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DEFORMATION OF CANONICAL METRICS I*

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1. Introduction. In this paper we study the deformation of canonical metrics associated to a family of complex manifolds. We describe a general method to establish the expansion of the Kähler forms of these metrics. Given a complex manifold X such that $c_1(X) < 0$, by Yau's work [13] we know that there exists a unique Kähler-Einstein metric on X. Similarly, if (X, L) is a polarized Calabi-Yau manifold, there exists a unique Ricci flat metric on X in the class $c_1(L)$. In this paper we study the deformation of these Kähler-Einstein metrics on a holomorphic family of such manifolds.

By the works of Donaldson [2], [3], in the case that $c_1(X) < 0$ there exists a unique balance metric and a V-balanced metric in $c_1(X)$ where V is the Kähler-Einstein volume form on X. The variation of these canonical metrics is also important in understanding geometry of the family of complex manifolds. In a sequel of this paper [10] we will study the deformation of these balanced metrics.

To compare the Kähler-Einstein metrics on different manifolds we need to identify these manifolds in C^∞ sense and thus we need to fix a gauge. In [6] Kuranishi introduced the Kuranishi gauge which is the most commonly used gauge later. However, in computing the deformation of Kähler-Einstein metrics and pluricanonical forms, by using the Kuranishi gauge we will have extra terms which do not vanish a priori. In Section 2 we define the divergence gauge. This gauge is equivalent to the Kuranishi gauge when we consider a holomorphic family of Kähler-Einstein manifolds of general type or a family of polarized Calabi-Yau manifolds.

Let $\pi : \mathfrak{X} \to B$ be a family of Kähler-Einstein manifolds of general type or a family of polarized Calabi-Yau manifolds. Let $X_t = \pi^{-1}(t)$ be the fiber. Let $\varphi(t) \subset A^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be a family of Beltrami differentials on the central fiber such that the complex structure on X_t is obtained by deforming the complex structure on X_0 via $\varphi(t)$. Let ω_0 be the Kähler form of the Kähler-Einstein metric on X_0. We have

THEOREM 1.1. *$\overline{\partial}^*\varphi(t) = 0$ if and only if $div(\varphi(t)) = 0$. Namely,*

$$\text{Kuranishi gauge} \iff \text{divergence gauge.}$$

Furthermore, we have $\varphi(t)\lrcorner\omega_0 = 0$ when either one of these gauges is imposed.

In Section 3 we describe a general method to find the Taylor expansion of the Kähler forms of the Kähler-Einstein metrics with respect to the Kuranishi-divergence gauge. We also give the explicit expansion up to order two.

THEOREM 1.2. *Let $\pi : \mathfrak{X} \to B$ be a family of Kähler-Einstein manifolds of general type. Let $X_t = \pi^{-1}(t)$ and let ω_t be the Kähler form of the Kähler-Einstein metric*

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on X_t. We assume that the complex structure on X_t is obtained by deforming the complex structure on X_0 by $\varphi(t) \in A^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ such that $\overline{\partial}^*\varphi(t) = 0$ where the operator $\overline{\partial}^*$ is the operator on X_0 with respect to the Kähler-Einstein metric. Then

$$\omega_t = \omega_0 + |t|^2 \left(\frac{\sqrt{-1}}{2}\partial\overline{\partial}\left((1-\Delta)^{-1}|\varphi_1|^2\right)\right) + O\left(|t|^3\right).$$

Let $K_{\mathfrak{X}/B}$ be the relative canonical bundle and let $E_m = R^0 K^m_{\mathfrak{X}/B}$ be the vector bundle over B. In Section 4 we use the Kuranishi-divergence gauge and the expansion of the Kähler forms of the Kähler-Einstein metrics to give a short proof of the curvature formula of the L^2 metric on E_m which was established in [7] and [1]. Also we showed

THEOREM 1.3. *The Ricci curvatures of the L^2 metrics on E_m converge to the Weil-Petersson metric on B after normalization.*

See Section 4 for details. Further discussions and applications of the methods in this paper can be found in [10].

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2. Complex structures of Kähler-Einstein manifolds. In this section we study the deformation of complex structures of Kähler-Einstein manifolds with respect to the Kuranishi gauge. We discuss a new gauge called the divergence gauge. We will show that the Kuranishi gauge is equivalent to the divergence gauge. As a consequence we show that the contraction of the Beltrami differentials with the Kähler form of the Kähler-Einstein metric on the central fiber vanish.

To setup the problem we consider a holomorphic family

$$\pi : \mathfrak{X} \to B \tag{2.1}$$

of complex manifolds. Here $B = B_\varepsilon \subset \mathbb{C}$ is the open disk of radius ε. Let t be the holomorphic coordinate on B. For each point $t \in B$ we let $X_t = \pi^{-1}(t)$ be the fiber. We assume that each fiber is connected and $\dim_{\mathbb{C}} X_t = n$.

By the Kodaira-Spencer theory [4], [5] and the work of Kuranishi [6], if we fix a Kähler metric on X_0 we can assume that the complex structure on X_t is obtained by deforming the complex structure on X_0 via a Beltrami differential $\varphi(t) \in A^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ such that

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \overline{\partial}^*\varphi(t) = 0 \end{cases} \tag{2.2}$$

where $\overline{\partial}$ and $\overline{\partial}^*$ are the operators on X_0 and $\overline{\partial}^*$ is defined with respect to the chosen Kähler metric on X_0. This means that we can take $\mathfrak{X} \cong X_0 \times B$ as a smooth manifold. For each point $p \in X_t$ we require

$$\Omega^{1,0}_p(X_t) = (I + \varphi(t)(p))\left(\Omega^{1,0}_p(X_0)\right)$$

where we view

$$\varphi(t)(p) : \Omega^{1,0}_p(X_0) \to \Omega^{0,1}_p(X_0)$$

as a linear map. Here we recall that the condition $\overline{\partial}^* \varphi(t) = 0$ is called the Kuranishi gauge.

Since $\varphi(t)$ depends on t holomorphicly, we have the convergent power series expansion $\varphi(t) = \sum_{k=1}^{\infty} t^k \varphi_k$. Then equation (2.2) can be rewritten as

$$\begin{cases} \overline{\partial}\varphi_i = \frac{1}{2}\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}] & \text{for any } i \geq 2 \\ \overline{\partial}^* \varphi_i = 0 & \text{for any } i \geq 2 \\ \varphi_1 \text{ is harmonic.} \end{cases} \tag{2.3}$$

When each fiber X_t is a Kähler-Einstein manifold of general type or a Calabi-Yau manifold, there is an equivalent formulation of the Kuranishi gauge which we will describe now. This new condition is convenient in studying the deformation of pluricanonical forms.

We first recall the divergence operator. Let (L, h) be a Hermitian line bundle over X_0. The divergence is the map

$$div = Tr \circ \nabla : A^{0,1}\left(X_0, T^{1,0}_{X_0} \otimes L\right) \to A^{0,1}\left(X_0, L\right).$$

Let $z_1, \cdots, z_n$ be local holomorphic coordinates on X_0 and let $\omega_g = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$ be the Kähler form of the chosen Kähler metric on X_0. Let e be a local holomorphic frame of L and let $h = h(e, e)$. If $\eta = \eta^i_{\overline{j}} d\overline{z}_j \otimes \frac{\partial}{\partial z_i} \otimes e \in A^{0,1}\left(X_0, T^{1,0}_{X_0} \otimes L\right)$ is a smooth section then $div\eta$ is given by

$$div\eta = \left(\partial_i \eta^i_{\overline{j}} + \eta^i_{\overline{j}} \partial_i \log(gh)\right) d\overline{z}_j \otimes e$$

where $g = \det[g_{i\overline{j}}]$.

Now we look at the case that each fiber X_t is a Kähler-Einstein manifold of general type. We have

THEOREM 2.1. *Let (X, ω_g) be a Kähler-Einstein manifold where $Ric(\omega_g) = -\omega_g$. Let*

$$\varphi(t) = \sum_{i=1}^{\infty} t^i \varphi_i \subset A^{0,1}\left(X, T^{1,0}_X\right)$$

be a holomorphic family of Beltrami differentials such that

$$\begin{cases} \overline{\partial}\varphi_i = \frac{1}{2}\sum_{j=1}^{i-1}[\varphi_j, \varphi_{i-j}] & \textit{for any } i \geq 2 \\ \overline{\partial}^* \varphi_i = 0 & \textit{for any } i \geq 2 \\ \varphi_1 \textit{ is harmonic.} \end{cases}$$

Then $div\varphi_k = 0$ and $\varphi_k \lrcorner \omega_g = 0$ for all $k \geq 1$.

To prove this theorem we need the following technical results. These results follow from direct computations.

LEMMA 2.1. *Let (X, ω_g) be a Kähler manifold where $\omega_g = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$ is the Kähler form. Let $\varphi, \psi \in A^{0,1}\left(X, T^{1,0}_X\right)$ be Beltrami differentials and let $\mu \in A^{1,1}(X)$ be a smooth form.*

1. $\overline{\partial}^*(div\varphi) = div\left(\overline{\partial}^*\varphi\right)$.
2. $\overline{\partial}(\varphi\lrcorner\mu) = \overline{\partial}\varphi\lrcorner\mu + \varphi\lrcorner\overline{\partial}\mu$.
3. *If* $\overline{\partial}^*\varphi = 0$ *then* $\overline{\partial}^*(\varphi\lrcorner\omega_g) = \frac{\sqrt{-1}}{4}div\varphi$.
4. *If* $\partial\mu = 0$ *then* $[\varphi,\psi]\lrcorner\mu = \varphi\lrcorner\partial(\psi\lrcorner\mu) + \psi\lrcorner\partial(\varphi\lrcorner\mu)$.
5. *If* $\overline{\partial}(\varphi\lrcorner\omega) = 0$ *and* $\overline{\partial}^*\varphi = 0$ *then*

$$\Box(\varphi\lrcorner\omega_g) = \frac{\sqrt{-1}}{4}div\left(\overline{\partial}\varphi\right) + \frac{1}{2}\varphi\lrcorner Ric(\omega_g)$$

where $\Box$ *is the Hodge Laplacian and* $Ric(\omega_g) = -\frac{\sqrt{-1}}{2}\partial_i\partial_{\overline{j}}\log\det[g_{i\overline{j}}]dz_i\wedge d\overline{z}_j$.

Now we prove Theorem 2.1.

Proof. We will prove this theorem by induction. For $k = 1$, we let $\psi = \varphi_1$ and we know that $\overline{\partial}\psi = 0$ and $\overline{\partial}^*\psi = 0$. By using Lemma 2.1 we know $\overline{\partial}(\psi\lrcorner\omega_g) = \overline{\partial}\psi\lrcorner\omega_g + \psi\lrcorner\overline{\partial}\omega_g = 0$. Also

$$\Box(\psi\lrcorner\omega_g) = \frac{\sqrt{-1}}{4}div\left(\overline{\partial}\psi\right) + \frac{1}{2}\psi\lrcorner Ric(\omega_g) = \frac{\sqrt{-1}}{4}div\left(\overline{\partial}\psi\right) - \frac{1}{2}\psi\lrcorner\omega_g = -\frac{1}{2}\psi\lrcorner\omega_g$$

which implies

$$0 \le (\Box(\psi\lrcorner\omega_g), \psi\lrcorner\omega_g) = -\frac{1}{2}\|\psi\lrcorner\omega_g\|^2_{L^2} \le 0.$$

Thus $\psi\lrcorner\omega_g = 0$, namely $\psi^i_{\overline{j}}g_{i\overline{l}} = \psi^i_{\overline{l}}g_{i\overline{j}}$. Since $\overline{\partial}^*\psi = 0$ we have

$$0 = \partial_k\left(\psi^i_{\overline{l}}g_{i\overline{j}}\right)g^{k\overline{l}} = \partial_k\left(\psi^i_{\overline{j}}g_{i\overline{l}}\right)g^{k\overline{l}} = \partial_k\psi^k_{\overline{j}} - \psi^i_{\overline{j}}g_{i\overline{l}}\partial_k g^{k\overline{l}} = \partial_k\psi^k_{\overline{j}} + \psi^k_{\overline{j}}\partial_k\log g$$

where $g = \det[g_{i\overline{j}}]$. This means $div\psi = 0$.

Now we assume $div\varphi_i = 0$ and $\varphi_i\lrcorner\omega_g = 0$ for all $i \le k-1$. For $i = k$ by using the equation of φ and Lemma 2.1 we have

$$\overline{\partial}(\varphi_k\lrcorner\omega_g) = \overline{\partial}\varphi_k\lrcorner\omega_g = \frac{1}{2}\sum_{i=1}^{k-1}[\varphi_i,\varphi_{k-i}]\lrcorner\omega_g = \sum_{i=1}^{k-1}(\varphi_{k-i}\lrcorner\partial(\varphi_i\lrcorner\omega_g)) = 0$$

since $\varphi_i\lrcorner\omega_g = 0$ for $i \le k-1$. Now we know that $\overline{\partial}^*\varphi_k = 0$, by Lemma 2.1 we have

$$\begin{aligned}\Box(\varphi_k\lrcorner\omega_g) =& \frac{\sqrt{-1}}{4}div\left(\overline{\partial}\varphi_k\right) + \frac{1}{2}\varphi_k\lrcorner Ric(\omega_g) = \frac{\sqrt{-1}}{8}div\left(\sum_{i=1}^{k-1}[\varphi_i,\varphi_{k-i}]\right) - \frac{1}{2}\varphi_k\lrcorner\omega_g \\ =& \frac{\sqrt{-1}}{4}\sum_{i=1}^{k-1}\varphi_{k-i}\lrcorner\partial(div\varphi_i) - \frac{1}{2}\varphi_k\lrcorner\omega_g = -\frac{1}{2}\varphi_k\lrcorner\omega_g\end{aligned}$$

since $div\varphi_i = 0$ for $i \le k-1$. Similar to the above proof we know that $\varphi_k\lrcorner\omega_g = 0$ and thus $div\varphi_k = 0$. We finished the proof. ∎

We call the condition $div(\varphi(t)) = 0$ the divergence gauge. The above theorem states that the Kuranishi gauge implies the divergence gauge. In fact the converse is also true. We have

THEOREM 2.2. *If (X, ω_g) is a Kähler-Einstein manifold of general type and $\varphi \in A^{0,1}\left(X, T_X^{1,0}\right)$ is a Beltrami differential such that $\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$ and $div\varphi = 0$, then $\overline{\partial}^*\varphi = 0$ and $\varphi \lrcorner \omega_g = 0$.*

Proof. Direct computations shows that

$$\begin{aligned} 0 =& \overline{\partial}\left(div\varphi\right) = div\left(\overline{\partial}\varphi\right) - 2\sqrt{-1}\varphi \lrcorner Ric\left(\omega_g\right) = \frac{1}{2}div[\varphi, \varphi] + 2\sqrt{-1}\varphi \lrcorner \omega_g \\ =& \varphi \lrcorner \partial\left(div\varphi\right) + 2\sqrt{-1}\varphi \lrcorner \omega_g = 2\sqrt{-1}\varphi \lrcorner \omega_g. \end{aligned}$$

This implies that $\varphi_{\bar{j}}^i g_{i\bar{l}} = \varphi_{\bar{l}}^i g_{i\bar{j}}$. Since $div\varphi = 0$ we have

$$0 = \partial_i \varphi_{\bar{j}}^i + \varphi_{\bar{j}}^i \partial_i \log g = \partial_i \left(\varphi_{\bar{l}}^k g_{k\bar{j}} g^{i\bar{l}}\right) + \varphi_{\bar{j}}^i \partial_i \log g = \partial_i \left(\varphi_{\bar{l}}^k g_{k\bar{j}}\right) g^{i\bar{l}}$$

which implies $\overline{\partial}^*\varphi = 0$. □

REMARK 1. *Theorem 2.1 and Theorem 2.2 imply that, if the fibers of the family* (2.1) *are Kähler-Einstein manifolds of general type, then*

Kuranishi gauge $\Longleftrightarrow$ divergence gauge.

Furthermore, we have $\varphi(t) \lrcorner \omega_{KE} = 0$ when either one of these gauges is imposed.

REMARK 2. *G. Schumacher showed that $\varphi_1 \lrcorner \omega_{KE} = 0$ by using the method of harmonic lift. See [8] for details.*

Now we look at the case when fibers are Calabi-Yau (CY) manifolds. We recall that a CY manifold of dimension n is a simply connected complex manifold X such that $c_1(X) = 0$ and $h^{k,0}(X) = 0$ for $1 \leq k \leq n-1$.

Results similar to Theorem 2.1 hold in this case which is based on Todorov's construction of flat coordinate system on the moduli space of polarized CY manifolds. We fix a polarized CY manifold (X_0, L) and let ω_0 be the CY metric in the class $c_1(L)$. Fix a holomorphic n-form $\Omega_0 \in H^{n,0}(X_0)$ such that $c_n \Omega_0 \wedge \overline{\Omega}_0 = \frac{\omega_0^n}{n!}$. Here $c_n = \left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}}$. We call such Ω_0 a normalized holomorphic n-form. It was proved in [12] that

LEMMA 2.2. *The contraction map $\iota : A^{0,1}\left(X_0, T_{X_0}^{1,0}\right) \to A^{n-1,1}(X_0)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega_0$ is a linear isometry. Furthermore, ι preserves the Hodge decomposition.*

Now we let $N = h^{n-1,1}(X_0)$ and let $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}\left(X_0, T_{X_0}^{1,0}\right)$ be a basis of harmonic Beltrami differentials. It was proved in [12] that

THEOREM 2.3. *There exists a unique convergent power series $\varphi(t) = \sum_{i=1}^N t_i \varphi_i + \sum_{|I| \geq 2} t^I \varphi_I$ of Beltrami differentials such that*

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \overline{\partial}^*\varphi(t) = 0 \\ \varphi_I \lrcorner \Omega_0 \text{ is } \partial-\text{exact for each } |I| \geq 2. \end{cases}$$

Similar to Theorem 2.1 we have

THEOREM 2.4. *Let $\varphi(t)$ be the family of Beltrami differentials constructed in Theorem 2.3. Then $div\varphi(t) = 0$ and $\varphi(t) \lrcorner \omega_0 = 0$.*

The proof of this theorem is similar to the proof of Theorem 2.1.

3. Deformation of Kähler-Einstein metrics. In this section we study the deformation of Kähler-Einstein metrics and their volume forms with respect to the Kuraniahi-divergence gauge.

As before we let $\pi : \mathfrak{X} \to B$ be a family of Kähler-Einstein manifolds of general type. Assume that the complex structure on $X_t = \pi^{-1}(t)$ is obtained by deforming the complex structure on X_0 via $\varphi(t) \in A^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ with respect to the Kuranishi gauge. Namely we have a power series $\varphi(t) = \sum_{k=1}^{\infty} t^k \varphi_k$ such that equations (2.3) hold. Let ω_t be the Kähler-Einstein metric on X_t whose Ricci curvature is -1. We let V_t be the volume form of the Kähler-Einstein metric ω_t.

Since we identified all fibers $\{X_t\}_{t\in B}$ with X_0 as smooth manifolds by using the Kuranishi gauge, we can view $\{V_t\}_{t\in B}$ as a family of volume forms on X_0. We first consider the Taylor expansion of this family.

THEOREM 3.1. *Let $\Delta = g^{i\overline{j}}\partial_i\partial_{\overline{j}}$ be the Laplace operator on $C^\infty(X_0)$. Then the volume forms V_t have the expansion*

$$V_t = \left(1 + |t|^2\Delta(1-\Delta)^{-1}\left(|\varphi_1|^2\right) + O\left(|t|^3\right)\right)V_0. \tag{3.1}$$

Proof. Let $\omega_0 = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$ be the Kähler form of the Kähler-Einstein metric on X_0, let $g = \det[g_{i\overline{j}}]$ and let $V_0 = c_n g dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ be the Kähler-Einstein volume form on X_0 where $c_n = \left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}}$.

We first construct an approximation of the volume form V_t using the Beltrami differential φ_t and the Kähler-Einstein metric ω_0 on the central fiber X_0. For each $t \in B$ we let $e_i(t) = dz_i + \varphi(t)(dz_i)$. By the Kodaira-Spencer theory we know that $\Omega^{1,0}(X_t)$ is spanned by $\{e_1(t), \cdots, e_n(t)\}$ locally. We let

$$\widetilde{V}_t = c_n g\ e_1(t) \wedge \cdots \wedge e_n(t) \wedge \overline{e_1(t)} \wedge \cdots \wedge \overline{e_n(t)}. \tag{3.2}$$

It is easy to check that $\widetilde{V}_t$ is a globally defined volume form on X_t. It follows from the definition (3.2) of $\widetilde{V}_t$ that

$$\widetilde{V}_t = \det\left(I - \varphi(t)\overline{\varphi(t)}\right)V_0. \tag{3.3}$$

Now we know that there exists a unique function

$$\rho = \rho(z, \overline{z}, t, \overline{t}) \in C^\infty(X_0 \times \Delta)$$

such that $V_t = e^\rho \widetilde{V}_t$ with $\rho(z, \overline{z}, 0) = 0$. Since V_t is the Kähler-Einstein volume form on X_t, it satisfies the Monge-Ampère equation

$$\left(\frac{\sqrt{-1}}{2}\partial_t\overline{\partial}_t \log V_t\right)^n = V_t \tag{3.4}$$

where ∂_t and $\overline{\partial}_t$ are the operators on X_t.

For each t We define the operator

$$T = T^t : C^\infty(X_0) \to A^{1,0}(X_0)$$

by $T(f) = \partial f - \overline{\varphi(t)} \lrcorner \overline{\partial} f$. Locally T is given by $T(f) = \sum_{i=1}^n T_i(f) dz_i$ where

$$T_i(f) = \partial_i f - \overline{\varphi_{\overline{i}}^{j}(t)}\partial_{\overline{j}} f.$$

Now we define the local matrix $B(t) = [B_{i\bar{j}}(t)]$ where

$$B_{i\bar{j}}(t) = g_{i\bar{j}} + \overline{T_j}\left(\left(I - \varphi(t)\overline{\varphi(t)}\right)^{pi} T_p(\rho)\right) - \partial_i \varphi^k_{\bar{j}}(t)\left(I - \varphi(t)\overline{\varphi(t)}\right)^{pk} T_p(\rho) \tag{3.5}$$

where $\left(I - \varphi(t)\overline{\varphi(t)}\right)^{pi}$ is the (p,i)-entry of the matrix $\left(I - \varphi(t)\overline{\varphi(t)}\right)^{-1}$.

By using these notations and Theorem 2.1 we know that the Monge-Ampère equation (3.4) can be written as

$$\log\det B = \rho + \log g + \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right). \tag{3.6}$$

By using formula (3.5) and the fact that $\rho\mid_{t=0} = 0$ we know that $B_{i\bar{j}}(0) = 0$ and

$$\frac{\partial B_{i\bar{j}}}{\partial t}(0) = \partial_i \partial_{\bar{j}}\left(\frac{\partial \rho}{\partial t}(0)\right).$$

Now we differentiate formula (3.6) and evaluate at $t = 0$ we get

$$\Delta\left(\frac{\partial \rho}{\partial t}(0)\right) = \frac{\partial \rho}{\partial t}(0)$$

which implies $\frac{\partial \rho}{\partial t}(0) = 0$. Similarly we have $\frac{\partial \rho}{\partial \bar{t}}(0) = 0$. Thus we know $\rho = O\left(|t|^2\right)$ and $\frac{\partial B_{i\bar{j}}}{\partial t}(0) = \frac{\partial B_{i\bar{j}}}{\partial \bar{t}}(0) = 0$.

By repeating the above argument we get

$$\frac{\partial^2 \rho}{\partial t^2}(0) = \frac{\partial^2 \rho}{\partial \bar{t}^2}(0) = 0$$

and

$$\frac{\partial^2 B_{i\bar{j}}}{\partial t^2}(0) = \frac{\partial^2 B_{i\bar{j}}}{\partial \bar{t}^2}(0) = 0.$$

A direct computation shows that

$$\frac{\partial^2}{\partial t \partial \bar{t}} \log\det\left(I - \varphi(t)\overline{\varphi(t)}\right)\Bigg|_{t=0} = -Tr\left(\varphi_1 \overline{\varphi_1}\right). \tag{3.7}$$

By using Theorem 2.1, since $\varphi_1 \lrcorner \omega_0 = 0$ we know that $-Tr\left(\varphi_1\overline{\varphi_1}\right) = -|\varphi_1|^2$. Similar to the above argument we have

$$\frac{\partial^2 B_{i\bar{j}}}{\partial t \partial \bar{t}}(0) = \partial_i \partial_{\bar{j}}\left(\frac{\partial^2 \rho}{\partial t \partial \bar{t}}(0)\right).$$

By differentiating formula (3.6) we get

$$\Delta\left(\frac{\partial^2 \rho}{\partial t \partial \bar{t}}(0)\right) = \frac{\partial^2 \rho}{\partial t \partial \bar{t}}(0) - |\varphi_1|^2.$$

This implies that

$$\rho = |t|^2 (1 - \Delta)^{-1}\left(|\varphi_1|^2\right) + O\left(|t|^3\right). \tag{3.8}$$

Formula (3.1) follows direct from formulas (3.3) and (3.8). □

REMARK 3. *We note that the first order term in the expansion* (3.1) *vanishes. This was proved by Schumacher before.*

By Yau's work we know that the deformation of the Kähler-Einstein metrics are governed by the deformation of corresponding volume forms. By using the above theorem and formula (3.4) we have

THEOREM 3.2. *With the above assumption, if we let ω_t be the Kähler form of the Kähler-Einstein metric on X_t then*

$$\omega_t = \omega_0 + |t|^2 \left(\frac{\sqrt{-1}}{2} \partial\overline{\partial} \left((1-\Delta)^{-1} |\varphi_1|^2 \right) \right) + O\left(|t|^3\right)$$

where ∂ and $\overline{\partial}$ are operators on X_0.

Now we look at the case that the fibers are polarized CY manifolds. Fix a polarized CY manifold (X_0, L_0) and let $N = h^{n-1,1}(X_0) = \dim_{\mathbb{C}} \mathcal{M}(X_0, L_0)$ be the dimension of the moduli space. Let $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be a basis of harmonic Beltrami differentials and let $\varphi(t)$ be the power series as described in Theorem 2.3 where $t_1, \cdots, t_N$ are the flat coordinates. In [12] Todorov proved that

THEOREM 3.3. *Let Ω_0 be a holomorphic n-form on X_0. Then $\Omega_t = e^{\varphi(t)} \lrcorner \Omega_0$ is a holomorphic n-form on X_t.*

The deformation of the volume forms of the polarized CY metrics follows directly from the Monge-Ampère equation and the above theorem.

COROLLARY 3.1. *Let V_t be the volume form of the polarized CY metric on X_t. Let Ω_0 be a normalized holomorphic n-form on X_0 and Ω_t be the holomorphic n-forms constructed in the above theorem. Let*

$$h_{i\overline{j}} = \int_{X_0} \langle \varphi_i, \varphi_j \rangle V_0$$

be the Weil-Petersson metric at 0 with respect to the flat coordinates t. Then

$$V_t = \frac{\int_{X_0} \Omega_0 \wedge \overline{\Omega}_0}{\int_{X_0} \Omega_t \wedge \overline{\Omega}_t} c_n \Omega_t \wedge \overline{\Omega}_t = \left(1 + \sum_{i,j} t_i \overline{t}_j \left(\frac{h_{i\overline{j}}}{W_0} - \langle \varphi_i, \varphi_j \rangle \right) + O\left(|t|^3\right) \right) V_0$$

where $W_0 = \pi^n \int_{X_0} c_1(L_0)^n$ is the volume.

Now we look at the deformation of the Kähler forms of the polarized CY metrics. By using the Calabi-Yau theorem and Corollary 3.1 we have

THEOREM 3.4. *Let ω_t be the Kähler form of the polarized CY metric on X_t. Then*

$$\omega_t = \omega_0 + \frac{\sqrt{-1}}{2} \sum_{i,j} \partial\overline{\partial} \left(\Delta^{-1} \left(\frac{h_{i\overline{j}}}{W_0} - \langle \varphi_i, \varphi_j \rangle \right) \right) t_i \overline{t}_j + O\left(|t|^3\right).$$

Here we note that, since the kernel of Δ consists of constant functions, the term

$$\partial\overline{\partial} \left(\Delta^{-1} \left(\frac{h_{i\overline{j}}}{W_0} - \langle \varphi_i, \varphi_j \rangle \right) \right)$$

is well defined.

Remark 4. *In fact we can establish the complete Taylor expansion of the function ρ in terms of $t, \overline{t}$ recursively from formula* (3.6)*. The recursive formula involve the operator* $(1-\Delta)^{-1}$*, contraction with φ_i and $\overline{\varphi}_j$ and the operator T.*

4. Curvature of the L^2 metrics. In this section we establish the curvature formula of the L^2 metrics of the direct images of pluricanonical bundles. We also show that, in the case that the fibers are Kähler-Einstein manifolds of general type, the Ricci curvatures of the L^2 metrics converge to the Weil-Petersson Kähler form on the base.

Similar to the setup in the above sections, we let $\pi : \mathfrak{X} \to B$ be a family of Kähler-Einstein manifolds of general type. Let $\varphi(t) = \sum_{k=1}^{\infty} t^k \varphi_k \subset A^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be a family of Beltrami differentials on the central fiber X_0 such that

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \overline{\partial}^* \varphi(t) = 0. \end{cases}$$

We require that the complex structure J_t on X_t is obtained by deforming the complex structure J_0 on X_0 via $\varphi(t)$. Here we use the Kähler-Einstein metric on X_0.

Let $K_{\mathfrak{X}/B} \to \mathfrak{X}$ be the relative canonical bundle over $\mathfrak{X}$. In this section we study the local holomorphic sections of the bundle $E_m = R^0 K^m_{\mathfrak{X}/B}$ over B for each $m \geq 1$. We first note that for any two points $t, t' \in B$ we have

$$rank\left(E_m(t)\right) = rank\left(E_m(t')\right).$$

This follows from Siu's work [9] directly. Alternatively, for $m = 1$ the above identity follows from the fact that

$$rank\left(E_m(t)\right) = h^{n,0}(X_t) = h^{n,0}(X_{t'}) = rank\left(E_m(t')\right).$$

For $m \geq 2$ this identity follows from Kodaira vanishing theorem and the Riemann-Roch theorem.

To compute the curvature of the L^2 metrics we first construct local holomorphic sections of E_m. For any $m \geq 1$ we define the map $\sigma = \sigma_t : A^0\left(X_0, K^m_{X_0}\right) \to A^0\left(X_t, K^m_{X_t}\right)$. For any smooth section $s \in A^0\left(X_0, K^m_{X_0}\right)$ we let $\sigma_t(s) = \left(e^{\varphi(t)} \lrcorner \left(s^{\frac{1}{m}}\right)\right)^m$. We note that, although $s^{\frac{1}{m}}$ is a multi-valued section, $\sigma_t(s)$ is well-defined. It is easy to see that the map σ_t is a linear isomorphism.

It follows from Lemma 4.1, Corollary 4.1 and Theorem 4.3 of [11] that for any given holomorphic pluricanonical form $s \in H^0\left(X_0, K^m_{X_0}\right)$ there is a unique convergent power series

$$s(t) = \sum_{k=0}^{\infty} t^k s_k \in A^0\left(X_0, K^m_{X_0}\right)$$

such that $s_0 = s$, s_i is $\overline{\partial}^*$-exact for each $i \geq 1$ and $\sigma_t\left(s(t)\right) \in H^0\left(X_t, K^m_{X_t}\right)$. In fact by the construction in Theorem 4.3 of [11] we have $s_1 = \overline{\partial}^* G\left(\varphi_1 \lrcorner \nabla s_0\right)$ where G is the Green operator on the space $A^{0,1}\left(X_0, K^m_{X_0}\right)$ with respect to the metric induced by the Kähler-Einstein metric.

For any $t \in B$ and smooth pluricanonical forms $s, s' \in A^0\left(X_t, K_{X_t}^m\right)$, the L^2 inner product is defined as

$$(s, s')(t) = \int_{X_t} \langle s, s' \rangle_{V_t^{-m}} V_t \tag{4.1}$$

where V_t is the Kähler-Einstein volume form on X_t and $\langle s, s' \rangle_{V_t^{-m}}$ is the pointwise inner product of s and s' induced by the Kähler-Einstein metric on X_t. Let $w_1, \cdots, w_n$ be any local holomorphic coordinates on X_t and assume $s = f(w)\left(dw_1 \wedge \cdots \wedge dw_n\right)^m$ and $s' = h(w)\left(dw_1 \wedge \cdots \wedge dw_n\right)^m$ locally. If $V_t = c_n g(t) dw_1 \wedge \cdots \wedge dw_n \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n$ then

$$\langle s, s' \rangle_{V_t^{-m}} = f(w)\overline{h(w)} g(t)^{-m}. \tag{4.2}$$

Now we assume $m \geq 2$. Let $N = N_n = h^0\left(X_0, K_{X_0}^m\right)$ and let $s_1, \cdots, s_N \in H^0\left(X_0, K_{X_0}^m\right)$ be a basis. For each $1 \leq \alpha \leq N$ we let $s_\alpha(t)$ be the power series described above. Let

$$h_{\alpha\overline{\beta}}(t) = \left(\sigma_t(s_\alpha(t)), \sigma_t(s_\beta(t))\right)(t). \tag{4.3}$$

We now compute the curvature of the metric $h_{\alpha\overline{\beta}}(t)$. We first note that

$$\langle \sigma_t(s_\alpha(t)), \sigma_t(s_\beta(t)) \rangle_{V_t^{-m}} = e^{-m\rho} \langle s_\alpha(t), s_\beta(t) \rangle_{V_0^{-m}}. \tag{4.4}$$

Now we let $\psi = \varphi_1$ be the harmonic Beltrami differential. By formulas (3.8) and (3.1) we have the following expansion

(4.5)

$$\begin{aligned}
&\langle \sigma_t(s_\alpha(t)), \sigma_t(s_\beta(t)) \rangle_{V_t^{-m}} V_t = e^{(1-m)\rho} \langle s_\alpha(t), s_\beta(t) \rangle_{V_0^{-m}} \det\left(I - \varphi(t)\overline{\varphi(t)}\right) V_0 \\
&= \left(\langle s_\alpha, s_\beta \rangle_{V_0^{-m}} + |t|^2 \left(\langle s_{\alpha,1}, s_{\beta,1} \rangle_{V_0^{-m}} - \langle s_\alpha, s_\beta \rangle_{V_0^{-m}} (m - \Delta)(1 - \Delta)^{-1} \left(|\psi|^2\right) \right) \right) V_0 \\
&+ \left(t \langle s_{\alpha,1}, s_\beta \rangle_{V_0^{-m}} + \overline{t} \langle s_\alpha, s_{\beta,1} \rangle_{V_0^{-m}} + t^2 \langle s_{\alpha,2}, s_\beta \rangle_{V_0^{-m}} + \overline{t}^2 \langle s_\alpha, s_{\beta,2} \rangle_{V_0^{-m}} + O\left(|t|^3\right) \right) V_0.
\end{aligned}$$

Since s_α is holomorphic and $s_{\alpha,i}$ is $\overline{\partial}^*$-exact for any $i \geq 1$ we know that

$$\left(s_{\alpha,i}, s_\beta\right)(0) = 0 = \left(s_\alpha, s_{\beta,i}\right)(0)$$

for each $i \geq 1$. It follows that

(4.6)

$$\begin{aligned}
h_{\alpha\overline{\beta}}(t) &= \int_{X_t} \langle \sigma_t(s_\alpha(t)), \sigma_t(s_\beta(t)) \rangle_{V_t^{-m}} V_t \\
&= h_{\alpha\overline{\beta}}(0) - |t|^2 \int_{X_0} \left(\langle s_\alpha, s_\beta \rangle_{V_0^{-m}} (m - \Delta)(1 - \Delta)^{-1} \left(|\psi|^2\right) - \langle s_{\alpha,1}, s_{\beta,1} \rangle_{V_0^{-m}} \right) V_0 + O\left(|t|^3\right).
\end{aligned}$$

Thus

$$\left.\frac{\partial h_{\alpha\overline{\beta}}(t)}{\partial t}\right|_{t=0} = \left.\frac{\partial h_{\alpha\overline{\beta}}(t)}{\partial \overline{t}}\right|_{t=0} = 0 \tag{4.7}$$

and

$$\left.\frac{\partial^2 h_{\alpha\overline{\beta}}(t)}{\partial t \partial \overline{t}}\right|_{t=0} = -\int_{X_0} \left(\langle s_\alpha, s_\beta \rangle_{V_0^{-m}} (m - \Delta)(1 - \Delta)^{-1} \left(|\psi|^2\right) - \langle s_{\alpha,1}, s_{\beta,1} \rangle_{V_0^{-m}} \right) V_0. \tag{4.8}$$

In order to control the term $\int_{X_0} \langle s_{\alpha,1}, s_{\beta,1}\rangle_{V_0^{-m}} V_0$, we need the following identities. To simplify computation we only state special cases which are adapted to our situation. First of all, by the construction we have

$$s_{\alpha,1} = \overline{\partial}^* G\left(\psi \lrcorner \nabla s_\alpha\right). \tag{4.9}$$

Lemma 4.1. *Let $\psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be a harmonic Beltrami differential and let $\eta \in A^0\left(X_0, K^m_{X_0}\right)$ be a smooth pluricanonical form. Then*

$$\psi \lrcorner \nabla \eta = div(\psi \otimes \eta).$$

Furthermore, if $\eta \in H^0\left(X_0, K^m_{X_0}\right)$ is holomorphic then $\overline{\partial}\left(div(\psi\otimes\eta)\right) = 0$.

Proof. Let z be any local holomorphic coordinates on X_0 and let $\chi = (dz_1 \wedge \cdots \wedge dz_n)^m$ be the corresponding local holomorphic frame of $K^m_{X_0}$. Let $\psi = \psi^i_{\overline{j}} d\overline{z}_j \otimes \frac{\partial}{\partial z_i}$ and let $\eta = f(z)\chi$. Let $\omega_0 = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$ and let $g = \det[g_{i\overline{j}}]$. Since $div\psi = 0$ we have $\partial_i \psi^i_{\overline{j}} = -\psi^i_{\overline{j}} \partial_i \log g$. Then we have

$$\begin{aligned} div(\psi\otimes\eta) &= \left(\partial_i\left(f\psi^i_{\overline{j}}\right) + f\psi^i_{\overline{j}}\partial_i \log g^{1-m}\right) d\overline{z}_j \otimes \chi \\ &= \left(\psi^i_{\overline{j}}\partial_i f + f\partial_i\psi^i_{\overline{j}} - (m-1)f\psi^i_{\overline{j}}\partial_i \log g\right) d\overline{z}_j \otimes \chi \\ &= \left(\psi^i_{\overline{j}}\partial_i f - f\psi^i_{\overline{j}}\partial_i \log g - (m-1)f\psi^i_{\overline{j}}\partial_i \log g\right) d\overline{z}_j \otimes \chi \\ &= \psi^i_{\overline{j}}\left(\partial_i f + f\partial_i \log g^{-m}\right) d\overline{z}_j \otimes \chi = \psi \lrcorner \nabla \eta. \end{aligned}$$

This proved the first claim. To prove the second claim we have

$$\overline{\partial}\left(div(\psi\otimes\eta)\right) = \overline{\partial}\left(\psi\lrcorner\nabla\eta\right) = \overline{\partial}\psi\lrcorner\nabla\eta - \psi\lrcorner\left(\nabla(\overline{\partial}\eta) + 2m\sqrt{-1}\omega_0\otimes\eta\right).$$

The second claim follows from the facts that $\overline{\partial}\psi = 0$, $\overline{\partial}\eta = 0$ and $\psi\lrcorner\omega_0 = 0$. ∎

Lemma 4.2. *Let $\eta \in A^{0,1}\left(X_0, K^m_{X_0}\right)$ be a smooth section such that $\overline{\partial}\eta = 0$. Then*

$$\Delta_{\overline{\partial}}\left(div^*\eta\right) - div^*\left(\Delta_{\overline{\partial}}\eta\right) = -(m-1)div^*\eta$$

where div^* is the adjoint operator of div.

Proof. Let $\eta = \eta_{\overline{l}} d\overline{z}_l \otimes \chi$. Then $\overline{\partial}\eta = 0$ implies that $\partial_{\overline{j}}\eta_{\overline{l}} = \partial_{\overline{l}}\eta_{\overline{j}}$. By using this we have

$$div^*\eta = -\left(\partial_{\overline{j}}\eta_{\overline{l}}g^{k\overline{j}} + \eta_{\overline{j}}\partial_{\overline{l}}g^{k\overline{j}}\right) d\overline{z}_l \otimes \frac{\partial}{\partial z_k} \otimes \chi = -\partial_{\overline{l}}\left(\eta_{\overline{j}}g^{k\overline{j}}\right) d\overline{z}_l \otimes \frac{\partial}{\partial z_k} \otimes \chi.$$

This implies that $\overline{\partial}\left(div^*\eta\right) = 0$. Thus $\Delta_{\overline{\partial}}\left(div^*\eta\right) - div^*\left(\Delta_{\overline{\partial}}\eta\right) = \overline{\partial}\overline{\partial}^* div^*\eta - div^*\overline{\partial}\overline{\partial}^*\eta$.

Now the Kähler-Einstein condition implies that $\partial_i\partial_{\overline{j}} \log g = g_{i\overline{j}}$. The lemma follows from the above formulas, the Kähler-Einstein condition and direct computations. ∎

Lemma 4.3. *Let $\psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be a harmonic Beltrami differential and let $\eta \in \mathbb{H}^0\left(X_0, K^m_{X_0}\right)$ be a holomorphic pluricanonical form. Then*

$$\Delta_{\overline{\partial}}\left(\psi\otimes\eta\right) = div^* \circ div\left(\psi\otimes\eta\right).$$

Proof. Let $\mu = \mu^i_{\bar j} d\bar z_j \otimes \frac{\partial}{\partial z_i} \otimes \chi = \psi \otimes \eta$. We have $\mu \lrcorner \omega_0 = (\psi \lrcorner \omega_0) \otimes \eta = 0$. This implies that $\mu^i_{\bar j} g_{i\bar l} = \mu^i_{\bar l} g_{i\bar j}$. Now we let z be normal coordinates of the Kähler-Einstein metric around some point $p \in X_0$. Then at p we have

$$\Delta_{\overline{\partial}} (\psi \otimes \eta) = -\left(\partial_k \partial_{\bar l} \mu^k_{\bar i} - (m-1)\mu^l_{\bar i}\right) d\bar z_l \otimes \frac{\partial}{\partial z_i} \otimes \chi = div^* \circ div\, (\psi \otimes \eta)\,.$$

□

We note that on the spaces $A^{p,q}\left(X_0, K^m_{X_0}\right)$ and $A^{p,q}\left(X_0, T^{1,0}_{X_0} \otimes K^m_{X_0}\right)$ there are natural metrics induced by the Kähler-Einstein metric on X_0. We will use these metrics in the following discussion and all operators and Green functions will be respect to these natural metrics. Now we look at the second term in the right side of formula (4.8).

Lemma 4.4. *Let $\varphi, \psi \in \mathbb{H}^{0,1}\left(X_0, T^{1,0}_{X_0}\right)$ be harmonic Beltrami differentials and let $\eta, \mu \in H^0\left(X_0, K^m_{X_0}\right)$ be holomorphic pluricanonical forms. Then*

$$\begin{aligned}&\int_{X_0} \langle \overline{\partial}^* G\,(\varphi \lrcorner \nabla \eta)\,, \overline{\partial}^* G\,(\psi \lrcorner \nabla \mu) \rangle V_0\\ =&\int_{X_0} \langle \varphi, \psi \rangle \langle \eta, \mu \rangle V_0 - (m-1) \int_{X_0} \langle (\Delta + m - 1)^{-1} (\varphi \otimes \eta)\,, \psi \otimes \mu \rangle V_0.\end{aligned}$$

Proof. By Lemma 4.1 we know $\varphi \lrcorner \nabla \eta = div\,(\varphi \otimes \eta)$ and $\psi \lrcorner \nabla \mu = div\,(\psi \otimes \mu)$. Thus

$$\begin{aligned}&\int_{X_0} \langle \overline{\partial}^* G\,(\varphi \lrcorner \nabla \eta)\,, \overline{\partial}^* G\,(\psi \lrcorner \nabla \mu) \rangle V_0 = \left(\overline{\partial}^* G\,(\varphi \lrcorner \nabla \eta)\,, \overline{\partial}^* G\,(\psi \lrcorner \nabla \mu)\right)\\ =&\left(G \circ div\,(\varphi \otimes \eta)\,, \overline{\partial}\overline{\partial}^* G \circ div\,(\psi \otimes \mu)\right) = \left(G \circ div\,(\varphi \otimes \eta)\,, \left(\Delta - \overline{\partial}^* \overline{\partial}\right) G \circ div\,(\psi \otimes \mu)\right).\end{aligned}$$

By Lemma 4.1 we know that

$$\overline{\partial} G \circ div\,(\psi \otimes \mu) = G\left(\overline{\partial}\,(div\,(\psi \otimes \mu))\right) = 0.$$

Similarly we know $\overline{\partial} G \circ div\,(\varphi \otimes \eta) = 0$. Since $H^{0,1}\left(X_0, K^m_{X_0}\right) = 0$ we know that $\Delta G = id$. It follows from the above formula, Lemma 4.2 and Lemma 4.3 that

$$\begin{aligned}&\int_{X_0} \langle \overline{\partial}^* G\,(\varphi \lrcorner \nabla \eta)\,, \overline{\partial}^* G\,(\psi \lrcorner \nabla \mu) \rangle V_0 = (G \circ div\,(\varphi \otimes \eta)\,, div\,(\psi \otimes \mu))\\ =&\,(div^* \circ G \circ div\,(\varphi \otimes \eta)\,, \psi \otimes \mu) = \left((\Delta + m - 1)^{-1} div^* \circ \Delta \circ G \circ div\,(\varphi \otimes \eta)\,, \psi \otimes \mu\right)\\ =&\left((\Delta + m - 1)^{-1} div^* \circ div\,(\varphi \otimes \eta)\,, \psi \otimes \mu\right) = \left((\Delta + m - 1)^{-1} \Delta\,(\varphi \otimes \eta)\,, \psi \otimes \mu\right)\\ =&\left(\varphi \otimes \eta - (m-1)(\Delta + m - 1)^{-1} (\varphi \otimes \eta)\,, \psi \otimes \mu\right)\\ =&\,(\varphi \otimes \eta, \psi \otimes \mu) - (m-1)\left((\Delta + m - 1)^{-1} (\varphi \otimes \eta)\,, \psi \otimes \mu\right).\end{aligned}$$

The lemma follows from that fact that

$$(\varphi \otimes \eta, \psi \otimes \mu) = \int \langle \varphi \otimes \eta, \psi \otimes \mu \rangle V_0 = \int_{X_0} \langle \varphi, \psi \rangle \langle \eta, \mu \rangle V_0.$$

□

To describe the Ricci curvature formula of the L^2 metric on E_m we need to use the Bergman kernel function of the Kähler-Einstein metric ω_0 on X_0. By our normalization the Ricci curvature of the Kähler-Einstein metric on X_0 is given by

$$R_{i\overline{j}} = -\partial_i\partial_{\overline{j}}\log g = -g_{i\overline{j}}.$$

This means $[\omega_0] = -\pi c_1(X_0)$. The Bergman kernel function $\tau_m = \tau_m(\omega_0)$ of the Kähler-Einstein metric ω_0 on X_0 is defined in the following way. Let $s_1, \cdots, s_{N_m}$ be an orthonormal basis of $H^0\left(X_0, K_{X_0}^m\right)$ with respect to the L^2 metric. Then

$$\tau_m = \sum_{\alpha=1}^{N_m} \langle s_\alpha, s_\alpha\rangle_{V_0^{-m}}. \tag{4.10}$$

By using the above normalization we have the following Tian-Yau-Zelditch expansion of the Bergman kernel

$$\tau_m \sim \frac{m^n}{\pi^n} - \frac{nm^{n-1}}{2\pi^n} + O\left(m^{n-2}\right). \tag{4.11}$$

THEOREM 4.1. *Let $R^m_{1\overline{1}}$ be the Ricci curvature of the L^2 metric on the bundle E_m. Then*

$$R^m_{1\overline{1}}(0) = (m-1)\left(\int_{X_0}\tau_m(1-\Delta)^{-1}\left(|\psi|^2\right)V_0 + h^{\alpha\overline{\beta}}(0)\int_{X_0}\langle(\Delta+m-1)^{-1}(\psi\otimes s_\alpha), \psi\otimes s_\beta\rangle V_0\right).$$

Proof. We fix m and let $s_1, \cdots, s_N$ be an orthonormal basis of $H^0\left(X_0, K_{X_0}^m\right)$. Let $h = \det[h_{\alpha\overline{\beta}}]$. By using formula (4.7) and the fact that $h_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}$ we have

$$R^m_{1\overline{1}}(0) = -\left.\frac{\partial^2}{\partial t\partial\overline{t}}\right|_{t=0}\log h = \sum_{\alpha=1}^{N}\left.\frac{\partial^2 h_{\alpha\overline{\alpha}}(t)}{\partial t\partial\overline{t}}\right|_{t=0}.$$

By using formulas (4.8), (4.9) and (4.10) we know

$$\begin{aligned} R^m_{1\overline{1}}(0) = &\int_{X_0}\tau_m(m-\Delta)(1-\Delta)^{-1}\left(|\psi|^2\right)V_0 \\ &-\sum_{\alpha=1}^{N}\int_{X_0}\langle\overline{\partial}^*G\left(\psi\lrcorner\nabla s_\alpha\right), \overline{\partial}^*G\left(\psi\lrcorner\nabla s_\alpha\right)\rangle V_0. \end{aligned} \tag{4.12}$$

By Lemma 4.4 we know

$$\begin{aligned} &\sum_{\alpha=1}^{N}\int_{X_0}\langle\overline{\partial}^*G\left(\psi\lrcorner\nabla s_\alpha\right), \overline{\partial}^*G\left(\psi\lrcorner\nabla s_\alpha\right)\rangle V_0 \\ =&\sum_{\alpha=1}^{N}\int_{X_0}\left(\langle s_\alpha, s_\alpha\rangle|\psi|^2 - (m-1)\langle(\Delta+m-1)^{-1}\left(\psi\otimes s_\alpha\right), \psi\otimes s_\alpha\rangle\right)V_0 \\ =&\int_{X_0}\tau_m|\psi|^2V_0 - (m-1)\sum_{\alpha=1}^{N}\int_{X_0}\langle(\Delta+m-1)^{-1}\left(\psi\otimes s_\alpha\right), \psi\otimes s_\alpha\rangle V_0. \end{aligned} \tag{4.13}$$

We also have

$$(m-\Delta)(1-\Delta)^{-1}\left(|\psi|^2\right) - |\psi|^2 = (m-1)(1-\Delta)^{-1}\left(|\psi|^2\right). \tag{4.14}$$

The theorem follows from formulas (4.12), (4.13) and (4.14). □

In fact the method we used in proving Theorem 4.1 directly gives the full curvature of the L^2 metric on the direct image sheaves of the relative pluricanonical bundles which was established by Schumacher [7] and Berndtsson [1].

Let X be a Kähler-Einstein manifold of general type and let $\mathcal{M}$ be its (course) moduli space. Assume its dimension is k. Let $p \in \mathcal{M}$ be a smooth point and let $t_1, \cdots, t_k$ be any local holomorphic coordinates around p. Let X_p be the Kähler-Einstein manifold corresponding to p.

Corollary 4.1. *Let $\varphi_1, \cdots, \varphi_k \in \mathbb{H}^{0,1}\left(X_p, T^{1,0}_{X_p}\right)$ be harmonic Beltrami differentials such that $[\varphi_i] = KS\left(\frac{\partial}{\partial t_i}\right)$ where KS is the Kodaira-Spencer map. Let $s_1, \cdots, s_N \in H^0\left(X_p, K^m_{X_p}\right)$ be any basis. Let $h_{\alpha\overline{\beta}} = \int_{X_p} \langle s_\alpha, s_\beta \rangle V_p$. Then the curvature of the L^2 metric on E_m is given by*

$$R_{\alpha\overline{\beta}i\overline{j}}(p) = (m-1)\left(\int_{X_p} \left(\langle s_\alpha, s_\beta\rangle (1-\Delta)^{-1}\left(\langle \varphi_i, \varphi_j\rangle\right) + \langle (\Delta+m-1)^{-1}\left(\varphi_i \otimes s_\alpha\right), \varphi_j \otimes s_\beta\rangle\right) V_p\right)$$

and the Ricci curvature is given by

$$\begin{aligned} R_{i\overline{j}}(p) =& (m-1)\int_{X_p} \tau_m (1-\Delta)^{-1}\left(\langle \varphi_i, \varphi_j\rangle\right) V_p \\ &+ (m-1) h^{\alpha\overline{\beta}} \int_{X_p} \langle (\Delta+m-1)^{-1}\left(\varphi_i \otimes s_\alpha\right), \varphi_j \otimes s_\beta\rangle V_p. \end{aligned}$$

Now we look at the Weil-Petersson metric on the base space B. Let μ be the WP metric on B. We have

Theorem 4.2. *The normalized Ricci curvatures of E_m converge to the Weil-Petersson Kähler form. Precisely we have*

$$\lim_{m\to\infty} \frac{\pi^n}{m^{n+1}} R^m_{1\overline{1}} = \mu_{1\overline{1}}.$$

Proof. The theorem follows from Theorem 4.1. At $0 \in \Delta$ for any fixed m we let $s_1, \cdots, s_{N_m}$ be an orthonormal basis of $H^0\left(X_0, K^m_{X_0}\right)$ with respect to the L^2 metric. By Theorem 4.1 we know that

$$R^m_{1\overline{1}}(0) = (m-1)\left(\int_{X_0} \tau_m (1-\Delta)^{-1}\left(|\psi|^2\right) V_0 + \sum_{\alpha=1}^{N_m} \int_{X_0} \langle (\Delta+m-1)^{-1}(\psi\otimes s_\alpha), \psi\otimes s_\alpha\rangle V_0\right).$$

Since the first eigenvalue of the operator $\Delta + m - 1$ is at least $m-1$ we have

$$\begin{aligned} 0 \le& \int_{X_0} \langle (\Delta+m-1)^{-1}(\psi\otimes s_\alpha), \psi\otimes s_\alpha\rangle V_0 \le \frac{1}{m-1}\int_{X_0} \langle \psi\otimes s_\alpha, \psi\otimes s_\alpha\rangle V_0 \\ =& \frac{1}{m-1}\int_{X_0} \langle s_\alpha, s_\alpha\rangle |\psi|^2 V_0 \end{aligned}$$

which implies that

$$0 \le \sum_{\alpha=1}^{N_m} \int_{X_0} \langle (\Delta+m-1)^{-1}(\psi\otimes s_\alpha), \psi\otimes s_\alpha\rangle V_0 \le \frac{1}{m-1}\int_{X_0} \tau_m |\psi|^2 V_0.$$

By combining the above formulas we have

$$(4.15)\quad \begin{aligned} &(m-1)\int_{X_0}\tau_m(1-\Delta)^{-1}\left(|\psi|^2\right)V_0\\ &\le R^m_{1\bar{1}}(0)\le (m-1)\int_{X_0}\tau_m\left((1-\Delta)^{-1}\left(|\psi|^2\right)+\frac{|\psi|^2}{m-1}\right)V_0. \end{aligned}$$

By the definition of the WP metric we know that

$$(4.16)\qquad \mu_{1\bar{1}}(0)=\int_{X_0}|\psi|^2V_0=\int_{X_0}(1-\Delta)^{-1}\left(|\psi|^2\right)V_0.$$

The theorem follows from the above definition, inequality (4.15) and the Bergman kernel expansion (4.11). □

Remark 5. *We note that, although we stated some results for one-parameter family of Kähler-Einstein manifolds, these formulas work in general with simple modifications. Furthermore, the methods we used in Section 3 on the deformation of Kähler-Einstein metrics can be generalized to study the deformation of other canonical metrics such as cscK metrics, ν-balanced metrics and balanced metrics. The study of these metrics will be in [10].*

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