $[C] \in M^0$ there is always a lifting of C to Y and π is étale along the lifting (note that we don't require $x \in \Sigma_C$ anymore).
Notations. By lifting $C \subset X$ to Y, we might get many unbalanced unbalanced components of very free rational curves on Y of the same generic splitting type of the normal bundle. Let M' be one of these components such that π is étale along C' for any $[C'] \in M'$. With respect to this M', we can do the same constructions on Y as in the previous section. We use the notation $\Pi_{C'}$ instead of $\Sigma_{C'}$ for the surface constructed from a general point $[C'] \in M'$. Similarly we will use Π' and $\tilde{\Pi}$ instead of Σ' and $\tilde{\Sigma}$.

Step 2. For a general $[C'] \in M'$, the pair $C' \subset \Pi'_{C'}$ is also equivalent to a conic on $\mathbb{P}^2$. Hence we also have the concept of pseudo-lines on Y.

Proof of step 2. Let $C \subset X$ be the image of C' via π. We know that π is étale along C'. The nearby deformations of C' in Y with three points fixed induce the nearby deformations of C in X with the three image points fixed. Hence π induces a morphism $\pi_{C'} : \Pi_{C'} \to \Sigma_C$ and $\pi'_{C'} : \Pi'_{C'} \to \Sigma'_C$. For any point $y_0 \in C' \subset \Pi'_{C'}$, let $x_0 = \pi'_{C'}(y_0) \in C \subset \Sigma'_C$ be the image. Consider the following diagram

$$\begin{tikzcd} \Pi' = \Pi'_{C'} \arrow[r] \arrow[d, "\pi'_{C'}"'] & Y \arrow[d, "\pi"] \\ \Sigma' = \Sigma'_C \arrow[r] & X \end{tikzcd}$$

Since both $T_{\Pi',y_0} \otimes k(y_0) \to T_{Y,y_0} \otimes k(y_0)$ and $T_{\Sigma',x_0} \otimes k(x_0) \to T_{X,x_0} \otimes k(x_0)$ are injective. Together with the fact that π is étale at y_0, we know $d\pi'_{C'} : T_{\Pi',y_0} \otimes k(y_0) \to T_{\Sigma',x_0} \otimes k(x_0)$ is an isomorphism. Hence $\pi'_{C'}$ is étale along C'. Hence $(\pi'_{C'})^{-1}(C)$ is a disjoint union of C' with some other divisor $D' \subset \Pi'$. We already know that as a divisor, C is nef and big on Σ'. This implies that $(\pi'_{C'})^{-1}(C)$ is also nef and big and hence connected. Thus we get $D' = \emptyset$. Hence $\pi'_{C'}$ is finite of degree 1, which means that it is isomorphism since Σ' is normal.

Step 3. The morphism π maps a general pseudo-line on Y to a pseudo-line on X.

Proof of step 3. Let $L' \subset Y$ be a general pseudo-line on Y. Then by definition, there is some general point $[C'] \in M'$ such that L' is the image of a line $L'_1 \subset \Pi'_{C'}$. Since $\pi'_{C'}$ is an isomorphism, $L_1 := \pi'_{C'}(L'_1)$ is a line on Σ'_C, where $C \subset X$ is the image of C'. Then $L = \pi(L')$, as the image of $L_1 \subset \Sigma'_C$, is a pseudo-line by definition.

Step 4. For a general pseudo-line $L \subset X$, $\pi^{-1}(L)$ is a disjoint union of d pseudo-lines on Y. On Y, there is a unique pseudo-line connecting a general pair of points.

Proof of step 4. It is easy to see from the definition of Y that the degree $d = \deg(\pi)$ is the number of pseudo-lines connecting a general pair of points on X. Let d' be the number of pseudo-lines on Y connecting a general pair of points. Let $(x, y) \in X \times X$ be a general pair of points on X and $L_1, \ldots, L_d$ be the pseudo-lines connecting them. Let $\pi^{-1}(x) = \{x_1, \ldots, x_d\}$ and $\pi^{-1}(y) = \{y_1, \ldots, y_d\}$. There are $d'd^2$ pseudo-lines L'_{ijk} connecting x_i and y_j, where $i, j = 1, \ldots, d$ and $k = 1, \ldots, d'$. Their images under π are exactly the pseudo-lines $\{L_i\}$ connecting x and y. On the other hand, the inverse image $\pi^{-1}(L_i)$ can contain at most d pseudo-lines for the degree reason. It follows that $d' = 1$ and $\pi^{-1}(L_i)$ consists of d pseudo-lines. These pseudo-lines are disjoint since π is étale along any of them.

Proof of Proposition. From Step 4 and Lemma 3.9, we know that $C' \subset Y$ is equivalent to a conic on $\mathbb{P}^3$ for general $[C'] \in M'$. Let $U \subset Y$ be the maximal open subset with an open immersion $\tilde{\rho} : U \to \mathbb{P}^3$ that realizes the above equivalence. Then a pseudo-line $L' \subset U$ corresponds to a line on $\mathbb{P}^3$ and hence we will call L' a line instead of a pseudo-line. We already see that for a general pseudo-line $L \subset X$, π is étale along $\pi^{-1}(L) \subset U$. Hence there is an open subscheme $V \subset X$ such that $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is étale and $\pi^{-1}(V) \subset U$. Define $\rho = \tilde{\rho}|_{\pi^{-1}(V)}$ then the proposition follows. □

3.2. Case II: Section of Hirzebruch surface.

Situation 3.11. In this whole subsection we will assume Situation 3.1 with one further assumption that $C \subset \Sigma'_C$ is equivalent to a positive section of a Hirzebruch surface F_n for general $[C] \in M$.

Recall that by definition, there is a natural fibration $\pi_n : F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n)) \to \mathbb{P}^1$. By blowing up at smooth points, we may assume that the above equivalence is given by a morphism $\sigma : \tilde{\Sigma} = \tilde{\Sigma}_C \to F_n$, which is an isomorphism on a neighborhood of C and the image of C is a positive section of π_n. On $\tilde{\Sigma}$, there is a distinguished divisor D that corresponds to the negative section of F_n with $D^2 = -n$. Note that D need not be irreducible but there is a unique component D_h which is a horizontal section. It is easy to see that C can only meet D at points of D_h since $\tilde{\Sigma} \to F_n$ is blowing up centered away from C. Let $F \subset \tilde{\Sigma}$ be a general fiber of $\pi_n \circ \sigma : \tilde{\Sigma} \to \mathbb{P}^1$. Then F is a smooth rational curve on $\tilde{\Sigma}$.

Definition 3.12. Let $\Gamma = \cup \Gamma_i$ be a nodal curve, and Y be a smooth projective variety. Let $\varphi : \Gamma \to Y$ be a morphism such that φ is an immersion on a neighborhood of each node and $d\varphi(x) : T_\Gamma \otimes k(x) \to T_Y \otimes k(\varphi(x))$ is injective for all smooth point x of Γ. We define the normal bundle of φ to be

$$\mathscr{N}_{\Gamma/Y} = \mathscr{N}_\varphi = [\ker(\varphi^*\Omega^1_Y \to \Omega^1_\Gamma)]^\vee$$

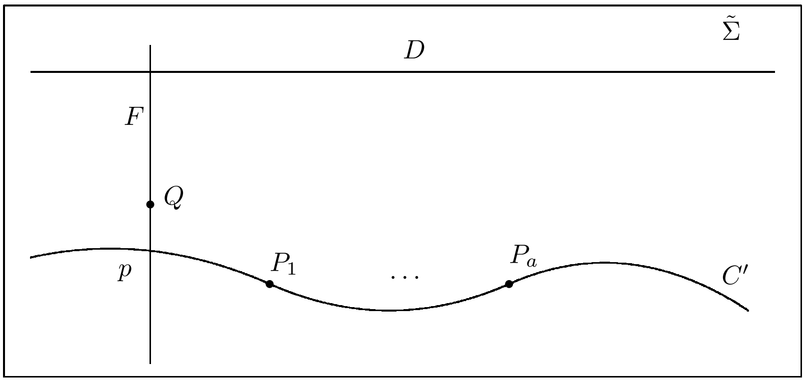
Note that the above definition agrees with Definition 2.6 when Γ is smooth. Since a nodal curve is always a local complete intersection, we know that $\mathscr{N}_{\Gamma/Y}$ is locally free. To better understand the normal bundle at the nodal points, let's assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ be a union to two smooth curves and let p be the nodal point. We always have the following exact sequence, [12] and [34],

$$0 \longrightarrow \mathscr{N}_{\Gamma_1/Y} \longrightarrow \mathscr{N}_{\Gamma/Y}|_{\Gamma_1} \longrightarrow k(p) \longrightarrow 0$$

This realizes $\mathscr{N}_{\Gamma/Y}|_{\Gamma_1}$ as the sheaf of sections of $\mathscr{N}_{\Gamma_1/Y}$ that are either regular or have a simple pole at p in the direction of $T_{\Gamma_1,p}$. A similar interpretation holds on Γ_2. The sheaf $\mathcal{T} := \mathcal{E}xt^1_{\mathcal{O}_\Gamma}(\Omega_\Gamma, \mathcal{O}_\Gamma)$ is a torsion sheaf supported at p whose fiber is canonically isomorphic to $\mathcal{T}|_p = T_{\Gamma_1,p} \otimes T_{\Gamma_2,p}$. The natural quotient map $\mathscr{N}_{\Gamma/Y} \twoheadrightarrow \mathcal{T}$ induce an isomorphism $(\mathscr{N}_{\Gamma/Y}|_{\Gamma_1})/\mathscr{N}_{\Gamma_1/Y} \cong \mathcal{T}$. With the above preparation, we are ready to prove the following

LEMMA 3.13. *Under Situation 3.11, the following does not happen: on $\tilde{\Sigma}$, the curve C meets D at least once and for a general fiber F we have $\mathscr{N}_{F/X} \cong \mathcal{O} \oplus \mathcal{O}(1)$.*

Proof. We prove the lemma by contradiction. So we assume the above situation happens. Since $C \cdot D \geq 1$, the curve C degenerates to a general fiber F and another positive section C'. We have the following picture.



Since $(C')^2 = (C-F)^2 = C^2 - 2(C \cdot F) = b-2$, we have $\mathscr{N}_{C'/\tilde{\Sigma}} \cong \mathcal{O}(b-2)$. Since $C \cdot D \geq 1$ and the tangent map $d\tilde{\phi}$ of $\tilde{\phi} : \tilde{\Sigma} \to X$ is injective along C, we know that $d\tilde{\phi}$ is also injective along a general fiber F. Hence we have the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{F/\tilde{\Sigma}} \longrightarrow \mathscr{N}_{F/X} \longrightarrow \mathscr{N}_{\tilde{\Sigma}/X}|_F \longrightarrow 0.$$

It follows, from the above sequence and the assumption, that $\mathscr{N}_{\tilde{\Sigma}/X}|_F \cong \mathcal{O}(1)$. Since $\mathscr{N}_{\tilde{\Sigma}/X}|_C \cong \mathcal{O}(a)$, see Corollary 2.7, we get that $\mathscr{N}_{\tilde{\Sigma}/X}|_{C'} \cong \mathcal{O}(a-1)$. Consider the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{C'/\tilde{\Sigma}} \longrightarrow \mathscr{N}_{C'/X} \longrightarrow \mathscr{N}_{\tilde{\Sigma}/X}|_{C'} \longrightarrow 0$$

and we get $\mathscr{N}_{C'/X} \cong \mathcal{O}(b-2) \oplus \mathcal{O}(a-1)$. Let $\Gamma = F \cup C' \subset \tilde{\Sigma}$ and let p be the nodal point. Consider the natural morphism $\varphi : \Gamma \to X$. The deformation problem of φ with only the target X being fixed is controlled by the cotangent complex of φ,

$$L^*_\varphi := \{0 \to \varphi^*\Omega^1_X \to \Omega^1_\Gamma \to 0\}.$$

Namely, the first order deformation is given by $\mathbb{E}\mathrm{xt}^1_\Gamma(L^*_\varphi, \mathcal{O}_\Gamma)$ and the obstruction space is in $\mathbb{E}\mathrm{xt}^2_\Gamma(L^*_\varphi, \mathcal{O}_\Gamma)$, see [26]. If we choose F and C' general, then $d\varphi$ is injective at all smooth points of Γ and φ is an immersion on an open neighborhood of the node

p. Hence we know that L^*_φ is quasi-isomorphic to $\mathscr{N}_\varphi^\vee$ centered at degree -1. Hence we have the following isomorphisms

$$\mathbb{E}\mathrm{xt}^1_\Gamma(L^*_\varphi, \mathcal{O}_\Gamma) \cong \mathrm{H}^0(\Gamma, \mathscr{N}_\varphi), \qquad \mathbb{E}\mathrm{xt}^1_\Gamma(L^*_\varphi, \mathcal{O}_\Gamma) \cong \mathrm{H}^1(\Gamma, \mathscr{N}_\varphi).$$

We pick $Q \in F$ and $P_1, \dots, P_a \in C'$ to be general points. Consider the same deformation problem while we require the deformations to pass through the points $\{Q, P_1, \dots, P_a\}$. Then the first order deformations and obstructions are given by $\mathrm{H}^0(\Gamma, \mathscr{E})$ and $\mathrm{H}^1(\Gamma, \mathscr{E})$ respectively, where $\mathscr{E} = \mathscr{N}_\varphi(-Q - \sum_{i=1}^a P_i)$. Since F has trivial normal bundle in $\tilde{\Sigma}$, the $\mathcal{O}(1)$ direction in $\mathscr{N}_{F/X}$ is pointing outside $\tilde{\Sigma}$. Recall that $\mathscr{E}|_F$ is the sheaf of sections of $\mathscr{N}_{F/X}(-Q)$ that are either regular or have a simple pole at p along the direction of $T_{C',p}$. This shows that the restriction morphism

$$\mathrm{H}^0(F, \mathscr{E}|_F) \longrightarrow \mathscr{E} \otimes k(p) = \mathscr{N}_\varphi \otimes k(p)$$

is surjective. In fact we have $\mathscr{N}_{F/X}(-Q) \cong \mathcal{O}(-1) \oplus \mathcal{O}$. The global section from the $\mathcal{O}$ factor and the rational section pointing to $T_{C',p}$ with a simple pole at p form a basis for $\mathrm{H}^0(F, \mathscr{E}|_F)$. They restrict to two linearly independent vectors in $\mathscr{N} \otimes k(p)$. To compute the cohomology groups of $\mathscr{E}$, we consider the following exact sequence.

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}|_F \oplus \mathscr{E}|_{C'} \longrightarrow \mathscr{E} \otimes k(p) \longrightarrow 0 \tag{6}$$

It follows easily from the interpretation of $\mathscr{E}|_F$ and $\mathscr{E}|_{C'}$ that

$$\mathrm{H}^1(F, \mathscr{E}|_F) = 0, \quad \mathrm{H}^1(C', \mathscr{E}|_{C'}) = 0$$

and

$$\dim \mathrm{H}^0(F, \mathscr{E}|_F) = 2 \quad \dim \mathrm{H}^0(C', \mathscr{E}|_{C'}) = b - a.$$

Hence the long exact sequence associated to (6) becomes

$$0 \longrightarrow \mathrm{H}^1(\Gamma, \mathscr{E}) \longrightarrow \mathrm{H}^0(F, \mathscr{E}|_F) \oplus \mathrm{H}^0(C', \mathscr{E}|_{C'}) \xrightarrow{\alpha} \mathscr{E} \otimes k(p)$$

$$\longrightarrow \mathrm{H}^1(\Gamma, \mathscr{E}) \longrightarrow 0$$

We already know that α is surjective. Hence we have

$$\dim \mathrm{H}^0(\Gamma, \mathscr{E}) = b - a, \qquad \dim \mathrm{H}^1(\Gamma, \mathscr{E}) = 0$$

So the deformation problem above is unobstructed. Note that the deformation that keeps the configuration $F \cup C'$ is $(b-a-1)$-dimensional and hence a general deformation smooth out the node and gives a curve $[C_1] \in M^0$ passing through $\{Q, P_1, \dots, P_a\}$. Now we deform $F \to X$ a little bit to get $F' \cong \mathbb{P}^1 \to X$ where the image of F' still passes through p and $T_{F',p}$ is not contained in $T_{\Sigma',p}$. Then we get a morphism $\varphi' : \Gamma' = F' \cup_{p'} C' \to X$. Pick $Q' \in F'$ we do the same deformation with respect to $\{Q', P_1, \dots, P_a\}$. Since the vanishing of obstruction is an open condition, we know that this new deformation problem is still unobstructed which gives a different curve $[C_2] \in M^0$. From C_1 we get the surface $\Sigma_1 = \Sigma$ and from C_2 we get a different surface Σ_2. Now since C' is component of the degeneration of both C_1 and

C_2 with $a+1$ points fixed, C' lies on both of Σ_1 and Σ_2. Consider the deformation of C' in X passing through $\{P_1,\dots,P_a\}$. If we consider C' as a curve on Σ_1', and we can do the deformation of C' in Σ_1'; Similarly we can also do the same deformation on Σ_2'. As a result, the curve C' can move along both of the directions $T_{F,p}$ and $T_{F',p}$ at the point p. But this is impossible since $\mathscr{N}_{C'/X}(-\sum_{i=1}^{a} P_i) \cong \mathcal{O}(-1)\oplus\mathcal{O}(b-a-2)$ is not globally generated. ∎

The main result of this subsection is the following

PROPOSITION 3.14. *Assume that the anti-canonical divisor $-K_X$ is nef together with Situation 3.11, then for a general fiber $F \subset \tilde{\Sigma}$, the morphism $F \to X$ has at worst nodal image and $\mathscr{N}_{F/X} = \mathcal{O}\oplus\mathcal{O}$.*

Proof. On $\tilde{\Sigma}$ we have the divisor class $C = D + cF$ for some integer c. We have the following basic relations

$$C^2 = D^2 + 2c(D\cdot F) = -n + 2c = b \quad\Rightarrow\quad c = \frac{b+n}{2} \tag{7}$$

$$C\cdot D = D^2 + c = -n + c \geq 0 \quad\Rightarrow\quad c \geq n \tag{8}$$

The above relations imply that $b \geq n$. We still use K_X to denote the pullback of K_X to $\tilde{\Sigma}$. From the following

$$a+b+2 = C\cdot(-K_X) = D\cdot(-K_X) + cF\cdot(-K_X)$$

and the assumption that $-K_X$ is nef, we get

$$F\cdot(-K_X) \leq \frac{a+b+2}{c} = \frac{2(a+b+2)}{b+n} \leq \frac{4b}{b} = 4. \tag{9}$$

By construction, a general F passes through a general point of X and hence F is free. As a result, the intersection number $F\cdot(-K_X)$ can only be 2, 3 or 4. To prove the proposition, we only need to rule out the cases $F\cdot(-K_X)$ being 3 or 4.

If $F\cdot(-K_X) = 4$ then $n = 0$, $b = a+2$ and $D\cdot(-K_X) = 0$. In this case, the divisor D is just the other ruling of $\mathbb{P}^1\times\mathbb{P}^1$ and hence a general member of the class of D is a rational curve that passes through a general point of X and hence is free. This implies that $D\cdot(-K_X)\geq 2$, which is a contradiction.

If $F\cdot(-K_X) = 3$ then we first show that $C\cdot D \geq 1$. In fact, if $C\cdot D = 0$, i.e. $D^2 + cF\cdot D = 0$, then $c = n$. From (7), we get $b = n$. Then in (9), we get

$$3 = F\cdot(-K_X) \leq \frac{2(a+b+2)}{b+n} = \frac{2(a+b+2)}{2b} = \frac{a+b+2}{b} \leq 2.$$

Hence we get contradiction again. Recall that there is an open neighborhood $\tilde{U}$ of C inside $\tilde{\Sigma}$ such that the morphism $\tilde{U}\to X$ has injective tangent map at each point. The fact that $C\cdot D\geq 1$ implies that $F\subset\tilde{U}$ for general F. This implies that $F\to X$ has injective tangent map at all points and hence $\mathscr{N}_{F/X}\cong\mathcal{O}\oplus\mathcal{O}(1)$, which is impossible by Lemma 3.13. ∎

3.3. Conclusion. Here we summarize the previous two subsections in the following theorem.

THEOREM 3.15. *Let $X/\mathbb{C}$ be a smooth projective variety with $\dim X = 3$. Let M be an unbalanced component of very free rational curves such that for general $[C]\in M$,*

C is a smooth rational curve on X with normal bundle $\mathscr{N}_{C/X} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $b - 2 \geq a \geq 1$. Let Σ be the surface swept out by deforming C with $a+1$ points fixed as before. Let Σ' be its normalization. Then we have one of the following to cases.

***Case I**: The pair $C \subset \Sigma'$ is equivalent to a conic in $\mathbb{P}^2$. In this case, there is a finite morphism $\pi : Y \to X$ and there is an open neighborhood $V \subset X$ of C such that $\pi^{-1}(V) \to V$ is étale. Furthermore, there is an open immersion $\rho : \pi^{-1}(V) \to \mathbb{P}^3$ such that any lift $C \to \pi^{-1}(V)$ is a conic on $\mathbb{P}^3$.*

***Case II**: The pair $C \subset \Sigma'$ is equivalent to a positive section of a Hirzebruch surface F_n. In this case, if we further assume that $-K_X$ is nef then a general fiber F of $\tilde{\Sigma}$ gives a free rational curve on X with trivial normal bundle, i.e. $\mathscr{N}_{F/X} \cong \mathcal{O} \oplus \mathcal{O}$. Let S be the component of the $\overline{\mathcal{M}}_{0,0}(X,\beta)$ that parameterizes such curves F. Then the natural morphism $\varphi : C \to S$ is a rational curve on S that connects a general pair of points on S. In particular, S is rationally connected and hence rational.*

Proof. The theorem is pretty much the combination of Proposition 3.3, Proposition 3.10 and Proposition 3.14. We only need to show that S is rationally connected in case II. Fix a general point $[F] \in S$, then there is some $C \subset \Sigma$ and $F \subset \Sigma$. Let x be the point that C meets F. By deforming C passing through the fixed point x, we get a family of Σ's. The fiber of any such Σ at the point x is always the fixed F. Hence we get a covering family of rational curves on S passing through the fixed point $[F]$. This means that S is rationally connected. $\square$

Definition 3.16. Let X and M be as in the theorem. When **Case I** happens, we say that M is an unbalanced component of *conic type*; when **Case II** happens, we say that M is of *fibration type.*

Corollary 3.17. *Let X be a smooth projective threefold of Picard number 1. If X has an unbalanced component M of very free rational curves of conic type, then $X \cong \mathbb{P}^3$ and M is the space of conics on X.*

Proof. Let $\pi : Y \to X$ be the finite morphism we get from the theorem. Then π is étale above $V \subset X$. Since V contains a general curve C and X has Picard number 1, the complement of V in X has dimension less than or equal to 1. Since X is simply connected (this follows from the fact that it is rationally connected), we get that V is also simply connected. This implies that $\deg(\pi) = 1$ and hence $V = \pi^{-1}(V) \subset \mathbb{P}^3$. We have the identification of the Picard groups $\mathrm{Pic}(X) = \mathrm{Pic}(V) = \mathbb{Z}H$. From the fact that $\mathcal{O}_X(-K_X)|_V \cong \Omega_V^3 = \mathcal{O}_V(-4H)$, we get $-K_X \cong 4H$. So X is Fano threefold of index 4, which implies that $X \cong \mathbb{P}^3$. $\square$

4. Non-triviality of the Abel-Jacobi mapping. In this section, we use the technique of intermediate Jacobian and the Abel-Jacobi mapping to prove Theorem 1.4.

4.1. Intermediate Jacobian and Abel-Jacobi mapping. The main reference to this section are [5], [36], [1], [2], [10] and [21].

Let X be a smooth projective variety over $\mathbb{C}$, $\dim X = 3$. We use $\mathrm{H}^*(X)$ to denote $\mathrm{H}^*(X,\mathbb{Z})/\text{torsion}$. We have the following Hodge decomposition.

$$\mathrm{H}^3(X) \otimes \mathbb{C} = \mathrm{H}^{3,0}(X) \oplus \mathrm{H}^{2,1}(X) \oplus \mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{0,3}(X)$$

Let $W(X) := \mathrm{H}^{1,2}(X) \oplus \mathrm{H}^{0,3}(X)$ and let $U(X) \subset W(X)$ be the lattice defined by the image of $\mathrm{H}^3(X)$ under the projection. We define a Hermitian form on $W(X)$ by

$$(\alpha,\beta) = h(\alpha,\beta) := 2i \int_X \alpha \wedge \bar{\beta}$$

Then the imaginary part of h restricts to an integral, unimodular, alternating form on $U(X)$.

DEFINITION 4.1. Let X be as above, the triple $(W(X), U(X), h)$ is called the *intermediate Jacobian* of X and denoted by $J(X)$.

PROPOSITION 4.2. *([5]) If* $\mathrm{H}^1(X) = 0$ *and* $\mathrm{H}^{0,3}(X) = 0$ *then* $J(X)$ *is a principally polarized abelian variety. In particular, if* X *is Fano then* $J(X)$ *is a principally polarized abelian variety.*

From now on, we always keep the assumption of the proposition above. The following proposition is well known, see [5] and [36].

PROPOSITION 4.3. *Let* X *be a smooth projective threefold with* $\mathrm{H}^1(X) = 0$ *and* $\mathrm{H}^{0,3}(X) = 0$*, and* $C \subset X$ *be a smooth curve on* X*. Let* $\tilde{X}$ *be the blow-up of* X *along the curve* C*, then we have canonical isomorphism*

$$J(\tilde{X}) \cong J(X) \oplus J(C)$$

as principally polarized abelian varieties, where $J(C)$ *is the jacobian of the curve.*

We also need the following basic property on the behavior of the intermediate Jacobian under the operation of a flop.

PROPOSITION 4.4. *Let* X *be a smooth projective threefold and let* $\chi : X \dashrightarrow X^+$ *be a flop of* (-2)*-curves. Then* $J(X^+) \cong J(X)$ *canonically.*

Proof. By the definition of a flop, we have a diagram

$$\begin{tikzcd} X \arrow[rr, dotted, "\chi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where both f and f^+ are small proper birational morphism. By a result of [30], χ is a composition of sequence of blow-ups and blow-downs centered along smooth rational curves. The proposition follows from the previous one. ∎

Let

$$\begin{tikzcd} \mathscr{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ S & \end{tikzcd}$$

be a family of curves on X, i.e. $\mathscr{C}_s$ is a curve on X for all $s \in S$. After fixing a general point $s_0 \in S$, we get a map

$$\Phi = \Phi_S : S \to J(X).$$

This is actually a morphism which induces

$$\Psi = \Psi_S : \mathrm{Alb}(S) \to J(X).$$

We refer to [5] and [36] for more details.

DEFINITION 4.5. Both Φ and Ψ are called the *Abel-Jacobi mapping* associate to the family $\mathscr{C} \to S$.

Now let's consider the infinitesimal version of the Abel-Jacobi mapping. Fix a smooth curve $C \subset X$, then we have the following exact sequence

$$0 \longrightarrow \mathscr{N}_{C/X}^{\vee} \longrightarrow \Omega_X^1|_C \longrightarrow \Omega_C^1 \longrightarrow 0 .$$

This induces an exact sequence as the following

$$0 \longrightarrow \wedge^2(\mathscr{N}_{C/X}^{\vee}) \longrightarrow \Omega_X^2|_C \longrightarrow \Omega_C^1 \otimes \mathscr{N}_{C/X}^{\vee} \longrightarrow 0 .$$

By taking the associated long exact sequence, we get an natural surjection

$$\alpha : \mathrm{H}^1(C, \Omega_X^2|_C) \longrightarrow \mathrm{H}^1(C, \Omega_C^1 \otimes \mathscr{N}_{C/X}^{\vee}) \cong \mathrm{H}^0(C, \mathscr{N}_{C/X})^{\vee},$$

where the isomorphism is Serre duality. Note that if $\mathrm{H}^1(C, \wedge^2 \mathscr{N}_{C/X}^{\vee}) = 0$ then α is an isomorphism. Let

$$r : \mathrm{H}^1(X, \Omega_X^2) \longrightarrow \mathrm{H}^1(C, \Omega_X^2|_C)$$

be the natural restriction map. Then the composition $\phi = \alpha \circ r$ is the dual of $d(\Phi_S)$ and the point $[C]$ when $\mathscr{C} \to S$ is the universal family. We call ϕ the *infinitesimal Abel-Jacobi mapping.*

PROPOSITION 4.6. *([37] Lemma 2.8) Suppose X can be embedded in a smooth 4 dimensional variety W. Then there is a commutative diagram as following*

$$\begin{tikzcd}
\mathrm{H}^0(X, \mathscr{N}_{X/W} \otimes \Omega_X^3) \arrow[r] \arrow[d, "r_C"'] & \mathrm{H}^1(X, \Omega_X^2) \arrow[d, "\phi"] \\
\mathrm{H}^0(C, \mathscr{N}_{X/W} \otimes \Omega_X^3 \otimes \mathcal{O}_C) \arrow[r, "\beta_C"] & \mathrm{H}^0(C, \mathscr{N}_{C/X})^{\vee}
\end{tikzcd}$$

Here the map β_C fits into the following long exact sequence

$$\mathrm{H}^0(C, \mathscr{N}_{X/W} \otimes \Omega_X^3 \otimes \mathcal{O}_C) \xrightarrow{\beta_C} \mathrm{H}^0(C, \mathscr{N}_{C/X})^{\vee} \longrightarrow$$

$$\to \mathrm{H}^1(C, \mathscr{N}_{C/W} \otimes \Omega_X^3) \longrightarrow \mathrm{H}^1(C, \mathscr{N}_{X/W} \otimes \Omega_X^3 \otimes \mathcal{O}_C) \longrightarrow 0$$

COROLLARY 4.7. *Notations and assumptions as above, if $\mathscr{N}_{C/X} \cong \mathcal{O} \oplus \mathcal{O}$ and the following two conditions hold, then the infinitesimal Abel-Jacobi mapping ϕ is nontrivial.*
(1) The restriction map $r_C : \mathrm{H}^0(X, \mathscr{N}_{X/W} \otimes \Omega_X^3) \to \mathrm{H}^0(C, \mathscr{N}_{X/W} \otimes \Omega_X^3 \otimes \mathcal{O}_C)$ is surjective;
(2) $h^1(C, \mathscr{N}_{C/W} \otimes \Omega_X^3) - h^1(C, \mathscr{N}_{X/W} \otimes \Omega_X^3 \otimes \mathcal{O}_C) \leq 1$.

4.2. Nontriviality of Abel-Jacobi mapping. To prove the main result of this section, we need a description of double covers. Let $\pi : X \to V$ be a double cover between smooth algebraic varieties, $R \subset X$ be the ramification locus and $B \subset V$ be the image of R. Then $R \cong B$ are smooth and there is a line bundle $\mathscr{L}$ on V such that $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_V(B)$. There is a section $\sigma \in \Gamma(V, \mathscr{L}^{\otimes 2})$ such that $B = \mathrm{div}(\sigma)$. We have the following diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\pi"'] & U \arrow[d, "p"] \\ & V \end{tikzcd} \tag{10}$$

where $U = \mathrm{Spec}_V(Sym^*(\mathscr{L}^{-1}))$ is the space of $\mathscr{L}$. On U, there is a canonical section $y \in \Gamma(U, p^*\mathscr{L})$ and $X = \mathrm{div}(y^2 - p^*\sigma)$. It is easy to see that $T_{U/V} = p^*\mathscr{L}$ and hence we have the following exact sequence

$$0 \longrightarrow p^*\mathscr{L} \longrightarrow T_U \longrightarrow p^*T_V \longrightarrow 0\,. \tag{11}$$

Then it is easy to see that $\mathscr{N}_{X/U} \cong p^*\mathscr{L}^{\otimes 2}$ and $\omega_X \cong \pi^*(\omega_V \otimes \mathscr{L})$.

Lemma 4.8. *Assume that $Q \subset \mathbb{P}^n$, $n \geq 4$, is a quadric hypersurface. Let $C \subset Q$ be a smooth conic rational curve in the smooth locus of Q. Let $\Pi = \Pi(C)$ be the plane spanned by C. Then $\mathscr{N}_{C/Q}$ has a direct summand of $\mathcal{O}_{\mathbb{P}^1}(4)$ if and only if Q contains Π.*

Proof. Consider the following short exact sequence

$$0 \longrightarrow \mathscr{N}_{C/Q} \longrightarrow \mathscr{N}_{C/\mathbb{P}^n} \longrightarrow \mathscr{N}_{Q/\mathbb{P}^n}|_C \longrightarrow 0$$

It is easy to see that $\mathscr{N}_{C/\mathbb{P}^n} \cong \mathcal{O}(4) \oplus \mathcal{O}(2)^{\oplus(n-2)}$ and the $\mathcal{O}(4)$ summand is canonically isomorphic to $\mathscr{N}_{C/\Pi}$. If $\mathscr{N}_{C/Q}$ contains an $\mathcal{O}(4)$ summand, then this summand has to map isomorphically onto the $\mathscr{N}_{C/\Pi}$ summand of $\mathscr{N}_{C/\mathbb{P}^n}$. This means that Π is tangent to Q along C. This can happen only if $\Pi \subset Q$. The other direction is easy. □

With the above preparations, we are ready to prove the following

Theorem 4.9. *Let X be a Fano threefold of index 1 or 2 and of Picard number 1. Assume that the intermediate Jacobian $J(X)$ is not zero. Let S be a component of rational curves on X with trivial normal bundle. Then the Abel-Jacobi mapping*

$$\Phi : S \to J(X)$$

is nontrivial. In particular, S is not rational.

Remark. The intermediate Jacobian $J(X)$ is trivial only when $X = X_5$ is of index 2 and degree 5 or when $X = X_{22}$ is of index 1 and genus 12.

Proof. We prove the theorem case by case. We use C to denote a general member of the family S.

First we consider the case when the index of X is 1 and in this case the rational curves with trivial normal bundle are conics on X. Recall that for those of high genus, we use the method of "double projection from a line" and get the following diagram, see [20], [18], [19] and [7].

$$\begin{tikzcd} \tilde{X} \arrow[rr, dotted, "\chi"] \arrow[dd, "\sigma"'] \arrow[dr, "\varphi'"] & & \tilde{X}^+ \arrow[dl, "\varphi^+"'] \arrow[dd, "\varphi"] \\ & \bar{X} & \\ X & & Y \end{tikzcd} \tag{12}$$

Recall that σ is blow-up along a line, χ is a flop of (-2)-curves and φ is an extremal contraction. Let $Z \subset S$ be the curve that parameterizes a component of the conics which meet l where l is a line on X and the center of the blow-up σ. Such Z always exists since we can pick l general. Let $\mathscr{C}_Z \to Z$ be the family over Z. After blowing up and the flop, this gives a family $\mathscr{C}_Z^+ \to Z$ of rational curves on $\tilde{X}^+$. Since χ is a flop of (-2)-curves, by Proposition 4.1, we know that there is a canonical isomorphism $J(X) \cong J(\tilde{X}^+)$ and we get the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\Phi_Z"] \arrow[d] & J(\tilde{X}^+) \arrow[d, "\cong"] \\ S \arrow[r, "\Phi_S"] & J(X) \end{tikzcd} \tag{13}$$

Note that all curve in the family $\mathscr{C}_Z^+ \to Z$ are contracted by φ and if we can show that Φ_Z is nontrivial then Φ_S is also nontrivial.

g=10: In this case, $\varphi : \tilde{X}^+ \to Y$ blows down a divisor onto a smooth curve of genus 2 on $Y \cong Q \subset \mathbb{P}^4$. This implies that Z is of genus 2 and $J(\tilde{X}^+) \cong J(Z)$ and hence Φ_Z is nontrivial.

g=9: In this case, $\varphi : \tilde{X}^+ \to Y$ blows down a divisor onto a smooth curve of genus 3 on $Y \cong \mathbb{P}^3$. This implies that Z is of genus 3 and $J(\tilde{X}^+) \cong J(Z)$ and hence Φ_Z is nontrivial.

g=8: In this case $\varphi : \tilde{X}^+ \to Y$ is a standard conic bundle over $Y \cong \mathbb{P}^2$ with discriminant $\Delta \subset \mathbb{P}^2$ being of degree 5. In this case $J(\tilde{X}^+)$ is the prim variety $Pr(\tilde{\Delta}/\Delta)$ of the double cover $\tilde{\Delta} \to \Delta$, see [10] and [1]. Then $Z \to \Delta_0$ is a double cover of a component Δ_0 of Δ. If $\deg \Delta_0 = 1$ then Z is an elliptic curve; If $\deg(\Delta_0) = 2$ then Z has genus 2; If $\deg(\Delta_0) = 3$ then Z has genus 2 or 3, depending on whether Δ_0 has a node or not; If $\deg(\Delta_0) = 4$, then Z has genus 7, 6, 5 or 4, depending on the number of nodes of Δ_0; If $\Delta_0 = \Delta$ then $Z \cong \tilde{\Delta}$. In any of the above cases, it is easy to check that the morphism Φ_Z is nontrivial. For example, if $\deg(\Delta_0) = 1$ then the double cover $Z \to \Delta_0$ ramifies at 4 points and $Pr(Z/\Delta_0) = J(Z)$ gives a factor of the $J(X) \cong Pr(\tilde{\Delta}/\Delta)$. The Abel-Jacobi mapping Φ_Z maps Z nontrivially to the factor $Pr(Z/\Delta_0)$. The other cases are similar.

g=7: In [17] (Proposition 2.2), it is proved that $S \cong \Gamma^{(2)}$ the symmetric product to a smooth curve Γ of genus 7. It is also known that the intermediate Jacobian of X is isomorphic to the Jacobian of Γ. Hence Φ_S is nontrivial.

For the remaining cases, we will use Corollary 4.7 to show the nontriviality of Abel-Jacobi mapping. We refer to the conditions in Corollary 4.7 as condition (1) and condition (2).

g=6: In this case, X is either (i)a section of the Grassmannian $Gr(2,5)$ embedded by Plücker into $\mathbb{P}^9$ by a linear $\mathbb{P}^7$ and a quadric or (ii)the section by a quadric of a cone $\tilde{V}_5 \subset \mathbb{P}^7$ over $V_5 \subset \mathbb{P}^6$ where V_5 is a Fano threefold of Picard number 1, index 2 and degree 5, see [20] §5.1.

In case (i), we take

$$C \subset X \subset W = G(2,5) \cap \mathbb{P}^7 = G(2,5) \cap H_1 \cap H_2$$

Then we have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(2H)$ and $\Omega^3_X \cong \mathcal{O}(-H)$. Consider the following natural commutative diagram.

$$\begin{tikzcd} \mathrm{H}^0(X, \mathcal{O}_X(H)) \arrow[r, "r_C"] & \mathrm{H}^0(C, \mathcal{O}(2)) \\ \mathrm{H}^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(H)) \arrow[u] \arrow[ur, "r'_C"'] & \end{tikzcd}$$

where r'_C is surjective. This implies that

$$r_C : \mathrm{H}^0(X, \mathscr{N}_{X/W} \otimes \Omega^3_X) \to \mathrm{H}^0(C, \mathscr{N}_{X/W} \otimes \Omega^3_X \otimes \mathcal{O}_C) \cong \mathrm{H}^0(C, \mathcal{O}(2))$$

is surjective and hence condition (1) holds. Set $G = G(2,5)$, then we have the following

$$0 \longrightarrow \mathscr{N}_{C/X} \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{X/W}|_C \longrightarrow 0$$

Note that $\mathscr{N}_{X/W}|_C \cong \mathcal{O}(4)$ and $\mathscr{N}_{C/X} \cong \mathcal{O} \oplus \mathcal{O}$. Then it is easy to see that if $\mathscr{N}_{C/W}$ does not have a summand $\mathcal{O}(4)$ then condition (2) also holds and hence we know that the Abel-Jacobi mapping is nontrivial. So we only need to prove that $\mathscr{N}_{C/W}$ can not have a summand of $\mathcal{O}(4)$. We prove this by contradiction. Assume that $\mathscr{N}_{C/W} \cong \mathcal{O}(4) \oplus \mathcal{O}(2)^{\oplus 2}$. It is well known that $G = Gr(2,5) \subset \mathbb{P}^2$ is cut out by quadrics, see [13]. Suppose Q is a quadric hypersurface of $\mathbb{P}^9$ that contains G. Since $\mathscr{N}_{C/G}$ injects into $\mathscr{N}_{C/Q}$, we know that if $\mathscr{N}_{C/G}$ has an $\mathcal{O}(4)$ summand then so does $\mathscr{N}_{C/Q}$. By Lemma 4.8, we have $\Pi = \Pi(C) \subset Q$, where $\Pi(C)$ is the plane spanned by C. Since Q is arbitrary, one sees that $\mathscr{N}_{C/G}$ contains an $\mathcal{O}(4)$ summand if and only if the plane $\Pi(C)$ is contained in G. From the following exact sequence

$$0 \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/G} \longrightarrow \mathcal{O}(2)^{\oplus 2} \longrightarrow 0$$

one sees easily that if $\mathscr{N}_{C/W} \cong \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(4)$, then $\mathscr{N}_{C/G}$ will have an $\mathcal{O}(4)$ summand. As a result, for a general conic C on X we have $\Pi(C) \subset G$ and hence $\Pi(C) \subset W$. This means that W has a 2-dimensional family of planes. However, it is known that the planes on W form a 1-dimensional family, see [27] (3.2).

In case (ii), the projection from the node of $\tilde{V}_5$ realizes X as a double cover of V_5 that ramifies along a smooth divisor $B \in |2H|$. Use the notations above for double covers, we take $W = U$ and then we have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(2H)$ and $\Omega^3_X \cong \mathcal{O}_X(-H)$. This gives the surjection condition (1) as before. To verify the condition (2), consider the following exact sequence.

$$0 \longrightarrow \mathscr{N}_{C/X} \longrightarrow \mathscr{N}_{C/V_5} \longrightarrow \mathfrak{Q} \longrightarrow 0$$

Note that $\mathscr{N}_{C/X} \cong \mathcal{O} \oplus \mathcal{O}$. The cokernel $\mathfrak{Q}$ is a skyscraper sheaf of degree 2 since $C \cdot R = 2$, where R is the ramification divisor. Then we get $\mathscr{N}_{C/V_5} \cong \mathcal{O} \oplus \mathcal{O}(2)$ or

$\mathcal{O}(1) \oplus \mathcal{O}(1)$. We also have the following short exact sequence

$$0 \longrightarrow T_{W/V_5}|_C \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/V_5} \longrightarrow 0$$

Since $T_{W/V_5} \cong \pi^*\mathcal{O}_{V_5}(1)$, we have $T_{W/V_5}|_C \cong \mathcal{O}(2)$. Then the sequence shows that $\mathscr{N}_{C/W} \cong \mathcal{O} \oplus \mathcal{O}(2)^{\oplus 2}$ or $\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$. Condition (2) holds in either case.

g=5: $X = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$ is a complete intersection of 3 quadrics. We take $W = Q_i \cap Q_j$ where $1 \le i \le j \le 3$. Then we have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(2H)$ and $\Omega_X^3 \cong \mathcal{O}_X(-H)$. Condition (1) is readily verified. For condition (2), we consider

$$0 \longrightarrow \mathscr{N}_{C/X} \cong \mathcal{O}^{\oplus 2} \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{X/W}|_C \cong \mathcal{O}(4) \longrightarrow 0$$

From this one sees that condition (2) holds if $\mathscr{N}_{C/W}$ does not have a summand of $\mathcal{O}(4)$. Now suppose that $\mathscr{N}_{C/W}$ has an $\mathcal{O}(4)$ summand for all possible choice of W, then the sequence

$$0 \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/Q_i} \longrightarrow \mathcal{O}(4) \longrightarrow 0$$

implies that $\mathscr{N}_{C/Q_i}$ also has a summand of $\mathcal{O}(4)$. By Lemma 4.8, the plane $\Pi(C)$ is contained in Q_i. This is true for all $i = 1, 2, 3$. Then X should contain a linear $\mathbb{P}^2$. This is impossible because by adjunction formula, any smooth surface on X is either $K3$ or of general type.

g=4: $X = Q \cap Y \subset \mathbb{P}^5$ is a complete intersection of a quadric and a cubic. Let's take $W = Q$ to be the quadric. We have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(3H)$ and $\Omega_X^3 \cong \mathcal{O}_X(-H)$. We can verify condition (1) easily. Consider the exact sequence

$$0 \longrightarrow \mathscr{N}_{C/X} \cong \mathcal{O}^{\oplus 2} \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathcal{O}(6) \longrightarrow 0$$

We easily see that condition (2) holds as long as $\mathscr{N}_{C/W} \not\cong \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(6)$. On the other hand we have

$$0 \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/\mathbb{P}^5} \cong \mathcal{O}(2)^{\oplus 3} \oplus \mathcal{O}(4) \longrightarrow \mathcal{O}(4) \longrightarrow 0$$

and this implies that $\mathscr{N}_{C/W}$ can not have a summand of degree greater than 4. Hence condition (2) holds.

g=3: $X = X_4 \subset \mathbb{P}^4$ and we take $W = \mathbb{P}^4$. We have $\mathscr{N}_{X/W} = \mathcal{O}(4H)$ and $\Omega_X^3 \cong \mathcal{O}_X(-H)$ and condition (1) follows easily. Condition (2) also easily follows from the fact that $\mathscr{N}_{C/W} \cong \mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(4)$.

g=3: $X \to Q \subset \mathbb{P}^4$ is a double cover of a quadric threefold that ramifies along a surface B of degree 8. With the notations for double covers, we take $W = U$ and here $V = Q$ and $\mathscr{L} = \mathcal{O}_V(2H)$. The we easily get $\Omega_X^3 \cong \pi^*\mathcal{O}_V(-H)$ and $\mathscr{N}_{X/W} \cong \pi^*\mathcal{O}_V(4H)$. Condition (1) is again easy to verify. For condition (2), we consider the following

$$0 \longrightarrow T_{W/V}|_C \cong \mathcal{O}(4) \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/V} \longrightarrow 0$$

On the quadric threefold V we always have $\mathscr{N}_{C/V} \cong \mathcal{O}(2) \oplus \mathcal{O}(2)$ and hence $\mathscr{N}_{C/W} \cong \mathcal{O}(4) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. Thus the condition (2) holds.

g=2: $X \to \mathbb{P}^3$ is double cover of $\mathbb{P}^3$ which ramifies along a smooth surface of degree 6. Take $W = U$, $V = \mathbb{P}^3$ and we get $\Omega_X^3 \cong \pi^*\mathcal{O}_V(-H)$ and $\mathscr{N}_{X/W} \cong \pi^*\mathcal{O}_V(6H)$. Hence condition (1) holds. The exact sequence

$$0 \longrightarrow T_{W/V}|_C \cong \mathcal{O}(6) \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/V} \cong \mathcal{O}(2) \oplus \mathcal{O}(4) \longrightarrow 0$$

shows that $\mathscr{N}_{C/W} \cong \mathcal{O}(6) \oplus \mathcal{O}(4) \oplus \mathcal{O}(2)$. Thus condition (2) also holds.

Now we consider the cases when the index of X is 2. We prove case by case according to the $d = H^3$. Note that in this case, the curve C is a line on X. We still use Corollary 4.7 to show nontriviality of Abel-Jacobi mapping.

d=4: $X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ is a complete intersection of two quadrics in $\mathbb{P}^5$. Take $W = Q_1$ and we have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(2H)$ and $\Omega_X^3 \cong \mathcal{O}_X(-2H)$. It is still easy to verify condition (1). We have the following two short exact sequences

$$0 \longrightarrow \mathscr{N}_{C/X} \cong \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathcal{O}(2) \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{N}_{C/W} \longrightarrow \mathscr{N}_{C/\mathbb{P}^5} \cong \mathcal{O}(1)^4 \longrightarrow \mathcal{O}(2) \longrightarrow 0$$

It follows easily that $\mathscr{N}_{C/W} \cong \mathcal{O}(1)^2 \oplus \mathcal{O}$. Hence condition (2) holds.

d=3: X is a smooth cubic threefold. This case is well known, see [5].

d=2: $X \to \mathbb{P}^3$ is a double cover of $\mathbb{P}^3$ that ramifies along a smooth surface of degree 4. This case is studied in [37], Proposition (2.13).

d=1: X is a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P} = \mathbb{P}(3, 2, 1, 1, 1)$ with weighted homogeneous coordinates $(x_0, x_1, x_2, x_3, x_4)$. Since X is smooth, it must be contained inside the smooth locus of $\mathbb{P}$. Let $pr : \mathbb{P} \dashrightarrow \mathbb{P}^2$ be the projection to the last three coordinates. Let $C \subset X$ be a general line on X. Consider

(14)
$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dotted, "pr|_X"'] & \mathbb{P} \arrow[dl, dashed, "pr"] \\ pr(C) \arrow[r] & \mathbb{P}^2 & \end{tikzcd}$$

This shows that the homomorphisms

$$\mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \longrightarrow \mathrm{H}^0(C, \mathcal{O}(n))$$

is surjective for all $n \geq 0$, if $pr(C)$ is not a single point. But it is clear that $pr(C)$ is not a single point for general C. Otherwise, a general fiber of $pr|_X$ always contains a line and hence reducible. But this is impossible by Bertini's theorem. Now we take $W = \mathbb{P}$ and we have $\mathscr{N}_{X/W} \cong \mathcal{O}_X(6)$ and $\Omega_X^3 \cong \mathcal{O}_X(-2)$. Then the above surjection implies condition (1). For condition (2), note that C is a line in the smooth locus of $\mathbb{P}$ and $pr(C) \neq pt$, we know that $\mathscr{N}_{C/W}$ is ample. Hence $\mathrm{H}^1(C, \mathscr{N}_{C/W} \otimes \Omega_X^3) = 0$, which implies condition (2). ☐

4.3. Proof of main theorem. Now we are ready to prove the main theorem of this article.

THEOREM 4.10. *Let X be a Fano threefold of Picard number 1. If X has an unbalanced component M of very free rational curves, then $X = \mathbb{P}^3$ and M is the space of conics on X.*

Proof. If M is of conic type, then by Corollary 3.17, we know that X is $\mathbb{P}^3$ and M is the space of conics on X. If M is of fibration type, then by Theorem 3.15, a component S of the space of rational curves with trivial normal bundle is rational. For index 3 and 4 cases, there is no such rational curves on X. For index 1 and 2 cases, the non-triviality of the associated Abel-Jacobi mapping implies that S is not rational unless $X = X_5$ or $X = X_{22}$.

When $X = X_5$, let S be the space of lines on X. Then by [11], we know $S = \mathbb{P}^2$ with the universal family being a projective bundle $\mathbb{P}(\mathscr{E})$ over S and the $(-1,1)$-curves corresponds to a conic curve $\Delta \subset S$. Consider the universal family

$$\begin{tikzcd} \mathbb{P}(\mathscr{E}) \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ S & \end{tikzcd}$$

The morphism f ramifies along $\pi^{-1}(\Delta)$. Let $B = f(\pi^{-1}(\Delta))$. Now let $[C] \in M$ be a general point. Then by constructing the surface Σ associated to C, we get a family of lines $\Sigma' \to \mathbb{P}^1$ with a section σ. This gives a morphism $\varphi : \mathbb{P}^1 \to S$ which has a lift $\sigma' : \mathbb{P}^1 \to \mathbb{P}(\mathscr{E})$.

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \mathbb{P}^1 \arrow[r, "\varphi"] \arrow[ur, "\sigma'"] & S & \end{tikzcd}$$

Where $f \circ \sigma'$ gives the curve C. Since $\sigma'(\mathbb{P}^1)$ meets $\pi^{-1}(\Delta)$, the curve C is always tangent to B. This is impossible since C is a general point in a component of very free rational curves.

The case $X = X_{22}$ can be ruled out similarly. In this case we also have $S \cong \mathbb{P}^2$ and the only difference is that f ramifies along $\pi^{-1}(\Delta)$ where Δ is a degree 4 divisor on S. See the appendix for details. ◻

Appendix A. Space of conics on X_{22}.

A.1. We work over the field $\mathbb{C}$ of complex numbers. Let $\mathscr{E}$ be a vector bundle on a variety Z, then we use $G(k, \mathscr{E})$ to denote the scheme that parameterizes k dimensional fiberwise subspaces of $\mathscr{E}$. Hence $G(k, \mathscr{E})$ is a Grassmannian bundle over Z. When k=1, it can also be written as $\mathbb{P}(\mathscr{E}^*)$. We similarly define $G(k_1, k_2, \ldots, k_r, \mathscr{E})$, $0 < k_1 < \cdots < k_r < \operatorname{rank}(\mathscr{E})$, to be the relative Flag variety over Z.

A.2. In the whole article, we fix $X = X_{22} \subset \mathbb{P}^{13}$ to be a prime Fano threefold of genus 12. In particular, this means that X is a smooth projective variety whose anti-canonical class $-K_X$ is very ample and generates $\operatorname{Pic}(X) \cong \mathbb{Z}$. The embedding $X \subset \mathbb{P}^{13}$ is given by the complete linear system $|-K_X|$ and the intersection of two general hyperplane sections gives a canonical curve of genus 12. To better understand the structure of X, we introduce several notations. Let V be a vector space. A *net* of alternating forms on V is a surjective homomorphism $\eta : \wedge^2 V \to N$ with $\dim N = 3$. We use $G(k, V; \eta)$ to denote $\{E \in G(k, V) : \eta(\wedge^2 E) = 0\}$. We have the following structure theorem.

Theorem A.3. *(Mukai [28]) Let $X = X_{22} \subset \mathbb{P}^{13}$ be a prime Fano threefold of genus 12. Then there is a 7 dimensional vector space V and a net of alternating forms, $\eta : \wedge^2 V \to N$, such that $X = G(3, V; \eta)$. Conversely, for a general such η, the variety $X = G(3, V; \eta)$ is prime Fano threefold of genus 12.*

A.4. From now on, we fix a 7 dimensional vector space V and a net of alternating forms η as above such that $X = G(3, V; \eta)$ is a Fano threefold of genus 12. We use $\mathscr{E}_3$ to denote the canonical rank 3 subbundle of the trivial bundle $V \otimes \mathcal{O}_X$. Let $C \cong \mathbb{P}^1 \subset X$ be a conic on X, then

$$\mathscr{E}_3|_C \cong \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2}, \quad V/\mathscr{E}_3 \cong \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}.$$

Associated to C, there are canonical subspaces $V_1 \subset V_5 \subset V$, such that V_1 is the intersection of $\mathscr{E}_3(x)$ as x runs through all points on C and that V_5 is generated by $\mathscr{E}_3(x)$ as x runs through all points on C. If we vary $C \subset X$, we get a line bundle $\mathscr{E}_1$ and a vector bundle $\mathscr{E}_5$ of rank 5 on S^0, where S^0 is space of smooth conics. Let

$$\begin{array}{ccc} \mathscr{C}^0 & \xrightarrow{f^0} & X \\ {\scriptstyle \pi^0}\big\downarrow & & \\ S^0 & & \end{array}$$

be the universal family. Then we have

$$(\pi^0)^*\mathscr{E}_1 \subset (f^0)^*\mathscr{E}_3 \subset (\pi^0)^*\mathscr{E}_5 \subset V$$

By abuse of notation, we omit the "pull-back" and write

$$\mathscr{E}_1 \subset \mathscr{E}_3 \subset \mathscr{E}_5 \subset V$$

Hence we have a canonical morphism

$$\varphi^0 : S^0 \to G(1, 5, V),$$

where $G(1, 5, V)$ is the flag variety. We still use $\mathscr{E}_1 \subset \mathscr{E}_5 \subset V$ to denote the canonical rank 1 and rank 5 subbundles of V on $G(1, 5, V)$. Note that η induces

$$\eta' : \mathscr{E}_1 \otimes (\mathscr{E}_5/\mathscr{E}_1) \to N$$

Let $S \subset G(1, 5, V)$ be the closed subscheme defined by $\eta' = 0$.

Lemma A.5. *Notations and assumptions as above, the following are true:*
(i) $S \supset \mathrm{Im}(\varphi^0)$;
(ii) The morphism $S^0 \to S$ induces inclusion $S^0(\mathbb{C}) \subset S(\mathbb{C})$.

Proof. Given a smooth conic $C \subset X$, one easily checks that $V_1 \otimes V_5/V_1 \to N$ vanishes. This proves (i). To prove (ii), let $V_1 \subset V_5 \subset V$ be a pair with $\eta(V_1 \otimes V_5/V_1) = 0$.

$$\begin{aligned} \{x \in X : V_1 \subset \mathscr{E}_3(x) \subset V_5\} &= \left\{V_3/V_1 \in G(2, V_5/V_1) : \wedge^2(V_3/V_1) \to N \text{ is } 0\right\} \\ &= G(2, V_5/V_1) \cap \mathbb{P}^2 \text{ in } \mathbb{P}^5 \\ &= \text{ conic on } X \end{aligned}$$

The last equality is because otherwise X contains a $\mathbb{P}^2$ which is impossible. Hence C is uniquely determined by the pair $V_1 \subset V_5$. ∎

Proposition A.6. *The following are true.*
(i) The scheme S has pure dimension 2. In particular, S is local complete intersection and hence reduced.
(ii) $S^0 \to S$ is open immersion and $S^0 \subset S$ is dense.
(iii) Over S there is a canonical family $\mathscr{C}$ of conics on X which is constructed in the following way:

$$\begin{tikzcd} X & \mathscr{C} \arrow[l, "f"'] \arrow[r] \arrow[d, "\pi"] & G(2, \mathscr{E}_5/\mathscr{E}_1) \arrow[r, "Plüker"] \arrow[d] & \mathbb{P}(\wedge^2(\mathscr{E}_5/\mathscr{E}_1)^*) \\ & S \arrow[r] & G(1,5,V) & \end{tikzcd}$$

where

$$\mathscr{C} = G(2, (\mathscr{E}_5/\mathscr{E}_1)|_S) \cap \{\lambda = 0\} \text{ in } \mathbb{P}(\wedge^2(\mathscr{E}_5/\mathscr{E}_1)^*|_S)$$

with $\lambda : \mathscr{L}_{taut}|_S \to \wedge^2(\mathscr{E}_5/\mathscr{E}_1)|_S \to N$ being the natural homomorphism. Furthermore, we have $\mathscr{C}^0 = \mathscr{C}|_{S^0}$.

Proof. The construction in (iii) is just the proof of the second part of Lemma A.5 in a family. The expected dimension of S is 2, hence $\dim S \geq 2$. If S is not of pure dimension 2, there would be a 3-dimensional family of broken conics on X which is impossible. Hence we proved (i). The fact that the broken conics on X form a 1-dimensional family implies that $S^0 \to S$ is open and dense. This proves (ii). ∎

A.7. Consider the natural morphism

$$\phi : S \hookrightarrow G(1,5,V) \to G(1,V) = \mathbb{P}(V^*) \cong \mathbb{P}^6$$

where the second morphism is the natural projection.

Proposition A.8. *We have the following.*
(i)The image of ϕ can be characterized in the following way

$$\begin{aligned} \mathrm{Im}(\phi) &= \{x \in \mathbb{P}(V^*) : \mathrm{rank}_x(\mathscr{E}_1 \otimes V/\mathscr{E}_1 \to N) \leq 2\} \\ &= \{x \in \mathbb{P}(V^*) : \mathrm{rank}_x(\mathscr{E}_1 \otimes V/\mathscr{E}_1 \to N) = 2\} \end{aligned}$$

(ii) ϕ is a closed immersion.
(iii) On S, we have

$$\mathscr{E}_1 \otimes (V/\mathscr{E}_5) \cong \mathscr{N}_2 \hookrightarrow N$$

is a rank 2 subbundle of N. This gives

$$\rho : S \to G(2,N) = \mathbb{P}(N) \cong \mathbb{P}^2.$$

Proof. If there is a 1-dimensional subspace $V_1 \subset V$ such that $\mathrm{rank}(V_1 \otimes V/V_1 \to N) = 1$, then there is a 6-dimensional subspace $V_6 \subset V$ such that

$$\eta(V_1 \otimes V_6/V_1) = 0$$

Then $G(2, V_6/V_1; \eta) \subset X$ where

$$\begin{aligned} G(2, V_6/V_1; \eta) &= \{E/V_1 \subset V_6/V_1 : \eta(\wedge^2 E) = 0 \text{ and } \dim E = 3\} \\ &= G(2,5) \cap H_1 \cap H_2 \cap H_3 \end{aligned}$$

This implies that $X = G(2,5) \cap H_1 \cap H_2 \cap H_3$ and hence X is a Fano threefold of index 2 and degree 5. This is a contradiction. Now suppose we are given $V_1 \subset V$ with $\operatorname{rank}(V_1 \otimes V/V_1 \to N) = 2$. Then there is a unique $V_5 \subset V$ such that $\eta(V_1 \otimes V_5/V_1) = 0$. This proves (i). Let $Z \subset \mathbb{P}(V^*)$ be the closed subscheme defined by the degeneration of the homomorphism $\mathscr{E}_1 \otimes V/\mathscr{E}_1 \to N$. The above argument also shows that $S \to Z$ is isomorphism hence we have (ii). The rank condition in (i) implies that $\mathscr{N}_2 = \operatorname{Im}(\mathscr{E}_1 \otimes V/\mathscr{E}_1 \to N)$ is a rank 2 subbundle of N. Hence (iii) follows easily. ◻

A.9. There is a natural linear map

$$\operatorname{Sym}^3(\wedge^2 V^*) \longrightarrow \wedge^6 V^* \cong V$$

This induces

$$V^* \xrightarrow{\tau} \operatorname{Sym}^3(\wedge^2 V) \xrightarrow{\operatorname{Sym}^3(\eta)} \operatorname{Sym}^3(N)$$

which induces

$$\phi' : \mathbb{P}(N) \cong \mathbb{P}^2 \longrightarrow \mathbb{P}(V^*) = G(1, V)$$

Eventually, we want to show that $S \xrightarrow{\rho} \mathbb{P}(N) \xrightarrow{\phi'} \mathbb{P}(V^*)$ is the same as ϕ.

A.10. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ be an ordered set of distinct symbols. A set $\Lambda = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ is called a *2-partition* of $\underline{\alpha}$ and we write $\Lambda \prec_2 \underline{\alpha}$, if $\Lambda_i = (\lambda_{i,1}, \lambda_{i,2})$ with $\lambda_{i,1} < \lambda_{i,2}$ and $\bigcup_{i=1}^n \Lambda_i = \underline{\alpha}$ as sets. Then we define the *sign* of Λ to be

$$\operatorname{sign}(\Lambda) = \operatorname{sign}_{\underline{\alpha}}(\Lambda) = \operatorname{sign}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{2n} \\ \lambda_{1,1} & \lambda_{1,2} & \lambda_{2,1} & \lambda_{2,2} & \cdots & \lambda_{n,2} \end{pmatrix}$$

Then the natural linear map $\tau : V^* \to \operatorname{Sym}^3(\wedge^2 V)$ is given by

$$\tau(e_i^*) = (-1)^{i-1} \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}(\Lambda) e_{\Lambda_1} e_{\Lambda_2} e_{\Lambda_3}$$

where $\{e_1, \ldots, e_7\}$ is a basis of V and $\{e_1^*, \ldots, e_7^*\}$ is the dual basis of V^*; $e_{\Lambda_i} = e_{\lambda_{i,1}} \wedge e_{\lambda_{i,2}}$. So the morphism $\phi'^* : V^* \to \operatorname{Sym}^3(N)$ is given by

$$e_i^* \mapsto (-1)^{i-1} \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}(\Lambda) \eta(e_{\Lambda_1}) \eta(e_{\Lambda_2}) \eta(e_{\Lambda_3})$$

Lemma A.11. *For any linear functional $l : N \to \mathbb{C}$, i.e. a point $[l] \in \mathbb{P}(N)$, let $v \in V$ be given by*

$$v = \sum_{i=1}^{7} (-1)^{i-1} e_i \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}(\Lambda) l \circ \eta(e_{\Lambda_1}) l \circ \eta(e_{\Lambda_2}) l \circ \eta(e_{\Lambda_3}) \tag{15}$$

Then $\phi'([l]) = [\mathbb{C}v]$ and $l \circ \eta(v' \wedge v) = 0$ for all $v' \in V$.

Proof. The fact that $\phi'([l]) = [\mathbb{C}v]$ follows directly from the above explicit computations. To prove the second equation, we only need to do so for $v' = e_j$. By symmetry, we only need to do the case $j = 1$. To make the computation easier to understand, we use the symbol $1'$ to replace $j = 1$.

$$
\begin{aligned}
& l \circ \eta(e_1 \wedge v) \\
&= l \circ \eta(e_{1'} \wedge v) \\
&= \sum_{i=1}^{7} (-1)^{i-1} l \circ \eta(e_{1'} \wedge e_i) \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}(\Lambda) l \circ \eta(e_{\Lambda_1}) l \circ \eta(e_{\Lambda_2}) l \circ \eta(e_{\Lambda_3}) \\
&= \sum_{i=1}^{7} \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} (-1)^{i-1} \operatorname{sign}\begin{pmatrix} 1' & i & 1 & \cdots & \hat{i} & \cdots & 7 \\ 1' & i & \lambda_{1,1} & \cdots & \cdot & \cdots & \lambda_{3,2} \end{pmatrix} \cdot l \circ \eta(e_{1'} \wedge e_i) \\
&\qquad \cdot l \circ \eta(e_{\Lambda_1}) \cdot l \circ \eta(e_{\Lambda_2}) \cdot l \circ \eta(e_{\Lambda_3}) \\
&= \sum_{i=1}^{7} \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}\begin{pmatrix} 1' & 1 & 2 & \cdots & i & \cdots & 7 \\ 1' & i & \lambda_{1,1} & & \cdots & & \lambda_{3,2} \end{pmatrix} l \circ \eta(e_{1'} \wedge e_i) \\
&\qquad \cdot l \circ \eta(e_{\Lambda_1}) \cdot l \circ \eta(e_{\Lambda_2}) \cdot l \circ \eta(e_{\Lambda_3}) \\
&= \sum_{\Lambda' \prec_2 (1',1,\ldots,7)} \operatorname{sign}(\Lambda') \prod_{i=1}^{4} l \circ \eta(e_{\Lambda'_i}) \\
&= - \sum_{\Lambda' \prec_2 (1,1',\ldots,7)} \operatorname{sign}(\Lambda') \prod_{i=1}^{4} l \circ \eta(e_{\Lambda'_i}) \\
&= -l \circ \eta(e_1 \wedge v)
\end{aligned}
$$

This implies that $l \circ \eta(e_1 \wedge v) = 0$. ∎

Proposition A.12. *The following are true.*

(i) The composition $S \xrightarrow{\rho} \mathbb{P}(N) \xrightarrow{\phi'} \mathbb{P}(V^*)$ is the same as $\phi : S \to \mathbb{P}(V^*)$.
(ii) The morphism $\rho : S \to \mathbb{P}(N)$ is isomorphism.

Proof. Let $s = [V_1 \subset V_5] \in S$ be an arbitrary closed point, then we get $N_2 = \eta(V_1 \otimes V/V_1) \subset N$ is a 2-dimensional subspace with $N_2 \cong V_1 \otimes V/V_5$. Then $\rho(s) = [l]$ where $l : N \to N/N_2 \cong \mathbb{C}$. We choose a basis of V such that $V_1 = \mathbb{C}e_1$ and $\{e_1, \ldots, e_5\}$ form a basis of V_5. Then by (15), we get

$$
\begin{aligned}
v &= \sum_{i=1}^{7} (-1)^{i-1} e_i \sum_{\Lambda \prec_2 (1,2,\ldots,\hat{i},\ldots,7)} \operatorname{sign}(\Lambda) l \circ \eta(e_{\Lambda_1}) l \circ \eta(e_{\Lambda_2}) l \circ \eta(e_{\Lambda_3}) \\
&= e_1 \sum_{\Lambda \prec_2 (2,3,\ldots,7)} \operatorname{sign}(\Lambda) l \circ \eta(e_{\Lambda_1}) l \circ \eta(e_{\Lambda_2}) l \circ \eta(e_{\Lambda_3})
\end{aligned}
$$

the second equality holds since $l \circ \eta(e_1 \wedge w) = 0$ for all $w \in V$. It follows that $\phi(s) = \phi' \circ \rho(s)$. Since S is reduced, we get $\phi = \phi' \circ \rho$ by Hilbert Nullstellensatz. This proves (i). Then $\rho : S \to \mathbb{P}(N)$ is bijective on points and has smooth image. Then ρ has to be an isomorphism and hence (ii). ∎

A.13. We have already constructed the canonical family $\mathscr{C}$ of conics on X over the base scheme S. Now we want to study more details about the conic bundle $\mathscr{C} \to S$. On S, we have an induced homomorphism $\wedge^2(\mathscr{E}_5/\mathscr{E}_1) \to N$. Let $\mathscr{F}$ be its kernel which is a rank 3 vector bundle on S. Then we have the following short exact sequence,

$$0 \longrightarrow \mathscr{F} \longrightarrow \wedge^2(\mathscr{E}_5/\mathscr{E}_1) \longrightarrow N \longrightarrow 0 \tag{16}$$

By construction, $\mathscr{C} = G(2, \mathscr{E}_5/\mathscr{E}_1) \cap G(1, \mathscr{F})$ in $\mathbb{P}(\wedge^2(\mathscr{E}_5/\mathscr{E}_1)^*) = G(1, \wedge^2(\mathscr{E}_5/\mathscr{E}_1))$. Then we have the following commutative diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}(\mathscr{F}^*) \arrow[dl] \\ S & \end{tikzcd}$$

The divisor $G(2, \mathscr{E}_5/\mathscr{E}_1)$ on $G(1, \wedge^2(\mathscr{E}_5/\mathscr{E}_1))$ is given be vanishing of the section

$$\sigma : \mathrm{Sym}^2(\mathscr{L}) \longrightarrow \mathrm{Sym}^2(\wedge^2(\mathscr{E}_5/\mathscr{E}_1)) \longrightarrow \wedge^4(\mathscr{E}_5/\mathscr{E}_1)$$

Here $\mathscr{L} \to \wedge^2(\mathscr{E}_5/\mathscr{E}_1)$ is the tautological rank 1 subbundle on the scheme $G(1, \wedge^2(\mathscr{E}_5/\mathscr{E}_1))$. Then we have

$$\begin{aligned} \sigma|_{\mathbb{P}(\mathscr{F}^*)} &\in \mathrm{H}^0(\mathbb{P}(\mathscr{F}^*), \mathscr{L}^{-2} \otimes \wedge^4(\mathscr{E}_5/\mathscr{E}_1)) \\ &= \mathrm{H}^0(S, \pi_*(\mathscr{L}^{-2}) \otimes \wedge^4(\mathscr{E}_5/\mathscr{E}_1)) \\ &= \mathrm{H}^0(S, \mathrm{Sym}^2(\mathscr{F}^*) \otimes \wedge^4(\mathscr{E}_5/\mathscr{E}_1)) \\ &\subset \mathrm{Hom}(\mathscr{F}, \mathscr{F}^* \otimes \wedge^4(\mathscr{E}_5/\mathscr{E}_1)) \end{aligned}$$

It is a basic fact, see [29], that the *degeneration divisor* or *discriminant* $\Delta \subset S$ is given by the vanishing of

$$\det(\sigma|_{\mathbb{P}(\mathscr{F}^*)}) \in \mathrm{H}^0(S, \det(\mathscr{F}^*)^{\otimes 2} \otimes \det(\mathscr{E}_5/\mathscr{E}_1)^{\otimes 3})$$

Namely, $\Delta = \mathrm{div}(\det(\sigma|_{\mathbb{P}(\mathscr{F}^*)}))$. This implies that the divisor class of Δ is $-2c_1(\mathscr{F}) + 3c_1(\mathscr{E}_5/\mathscr{E}_1)$. By (16), we know that $c_1(\mathscr{F}) = c_1(\wedge^2(\mathscr{E}_5/\mathscr{E}_1)) = 3c_1(\mathscr{E}_5/\mathscr{E}_1)$. Hence we eventually have

$$\Delta \sim -3c_1(\mathscr{E}_5/\mathscr{E}_1). \tag{17}$$

Lemma A.14. *On $S \cong \mathbb{P}(N) \cong \mathbb{P}^2$ we have the following relations on divisor classes:*

$$c_1(\mathscr{E}_1) = -3h, \quad c_1(\mathscr{E}_5) = -5h$$

where h is the class of a line on $\mathbb{P}^2$.

Proof. Since $\mathscr{E}_1 \cong \phi^*\mathcal{O}_{\mathbb{P}(V^*)}(-1) \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, we get $c_1(\mathscr{E}_1) = -3h$. We also have $\mathscr{E}_1 \otimes V/\mathscr{E}_5 \cong \mathscr{N}_2 \subset N$, this implies that $c_1(\mathscr{E}_5) = -5h$. ∎

Proposition A.15. *The degeneration divisor Δ of the conic bundle $\pi : \mathscr{C} \to S$ is equivalent to $6\,h$, where h is the class of a line on $S \cong \mathbb{P}^2$.*

A.16. In this section, we would like to study the ramification of the natural map $f : \mathscr{C} \to X$. To do this, we consider the following diagram.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "j''"] \arrow[d, "i_{\mathscr{C}}"] & Y \arrow[r, "p_2'"] \arrow[d, "i_Y"] & X \arrow[d, "i_X"] \\
G(2, \mathscr{E}_5/\mathscr{E}_1) \arrow[r, "j'"] \arrow[d, "p_{13}'"] & G(1,3,5,V) \arrow[r, "p_2"] \arrow[d, "p_{13}"] & G(3,V) \\
S \arrow[r, "j"] & G(1,5,V) &
\end{tikzcd} \tag{18}$$

where all squares are fiber-product squares. The closed immersion i_X gives the following short exact sequence

$$0 \longrightarrow \wedge^2 \mathscr{E}_3|_X \otimes N^* \longrightarrow \Omega^1_{G(3,V)}|_X \xrightarrow{i_X^*} \Omega^1_X \longrightarrow 0 \tag{19}$$

The morphism p_2' realizes Y as a $G(1,3) \times G(2,4)$-bundle over X and hence Y is smooth. We have a similar sequence for i_Y.

$$0 \longrightarrow \wedge^2 \mathscr{E}_3|_Y \otimes N^* \longrightarrow \Omega^1_{G(1,3,5,V)} \xrightarrow{i_Y^*} \Omega^1_Y \longrightarrow 0 \tag{20}$$

Note that $j'' : \mathscr{C} \to Y$ is given by the vanishing of $\mathscr{E}_1 \otimes (\mathscr{E}_5/\mathscr{E}_3) \to N$. This gives

$$0 \longrightarrow \mathscr{E}_1 \otimes (\mathscr{E}_5/\mathscr{E}_3)|_{\mathscr{C}} \otimes N^* \longrightarrow \Omega^1_Y|_{\mathscr{C}} \xrightarrow{j''^*} \Omega^1_{\mathscr{C}} \longrightarrow 0\ . \tag{21}$$

We put all the above sequences together and get the following commutative diagram.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \wedge^2 \mathscr{E}_3|_{\mathscr{C}} \otimes N^* \arrow[r] \arrow[d] & \Omega^1_{G(3,V)}|_{\mathscr{C}} \arrow[r] \arrow[d] & \Omega^1_X|_{\mathscr{C}} \arrow[r] \arrow[d, "f^*"] & 0 \\
0 \arrow[r] & \mathscr{K} \arrow[r] \arrow[d] & \Omega^1_{G(1,3,5,V)}|_{\mathscr{C}} \arrow[r] \arrow[d] & \Omega^1_{\mathscr{C}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}_1 \otimes (\mathscr{E}_5/\mathscr{E}_3)|_{\mathscr{C}} \otimes N^* \arrow[r, "q"] \arrow[d] & \mathscr{E}_1 \otimes (\mathscr{E}_3/\mathscr{E}_1)^* \oplus (\mathscr{E}_5/\mathscr{E}_3) \otimes (V/\mathscr{E}_5)^* \arrow[r] \arrow[d] & \operatorname{coker} f^* \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

It follows that the ramification divisor R of the morphism of $f : \mathscr{C} \to X$ is given by $R = \operatorname{div}(\det(q))$, where

$$\det(q) \in \mathrm{H}^0\left(\mathscr{C}, \det((\mathscr{E}_1 \otimes \mathscr{E}_5/\mathscr{E}_3)^*|_{\mathscr{C}} \otimes N) \otimes \det(\mathscr{E}_1 \otimes (\mathscr{E}_3/\mathscr{E}_1)^* \oplus (\mathscr{E}_5/\mathscr{E}_3) \otimes (V/\mathscr{E}_5)^*)\right)$$

It follows easily that $R \sim \pi^*(-3c_1(\mathscr{E}_1) + c_1(\mathscr{E}_5))$.

Proposition A.17. *The ramification divisor R of the morphism $f : \mathscr{C} \to X$ can be written as $R = \pi^* \operatorname{div}(s_0)$ as divisors. Here we identify S with $\mathbb{P}^2$ and $s_0 \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$.*

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