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A NEW PINCHING THEOREM FOR CLOSED HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN S^{n+1}*

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Abstract. We investigate the generalized Chern conjecture, and prove that if M is a closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature, then there exists an explicit positive constant $C(n)$ depending only on n such that if $|H| < C(n)$ and $S > \beta(n,H)$, then $S > \beta(n,H) + \frac{3n}{7}$, where $\beta(n,H) = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$.

Key words. Closed hypersurface, Pinching phenomenon, Mean curvature, Scalar curvature, Second fundamental form.

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1. Introduction. Let M^n be an n-dimensional closed hypersurface in the unit sphere S^{n+1} with constant scalar curvature and constant mean curvature. Denote by H and S the mean curvature and the squared norm of the second fundamental form of M, respectively. It follows from the Gauss equation that the scalar curvature of M is given by $R = n(n-1) + n^2H^2 - S$.

A famous rigidity theorem due to Simons [15], Lawson [10] and Chern-do Carmo-Kobayashi [8] says that if M is a closed minimal hypersurface in S^{n+1} satisfying $S \le n$, then $S \equiv 0$ and M is the great sphere S^n, or $S \equiv n$ and M is the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$. Afterward, Li-Li [11] improved Simons' pinching constant for n-dimensional closed minimal submanifolds in S^{n+p} to $\max\{\frac{n}{2-1/p}, \frac{2}{3}n\}$. Peng-Terng [13] proved that there exists a positive constant $\delta(n)$ depending only on n such that if $n \le S \le n + \delta(n)$, then $S = n$. Later Yang-Cheng [25] improved the pinching constant $\delta(n)$ to $\frac{n}{3}$. In 1993, S. P. Chang [2, 7] solved Chern's conjecture in dimension 3. For closed minimal hypersurfaces in S^{n+1}, the scalar curvature pinching phenomenon without the assumption of constant scalar curvature was also investigated by several authors [5, 9, 14, 17, 23] etc.

More generally, we would like to propose the following

THE GENERALIZED CHERN CONJECTURE. *For closed hypersurfaces in the unit sphere S^{n+1} with constant scalar curvature and constant mean curvature, the values S of the squared norm of the second fundamental forms must be discrete.*

Set $\alpha(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$. In 1990, the first author [18] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

THEOREM A. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \le C_1(n,p,H)$, then M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an*

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$(n+1)$-sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C_1(n,p,H)$ is defined by

$$C_1(n,p,H) = \begin{cases} \alpha(n,H), & p=1, \text{ or } p=2 \text{ and } H\neq 0,\\ \frac{n}{2-\frac{1}{p}}, & p\geq 2 \text{ and } H=0,\\ \min\Big\{\alpha(n,H), \frac{n+nH^2}{2-\frac{1}{p-1}}+nH^2\Big\}, & p\geq 3 \text{ and } H\neq 0. \end{cases}$$

Consequently, we have the following

Corollary A (also see [1, 6]). *Let M be an n-dimensional closed hypersurface with constant mean curvature ($H\neq 0$) in S^{n+1}. If $S\leq \alpha(n,H)$, then M is either a totally umbilical sphere $S^n\big(\frac{1}{\sqrt{1+H^2}}\big)$, or a Clifford hypersurface $S^{n-1}\big(\frac{1}{\sqrt{1+\lambda^2}}\big)\times S^1\big(\frac{\lambda}{\sqrt{1+\lambda^2}}\big)$. Here $\lambda=\frac{1}{2(n-1)}[n|H|+\sqrt{n^2H^2+4(n-1)}]$.*

In [19], H. W. Xu improved the pinching constant $C_1(n,p,H)$ in Theorem A to

$$C_2(n,p,H) = \begin{cases} \alpha(n,H), & p=1, \text{ or } p=2 \text{ and } H\neq 0,\\ \min\Big\{\alpha(n,H), \frac{1}{3}(2n+5nH^2)\Big\}, & \text{otherwise.} \end{cases}$$

In [3], S. P. Chang showed that any closed hypersurface in S^4 with constant scalar curvature and constant mean curvature must be an isoparametric hypersurface. Recently, Suh and Yang [16] improved Yang and Cheng's pinching theorem for closed minimal hypersurfaces, and proved the following

Theorem B. *Let M^n ($n\geq 4$) be an n-dimensional closed minimal hypersurface with constant scalar curvature in S^{n+1}. If $S>n$, then $S>n+\frac{3}{7}n$.*

In this paper, we extend the result of Suh and Yang to the case where M is a closed hypersurface with constant scalar curvature and small constant mean curvature. Putting $\beta(n,H)=n+\frac{n^3H^2}{2(n-1)}+\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$, we prove the following

Main Theorem. *Let M^n ($n\geq 4$) be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature. Then there exists an explicit positive constant $C(n)$ depending only on n, such that if $|H|<C(n)$ and $S>\beta(n,H)$, then $S>\beta(n,H)+\frac{3n}{7}$.*

We have the following corollary immediately.

Corollary. *Let M^n ($n\geq 4$) be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature ($H\neq 0$). If $|H|<C(n)$ and $\beta(n,H)\leq S\leq \beta(n,H)+\frac{3n}{7}$, then $S=\beta(n,H)$ and M is an isoparametric hypersurface $S^1\big(\frac{1}{\sqrt{1+\mu^2}}\big)\times S^{n-1}\big(\frac{\mu}{\sqrt{1+\mu^2}}\big)$, where $\mu=\frac{1}{2}[n|H|+\sqrt{n^2H^2+4(n-1)}]$.*

The following problem seems very interesting.

Open Problem. *For an n-dimensional closed hypersurface M in S^{n+1} with constant scalar curvature R and constant mean curvature H ($H\neq 0$), set $\mu_k=\frac{n|H|+\sqrt{n^2H^2+4k(n-k)}}{2k}$. Suppose that $\alpha(n,H)\leq S\leq \beta(n,H)$. Can one prove that M must be the isoparametric hypersurface $S^k\big(\frac{1}{\sqrt{1+\mu_k^2}}\big)\times S^{n-k}\big(\frac{\mu_k}{\sqrt{1+\mu_k^2}}\big)$, $k=1,2,\cdots,n-1$?*

It should be mentioned that the scalar curvature pinching phenomenon for closed constant mean curvature hypersurfaces in S^{n+1} has been investigated in [4, 21, 22] etc.

2. Fundamental formulas. Throughout this paper let M^n be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1, \quad 1 \leq i, j, k, \cdots \leq n.$$

Choose a local orthonormal frame field $\{e_A\}$ in S^{n+1} such that, restricted to M, $\{e_i\}$ are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection 1-forms of S^{n+1}, respectively. Restricting these forms to M, we have

$$\omega_{n+1} = 0, \quad \omega_{n+1i} = \sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

The second fundamental form of M can be written as $h = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}$. Denote by H and S the mean curvature of M and the squared norm of h, respectively. Then

$$S = \sum_{i,j=1}^{n} h_{ij}^2, \quad H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}.$$

For an arbitrary fixed point $p \in M$, choose $\{e_A\}$ such that

$$H \geq 0, \quad h_{ij} = \lambda_i \delta_{ij}, \quad S = \sum_{i=1}^{n} \lambda_i^2.$$

Set

$$f_k := \sum_i \lambda_i^k,$$

$$h_{ijk} := \nabla_k h_{ij}, \; h_{ijkl} := \nabla_l \nabla_k h_{ij}, \; h_{ijkls} := \nabla_s \nabla_l \nabla_k h_{ij}.$$

The Gauss equation, Codazzi equation and Ricci formula can be written as

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk},$$

$$h_{ijk} = h_{ikj},$$

$$h_{ijkl} - h_{ijlk} = \sum_m (h_{mj}R_{mikl} + h_{im}R_{mjkl}),$$

$$h_{ijkls} - h_{ijksl} = \sum_m (h_{mjk}R_{mils} + h_{imk}R_{mjls} + h_{ijm}R_{mkls}),$$

where R_{ijkl} is the curvature tensor of M. So the scalar curvature of M is given by $R = n(n-1) + n^2H^2 - S$.

When H and S are constants, by a similar computation as in [13], we get

$$\sum_{i,j,k} h_{ijk}^2 = S(S-n) + n^2H^2 - nHf_3, \tag{2.1}$$

$$\sum_k h_{ijkk} = (n-S)h_{ij} + (\sum_m h_{im}h_{mj} - \delta_{ij})nH, \tag{2.2}$$

$$h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j). \tag{2.3}$$

Following [20], we have

$$\frac{1}{3}\triangle f_3 = (n-S)f_3 + nH(f_4 - S) + 2\sum_{i,j,k}\lambda_i h_{ijk}^2, \tag{2.4}$$

$$\frac{1}{4}\triangle f_4 = (n-S)f_4 + nH(f_5 - f_3) + (2A + B), \tag{2.5}$$

$$\begin{aligned}\frac{1}{6}\triangle f_3^2 =&(n-S)f_3^2 + nHf_3(f_4 - S) + 2f_3\sum_{i,j,k}\lambda_i h_{ijk}^2\\ &+ 3\sum_j(\sum_i \lambda_i^2 h_{iij})^2,\end{aligned} \tag{2.6}$$

where

$$A = \sum_{i,j,k}\lambda_i^2 h_{ijk}^2, \quad B = \sum_{i,j,k}\lambda_i\lambda_j h_{ijk}^2.$$

By (2.1), (2.4) and the Laplacian of $\sum h_{ijk}^2$, we obtain the following

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 =&(S - 2n - 3)(S^2 - nS + n^2H^2 - nHf_3)\\ &- \frac{3nH}{2}\big[(n-S)f_3 - nSH + nHf_4\big]\\ &+ 3(A - 2B) - 6nH\sum_{i,j,k}\lambda_i h_{ijk}^2.\end{aligned} \tag{2.7}$$

On the other hand, it follows from (2.3) that

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 \geq& \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{4}\sum_{i\neq j}(h_{ijij} - h_{jiji})^2\\ =& \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS - n^2H^2 + 2nHf_3),\end{aligned} \tag{2.8}$$

where

$$u_{ijkl} := \frac{1}{4}(h_{ijkl} + h_{jkli} + h_{klij} + h_{lijk}).$$

3. Several lemmas. To prove our main result, we need several lemmas. From now on, we take $\tilde{S} = S - nH^2$, $\sum_{i,j,k} h_{ijk}^2 = t\tilde{S}^2$. Because of (2.1), it is easy to see that

$$t = \frac{\tilde{S} - n}{\tilde{S}} + \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2}. \tag{3.1}$$

Put

$$\begin{aligned} f =& \sum_i \left(\lambda_i^2 - \frac{f_3 - HS}{\tilde{S}}\lambda_i - \frac{S^2 - nHf_3}{n\tilde{S}}\right)^2 \\ =& f_4 - \frac{(f_3 - HS)^2}{\tilde{S}} - \frac{S^2}{n}, \end{aligned} \tag{3.2}$$

and $\tilde{\lambda}_i = \lambda_i - H$, $i = 1, \cdots, n$. At any point $p \in M$, let $\lambda_1 = \max_i\{\lambda_i\}$, $\lambda_2 = \min_i\{\lambda_i\}$.

Lemma 1. *Let M^n be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature. Then at any point $p \in M$, we have*

$$f \geq \frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2}\left(\tilde{\lambda}_1\tilde{\lambda}_2 + \frac{\tilde{S}}{n}\right)^2, \tag{3.3}$$

$$A - B \leq \frac{1-\alpha}{3}(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 t\tilde{S}^2, \tag{3.4}$$

$$\sum_j (\sum_i \lambda_i^2 h_{iij})^2 \leq \frac{1+2\alpha}{3} t\tilde{S}^2 f, \tag{3.5}$$

$$\left(\sum_{i,j,k} \lambda_i h_{ijk}^2\right)^2 \leq \left\{\frac{1}{3}(A+2B) - \frac{4}{3}\sum_k \frac{(n+2)(\sum_i \lambda_i^2 h_{iik})^2}{(n+2)\tilde{S} + 2n\tilde{\lambda}_k^2}\right\} t\tilde{S}^2, \tag{3.6}$$

where $\alpha \sum_{i,j,k} h_{ijk}^2 = \sum_i h_{iii}^2$ and α satisfies $0 \leq \alpha \leq 1$.

Proof. (3.3), (3.4) and (3.5) can be proved by using the same method as in [24]. We only need to prove (3.6). Consider equations

$$\begin{cases} \sum_{i,j} P_{ijk} h_{ij} = 0 \\ \sum_{i,j} P_{ijk}\delta_{ij} = 0, \end{cases}$$

where $P_{ijk} := (\lambda_i + \lambda_j + \lambda_k)h_{ijk} - (h_{ij}y^k + h_{ik}y^j + h_{jk}y^i) - (\delta_{ij}x^k + \delta_{ik}x^j + \delta_{jk}x^i)$. This together with the fact that H and S are constants implies

$$\begin{cases} 2\sum_i \lambda_i^2 h_{iik} - (2\lambda_k^2 + S)y^k - (2\lambda_k + nH)x^k = 0 \\ (2\lambda_k + nH)y^k + (2+n)x^k = 0. \end{cases}$$

We then have the following solution.

$$x^k = -\frac{2(2\lambda_k + nH)\sum_i \lambda_i^2 h_{iik}}{2n(\lambda_k - H)^2 + (n+2)\tilde{S}}, y^k = \frac{2(n+2)\sum_i \lambda_i^2 h_{iik}}{2n(\lambda_k - H)^2 + (n+2)\tilde{S}}, k = 1, \cdots, n.$$

By a similar argument as in [24], we get

$$\begin{aligned}&(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2=\frac{1}{9}(\sum_{i,j,k}P_{ijk}h_{ijk})^2\\ \leq&\frac{1}{9}\sum_{i,j,k}P_{ijk}^2\sum_{i,j,k}h_{ijk}^2\\ =&\left\{\frac{1}{3}(A+2B)-\frac{4}{3}\sum_k\frac{(n+2)(\sum_i\lambda_i^2h_{iik})^2}{(n+2)\tilde{S}+2n\tilde{\lambda}_k^2}\right\}t\tilde{S}^2.\end{aligned}$$

LEMMA 2. *Let M^n be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature. Then*

$$f\leq\frac{\big(\tilde{\lambda}_1-\tilde{\lambda}_2\big)^2}{4}\tilde{S}-\frac{\tilde{S}^2}{n}. \tag{3.7}$$

Proof. Put $\tilde{f}_3=\sum_i\tilde{\lambda}_i^3$, $\tilde{f}_4=\sum_i\tilde{\lambda}_i^4$. It follows from (3.2) that

$$f=\sum_i\left(\tilde{\lambda}_i^2-\frac{\tilde{f}_3}{\tilde{S}}\tilde{\lambda}_i-\frac{\tilde{S}}{n}\right)^2=\tilde{f}_4-\frac{\tilde{f}_3^2}{\tilde{S}}-\frac{\tilde{S}^2}{n}.$$

By the definitions of λ_1 and λ_2, we have

$$\sum_i\tilde{\lambda}_i^2\big(\tilde{\lambda}_i-\tilde{\lambda}_1\big)\big(\tilde{\lambda}_i-\tilde{\lambda}_2\big)\leq 0.$$

Then

$$\tilde{f}_4\leq\big(\tilde{\lambda}_1+\tilde{\lambda}_2\big)\tilde{f}_3-\tilde{\lambda}_1\tilde{\lambda}_2\tilde{S}.$$

Thus

$$\begin{aligned}f\leq&\big(\tilde{\lambda}_1+\tilde{\lambda}_2\big)\tilde{f}_3-\frac{\tilde{f}_3^2}{\tilde{S}}-\tilde{\lambda}_1\tilde{\lambda}_2\tilde{S}-\frac{\tilde{S}^2}{n}\\ \leq&\frac{\big(\tilde{\lambda}_1+\tilde{\lambda}_2\big)^2}{4}\tilde{S}-\tilde{\lambda}_1\tilde{\lambda}_2\tilde{S}-\frac{\tilde{S}^2}{n}=\frac{\big(\tilde{\lambda}_1-\tilde{\lambda}_2\big)^2}{4}\tilde{S}-\frac{\tilde{S}^2}{n}.\end{aligned}$$

LEMMA 3. *Let M^n be an n-dimensional closed hypersurface in S^{n+1} with constant scalar curvature and constant mean curvature. Then*

$$\int_{M^n}(A-2B)dv=\int_{M^n}\left(\tilde{S}f+\frac{t\tilde{S}^3}{n}+g_1\right)dv, \tag{3.8}$$

where $g_1=H^2(nf+t\tilde{S}^2)-Ht\tilde{S}(f_3-HS)$.

Proof. Note that

$$\begin{aligned}
&\sum_{i,j,k,l,m} h_{lmik}h_{ij}h_{jm}h_{lk} \\
&= \sum_{i,j,k,l,m}(h_{lmi}h_{ij}h_{jm}h_{lk})_k - \sum_{i,j,k,l,m} h_{lmi}h_{ijk}h_{jm}h_{lk} \\
&\quad - \sum_{i,j,k,l,m} h_{lmi}h_{ij}h_{jmk}h_{lk} \\
&=\frac{1}{3}\sum_{k,l}((f_3)_l h_{lk})_k - 2B.
\end{aligned} \tag{3.9}$$

Since H and S are constants, we have

$$\begin{aligned}
&\sum_{i,j,k,l,m} h_{lmik}h_{ij}h_{jm}h_{lk} \\
&= \sum_{i,j,k,l,m}(h_{lmki} + \sum_r h_{rm}R_{rlik} + \sum_r h_{lr}R_{rmik})h_{ij}h_{jm}h_{lk} \\
&= \sum_{i,j,k,l,m}(h_{lmk}h_{ij}h_{jm}h_{lk})_i - \sum_{i,j,k,l,m} h_{lmk}h_{iji}h_{jm}h_{lk} \\
&\quad - \sum_{i,j,k,l,m} h_{lmk}h_{ij}h_{jmi}h_{lk} - \sum_{i,j,k,l,m} h_{lmk}h_{ij}h_{jm}h_{lki} \\
&\quad + nHf_3 + Sf_4 - S^2 - f_3^2 \\
&= - A + nHf_3 + Sf_4 - S^2 - f_3^2.
\end{aligned} \tag{3.10}$$

Putting $X = \sum_k \big(\sum_l (f_3)_l h_{lk}\big)e_k$, we get $\operatorname{div}(X) = \sum_{k,l}((f_3)_l h_{lk})_k$. By (3.9), (3.10) and the divergence theorem, we obtain

$$\int_{M^n}(A - 2B)dv = \int_{M^n}(Sf_4 - f_3^2 + nHf_3 - S^2)dv.$$

This together with (3.1) and the definitions of f and $\tilde{S}$ implies (3.8).

4. Curvature estimates. The crucial point of the proof of Main Theorem is to give a proper estimate of the RHS of (2.8).

We begin with an estimate of $\sum_{i,j,k,l} u_{ijkl}^2$. Define

$$f_{ij} = \sum_k h_{ik}h_{kj} - \frac{f_3 - HS}{\tilde{S}}h_{ij} - \frac{S^2 - nHf_3}{n\tilde{S}}\delta_{ij}, \quad i,j = 1,\cdots,n, \tag{4.1}$$

which satisfy

$$\sum_{i,j} h_{ij}f_{ij} = 0, \quad \sum_{i,j}\delta_{ij}f_{ij} = 0, \quad \sum_{i,j} f_{ij}^2 = f.$$

Putting

$$F := -A + \frac{1}{2}Sf + \frac{f_3 - HS}{\tilde{S}}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - t\tilde{S}^2 H) + \frac{t\tilde{S}^2 S}{n},$$

we have

$$\sum_{i,j,k,l} u_{ijkl}h_{ij}f_{kl} = F, \tag{4.2}$$

$$\sum_{i,j,k} u_{ijkl}h_{ijk} = -\frac{3}{2}\sum_i \lambda_i^2 h_{iil}\lambda_l - \frac{3nH}{2}\sum_i \lambda_i^2 h_{iil} = -\frac{1}{2}(f_3)_l(\lambda_l + nH), \tag{4.3}$$

$$\sum_{i,j,k,l} u_{ijkl}h_{ij}\delta_{kl} = -\frac{1}{2}t\tilde{S}^2, \quad \sum_{i,j,k} h_{ijk}f_{ij}h_{kl} = \sum_i \lambda_i^2 h_{iil}\lambda_l = \frac{1}{3}(f_3)_l\lambda_l. \tag{4.4}$$

Set

$$\begin{aligned} U_{ijkl} :=& u_{ijkl} - \alpha_1(h_{ij}f_{kl} + f_{ij}h_{kl}) + (\beta_i h_{jkl} + \beta_j h_{ikl} + \beta_k h_{lij} + \beta_l h_{ijk}) \\ &+ \gamma h_{ij}h_{kl} + \delta(h_{ij}\delta_{kl} + \delta_{ij}h_{kl}) + \zeta\delta_{ij}\delta_{kl}. \end{aligned} \tag{4.5}$$

For any $\alpha_1, \beta_i, \gamma, \delta, \zeta \in \mathbf{R}$, $\sum_{i,j,k,l} U_{ijkl}^2 \geq 0$. Choose

$$\begin{aligned} \alpha_1 =& \frac{1}{Sf - \frac{1}{18t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j^2}\left[F + \frac{1}{12t\tilde{S}^2}\sum_j (f_3)_j^2(\lambda_j^2 + nH\lambda_j)\right], \\ \beta_i =& \frac{1}{8t\tilde{S}^2}(f_3)_i\left(\frac{2\alpha_1 + 3}{3}\lambda_i + nH\right), \\ \gamma =& -tH + \frac{\sum_{i,j,k}\lambda_i h_{ijk}^2}{\tilde{S}^2}, \\ \delta =& -\frac{\sum_{i,j,k}\lambda_i h_{ijk}^2}{\tilde{S}^2}H + \frac{t(S + nH^2)}{2n}, \\ \zeta =& \frac{\sum_{i,j,k}\lambda_i h_{ijk}^2}{\tilde{S}^2}H^2 - \frac{tS}{n}H, \end{aligned}$$

respectively. By the symmetry of $U_{ijkl} = U_{klij}$, $U_{ijkl} = U_{jikl}$ and (4.2)-(4.5), substituting these values into the inequality $\sum_{i,j,k,l} U_{ijkl}^2 \geq 0$, we obtain

$$\begin{aligned} \sum_{i,j,k,l} u_{ijkl}^2 \geq& \frac{1}{2Sf - \frac{1}{9t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j^2}\left[2F + \frac{1}{6t\tilde{S}^2}\sum_j (f_3)_j^2(\lambda_j^2 + nH\lambda_j)\right]^2 \\ &+ \frac{1}{4t\tilde{S}^2}\sum_j (f_3)_j^2(\lambda_j + nH)^2 \\ &+ \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2)^2 + \frac{t^2\tilde{S}^3}{2n}. \end{aligned}$$

Noting the definition of F, we have

$$
\begin{aligned}
\sum_{i,j,k,l} u_{ijkl}^2 \geq & \frac{1}{2\big[Sf - \frac{1}{18t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j^2\big]}\Bigg\{\Bigg[Sf - \frac{1}{18t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j^2\Bigg] \\
& + \Bigg[-2A + \frac{2(f_3 - HS)}{\tilde{S}}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2) + \frac{2t\tilde{S}^2 S}{n} \\
& + \frac{2}{9t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j^2 + \frac{nH}{6t\tilde{S}^2}\sum_j (f_3)_j^2\lambda_j\Bigg]\Bigg\}^2 \\
& + \frac{1}{4t\tilde{S}^2}\sum_j (f_3)_j^2(\lambda_j + nH)^2 \\
& + \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2)^2 + \frac{t^2\tilde{S}^3}{2n}
\end{aligned}
$$

$$
\begin{aligned}
\geq & -2A + \frac{1}{2}Sf + \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2)^2 \\
& + \frac{2(f_3 - HS)}{\tilde{S}}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2) + \frac{t\tilde{S}^2}{n}(2S + \frac{t\tilde{S}}{2}) \\
& + \frac{1}{36t\tilde{S}^2}\sum_j (f_3)_j^2\big[7\lambda_j^2 + 9(\lambda_j + nH)^2 + 6nH\lambda_j\big] \\
= & -2A + \frac{1}{2}Sf + \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2)^2 \\
& + \frac{2(f_3 - HS)}{\tilde{S}}(\sum_{i,j,k}\lambda_i h_{ijk}^2 - tH\tilde{S}^2) + \frac{t\tilde{S}^2}{n}(2S + \frac{t\tilde{S}}{2}) \\
& + \frac{1}{36t\tilde{S}^2}\sum_j (f_3)_j^2(4\lambda_j + 3nH)^2. \qquad (4.6)
\end{aligned}
$$

It follows from (2.7), (2.8), (4.6) and $S = \tilde{S} + nH^2$ that

$$
\begin{aligned}
& \frac{t\tilde{S}^2}{2}\Bigg\{5\tilde{S} - 7n - 6 - \frac{(4+3n)t\tilde{S}}{n} - \frac{4\tilde{S}}{n}\Bigg\} + (5A - 6B) \\
\geq & 2\tilde{S}f + \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2 + \frac{2(f_3 - HS)}{\tilde{S}}\sum_{i,j,k}\lambda_i h_{ijk}^2 \\
& + \frac{1}{36t\tilde{S}^2}\sum_j (f_3)_j^2(4\lambda_j + 3nH)^2 + \tilde{g}_2, \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{g}_2 = & \Bigg((6n - 2t)\sum_{i,j,k}\lambda_i h_{ijk}^2 - 2t\tilde{S}(f_3 - HS)\Bigg)H \\
& + \Bigg(2nf + \frac{3n^2 f}{2} + t^2\tilde{S}^2 - \frac{5n-4}{2}t\tilde{S}^2\Bigg)H^2.
\end{aligned}
$$

From (3.1) and (4.7), we have

$$-6=-\frac{6n}{n}=-\frac{6}{n}\Big\{(1-t)\tilde{S}+\frac{nH(H\tilde{S}-(f_3-HS))}{\tilde{S}}\Big\}$$

and

$$\begin{aligned} & t\tilde{S}^2\Big\{\tilde{S}-2n-(5-t)\frac{\tilde{S}}{n}\Big\}+(5A-6B)-2\tilde{S}f \\ \geq & \frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2+\frac{2(f_3-HS)}{\tilde{S}}\sum_{i,j,k}\lambda_i h_{ijk}^2 \\ & +\frac{1}{36t\tilde{S}^2}\sum_j (f_3)_j^2(4\lambda_j+3nH)^2+g_2, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} g_2 = & \Big(2(3n-t)\sum_{i,j,k}\lambda_i h_{ijk}^2-\frac{10+3n}{2}t\tilde{S}(f_3-HS)\Big)H \\ & +\Big(\frac{4+3n}{2}nf+(5-n+t)t\tilde{S}^2\Big)H^2. \end{aligned}$$

Now we estimate the right hand side of (4.8). We first consider the following integral equality.

$$\begin{aligned} 0= & \int_{M^n}\Big\{\frac{1}{4}\triangle f_4-\frac{1+c_0}{6\tilde{S}}\triangle f_3^2+\frac{(1+c_0)(1+c_2)H}{3}\triangle f_3\Big\} \\ & -\int_{M^n}c_1\Big\{(A-2B)-\Big(\tilde{S}f+\frac{t\tilde{S}^3}{n}+g_1\Big)\Big\}. \end{aligned}$$

Using (2.4), (2.5), (2.6) and (3.8), we see that there is a point $p_0\in M$ such that at p_0, for any $c_0\geq 0$ and $u>0$,

$$\begin{aligned} & (1-c_1)u\frac{t\tilde{S}^3}{n}-(2-c_1)uA-(1+2c_1)uB+(t-c_1)u\tilde{S}f \\ = & -\frac{2u(c_0+1)}{\tilde{S}}(f_3-HS)\sum_{i,j,k}\lambda_i h_{ijk}^2+c_0ut(f_3-HS)^2 \\ & -\frac{u(c_0+1)}{3\tilde{S}}\sum_j (f_3)_j^2+g_3, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} g_3 = & -\frac{uH}{\tilde{S}^2}\Big\{\tilde{S}^2H\big[(\tilde{S}+nH^2)^2-t\tilde{S}^2(c_1-1)\big]+nH\tilde{S}f\xi \\ & +\tilde{S}\big[-\tilde{S}[-2(\tilde{S}+nH^2)+t(\tilde{S}+\xi)]+(2+c_0)nf\big](f_3-HS) \\ & -nH\tilde{S}(f_3-HS)^2+2\tilde{S}(1+c_0)(nH^2-c_2\tilde{S})\sum_{i,j,k}\lambda_i h_{ijk}^2 \\ & -n\big(\tilde{S}^2f_5-(f_3-HS)^3\big)\Big\}, \\ \xi = & nH^2(1+c_0)-\tilde{S}(c_1+(1+c_0)c_2). \end{aligned}$$

Adding (4.9) to (4.8), we obtain

$$
\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S}-2n-[5-t-(1-c_1)u]\frac{\tilde{S}}{n}\Big\}+[5-(2-c_1)u]A\\
&-[6+(1+2c_1)u]B-[2-(t-c_1)u]\tilde{S}f\\
\geq&\frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2+\frac{2(f_3-HS)}{\tilde{S}}\sum_{i,j,k}\lambda_i h_{ijk}^2[1-(c_0+1)u]+c_0ut(f_3-HS)^2\\
&+\sum_j (f_3)_j^2\Big\{\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\Big\}+g_2+g_3\\
=&\frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2\Big\{1-\frac{[1-(c_0+1)u]^2}{c_0ut}\Big\}\\
&+c_0ut\Big\{(f_3-HS)+\frac{1-(c_0+1)u}{c_0ut\tilde{S}}\sum_{i,j,k}\lambda_i h_{ijk}^2\Big\}^2\\
&+\sum_j (f_3)_j^2\Big\{\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\Big\}+g_2+g_3\\
\geq&\frac{1}{\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2\Big\{1-\frac{[u(c_0+1)-1]^2}{c_0ut}\Big\}\\
&+\sum_j (f_3)_j^2\Big\{\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\Big\}+g_2+g_3.
\end{aligned}
$$

Setting $\beta=\frac{\{(c_0+1)u-1\}^2}{c_0u}-t$, we choose c_0, u such that $\beta\geq 0$. By (3.6), the inequality above becomes

$$
\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S}-2n-[5-t-(1-c_1)u]\frac{\tilde{S}}{n}\Big\}+[5-(2-c_1)u]A\\
&-[6+(1+2c_1)u]B-[2-(t-c_1)u]\tilde{S}f\\
\geq&-\frac{\beta}{t\tilde{S}^2}(\sum_{i,j,k}\lambda_i h_{ijk}^2)^2+\sum_j (f_3)_j^2\Big\{\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\Big\}+g_2+g_3\\
\geq&-\beta\Big\{\frac{1}{3}(A+2B)-\frac{4}{27}\sum_j\frac{(n+2)(f_3)_j^2}{(n+2)\tilde{S}+2n\tilde{\lambda}_j^2}\Big\}\\
&+\sum_j (f_3)_j^2\Big\{\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\Big\}+g_2+g_3.
\end{aligned}
$$

Hence we have

$$\begin{aligned}
&t\tilde{S}^2\left\{\tilde{S}-2n-[5-t-(1-c_1)u]\frac{\tilde{S}}{n}\right\}+\left\{5-(2-c_1)u+\frac{\beta}{3}\right\}A\\
&-\left\{6+(1+2c_1)u-\frac{2\beta}{3}\right\}B-[2-(t-c_1)u]\tilde{S}f\\
&\geq\sum_j (f_3)_j^2\left\{\frac{4}{27}\cdot\frac{(n+2)\beta}{(n+2)\tilde{S}+2n\tilde{\lambda}_j^2}+\frac{(4\lambda_j+3nH)^2}{36t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\right\}\\
&+g_2+g_3\\
&=\sum_j (f_3)_j^2\left\{\frac{4}{27}\cdot\frac{(n+2)\beta}{(n+2)\tilde{S}+2n\tilde{\lambda}_j^2}+\frac{4\tilde{\lambda}_j^2}{9t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\right\}\\
&+g_2+g_4 \qquad (4.10)\\
&\geq\frac{1}{9\tilde{S}}\sum_j (f_3)_j^2\left\{4\sqrt{\frac{2(n+2)\beta}{3nt}}-\frac{2(n+2)}{nt}-3u(c_0+1)\right\}+g_2+g_4, \qquad (4.11)
\end{aligned}$$

where

$$g_4=g_3+\frac{4+3n}{9t\tilde{S}^2}\sum_j (f_3)_j^2\left(2\lambda_j+(\frac{3}{4}n-1)H\right)H.$$

5. Proof of the main theorem. Making use of the lemmas and estimates above, we shall prove our main theorem. We first show $t\geq\frac{1}{4}$. Suppose that $t\in[\frac{1}{9},\frac{1}{4})$. From (4.10) and $\frac{(n+2)\beta}{(n+2)\tilde{S}+2n\tilde{\lambda}_j^2}\geq\frac{\beta}{\tilde{S}+2\tilde{\lambda}_j^2}$, we obtain

$$\begin{aligned}
&t\tilde{S}^2\left\{\tilde{S}-2n-[5-t-(1-c_1)u]\frac{\tilde{S}}{n}\right\}+\left\{5-(2-c_1)u+\frac{\beta}{3}\right\}A\\
&-\left\{6+(1+2c_1)u-\frac{2\beta}{3}\right\}B-[2-(t-c_1)u]\tilde{S}f\\
&\geq\sum_j (f_3)_j^2\left\{\frac{4}{27}\cdot\frac{\beta}{\tilde{S}+2\tilde{\lambda}_j^2}+\frac{4\tilde{\lambda}_j^2}{9t\tilde{S}^2}-\frac{u(c_0+1)}{3\tilde{S}}\right\}+g_2+g_4\\
&\geq\frac{1}{9\tilde{S}}\sum_j (f_3)_j^2\left\{8\sqrt{\frac{\beta}{6t}}-\frac{2}{t}-3u(c_0+1)\right\}+g_2+g_4.
\end{aligned}$$

Taking $u=\frac{13}{3}$ and $\beta=14$ in the inequality above, we get $14=\frac{\{\frac{13}{3}(c_0+1)-1\}^2}{\frac{13}{3}c_0}-t$ and

$$\begin{aligned}
&t\tilde{S}^2\left\{\tilde{S}-2n-[5-t-\frac{13}{3}(1-c_1)]\frac{\tilde{S}}{n}\right\}+\left\{5-\frac{13}{3}(2-c_1)+\frac{14}{3}\right\}A\\
&-\left\{6+\frac{13}{3}(1+2c_1)-\frac{28}{3}\right\}B-\left\{2-\frac{13}{3}(t-c_1)\right\}\tilde{S}f\\
&\geq\frac{1}{\tilde{S}}\sum_j(\sum_i\lambda_i^2h_{iij})^2\left\{8\sqrt{\frac{7}{3t}}-\frac{2}{t}-13(c_0+1)\right\}+g_2+g_4.
\end{aligned}$$

Set

$$\frac{13}{3}\theta_1(t)=8\sqrt{\frac{7}{3t}}-\frac{2}{t}-13(c_0+1).$$

Because $\frac{d\theta_1(t)}{dt} < 0$ for $\frac{1}{9} < t < \frac{1}{4}$, $\theta_1(t) \geq \theta_1(\frac{1}{4}) \geq -\frac{3}{5}$. It follows from (3.5) and $t < \frac{1}{4}$ that

$$\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{13}{3}c_1 - \frac{2}{3}\big)\frac{\tilde{S}}{n}\Big\} \\
\geq& -\big(1 + \frac{13}{3}c_1\big)A + \big(1 + \frac{26}{3}c_1\big)B + \big(2 + \frac{13}{3}c_1 - 4t\frac{13}{10}(1 + \frac{\alpha}{3})\big)\tilde{S}f + g_2 + g_4 \\
\geq& -\big(1 + \frac{13}{3}c_1\big)A + \big(1 + \frac{26}{3}c_1\big)B + \big(2 + \frac{13}{3}c_1 - \frac{13}{10}(1 + \frac{\alpha}{3})\big)\tilde{S}f + g_2 + g_4 \\
=& -\big(1 + \frac{13}{3}c_1\big)A + \big(1 + \frac{26}{3}c_1\big)B + \big(\frac{7}{10} + \frac{13}{3}(c_1 - \frac{\alpha}{10})\big)\tilde{S}f + g_2 + g_4.
\end{aligned}$$

Taking $c_1 = \frac{\alpha}{10}$ in above inequality, we see $c_1 \in [0, \frac{1}{10}]$ from the definition of α. By (3.3), (3.4) and $A + 2B = \frac{1}{3}\sum_{i,j,k}(\lambda_i + \lambda_j + \lambda_k)^2 h_{ijk}^2 \geq 0$, we have

$$\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{2}{3} - \frac{13}{30}\alpha\big)\frac{\tilde{S}}{n}\Big\} \\
\geq& -\big(1 + \frac{13}{30}\alpha\big)A + \big(1 + \frac{26}{30}\alpha\big)B + \frac{7}{10}\tilde{S}f + g_2 + g_4 \\
=& -\big(1 + \frac{13}{30}\alpha \cdot \frac{4}{3}\big)(A - B) + \frac{13}{30}\alpha \cdot \frac{1}{3}(A + 2B) + \frac{7}{10}\tilde{S}f + g_2 + g_4 \\
\geq& -\big(1 + \frac{26}{45}\alpha\big)(A - B) + \frac{7}{10}\tilde{S}f + g_2 + g_4 \\
\geq& -\big(1 + \frac{26}{45}\alpha\big)\frac{1-\alpha}{3}(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 t\tilde{S}^2 + \frac{7}{10}\tilde{S}\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2}\big(\tilde{\lambda}_1\tilde{\lambda}_2 + \frac{\tilde{S}}{n}\big)^2 + g_2 + g_4.
\end{aligned}$$

By $\sum_i \tilde{\lambda}_i = 0$ and $\sum_i \tilde{\lambda}_i^2 = \tilde{S}$, we get $\tilde{\lambda}_1\tilde{\lambda}_2 \leq 0$ and $\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2} \geq 1$. Since $-\big(1 + \frac{26}{45}\alpha\big)(1 - \alpha) = \frac{26}{45}(\alpha - 1)\big(\alpha + \frac{45}{26}\big) \geq -1$ $(0 \leq \alpha \leq 1)$, we have

$$\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{2}{3} - \frac{13}{30}\alpha\big)\frac{\tilde{S}}{n}\Big\} \\
\geq& -\frac{1}{3}(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2)t\tilde{S}^2 + \frac{7}{10}\tilde{S}\big(\tilde{\lambda}_1\tilde{\lambda}_2 + \frac{\tilde{S}}{n}\big)^2 + g_2 + g_4 \\
\geq& -\frac{1}{3}\big(\tilde{S} + 2x\tilde{S}\big)t\tilde{S}^2 + \frac{2}{3}\tilde{S}^3\big(\frac{1}{n} - x\big)^2 + g_2 + g_4 \qquad \big(x := -\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\tilde{S}}\big) \\
=& -\frac{1}{3}t\tilde{S}^3\big(1 + \frac{2}{n}\big) + \frac{2}{3}\tilde{S}^3\Big[\big(x - \frac{1}{n}\big)^2 - \big(x - \frac{1}{n}\big)t\Big] + g_2 + g_4 \\
\geq& -\frac{1}{3}t\tilde{S}^3\big(1 + \frac{2}{n}\big) - \frac{1}{6}t^2\tilde{S}^3 + g_2 + g_4. \qquad (5.1)
\end{aligned}$$

On the other hand,

$$t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{2}{3} - \frac{13}{30}\alpha\big)\frac{\tilde{S}}{n}\Big\} \leq t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{2}{3}\big)\frac{\tilde{S}}{n}\Big\}.$$

This implies

$$t\tilde{S}^2\Big\{\tilde{S} - 2n + \big(t - \frac{2}{3}\big)\frac{\tilde{S}}{n} + \frac{1}{3}\big(1 + \frac{2}{n}\big)\tilde{S} + \frac{1}{6}t\tilde{S}\Big\} \geq g_2 + g_4.$$

It follows from (3.1) that

$$n = \tilde{S}(1-t) + \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}}. \tag{5.2}$$

Hence

$$\begin{aligned} t\tilde{S}^2\Big\{&\tilde{S}\big[1 - 2(1-t) + (t - \frac{2}{3})\frac{1}{n} + \frac{1}{3}(1 + \frac{2}{n}) + \frac{1}{6}t\big] \\ &- 2\frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}}\Big\} \\ =& t\tilde{S}^2\Big\{\tilde{S}\big[(\frac{13}{6} + \frac{1}{n})t - \frac{2}{3}\big] - 2\frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}}\Big\} \geq g_2 + g_4, \end{aligned}$$

i.e.,

$$(\frac{13}{6} + \frac{1}{n})t - \frac{2}{3} \geq \frac{g_2 + g_4}{t\tilde{S}^3} + 2\frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2}.$$

Using $\big(\frac{13}{6} + \frac{1}{n}\big)t - \frac{2}{3} \leq \big(\frac{13}{6} + \frac{1}{4}\big)t - \frac{2}{3} = \frac{29}{12}\big(t - \frac{8}{29}\big)$, we obtain

$$t \geq \frac{8}{29} + \frac{12}{29}\Big\{\frac{g_2 + g_4}{t\tilde{S}^3} + 2\frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2}\Big\}.$$

Therefore, there exists a positive constant $C_1(n)$ depending only on n such that if $|H| < C_1(n)$, then

$$\left|\frac{g_2 + g_4}{t\tilde{S}^3} + 2\frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2}\right| \leq \frac{1}{16}$$

and $t \geq \frac{1}{4}$. This contradicts with $t \in [\frac{1}{9}, \frac{1}{4})$. So we have $t \geq \frac{1}{4}$ when $|H| < C_1(n)$.

Secondly we prove that $t \geq \frac{3}{10} + \varepsilon(n)$, where

$$\varepsilon(n) = \begin{cases} 0.0026, & \text{n>6;} \\ 0.0008, & \text{n=6;} \\ 0.0016, & \text{n=5;} \\ 0.0023, & \text{n=4.} \end{cases}$$

Suppose that $t \in [\frac{1}{4}, \frac{3}{10} + \varepsilon(n)]$. In (4.11), take $u = \frac{17}{5}$, $c_0 = \frac{6}{17}$ and let

$$5 - 2u + c_1 u + \frac{\beta}{3} = 6 + u + 2c_1 u - \frac{2\beta}{3}.$$

This together with the definition of β gives $\beta = \frac{54}{5} - t$ and $c_1 = -\frac{2}{17} - \frac{5}{17}t$.

Thus (4.11) can be written as

$$\begin{aligned} &t\tilde{S}^2\Big\{\tilde{S} - 2n - (\frac{6}{5} - 2t)\frac{\tilde{S}}{n}\Big\} \\ \geq& -(\frac{7}{5} - \frac{4t}{3})(A - B) + (\frac{8}{5} - \frac{22t}{5})\tilde{S}f + \frac{\theta_2(t)}{\tilde{S}}\sum_j(\sum_i \lambda_i^2 h_{iij})^2 + g_2 + g_4, \end{aligned}$$

where

$$\theta_2(t) = 4\sqrt{\frac{2(n+2)}{3nt}\big(\frac{54}{5} - t\big)} - \frac{2(n+2)}{nt} - \frac{69}{5}.$$

It is easy to see that $\frac{d\theta_2(t)}{dt} < 0$, for $\frac{1}{4} \leq t \leq 0.3026$. Hence $\theta_2(t)$ is decreasing for $t \in [\frac{1}{4}, 0.3026]$. We consider the following cases.

Case(a) $n > 6$. Since $\theta_2(t) \leq \theta_2(\frac{1}{4}) < 0$ holds for $n > 6$, by (3.5), we have

$$\begin{aligned} &t\tilde{S}^2\Big\{\tilde{S} - 2n - \big(\frac{6}{5} - 2t\big)\frac{\tilde{S}}{n}\Big\} \\ \geq& -\big(\frac{7}{5} - \frac{4t}{3}\big)(A-B) + \big(a_2(t) - b_2(t)\alpha\big)t\tilde{S}f + g_2 + g_4, \end{aligned}$$

where

$$a_2(t) = \frac{8-22t}{5t} + \frac{1}{3}\theta_2(t),\ b_2(t) = -\frac{2}{3}\theta_2(t).$$

When $a_2(0.3026) - b_2(0.3026)\alpha > 0$, from the monotonicity of $\theta_2(t)$ for $t \in [\frac{1}{4}, 0.3026]$, we see that $a_2(t) - b_2(t)\alpha \geq a_2(0.3026) - b_2(0.3026)\alpha > 0$. Using (3.3), (3.4) and $x = -\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\tilde{S}} \in [0, \frac{1}{2}]$ as in (5.1), we have

$$\begin{aligned} &\tilde{S} - 2n - \big(\frac{6}{5} - 2t\big)\frac{\tilde{S}}{n} \\ \geq& -\big(\frac{7}{5} - \frac{4t}{3}\big)\frac{1-\alpha}{3}(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 + \big(a_2(t) - b_2(t)\alpha\big)\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2}\Big(\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\tilde{S}} + \frac{1}{n}\Big)^2\tilde{S} \\ &+ \frac{g_2 + g_4}{t\tilde{S}^2} \\ \geq& -\big(\frac{7}{15} - \frac{4t}{9}\big)(1-\alpha)(1+2x)\tilde{S} + \big(a_2(t) - b_2(t)\alpha\big)(1+2x)\big(\frac{1}{n} - x\big)^2\tilde{S} \\ &+ \frac{g_2 + g_4}{t\tilde{S}^2} \\ =&\big(\rho(x,n) + \delta(x,n)\alpha\big)\tilde{S} + \frac{g_2 + g_4}{t\tilde{S}^2}, \end{aligned}$$

where

$$\rho(x,n) = -\big(\frac{7}{15} - \frac{4t}{9}\big)(1+2x) + a_2(t)(1+2x)\big(\frac{1}{n} - x\big)^2,$$

$$\delta(x,n) = \big(\frac{7}{15} - \frac{4t}{9}\big)(1+2x) - b_2(t)(1+2x)\big(\frac{1}{n} - x\big)^2.$$

By the monotonicity of $\rho(x,n)$ and $\delta(x,n)$ (see [16]), we get

$$\delta(x,n) \geq \min\big\{\delta(0,n), \delta(\frac{1}{2}, n)\big\} > 0,$$

$$\rho(x,n) + \big(\frac{6}{5} - 2t\big)\frac{1}{n} \geq \min\Big\{\rho(x,n) + \big(\frac{6}{5} - 2t\big)\frac{1}{n}\Big\}\Big|_{t=0.3026} = -0.3943.$$

So

$$\tilde{S} - 2n \geq -0.3943\tilde{S} + \frac{g_2 + g_4}{t\tilde{S}^2}.$$

It follows from (5.2) that

$$t \geq 0.3028 + \frac{g_2 + g_4}{2t\tilde{S}^3} + \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2}. \tag{5.3}$$

When $a_2(0.3026) - b_2(0.3026)\alpha \leq 0$, i.e., $\frac{a_2(0.3026)}{b_2(0.3026)} \leq \alpha \leq 1$, it follows from (3.4), (3.7) and $a_2(t) - b_2(t)\alpha \geq a_2(0.3026) - b_2(0.3026)\alpha$ that

$$\begin{aligned}
&t\tilde{S}^2\Big\{\tilde{S} - 2n - (\frac{6}{5} - 2t)\frac{\tilde{S}}{n}\Big\}\\
\geq& -(\frac{7}{15} - \frac{4t}{9})(1-\alpha)(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 t\tilde{S}^2\\
&+ \big(a_2(0.3026) - b_2(0.3026)\alpha\big)\big[\frac{(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2}{4} - \frac{\tilde{S}}{n}\big]t\tilde{S}^2 + g_2 + g_4.
\end{aligned}$$

Noting that $(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 \leq 2\tilde{S}$, we get

$$\begin{aligned}
&\tilde{S} - 2n - (\frac{6}{5} - 2t)\frac{\tilde{S}}{n}\\
\geq& \Big\{-2(\frac{7}{15} - \frac{4t}{9}) + \frac{n-2}{2n}a_2(0.3026) + \big[2(\frac{7}{15} - \frac{4t}{9}) - \frac{n-2}{2n}b_2(0.3026)\big]\alpha\Big\}\tilde{S}\\
&+ \frac{g_2 + g_4}{t\tilde{S}^2}.
\end{aligned}$$

Since $b_2(0.3026) < 0.7821$ and the assumption of $t < 0.3026$, we have $2\big(\frac{7}{15} - \frac{4t}{9}\big) - \frac{n-2}{2n}b_2(0.3026) > 0$. Using $\alpha \geq \frac{a_2(0.3026)}{b_2(0.3026)}$, we obtain

$$\tilde{S} - 2n - (\frac{6}{5} - 2t)\frac{\tilde{S}}{n} \geq -2(\frac{7}{15} - \frac{4t}{9})\big[1 - \frac{a_2(0.3026)}{b_2(0.3026)}\big]\tilde{S} + \frac{g_2 + g_4}{t\tilde{S}^2}.$$

By a direct computation, one see that $\frac{a_2(0.3026)}{b_2(0.3026)} \geq 0.6348$. Then

$$\begin{aligned}
\tilde{S} - 2n - (\frac{6}{5} - 2t)\frac{\tilde{S}}{n} \geq& -2(\frac{7}{15} - \frac{4t}{9})0.3652\tilde{S} + \frac{g_2 + g_4}{t\tilde{S}^2}\\
=& -\frac{0.7303}{45}\tilde{S} - \frac{2.9215}{9}n + g_5,
\end{aligned} \tag{5.4}$$

where

$$g_5 = \frac{g_2 + g_4}{t\tilde{S}^2} + \frac{2.9215nH[H\tilde{S} - (f_3 - HS)]}{9\tilde{S}}.$$

For $t \in [\frac{1}{4}, 0.3026]$, we have $\big(\frac{6}{5} - 2t\big)\frac{\tilde{S}}{n} > 0$. It is seen from (5.4) that

$$\tilde{S} - 2n > -\frac{0.7303}{45}\tilde{S} - \frac{2.9215}{9}n + g_5.$$

Therefore, using (5.2), we get

$$t > 0.3934 + \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2} + \frac{0.5968 g_5}{\tilde{S}}. \tag{5.5}$$

From (5.3) and (5.5), we see that there is a positive constant $C_2(n)$ $(n > 6)$ such that if $|H| < C_2(n)$, then

$$\left| \frac{g_2 + g_4}{2t\tilde{S}^3} + \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2} \right| < 0.0002,$$

and

$$\left| \frac{nH[H\tilde{S} - (f_3 - HS)]}{\tilde{S}^2} + \frac{0.5968 g_5}{\tilde{S}} \right| < 0.0908,$$

which imply $t > 0.3026$. This makes a contradiction. Hence we have $t > 0.3026$ when $|H| < C_2(n)$ $(n > 6)$.

Case(b) $4 \le n \le 6$. We take $u = \frac{7}{2}$, $c_0 = \frac{61 - \sqrt{1221}}{70}$ in (4.11), and get

$$\begin{aligned} &t\tilde{S}^2 \Big\{ \tilde{S} - 2n - \big(\frac{11}{10} - 2t\big)\frac{\tilde{S}}{n} \Big\} \\ &\ge -\big(\frac{13}{10} - \frac{4t}{3}\big)(A - B) + \big(\frac{8}{5} - \frac{9t}{2}\big)\tilde{S}f + \frac{\theta_3(t)}{\tilde{S}} \sum_j \big(\sum_i \lambda_i^2 h_{iij}\big)^2 + g_2 + g_4, \end{aligned}$$

where

$$\theta_3(t) = 4\sqrt{\frac{2(n+2)}{3nt}\big(\frac{111}{10} - t\big)} - \frac{2(n+2)}{nt} - \frac{393 - 3\sqrt{1221}}{20}.$$

One can see that $\theta_3(t)$ is decreasing for $t \in [\frac{1}{4}, 0.3023]$ and $\theta_3(t) \le \theta_3(\frac{1}{4}) < 0$ holds for $4 \le n \le 6$. It follows from (3.5) that

$$\begin{aligned} &t\tilde{S}^2 \Big\{ \tilde{S} - 2n - \big(\frac{11}{10} - 2t\big)\frac{\tilde{S}}{n} \Big\} \\ &\ge -\big(\frac{13}{10} - \frac{4t}{3}\big)(A - B) + \big(a_3(t) - b_3(t)\alpha\big)t\tilde{S}f + g_2 + g_4, \end{aligned}$$

where

$$a_3(t) = \frac{8}{5t} - \frac{9}{2} + \frac{1}{3}\theta_3(t), \ \ b_3(t) = -\frac{2}{3}\theta_3(t).$$

For $4 \le n \le 6$ and $\frac{1}{4} \le t \le 0.3061$, we have $a_3(t) > b_3(t) > 0$. Thus $a_3(t) - b_3(t)\alpha \ge a_3(t) - b_3(t) > 0$. By (3.3) and (3.4), we repeat the computational procedure in Case(a) for $n \in \{4, 5, 6\}$, respectively. From the assumption of $t \le \frac{3}{10} + \varepsilon(n)$, we obtain

$$\tilde{S} - 2n > -\big(\frac{4}{10} - 2\varepsilon'(n)\big)\tilde{S} + \frac{g_2 + g_4}{t\tilde{S}^2},$$

where

$$\varepsilon'(n)=\begin{cases}0.0009, & \text{n=6;}\\ 0.0017, & \text{n=5;}\\ 0.0025, & \text{n=4.}\end{cases}$$

It follows from (5.2) that

$$t>\frac{3}{10}+\varepsilon'(n)+\frac{g_2+g_4}{2t\tilde{S}^3}+\frac{nH[H\tilde{S}-(f_3-HS)]}{\tilde{S}^2}.$$

Then there exists a positive constant $C_2(n)$ (n=4,5,6) such that $t>\frac{3}{10}+\varepsilon(n)$ when $|H|<C_2(n)$. This is a contradiction. Hence we have $t>\frac{3}{10}+\varepsilon(n)$ for $n=4,5,6$.

It is known in [12] that

$$\tilde{S}+\frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{\tilde{S}}-n-nH^2\geq t\tilde{S}.$$

So

$$\begin{aligned}\tilde{S}\geq& n\sigma+\frac{nH^2}{2(n-1)}\sigma\left((n-2)^2\sigma+2(n-1)\right)\\ &-\frac{n(n-2)H}{2(n-1)}\sigma^2\sqrt{(n-2)^2H^2+\frac{4(H^2+1)(n-1)}{\sigma}}\\ \geq& n\sigma+nH^2\left(\frac{n^2}{2(n-1)}-1\right)-\frac{n(n-2)H}{n-1}\sqrt{n^2H^2+4(n-1)},\end{aligned}$$

where $\sigma=\frac{1}{1-t}$, and $1<\sigma<2$ for $0<t<\frac{1}{2}$. Then

$$\begin{aligned}S-\beta(n,H)>&nH^2-\beta(n,H)+n\sigma+nH^2\left(\frac{n^2}{2(n-1)}-1\right)\\ &-\frac{n(n-2)H}{n-1}\sqrt{n^2H^2+4(n-1)}\\ =&nH^2-n-\frac{n^3H^2}{2(n-1)}+n\sigma+nH^2\left(\frac{n^2}{2(n-1)}-1\right)\\ &-\frac{3n(n-2)H}{2(n-1)}\sqrt{n^2H^2+4(n-1)}\\ =&n(\sigma-1)-\frac{3n(n-2)H}{2(n-1)}\sqrt{n^2H^2+4(n-1)}.\end{aligned}$$

If $H\leq\frac{2}{n}$ and $t>\frac{3}{10}+\varepsilon(n)$, we have

$$S-\beta(n,H)>\frac{nt}{1-t}-\frac{3(n-2)n^{\frac{3}{2}}}{n-1}H>\frac{3n}{7}+\frac{100n}{49}\varepsilon(n)-\frac{3(n-2)n^{\frac{3}{2}}}{n-1}H.$$

Set $C(n)=\min\left\{C_1(n),C_2(n),\frac{2}{n},\frac{100(n-1)}{147(n-2)\sqrt{n}}\varepsilon(n)\right\}$ and assume $|H|<C(n)$. We conclude that if $S>\beta(n,H)$, then $S>\beta(n,H)+\frac{3}{7}n$. This completes the proof of our main theorem.

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